

Chapter 2

Kernel Density Estimation and Local Time

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Abstract In this paper we develop an asymptotic theory for some regression models involving standard Brownian motion and standard Brownian sheet.

2.1 Introduction

The motivation of our work comes from the econometric theory. Consider a regression model of the form

$$y_i = f(x_i) + u_i, \quad i \geq 0 \quad (2.1)$$

where $(u_i)_{i \geq 0}$ is the “error” and $(x_i)_{i \geq 0}$ is the regressor. The purpose is to estimate the function f based on the observation of the random variables y_i , $i \geq 0$. The conventional kernel estimate of $f(x)$ is

$$\hat{f}(x) = \frac{\sum_{i=0}^n K_h(x_i - x)y_i}{\sum_{i=0}^n K_h(x_i - x)}$$

where K is a nonnegative real kernel function satisfying $\int_{\mathbb{R}} K^2(y)dy = 1$ and $\int_{\mathbb{R}} yK(y)dy = 0$ and $K_h(s) = \frac{1}{h}K(\frac{s}{h})$. The bandwidth parameter $h \equiv h_n$ satisfies $h_n \rightarrow 0$ as $n \rightarrow \infty$. We will choose in our work $h_n = n^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$. The asymptotic behavior of the estimator $\hat{f}$ is usually related to the behavior of the sequence

$$V_n = \sum_{i=1}^n K_h(x_i - x)u_i.$$

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The limit in distribution as $n \rightarrow \infty$ of the sequence V_n has been widely studied in the literature in various situations. We refer, among others, to [5] and [6] for the case where x_t is a recurrent Markov chain, to [12] for the case where x_t is a partial sum of a general linear process, and [13] for a more general situation. See also [9] or [10]. An important assumption in the main part of the above references is the fact that u_i is a martingale difference sequence. In our work we will consider the following situation: first the error u_i is chosen to be $u_i = W_{i+1} - W_i$ for every $i \geq 0$, where $(W_t)_{t \geq 0}$ denotes a standard Wiener process and $x_i = W_i$ for $i \geq 0$. Note that in this case, although for every i the random variables u_i and x_i are independent, there is not global independence between the regressor $(x_i)_{i \geq 0}$ and $(u_i)_{i \geq 0}$. However, this case has been already treated in previous works (see e.g. [12, 13]). See also [2] for models related with fractional Brownian motion. In this case, the sequence V_n reduces to (we will also restrict to the case $x = 0$ because the estimation part is not addressed in this paper)

$$S_n = \sum_{i=0}^{n-1} K(n^\alpha W_i) (W_{i+1} - W_i). \quad (2.2)$$

The second case we consider concerns a two-parameter model:

$$y_{i,j} = f(x_{i,j}) + e_{i,j}, \quad i, j \geq 0 \quad (2.3)$$

where $e_{i,j} = W_{i+1,j+1}^{(2)} - W_{i+1,j}^{(2)} - W_{i,j+1}^{(2)} + W_{i,j}^{(2)}$ are the rectangular increments of a Wiener sheet $W^{(2)}$ (see Sect. 2 for the definition of the Wiener sheet). This case seems to be new in the literature. But in this situation, because of the complexity of the stochastic calculus for two-parameter processes, we will restrict ourselves to case when the regressor $x_{i,j}$ is independent by the error $u_{i,j}$. That is, we assume that $x_{i,j} = W_{i,j}^{(1)}$ where $W^{(1)}$ is a Wiener sheet independent by $W^{(2)}$. The model (2.3) leads to the study of the sequence

$$T_n = \sum_{i,j=0}^{n-1} K\left(n^\alpha W_{i,j}^{(1)}\right) \left(W_{i+1,j+1}^{(2)} - W_{i+1,j}^{(2)} - W_{i,j+1}^{(2)} + W_{i,j}^{(2)}\right).$$

We will assume that the kernel K is the standard Gaussian kernel

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \quad (2.4)$$

The limits in distribution of S_n and T_n will be $c_1 \beta_{L^W(1,0)}$ and $c_2 \beta_{L^{W^{(1)}}(\underline{1},0)}$ respectively, where L^W and $L^{W^{(1)}}$ denote the local time of W and $W^{(1)}$ respectively, β is a Brownian motion independent by W and $W^{(1)}$ and c_1, c_2 are explicit positive constants.

2.2 The One Parameter Case

Let $(W_t)_{t \geq 0}$ be a standard Brownian motion on a standard probability space $(\Omega, \mathcal{F}, P)$ and let us consider the sequence S_n given by (2.2) with $0 < \alpha < \frac{1}{2}$ and the kernel function K given by (2.4). Denote by $\mathcal{F}_t$ the filtration generated by W . Our first step is to estimate the L^2 mean of S_n .

Lemma 2.1. *As $n \rightarrow \infty$ it holds that*

$$n^{\alpha - \frac{1}{2}} \mathbf{E} S_n^2 \rightarrow C = \frac{\sqrt{2}}{2\pi}.$$

Proof. Recall that, if Z is a standard normal random variable, and if $1 + 2c > 0$

$$\mathbf{E} \left(e^{-cZ^2} \right) = \frac{1}{\sqrt{1 + 2c}}. \quad (2.5)$$

Since the increments of the Brownian motion are independent and $W_{i+1} - W_i$ is independent by $\mathcal{F}_i$ for every i , it holds that (here Z denotes a standard normal random variable)

$$\begin{aligned} \mathbf{E} S_n^2 &= \mathbf{E} \sum_{i=0}^{n-1} K^2(n^\alpha W_i) (W_{i+1} - W_i)^2 = \mathbf{E} \sum_{i=0}^{n-1} K^2(n^\alpha W_i) \\ &= \frac{1}{2\pi} \sum_{i=0}^{n-1} \mathbf{E} e^{-n^{2\alpha} i Z^2} = \frac{1}{2\pi} \sum_{i=0}^{n-1} (1 + 2n^{2\alpha} i)^{-\frac{1}{2}} \end{aligned}$$

and this behaves as $\frac{\sqrt{2}}{2\pi} n^{-\alpha + \frac{1}{2}}$ when n tends to infinity. $\square$

In the following our aim is to prove that the sequence $n^{-\frac{1}{4} + \frac{\alpha}{2}} S_n$ converges in distribution to a non-trivial limit. Note that the sequence S_n can be written as

$$\begin{aligned} S_n &= \sum_{i=0}^{n-1} K(n^\alpha W_i) (W_{i+1} - W_i) = \sum_{i=0}^{n-1} \int_i^{i+1} K(n^\alpha W_i) dW_s \\ &= \sum_{i=0}^{n-1} \int_i^{i+1} K(n^\alpha W_{[s]}) dW_s = \int_0^n K(n^\alpha W_{[s]}) dW_s \end{aligned}$$

where $[s]$ denotes the integer part of the real number s . Define, for every $t \geq 0$,

$$S_t^n = \int_0^t K(n^\alpha W_{[s]}) dW_s. \quad (2.6)$$

Then for every $n \geq 1$ the process $(S_t^n)_{t \geq 0}$ is a $\mathcal{F}_t$ martingale (recall that $\mathcal{F}_t$ denotes the sigma algebra generated by the Wiener process W). The bracket of the martingale $(S_t)_{t \geq 0}$ will be given by, for every $t \geq 0$

$$\langle S^n \rangle_t = \int_0^t K^2(n^\alpha W_{[s]}) ds.$$

This bracket plays a key role in order to understand the behavior of S_n . Let us first understand the limit of the sequence $\langle S^n \rangle_n$. Its asymptotic behavior is related to the local time of the Brownian motion. We recall its definition. For any $t \geq 0$ and $x \in \mathbb{R}$ we define $L^W(t, x)$ as the density of the occupation measure (see [1, 21])

$$\mu_t(A) = \int_0^t 1_A(W_s) ds, \quad A \in \mathcal{B}(\mathbb{R}).$$

The local time $L^W(t, x)$ satisfies the occupation time formula

$$\int_0^t f(W_s) ds = \int_{\mathbb{R}} L^W(t, x) f(x) dx \quad (2.7)$$

for any measurable function f . The local time is Hölder continuous with respect to t and with respect to x . Moreover, it admits a bicontinuous version with respect to (t, x) .

We will denote by p_ε the Gaussian kernel with variance $\varepsilon > 0$ given by $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$. Note that

$$n^{\alpha+\frac{1}{2}} K^2(n^{\alpha+\frac{1}{2}} W_t) = \frac{1}{2\sqrt{\pi}} p_{\frac{1}{2}n^{-\alpha-\frac{1}{2}}}(W_t)$$

and by the scaling property of the Brownian motion

$$\begin{aligned} n^{-\frac{1}{2}+\alpha} \langle S^n \rangle_n &= n^{-\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} K^2(n^\alpha W_i) \\ &= {}_{(d)} n^{-\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} K^2\left(n^{\alpha+\frac{1}{2}} W_{\frac{i}{n}}\right) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{2\sqrt{\pi}} p_{\frac{1}{2}n^{-2\alpha-1}}\left(W_{\frac{i}{n}}\right). \end{aligned}$$

where $=_{(d)}$ means the equality in distribution. A key point of our paper is the following result which gives the convergence of the “bracket.”

Lemma 2.2. *The sequence $\frac{1}{n} \sum_{i=0}^{n-1} p_{\frac{1}{2}n^{-\alpha-\frac{1}{2}}}(W_{\frac{i}{n}})$ converges in $L^2(\Omega)$, as $n \rightarrow \infty$ to $L^W(1, 0)$.*

Proof. Let us recall that $\int_0^1 p_\varepsilon(W_s) ds$ converges as $\varepsilon \rightarrow 0$ to $L^W(1, 0)$ in $L^2(\Omega)$ and almost surely (see e.g. [8]). Using this fact, it suffices to show that the quantity

$$I_n := \mathbf{E} \left(\int_0^1 \left(p_{\alpha_n}(W_s) - p_{\alpha_n} \left(W_{\frac{[ns]}{n}} \right) \right) ds \right)^2 \quad (2.8)$$

converges to zero as $n \rightarrow \infty$, where we denoted by $\alpha_n = \frac{1}{2}n^{-\alpha-\frac{1}{2}}$. We have

$$\begin{aligned} I_n &= \mathbf{E} \int_0^1 \int_0^1 ds dt \left(p_{\alpha_n}(W_s) - p_{\alpha_n} \left(W_{\frac{[ns]}{n}} \right) \right) \left(p_{\alpha_n}(W_t) - p_{\alpha_n} \left(W_{\frac{[nt]}{n}} \right) \right) \\ &= 2\mathbf{E} \int_0^1 dt \int_0^t ds \left(p_{\alpha_n}(W_s) - p_{\alpha_n} \left(W_{\frac{[ns]}{n}} \right) \right) \left(p_{\alpha_n}(W_t) - p_{\alpha_n} \left(W_{\frac{[nt]}{n}} \right) \right). \end{aligned}$$

Notice that for every $s, t \in [0, 1]$, $s \leq t$,

$$\begin{aligned} \mathbf{E} p_\varepsilon(W_s) p_\varepsilon(W_t) &= \mathbf{E} (\mathbf{E} [p_\varepsilon(W_s) p_\varepsilon(W_t) | \mathcal{F}_s]) \\ &= \mathbf{E} (p_\varepsilon(W_s) \mathbf{E} [p_\varepsilon(W_t) | \mathcal{F}_s]) \\ &= \mathbf{E} (p_\varepsilon(W_s) \mathbf{E} [p_\varepsilon(W_t - W_s + W_s) | \mathcal{F}_s]). \end{aligned}$$

By the independence of $W_t - W_s$ and $\mathcal{F}_s^W$ we get

$$\mathbf{E} [p_\varepsilon(W_t - W_s + W_s) | \mathcal{F}_s] = (\mathbf{E} p_\varepsilon(W_t - W_s + x))_{x=W_s} = p_{\varepsilon+t-s}(W_s).$$

We will obtain

$$\begin{aligned} \mathbf{E} p_\varepsilon(W_s) p_\varepsilon(W_t) &= \mathbf{E} p_\varepsilon(W_s) p_{\varepsilon+t-s}(W_s) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{s}{\varepsilon s + \varepsilon(t-s+\varepsilon) + s(t-s+\varepsilon)} \right)^{\frac{1}{2}}. \end{aligned} \quad (2.9)$$

This sequence converges to $\frac{1}{\sqrt{2\pi s(t-s)}}$ as $\varepsilon \rightarrow 0$. If we replace s or t by $\frac{[ns]}{n}$ or $\frac{[nt]}{n}$ respectively, we get the same limit.

As a consequence of the Lemma 2.2 we obtain

Proposition 1. *The sequence $n^{-\frac{1}{2}+\alpha} \langle S^n \rangle_n$ converges in distribution, as $n \rightarrow \infty$, to*

$$\left(\int_{\mathbb{R}} K^2(y) dy \right) L^W(1, 0) = \frac{1}{2\sqrt{\pi}} L^W(1, 0)$$

where L^W denotes the local time of the Brownian motion W .

Proof. The conclusion follows because

$$n^{-\frac{1}{2}+\alpha} \langle S^n \rangle_n = n^{-\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} K^2(n^\alpha W_i) =_{(d)} n^{-\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} K^2(n^{\alpha+\frac{1}{2}} W_{\frac{i}{n}})$$

and this converges to $L^W(1, 0)$ in $L^2(\Omega)$ from Lemma 2.2.

Remark 2.1. Intuitively, the result in Proposition 1 follows because

$$\begin{aligned} n^{-\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} K^2(n^\alpha W_i) &=_{(d)} n^{-\frac{1}{2}+\alpha} \sum_{i=0}^{n-1} K^2(n^{\alpha+\frac{1}{2}} W_{\frac{i}{n}}) \\ &\sim n^{\frac{1}{2}+\alpha} \int_0^1 K^2(n^{\alpha+\frac{1}{2}} W_s) ds = \int_{\mathbb{R}} K^2(n^{\alpha+\frac{1}{2}} x) L^W(1, x) dx \\ &= \int_{\mathbb{R}} dy K^2(y) L^W\left(1, \frac{y}{n^{\alpha+\frac{1}{2}}}\right) \end{aligned}$$

where we used the occupation time formula (2.7). The bicontinuity of the local time implies that this last expression converges to the limit in Proposition 1.

We state the main result of this part.

Theorem 2.1. *Let S_n be given by (2.2). Then as $n \rightarrow \infty$, the sequence $n^{\frac{\alpha}{2}-\frac{1}{4}} S_n$ converges in distribution to*

$$\left(\left(\int_{\mathbb{R}} K^2(y) dy \right) L^W(1, 0) \right)^{\frac{1}{2}} Z$$

where Z is a standard normal random variable independent by $L^W(1, 0)$.

Proof. A similar argument has already been used in [4]. Obviously,

$$S_n =_{(d)} n^{\frac{1}{4}+\frac{\alpha}{2}} \int_0^1 K\left(n^{\alpha+\frac{1}{2}} W_{\frac{[ns]}{n}}\right) dW_s := T_n.$$

Let

$$T_t^n = n^{\frac{1}{4}+\frac{\alpha}{2}} \int_0^t K\left(n^{\alpha+\frac{1}{2}} W_{\frac{[ns]}{n}}\right) dW_s.$$

Then T_t^n is a martingale with respect to the filtration of W . We can show that $\langle T^n, W \rangle$ converges to zero in probability as $n \rightarrow \infty$. Indeed,

$$\langle T^n, W \rangle_t = n^{\frac{1}{4}+\frac{\alpha}{2}} \int_0^t K\left(n^{\alpha+\frac{1}{2}} W_{\frac{[ns]}{n}}\right) ds$$

and this clearly goes to zero using formula (2.5). It is not difficult to see that the convergence is uniform on compact sets. On the other hand $\langle T^n \rangle_1$ converges to $(\int_{\mathbb{R}} K^2(y) dy) L^W(1, 0)$ in $L^2(\Omega)$ from Lemma 2. The result follows immediately from the asymptotic Knight theorem (see [11], Theorem 2.3 page 524, see also [4]). $\square$

2.3 The Multiparameter Settings

This part concerns the two-parameter model (2.3) defined in the introduction. Let $W^{(1)}$ and $W^{(2)}$ denote two independent Wiener sheets on a probability space $(\Omega, \mathcal{F}, P)$. Recall that a Brownian sheet $(W_{u,v})_{u,v \geq 0}$ is defined as a centered two-parameter Gaussian process with covariance function

$$\mathbf{E}(W_{s,t} W_{u,v}) = (s \wedge u)(t \wedge v)$$

for every $s, t, u, v \geq 0$. The model (2.3) leads to the study of the sequence

$$T_n = \sum_{i,j=0}^{n-1} K\left(n^\alpha W_{i,j}^{(1)}\right) \left(W_{i+1,j+1}^{(2)} - W_{i+1,j}^{(2)} - W_{i,j+1}^{(2)} + W_{i,j}^{(2)}\right). \quad (2.10)$$

As in the previous section, we will first give the renormalization of the L^2 norm of T_n as $n \rightarrow \infty$.

Proposition 2. *We have*

$$\mathbf{E}\left(n^{\alpha-1} T_n\right)^2 \rightarrow_{n \rightarrow \infty} \frac{\sqrt{2}}{\pi}.$$

Proof. By the independence of $W^{(1)}$ and $W^{(2)}$ and by the independence of the increments of the Brownian sheet $W^{(2)}$ we have, using (2.5)

$$\mathbf{E} T_n^2 = \sum_{i,j=0}^{n-1} \mathbf{E} \left(K^2(n^\alpha W_{i,j}^{(1)}) \right) = \frac{1}{2\pi} \sum_{i,j=0}^{n-1} \mathbf{E} \left(e^{-n^{2\alpha} (W_{i,j}^{(1)})^2} \right) = \frac{1}{2\pi} \sum_{i,j=0}^{n-1} \frac{1}{\sqrt{1+2n^{2\alpha} i j}}$$

and the conclusion follows because $\sum_{i=0}^{n-1} \frac{1}{\sqrt{i}}$ behaves, when $n \rightarrow \infty$ as $2\sqrt{n}$. $\square$

We will first study the “bracket” $\langle T \rangle_n = \sum_{i,j=0}^{n-1} K^2(n^\alpha W_{i,j}^{(1)})$ which is in some sense the analogous of the bracket of S_n defined in the one-dimensional model. For simplicity, we will still use the notation $\langle T \rangle_n$ even if it is not anymore a true martingale bracket (the stochastic calculus for two parameter martingales is more

complex, see e.g. [7]). By the scaling property of the Brownian sheet, the sequence $n^{\alpha-1} \langle T \rangle_n$ has the same distribution as

$$n^{\alpha-1} \sum_{i,j=0}^{n-1} K^2 \left(n^{\alpha+1} W_{\frac{i}{n}, \frac{j}{n}}^{(1)} \right).$$

Note that for every $u, v \geq 0$ we can write

$$\sqrt{\pi} n^{\alpha+1} K^2 \left(n^{\alpha+1} W_{u,v}^{(1)} \right) = \frac{1}{2} p_{\frac{1}{2n^{2(1+\alpha)}}} (W_{u,v}^{(1)}).$$

As a consequence $n^{\alpha-1} \langle T \rangle_n$ has the same law as

$$\frac{1}{2\sqrt{\pi}} \frac{1}{n^2} \sum_{i,j=0}^{n-1} p_{\frac{1}{2n^{2(1+\alpha)}}} \left(W_{\frac{i}{n}, \frac{j}{n}}^{(1)} \right).$$

In the limit of the above sequence, the local time of the Brownian sheet $W^{(1)}$ will be involved. This local time can be defined as in the one-dimensional case. More precisely, for any $s, t \geq 0$ and $x \in \mathbb{R}$ the local time $L^{W^{(1)}}((s, t), x)$ is defined as the density of the occupation measure (see [1, 21])

$$\mu_{s,t}(A) = \int_0^t \int_0^s 1_A(W_{u,v}) du dv, \quad A \in \mathcal{B}(\mathbb{R}).$$

and it satisfies the occupation time formula: for any measurable function f

$$\int_0^t \int_0^s f(W_{u,v}^{(1)}) du dv = \int_{\mathbb{R}} L^{W^{(1)}}((s, t), x) f(x) dx. \quad (2.11)$$

The following lemma is the two-dimensional counterpart of Lemma 2.2.

Lemma 2.3. *The sequence $\frac{1}{n^2} \sum_{i,j=0}^{n-1} p_{\frac{1}{2n^{2\alpha+2}}} \left(W_{\frac{i}{n}, \frac{j}{n}}^{(1)} \right)$ converges in $L^2(\Omega)$ as $n \rightarrow \infty$ to $L^{W^{(1)}}(\underline{1}, 0)$ where $L^{W^{(1)}}(\underline{1}, 0)$ denotes the local time of the Brownian sheet $W^{(1)}$, where $\underline{1} = (1, 1)$.*

Proof. This proof follows the lines of the proof of Lemma 2.2. Since $\int_0^1 \int_0^1 p_\varepsilon(W_{u,v}) du dv$ converges to $L^{W^{(1)}}(\underline{1}, 0)$ as $\varepsilon \rightarrow 0$ (in $L^2(\Omega)$ and almost surely, it suffices to check that

$$J_n := \mathbf{E} \left(\int_0^1 \int_0^1 \left(p_{\alpha_n}(W_{u,v}) - p_{\alpha_n} \left(W_{\frac{[un]}{n}, \frac{[vn]}{n}} \right) \right) du dv \right)^2$$

converges to zero as $n \rightarrow \infty$ with $\alpha_n = \frac{1}{2}n^{-2\alpha-2}$. This follows from the formula, for every $a \geq u$ and $b \geq v$

$$\mathbf{E}(p_\varepsilon(W_{a,b} - W_{u,v})p_\varepsilon(W_{u,v})) = \mathbf{E}(p_\varepsilon(W_{u,v})p_{\varepsilon+ab-uv}(W_{u,v}))$$

and relation (2.9). $\square$

Let us now state our main result of this section.

Theorem 2.2. *As $n \rightarrow \infty$, the sequence $n^{\frac{\alpha}{2}-\frac{1}{2}}T_n$ converges in distribution to $(c_0 L^{W^{(1)}}(\underline{1}, 0))^{\frac{1}{2}} Z$ where $L^W(1, 0)$ is the local time of the Brownian sheet $W^{(1)}$, $c_0 = \frac{1}{2\sqrt{\pi}}$ and Z is a standard normal random variable independent by $W^{(1)}$.*

Proof. We will compute the characteristic function of the T_n . Let $\lambda \in \mathbb{R}$. Since the conditional law of T_n given $W^{(1)}$ is Gaussian with variance $\sum_{i,j=0}^{n-1} K^2(n^\alpha W_{i,j}^{(1)})$ we can write

$$\begin{aligned} \mathbf{E}\left(e^{i\lambda n^{\frac{\alpha}{2}-\frac{1}{2}}T_n}\right) &= \mathbf{E}\left(\mathbf{E}\left(e^{i\lambda n^{\frac{\alpha}{2}-\frac{1}{2}}T_n} \mid W^{(1)}\right)\right) \\ &= \mathbf{E}\left(e^{-\frac{\lambda^2}{2}n^{\alpha-1}\sum_{i,j=0}^{n-1}K^2(n^\alpha W_{i,j}^{(1)})}\right) = \mathbf{E}\left(e^{-\frac{\lambda^2}{2}n^{\alpha-1}\langle T \rangle_n}\right). \end{aligned}$$

By the scaling property of the Brownian sheet, the sequence

$$n^{\alpha-1}\langle T \rangle_n =_{(d)} n^{\alpha-1} \sum_{i,j=0}^{n-1} K^2\left(n^{\alpha+1}W_{\frac{i}{n}, \frac{j}{n}}^{(1)}\right) = \frac{1}{2\sqrt{\pi}} \frac{1}{n^2} \sum_{i,j=0}^{n-1} P_{\frac{1}{2n^2(1+\alpha)}}\left(W_{\frac{i}{n}, \frac{j}{n}}^{(1)}\right).$$

The result follows from Lemma 2.3. $\square$

Remark 2.2. A similar remark as Remark 1 is available in the two-parameter settings. Indeed, the basic idea of the result is that

$$\begin{aligned} n^{\alpha-1}T_n &=_{(d)} n^{\alpha-1} \sum_{i,j=0}^{n-1} K^2\left(n^{\alpha+1}W_{\frac{i}{n}, \frac{j}{n}}^{(1)}\right) \sim n^{\alpha+1} \int_0^1 \int_0^1 K^2(n^{\alpha+1}W_{u,v}^{(1)}) du dv \\ &= n^{\alpha+1} \int_{\mathbb{R}} K^2(n^{\alpha+1}x) L^{W^{(1)}}(\underline{1}, x) dx \\ &= \int_{\mathbb{R}} K^2(y) L^{W^{(1)}}\left(1, \frac{y}{n^{\alpha+1}}\right) dy \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} K^2(y) dy L^{W^{(1)}}(\underline{1}, 0) \end{aligned}$$

by using (2.11) and the bicontinuity of the local time.

As a final remark, let us mention that above result (and) the model (2.3) can be relatively easily extended to the case of N -parameter Brownian motion, with $N \geq 2$.

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