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	<b><math>\{\mathbf{Ax} = \mathbf{y}   \mathbf{A} \in \mathbb{R}^{n \times m}, \mathbf{y} \in \mathcal{R}(\mathbf{A}) \sim \text{rk } \mathbf{A} = n = \dim \mathbb{Y}\}</math>, MINOS</b>	<b>1</b>
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