Preface

SUPPOSE you were asked if the so-called Fundamental Theorem of Algebra is a result of Algebra or of Analysis. What would you answer? And again, among the arguments proving that the complex field is algebraically closed, which would you choose? A proof making use of the concept of continuity? Or of analytic functions, even? Or of Galois theory, instead?

The central topic of this book, the mathematical result named after the mathematicians *Henry Frederick Baker*, *John Edward Campbell*, *Eugene Borisovich Dynkin* and *Felix Hausdorff*, shares – together with the Fundamental Theorem of Algebra – the remarkable property to cross, by its nature, the realms of many mathematical disciplines: Algebra, Analysis, Geometry.

As for the proofs of this theorem (named henceforth CBHD, in chronological order of the contributions), it will be evident in due course of the book that the intertwining of Algebra and Analysis is especially discernible: We shall present arguments making use – all at the same time – of topological algebras, the theory of power series, ordinary differential equations techniques, the theory of Lie algebras, metric spaces; and more.

If we glance at the fields of application of the CBHD Theorem, it is no surprise that so many different areas are touched upon: the theory of Lie groups and Lie algebras; linear partial differential equations; quantum and statistical mechanics; numerical analysis; theoretical physics; control theory; sub-Riemannian geometry; and more.

Curiously, the CBHD Theorem crosses our path already at an early stage of our secondary school education ($e^x e^y = e^{x+y}$, $x, y \in \mathbb{R}$); then it reappears as a nontrivial problem at the very beginning of our university studies, yet in the simplest of non-Abelian contexts (is there a formula for $e^A e^B$ when A, B are square matrices?); then, as our mathematical background progresses, we become acquainted with deeper and more natural settings where this theorem plays a rôle (for example if we face the problem of writing $\mathrm{Exp}(X) \cdot \mathrm{Exp}(Y)$ in logarithmic coordinates, when X, Y belong to the Lie algebra of a Lie group); finally, when our undergraduate studies are complete, we may happen – in all likelihood – to meet the CBHD Theorem again if we are researchers into one of the mathematical fields mentioned a few lines above.

Since the early 1897-98 studies by Campbell on this problem, more than 110 years have passed. Still, the problem of the "multiplication of two exponentials" (whatever the context) has not ceased to provide sources for new questions. Take, for instance, the problem of finding the optimal domain of convergence for the series naturally attached to $\log(e^x e^y)$ (when x, y belong to an arbitrary non-Abelian Banach algebra), a problem which is still not solved in complete generality. Or consider the question of finding more natural and more fitting proofs of the CBHD Theorem, a question which has been renewed – at repeated intervals – in the literature. Indeed, mathematicians have gone on feeling the need for new and simpler proofs of the CBHD Theorem throughout the last century: See for example the papers [183, *1937*], [33, *1956*], [59, *1968*], [48, *1975*], [174, *1980*], [169, *2004*]; and many books may be cited: [99, *1962*], [159, *1964*], [85, *1965*], [79, *1968*], [27, *1972*], [151, *1973*], [171, *1974*], [70, *1982*], [84, *1991*], [144, *1993*], [72, *1997*], [91, *1998*], [52, *2000*], [149, *2002*], [77, *2003*], [1, *2007*], [158, *2007*] – and this is just a small sample; an exhaustive list would be very much longer.

The interest in the CBHD Theorem straddling the decades, the very nature of this result ranging over Algebra, Analysis and Geometry, its fields of application stretching across so many branches of Mathematics and Physics, the proofs so variegated and rich in ideas, the engrossing history of the early contributions: all these facts have seemed to us a sufficient incentive and stimulus in devoting this monograph to such a fascinating Theorem.

This book is intended to present in a unified and *self-contained* way the natural context in which the CBHD Theorem can be formulated and proved. This context is purely algebraic, but the proofs – as mentioned – are very rich and diversified. Also, in order to understand and appreciate the varied arguments attacking the proof of the CBHD Theorem, a historical overview of its early proofs is also needed, without forgetting – in due course later in the monograph – to catch a glimpse of more modern studies related to it, the current state-of-the-art and some open problems.

Most importantly, our aim is to look ahead to applications of the CBHD Theorem. In order to arrive at these applications, it is first necessary to deal with the statements and proofs of the CBHD Theorem in the domain of *Algebra*. Then the applications in *Geometry* and in *Analysis* will eventually branch off from this algebraic setting.

Since this book may be used by a non-specialist in Algebra (as he may be, for example, a researcher in PDEs or a quantum physicist who has felt the need for a deeper understanding of a theorem which has been a cornerstone for some part of his studies), our biggest effort here is to furnish an exposition complete with all details and prerequisites. Any Reader, more or less acquainted with the algebraic background, will be free to skip those details he feels fully conversant with.

Now, before revealing the detailed contents of this book (and to avoid singing further praises of the CBHD Theorem), we shall pass on to some brief historical notes about this theorem.

§1. A Brief Historical *Résumé*. A more exhaustive historical overview is provided in Chapter 1. Here we confine ourselves to disclosing some forgotten facts about the history of the theorem we are concerned with.

Though the exponential nature of the composition of two 'exponential transformations' is somehow implicit in Lie's original theory of finite continuous groups (tracing back to the late nineteenth century), the need for an autonomous study of the symbolic identity "$e^x e^y = e^z$" becomes prominent at the very beginning of the twentieth century.

In this direction, F.H. Schur's papers [154]–[157] present some explicit formulas containing – in a very "quantitative" fashion – the core of the Second and Third Fundamental Theorems of Lie: given a Lie algebra, he exhibits suitable multivariate series expansions, only depending on the structure constants of the algebra and on some universal numbers (Bernoulli's), reconstructing a (local) Lie group with prescribed structure. *This is a precursor of the CBHD Theorem*.

Meanwhile, in 1897 [28] – motivated by group theory – Campbell takes up the study of the existence of an element z such that the composition of two finite transformations e^x, e^y of a continuous transformation group satisfies the identity $e^x \circ e^y = e^z$. By means of a not completely transparent series expansion [30], Campbell solves this problem with little reference (if any) to group theory, showing that z can be expressed as a *Lie series* in x, y.

Using arguments not so far from those of Schur, and inspired by the same search for direct proofs of the Fundamental Theorems of Lie, J.H. Poincaré and E. Pascal attack "Campbell's problem" by the use of suitable algebraic manipulations of polynomials (around 1900-1903).

For instance, in studying the identity $e^x e^y = e^z$ from a more symbolic point of view [142], Poincaré brilliantly shapes a tool (he invents the universal enveloping algebra!) allowing him to manage both algebraic and analytic aspects of the problem. Aiming to give full analytical meaning to his formulas, Poincaré introduces a successful *ODE technique*: he derives an ordinary differential equation for $z(t)$ equivalent to the identity $e^{z(t)} = e^x e^{ty}$, whose solution (at $t = 1$) solves Campbell's problem and, at the same time, the Second and Third Fundamental Theorems of Lie. In fact, $z(t)$ can be expressed, by means of the residue calculus, in a suitable (integral) form exhibiting its Lie-series nature. Although Poincaré's contribution is decisive for the late history of the CBHD Theorem, his latent appeal to group theory and the lack of a formula expressing z as a *universal* Lie series in the symbols x, y probably allowed this contribution to die out amid the twists and turns of mathematical history.

Much in the spirit of Poincaré, but with the use of more direct – and more onerous – computations, Pascal pushes forward Poincaré's "symmetrization" of polynomials, in such a way that he is able to rebuild the formal power series $\sum_{m,n\geq 0} \frac{x^m\, y^n}{m!\, n!}$ as a pure exponential $\sum_{k\geq 0} \frac{z(x,y)^k}{k!}$, where $z(x,y)$ is a Lie series in x, y involving the Bernoulli numbers. Furthermore, Pascal sketches the way the commutator series for $z(x,y)$ can be produced: after the in-embryo formula by Campbell, Pascal's fully fledged results point out (for the first time) the universal Lie series expansibility of z in terms of x, y, a fact which had escaped Poincaré's notice. Though Pascal's papers [135]–[140] will leave Hausdorff and Bourbaki unsatisfied (mostly for the failure to treat the convergence issue and for the massive computations), the fact that in modern times Pascal's contribution to the CBHD Theorem has been almost completely forgotten seems to us to be highly unwarranted.

The final impulse towards a completely *symbolical* version of the Exponential Theorem $e^x e^y = e^z$ is given by Baker [8] and by Hausdorff [78]. The two papers (reasonably independent of each other, for Hausdorff's 1906 paper does not mention Baker's of 1905) use the same technique of 'polar differentiation' to derive for $z(x,y)$ suitable recursion formulas, exhibiting its Lie-series nature. Both authors obtain the same expansion $z = \exp(\delta)(y)$ in terms of a "PDE operator" $\delta = \omega(x,y)\, \frac{\partial}{\partial y}$, where

$$\omega(x,y) = x + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1}(x).$$

The Lie series $\omega(x,y)$, besides containing the Bernoulli numbers[1] (reappearing, after Schur, in every proof mentioned above), is nothing but the subseries obtained by collecting – from the expansion of $\log(e^x e^y)$ – the summands containing x precisely once. The same series appeared clearly in Pascal, implicitly in Campbell and in an integral form in Poincaré: it can be rightly considered as the *fil rouge* joining all the early proofs cited so far.

In proving the representation $z = \exp(\delta)(y)$, Baker makes use of quite a puzzling formalism on Lie polynomials, but he is able to draw out of his machinery such abundance of formulas, that it is evident that this is much more than a pure formalism.

However, Hausdorff's approach in proving the formula $z = \exp(\delta)(y)$ is so efficacious and authoritative that it became the main source for future work on the exponential formula, to such an extent that Baker's contribution went partly – but undeservedly – forgotten. (As a proof of this fact we must

[1]Indeed, $\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n$.

recall that, in a significant part of the related literature, the Exponential Theorem is named just "Campbell-Hausdorff".) This is particularly true of the commentary of Bourbaki on the early history of this formula, citing Hausdorff as the only "perfectly precise" and reliable source. Admittedly, Hausdorff must be indeed credited (together with fruitful recursion formulas for the coefficients of z) for providing the long awaited *convergence* argument in the set of Lie's transformation groups.

After Hausdorff's 1906 paper, about 40 years elapsed before Dynkin [54] proved another long awaited result: *an explicit presentation for the commutator series of* $\log(e^x e^y)$. Dynkin's paper indeed contains much more: thanks to the explicitness of his representation, Dynkin provides a direct estimate – the first in the history of the CBHD Theorem – for the convergence domain, more general than Hausdorff's. Moreover, the results can be generalized to the infinite-dimensional case of the so-called *Banach-Lie algebras* (to which Dynkin extensively returns in [56]). Finally, Dynkin's series allows us to prove Lie's Third Theorem (a concern for Schur, Poincaré, Pascal and Hausdorff) in an incredibly simple way.

Two years later [55], Dynkin will give another proof of the Lie-series nature of $\log(e^x e^y)$, independently of all his predecessors, a proof disclosing all the *combinatorial* aspects behind the exponential formula. As for the history of the CBHD Theorem, Dynkin's papers [54]–[56] paved the way, by happenstance, for the study of other possible presentations of $\log(e^x e^y)$ and consequently for the problem of further *improved domains of convergence*, dominating the "modern era" of the CBHD Theorem (from 1950 to present days).

This modern age of the Exponential Theorem begins with the first applications to Physics (dating back to the 1960-70s) and especially to Quantum and Statistical Mechanics (see e.g., [51, 64, 65, 104, 119, 128, 178, 180, 181]).

In parallel, a rigorous mathematical formalization of the CBHD Theorem became possible thanks to the Bourbakist refoundation of Mathematics, in particular of Algebra. Consequently, the new proofs of the CBHD Theorem (rather than exploiting *ad hoc* arguments as in all the contributions we cited so far) are based on very general algebraic tools. For example, they can be based on characterizations of Lie elements (as given e.g. by Friedrichs's criterion for primitive elements) as in Bourbaki [27], Hochschild [85], Jacobson [99], Serre [159]. As a consequence, the CBHD Theorem should be regarded (mathematically and historically) as a result from noncommutative algebra, rather than a result of Lie group theory, as it is often popularized in undergraduate university courses.

As a matter of fact, this popularization is caused by the remarkable application of the Exponential Theorem to the structure theory of Lie groups. Indeed, as is well known, in this context the CBHD Theorem allows us to prove a great variety of results: the effective analytic regularity of all smooth Lie groups (an old result of Schur's!), the local "reconstruction" of the group law via the bracket in the Lie algebra, many interesting results on the duality group/algebra homomorphisms, the classifying of the simply connected Lie groups by their Lie algebras, the local version of Lie's Third Theorem, and many others.

For this reason, all major books in Lie group theory starting from the 1960s comprise the CBHD Theorem (mainly named after Campbell, Hausdorff or Baker, Campbell, Hausdorff): See e.g., the classical books (ranging over the years sixties–eighties) Bourbaki [27], Godement [70], Hausner, Schwartz [79], Hochschild [85], Jacobson [99], Sagle, Walde [151], Serre [159], Varadarajan [171]; or the more recent books Abbaspour, Moskowitz [1], Duistermaat, Kolk [52], Gorbatsevich, Onishchik, Vinberg [72], Hall [77], Hilgert, Neeb [84], Hofmann, Morris [91], Rossmann [149], Sepanski [158]. (Exceptions are Chevalley [38], which is dated 1946, and Helgason [81], where only expansions up to the second order are used.)

A remarkable turning point in the history of the CBHD Theorem is provided by Magnus's 1954 paper [112]. In studying the exponential form $\exp(\Omega(t))$ under which the solution $Y(t)$ to the nonautonomous linear ODE system $Y'(t) = A(t)Y(t)$ can be represented, Magnus introduced a formula – destined for a great success – for expanding $\Omega(t)$, later referred to also as the *continuous Campbell-Baker-Hausdorff Formula*. (See also [10, 37, 119, 160, 175].) In fact, in proper contexts and when $A(t)$ has a suitable form, a certain evaluation of the expanded Ω gives back the CBHD series. For a comprehensive treatise on the Magnus expansion, the Reader is referred to Blanes, Casas, Oteo, Ros, 2009 [16] (and to the detailed list of references therein).

Here we confine ourselves to pointing out that the modern literature (mainly starting from the 1980s) regarding the CBHD Theorem has mostly concentrated on the problem of *improved domains of convergence* for the possible different presentations of $\log(e^x e^y)$, both in commutator or non-commutator series expansions (for the latter, see the pioneering paper by Goldberg [71]). Also, the problem of efficient algorithms for computing the terms of this series (in suitable bases for free Lie algebras and with minimal numbers of commutators) has played a major rôle. For this and the above topics, see [9, 14–16, 23, 35, 36, 46, 53, 61, 103, 106, 107, 109, 118, 122, 123, 131, 134, 145, 146, 166, 177].

In the study of convergence domains, the use of the Magnus expansion has proved to be a very useful tool. However, the problem of the best domain of convergence of the CBHD series in the setting of general *Banach algebras* and of general *Banach-Lie algebras* is still open, though many optimal results exist for matrix algebras and in the setting of Hilbert spaces (see the references in Section 5.7 on page 359).

In parallel, starting from the mid seventies, the CBHD Theorem has been crucially employed in the study of wide classes of PDEs, especially of subelliptic type, for example those involving the so-called *Hörmander operators* (see Folland [62], Folland, Stein [63], Hörmander [94], Rothschild, Stein [150], Nagel, Stein, Wainger [129], Varopoulos, Saloff-Coste, Coulhon [172]).

The rôle of the CBHD Theorem is not only prominent for usual (finite-dimensional) Lie groups, but also within the context of *infinite dimensional* Lie groups (for a detailed survey, see Neeb [130]). For example, among infinite dimensional Lie groups, the so-called BCH-groups (Baker-Campbell-Hausdorff groups) are particularly significant. For some related topics (a comprehensive list of references on infinite-dimensional Lie groups being out of our scope), see e.g., [12, 13, 24, 42, 43, 56, 66–69, 73, 83, 86, 87, 92, 93, 130, 133, 147, 148, 152, 170, 173, 182].

Finally, the early years of the twenty-first century have seen a renewed interest in CBHD-type theorems (both continuous and discrete) within the field of numerical analysis (specifically, in *geometric integration*) see e.g. [76, 97, 98, 101, 114].

§ **2. The Main Contents of This book.** We furnish a very brief digest of the contents of this book. After a historical preamble given in Chapter 1 (also containing reference to modern applications of the CBHD Theorem), the book is divided into two parts.

Part I (Chapters 2–6) begins with an introduction of the background algebra (comprehensive of all the involved notations) which is a prerequisite to the rest of the book. Immediately after such preliminaries, we jump into the heart of the subject. Indeed, Part I treats widely all the qualitative properties and the problems arising from the statement of the CBHD Theorem and from its various proofs, such as the well-posedness of the 'CBHD operation', its associativity and convergence, or the relationship between the CBHD Theorem, the Theorem of Poincaré, Birkhoff and Witt and the existence of free Lie algebras. The results given in Chapter 2, although essential to the stream of the book, would take us a long distance away if accompanied by their proofs. For this reason they are simply stated in Part I, while all the missing proofs can be found in **Part II** (Chapters 7-10).

Let us now have a closer look at the contents of each chapter.

Chapter 2 is entirely devoted to recalling algebraic prerequisites and to introduce the required notations. Many essential objects are introduced, such as tensor algebras, completions of graded algebras, formal power series, free Lie algebras, universal enveloping algebras. Some of the results are demonstrated in Chapter 2 itself, but most of the proofs are deferred to Chapter 7. Chapter 2 is meant to provide the necessary algebraic background to non-specialist Readers and may be skipped by those trained in Algebra. Section 2.3 also contains some needed results (on metric spaces) from Analysis.

Chapter 3 illustrates a complete proof of the CBHD Theorem, mainly relying on the book by Hochschild [85, Chapter X]. The proof is obtained from general results of Algebra, such as Friedrichs's characterization of Lie elements and the use of the Hausdorff group. Afterwards, Dynkin's Formula is produced. This result is conceptually subordinate to the so called Campbell-Baker-Hausdorff Theorem, and it is based, as usual, on the application of the Dynkin-Specht-Wever Lemma.

Our inquiry into the meaningful reasons why the CBHD Theorem holds widens in **Chapter 4**, where several shorter (but more specialized) proofs of the Theorem, differing from each other and from the one given in Chapter 3, are presented. We deal here with the works by M. Eichler [59], D. Ž. Djoković [48], V. S. Varadarajan [171], C. Reutenauer [144], P. Cartier [33].

Eichler's argument is the one most devoid of prerequisites, though crucially *ad hoc* and tricky; Djoković's proof (based on an "ODE technique", partially tracing back to Hausdorff) has the merit to rediscover early arguments in a very concise way; Varadarajan's proof (originally conceived for a Lie group context) completes, in a very effective fashion, Djoković's proof and allows us to obtain recursion formulas perfectly suited for convergence questions; Reutenauer's argument fully formalizes the early approach by Baker and Hausdorff (based on so-called polar differentiation); Cartier's proof, instead, differs from the preceding ones, based as it is on a suitable characterization of Lie elements, in line with the approach of Chapter 3. Each of the strategies presented in Chapter 4 has its advantages, so that the Reader has the occasion to compare them thoroughly and to go into the details for every different approach (and, possibly, to choose the one more suited for his taste or requirements).

In **Chapter 5** the convergence of the Dynkin series is studied, in the context of finite-dimensional Lie algebras first, and then in the more general setting of normed Banach-Lie algebras. Besides, the "associativity" of the operation defined by the Dynkin series is afforded. Throughout this chapter, we shall be exploiting identities implicitly contained (and hidden) in the CBHD Theorem. As a very first taste of the possible (geometrical) applications of the results presented here, we shall have the chance to prove – in a direct and natural fashion – the Third Fundamental Theorem of Lie for finite-dimensional nilpotent Lie algebras (and more). Finally, the chapter closes with a long list (briefly commented, item by item) of modern bibliography on the convergence problem and on related matters.

Chapter 6 clarifies the deep and – in some ways – surprising intertwinement occurring between the CBHD and PBW Theorems (PBW is short for Poincaré-Birkhoff-Witt). As it arises from Chapter 3, CBHD is classically derived by PBW, although other strategies are possible. In Chapter 6 we will show how the opposite path may be followed, thus proving the PBW Theorem by means of CBHD. This less usual approach was first provided by Cartier, whose work [33] is at the basis of the chapter. An essential tool is represented by free Lie algebras, whose rôle – in proving CBHD and PBW – is completely clarified here.

Chapter 7 consists of a collection of the missing proofs from Chapter 2.

Chapter 8 is intended to complete those results of Chapter 2 which deal with the existence of the free Lie algebra $\mathrm{Lie}(X)$ related to a set X. The characterization of $\mathrm{Lie}(X)$ as the Lie subalgebra (contained in the algebra of the polynomials in the elements of X) consisting of Lie-polynomials is also given in detail (without requiring the PBW Theorem or any of its corollaries). Furthermore, some results about free nilpotent Lie algebras are presented here, helpful e.g., in constructing free Carnot groups (as in [21, Chapters 14, 17]).

An algebraic approach to formal power series can be found in **Chapter 9**.

Finally, **Chapter 10** contains all the machinery about symmetric algebras which is needed in Chapter 6.

§ **3. How to Read This Book.** Since this book is intended for a readership potentially not acquainted with graduate level Algebra, the main effort is to make the presentation *completely self-contained*. The only prerequisites are a basic knowledge of Linear Algebra and undergraduate courses in Analysis and in Algebra. The book opens with a historical overview, Chapter 1, of the early proofs of the CBHD Theorem and a glimpse into more modern results: it is designed not only for historical scholars, but also for the Reader who asked himself the question "Campbell, Baker, Hausdorff, Dynkin: who proved what?".

The algebraic prerequisites are collected in Chapter 2, where the notations used throughout are also collected. The Reader interested in the corresponding proofs will find them in Part II (Chapters from 7 to 10). Chapter 2 and Part II can be skipped by the Reader fully conversant with the algebraic prerequisites. In any case, Chapter 2 must be used as a complete reference for the notations. The Reader interested in the proofs of the CBHD Theorem can directly refer to Chapter 3 (for a more elaborate proof, making use of general algebraic results, in the spirit of the Bourbaki exposition of the subject) or to Chapter 4 (where shorter proofs are presented, but with more *ad hoc* arguments). These chapters require the background results of Chapter 2.

Once the main CBHD Theorem has been established, Chapter 5 presents a primer on the convergence question. The Reader will also find an extended list of references on related topics. Chapter 5 can also be read independently of the preceding chapters, once any proof of the CBHD Theorem is taken for granted. Analogously, Chapter 6 can be read on its own, only requiring some theory of free Lie algebras and of symmetric algebras (coming from Chapters 8 and 10, respectively). A synopsis of the book structure together with the interdependence of the different chapters is given in Figure 1 below.

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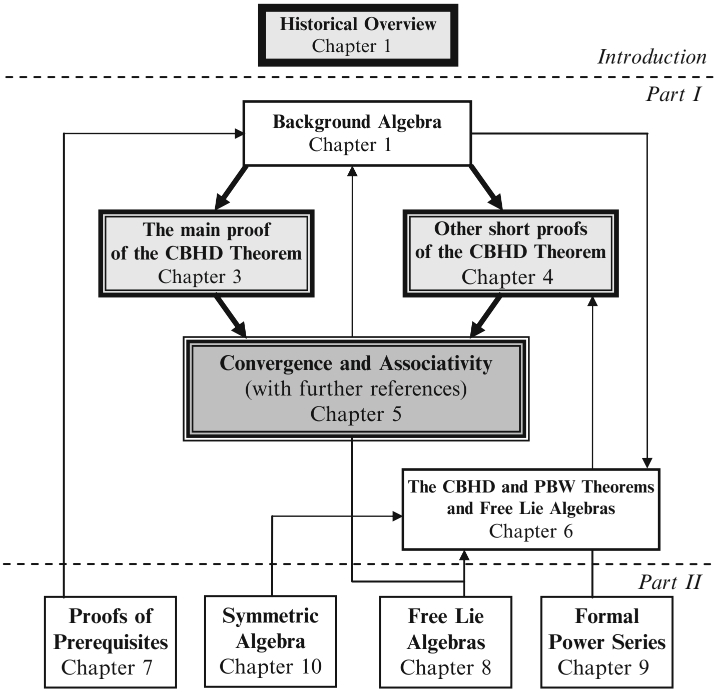

Fig. 1 Synopsis of the book structure

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