

Chapter 2

The Mirror Problem

2.1 Introduction

Mirror, mirror on the wall,
Who is the fairest one of all?

asks the jealous queen in the fairy tale “Snow White” and gets an answer she doesn’t like.

Since time immemorial, looking into a mirror has been very popular. An amusing question which keeps popping up at regular intervals in magazines is the following:

How large does a mirror have to be so that a person can see themselves completely in it?

Here are some typical answers:

1. It depends on how far I stand from the mirror.
2. The mirror has to be as tall as I am.

These seem to make sense; but, unfortunately, both answers are wrong.

2.2 The Mirror Problem for Individuals

Let’s look first at a person who wants to buy a mirror.

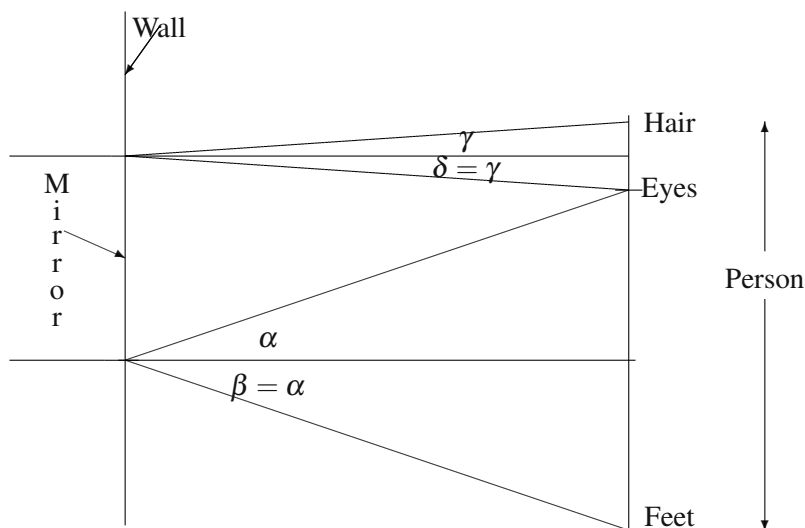


Fig. 2.1 A person standing in front of a mirror has only to look down halfway to their feet and look up halfway to their hair in order to see themselves completely; that's why the mirror – if attached correctly – has to be only half as tall as the person.

The image in [Figure 2.1](#) will help explain the situation.

According to the law of reflection, the angle of incidence α equals the angle of reflection β and, likewise, the angle of incidence γ equals the angle of reflection δ . Now, if I look into a mirror hanging vertically on the wall directly opposite me; and I want to see my feet, then I have to lower my eyes only halfway between my eyes and the floor. My line of sight hits the mirror at the angle α . The angle of reflection β has the same size and automatically directs my line of sight towards my feet.

The same thing happens if I want to look at my hair. My line of sight needs to only be lifted halfway between my eyes and my hair, allowing me to see that my gray hair doesn't get any darker in the mirror either.

Thus, we can summarize this as follows:

1. Seeing yourself completely in a mirror doesn't depend on the distance you are from the mirror.

2. The mirror needs to be exactly half as tall as you are.

Some care needs to be exercised when hanging up the mirror. As mathematicians, we can summarize this in an algorithm.

Algorithm for Attaching a mirror for an Individual

1. The mirror has to hang on a vertical wall.
2. The mirror has to be half as tall as I am.
3. The mirror has to be hung in such a way that I can just see my hair at the upper edge of the mirror. The upper edge of the mirror thus has to be mounted at a height equal to

my eye level + half the distance to my eyes — half the distance to
my hair

2.3 The Mirror Problem for Groups

If several persons within a group, for example, a family or a bowling club, all want to look at themselves in the same mirror from top to toe, then we have to plan a little bit more carefully.

If you take the tallest person in the group, then you might be able to align the mirror at its upper edge with this person. The bottom edge is determined by half the height of this person. But if the mirror is actually attached this way, then a shorter person won't be able to see their feet. So if we don't want to attach the mirror to a movable device so that it can be pulled up and down like a hanging lamp, then we have to give the sales clerk some more money to buy a larger mirror.

The tallest person determines where the mirror is mounted at the top.

The length downwards from this point is defined by the shortest person who is actually the person with the lowest eye level. At the bottom, the mirror has to reach the halfway point of this person's eye level. This results in the total length of the mirror being as follows (figure 2.2:

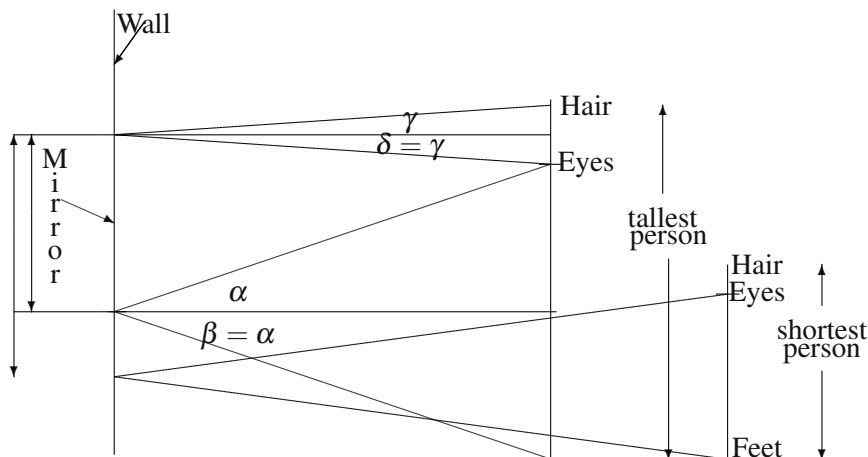


Fig. 2.2 Here, we've placed the shortest person in the group next to the tallest person. We can see that this person has to look down beyond the bottom edge of the mirror so that they can see themselves completely. Thus, the mirror has to be taller. The arrows on the left indicate the length of the mirror for an individual and, further left, the length of the mirror for a group.

Total Length of Group's Mirror

eye level + half the distance to the hair to eye level of the tallest person
 – halfway to the eye level of the smallest person.

Algorithm for Attaching a Mirror for a Group

1. The mirror has to hang on a vertical wall.
2. The mirror has to be attached in such a way that the tallest member of the group is able to just see their own hair in the mirror.
3. Thus, the upper edge of the mirror has to be mounted at a height equal to

eye level + half the distance to the eyes – half the distance to the hair of the tallest person

Well, this was just a rehearsal, which has hopefully triggered our interest in the following, more difficult task.

2.4 The Problem

The question we're now going to ask is a more puzzling one:

The Mirror Problem

If I shake my right hand, my mirror image shakes its left hand. But if I shake my head, my mirror image doesn't shake its feet. So why does a mirror exchange right and left, but not top and bottom?

We haven't found anyone yet who didn't find this question interesting. But what is the answer? To find it, we have to examine the mirror a bit more precisely; actually, mathematically.

Those of my dear readers who are not familiar with vectors in a plane might want to skip the following section. We'll summarize the result in Section 2.6. That's where those readers can log on again, because the result is understandable even to people who are mathematical amateurs.

2.5 The Mirror Problem Expressed Mathematically

We now enter the plane to look at vectors.

So that we don't have to carry too much of a burden with us, which, in turn, could cause us to lose sight of the essentials, we'll choose a situation where the mirror's axis is the y axis. The diagram in [Figure 2.3](#) shows that a random point (x,y) , which we identify with its position vector $\mathbf{a} = (x,y)$, is depicted in the mirror at a point $\mathbf{a}^* = (-x,y)$.

Such a representation has very simple properties. If we take a multiple of the vector \mathbf{a} , we can form the image of the vector \mathbf{a} first, followed by multiplication of the image vector, which we can express mathematically as follows if we abbreviate "reflection" with the symbol RE :

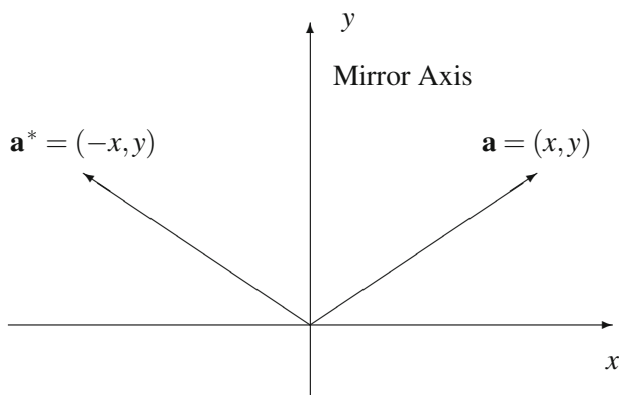


Fig. 2.3 Reflection at the y Axis

$$RE(k \cdot \mathbf{a}) = k \cdot RE(\mathbf{a}).$$

We can play the same little game when we add two vectors together. We can obtain the image of the sum if we first form the images of the two vectors separately and then add the images together, which is expressed mathematically as follows:

$$RE(\mathbf{a} + \mathbf{b}) = RE(\mathbf{a}) + RE(\mathbf{b}).$$

Such representations are called “linear.”

Now, math teaches us that such a representation can best be described by a matrix. A matrix is a square field into which we enter numbers. Since we’re looking at vectors in the plane, we take a 2×2 field. This field describes the representation. In order to characterize it as a mathematical rule, we enclose it in parentheses. This is done as follows:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

In order to extract the mapping rule, we look at the two unit vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ and image them:

$$\mathbf{e}_1 = (1, 0) \rightarrow (-1, 0), \quad \mathbf{e}_2 = (0, 1) \rightarrow (0, 1).$$

We've written this down in a relaxed manner. But take a look again at the drawing in [Figure 2.3](#). The first unit vector $\mathbf{e}_1 = (1, 0)$ points exactly to the right from the origin. If it's mirrored in the y axis, then its image vector points exactly to the left from the origin. That's the vector $-\mathbf{e}_1 = -(1, 0) = (-1, 0)$. The second unit vector, $\mathbf{e}_2 = (0, 1)$ points vertically upward. It is located directly on the y axis. That's why its reflection stays where it is.

Now comes the rule for the calculation scheme; that is, the matrix of the representation.

We enter the image vectors into a (2×2) matrix as columns:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.1)$$

Thus, the representation is written in the form of a matrix as

$$\mathbf{x} \rightarrow \mathbf{x}^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

Now, we have to introduce another term, which is, in principle, quite difficult to explain. Since we want to linger only in this plane, though, we've only got to consider 2×2 matrices. Our new term is in fact quite simple.

Definition 2.1 *By the determinant of a 2×2 matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we mean the expression

$$\det(A) = a \cdot d - b \cdot c.$$

That's just a number that's a little bit hard to calculate. Let's use an example:

$$A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} \Rightarrow \det(A) = 2 \cdot 1 - 4 \cdot (-3) = 14.$$

The next example shows that the determinant of a matrix can become negative:

$$B = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \Rightarrow \det(A) = 2 \cdot 1 - 4 \cdot 3 = -10.$$

The sign of the determinant is actually the distinctive feature that we are interested in, because mathematicians have found the following:

Theorem 2.1 *Linear representations whose corresponding matrices have a positive determinant retain orientations whereas linear representations whose matrices have a negative determinant reverse orientations.*

Well, it's slowly becoming clearer. It's all about orientation.

A brief comment for purists:

All of this shouldn't anger serious mathematicians. We have just given some hints here and there; those who wish to get more information are advised to look in specialist textbooks on linear algebra.

2.6 Results of Analysis of the Mirror Problem

First, we note that there are significant differences between the pairs "right and left" and "top and bottom."

The terms "right" and "left" have a significance which depends on the respective person. What is "right" for me is "left" for my counterpart, and vice versa. It is my personal orientation that determines where right and left are.

In contrast, the terms "top" and "bottom" are the same for everyone, at least for those who are in our immediate vicinity, for example all Americans. We all look up towards the clouds when someone talks to us from above.

And here's an important insight:

For our reflection matrix A in equation (2.1), we get

$$\det(A) = (-1) \cdot 1 - 0 \cdot 0 = -1.$$

So the determinant is negative! And this means:

The orientation is reversed when mirrored.

The best way to grasp this concept is by looking at a circle and its mirror image. When we move a pen along a circular line in a clockwise direction from the top, our mirror image moves the pen in a counterclockwise direction. If I were to show my mirror image a clock face, then the clock would always run in the wrong direction, regardless of where I was standing or where the mirror was located. That's what happens with orientation.

The mirror doesn't actually swap right and left; instead, it reverses the orientation. Orientation is thus associated directly with persons. My orientation tells me where to find the right side. The mirror reverses this image, which explains why my mirror image sees this quite differently and calls it left.

"Top" and "bottom," however, are terms of an entirely different nature. "Top" and "bottom" are the same for everyone in our vicinity. They are objective terms, which are not associated with our personal orientation. These terms are, thus, not swapped by a mirror. In fact, this applies to the terms "east" and "west" as well. If I point towards the east, my mirror image also points towards the east in the mirror realm. And if I ask my mirror image, when I am standing in Berlin, to point towards the Eiffel Tower, then my mirror image will point in the mirror in exactly the same direction as I am pointing: towards Paris and not towards Moscow.



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