# Chapter 2
# Background Algebra

The aim of this chapter is to recall the main algebraic prerequisites and all the notation and definitions used throughout the Book. All main proofs are deferred to Chap. 7. This chapter (and its counterpart Chap. 7) is intended for a Reader having only a basic undergraduate knowledge in Algebra; a Reader acquainted with a more advanced knowledge of Algebra may pass directly to Chap. 3.

Our main objects of interest for this chapter are:

– Free vector spaces, unital associative algebras, tensor products
– Free objects over a set $X$: the free magma, the free monoid, the free (associative and non-associative) algebra over $X$
– Free Lie algebras
– Completions of metric spaces and of graded algebras; formal power series
– The universal enveloping algebra of a Lie algebra

## 2.1 Free Vector Spaces, Algebras and Tensor Products

### *2.1.1 Vector Spaces and Free Vector Spaces*

Throughout this section, $\mathbb{K}$ will denote a field, while $V$ will denote a vector space over $\mathbb{K}$. Moreover, when referring to linear maps, spans, basis, generators, linear independence, etc., we shall tacitly mean[1] "with respect to $\mathbb{K}$".

[1]For instance, "let $U, V$ be vector spaces" means that both $U$ and $V$ are vector spaces over the *same* field $\mathbb{K}$.

A. Bonfiglioli and R. Fulci, *Topics in Noncommutative Algebra*, Lecture Notes in Mathematics 2034, DOI 10.1007/978-3-642-22597-0_2,

We recall the well known fact that any vector space possesses a basis. More generally, we shall have occasion to apply the following result, which can be easily proved by means of Zorn's Lemma (as in [108, Theorem 5.1]):

*Let $V \neq \{0\}$ be a vector space. Let $I, G$ be subsets of $V$ such that $I \subseteq G$, $I$ is linearly independent and $G$ generates $V$. Then there exists a basis $\mathcal{B}$ of $V$ with $I \subseteq \mathcal{B} \subseteq G$.*

Bases of vector spaces will always assumed to be *indexed*. Let $\mathcal{B} = \{v_i\}_{i \in \mathcal{I}}$ be a basis of $V$ (indexed over the nonempty set $\mathcal{I}$). Then for every $v \in V$ there exists a *unique* family $\{c_i(v)\}_{i \in \mathcal{I}} \subset \mathbb{K}$ such that $c_i(v) \neq 0$ for all but finitely many indices $i$ in $\mathcal{I}$ and such that $v = \sum_{i \in \mathcal{I}} c_i(v)\, v_i$ (the sum being well posed since it runs over a finite set). Occasionally, the subset $\mathcal{I}' \subseteq \mathcal{I}$ such that $c_i(v) \neq 0$ for every $i \in \mathcal{I}'$ will be denoted by $\mathcal{I}(v)$. When $v = 0$, or equivalently $\mathcal{I}(v) = \emptyset$, the notation $\sum_{i \in \emptyset} c_i\, v_i := 0$ applies. Note that, for every fixed $v \in V$, the following formula

$$c : \mathcal{I} \to \mathbb{K}, \quad i \mapsto c_i(v)$$

defines a well posed *function*, uniquely depending on $v$.

We obviously have the following result.

**Proposition 2.1.** *Let $V$ be a vector space and let $\mathcal{B}$ be a basis of $V$. Then for every vector space $X$ and every function $L : \mathcal{B} \to X$, there exists a unique linear map $\overline{L} : V \to X$ prolonging $L$.*

If $\mathcal{B} = \{v_i\}_{i \in \mathcal{I}}$, it suffices to set

$$\overline{L}(v) := \textstyle\sum_{i \in \mathcal{I}(v)} c_i(v)\, L(v_i).$$

The above proposition asserts that there always exists a unique linear map $\overline{L}$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "L"] \arrow[d, hook, "\iota"'] & X \\
V \arrow[ur, "\overline{L}"'] &
\end{tikzcd}$$

Here and in the sequel, when the context is understood, $\iota$ will always denote the *inclusion* map of a set $A \subseteq B$ into a set $B$.

The following are well known standard facts from Linear Algebra and are stated without proofs for the sake of future reference.

**Proposition 2.2.** (i). *Let $V, X$ be vector spaces and let $W$ be a vector subspace of $V$. Suppose also that $L : V \to X$ is a linear map such that $W \subseteq \ker(L)$ and let $\pi : V \to V/W$ denote the natural projection map.*

*Then there exists a unique linear map* $\widetilde{L} : V/W \to X$ *such that*

$$\widetilde{L}(\pi(v)) = L(v) \quad \textit{for every } v \in V, \tag{2.1}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] & X \\ V/W \arrow[ur, "\widetilde{L}"'] & \end{tikzcd}$$

(ii). *Let* $V, X$ *be vector spaces and let* $L : V \to X$ *be a linear map. Then the map*

$$\widetilde{L} : V/\mathrm{ker}(L) \to L(V), \quad [v]_{\mathrm{ker}(L)} \mapsto L(v)$$

*is an isomorphism of vector spaces.*

Actually, (2.1) also defines $\widetilde{L}$ uniquely, the definition being well posed thanks to the hypothesis $W \subseteq \ker(L)$ (indeed, $\pi(v) = \pi(v')$ iff $v - v' \in W$, so that $\pi(v) = \pi(v - v') + \pi(v') = \pi(v')$).

**Definition 2.3 (Free Vector Space).** Let $S$ be any nonempty set. We denote by $\mathbb{K}\langle S\rangle$ the vector space of the $\mathbb{K}$-valued functions on $S$ non-vanishing only on a finite (possibly empty) subset of $S$. The set $\mathbb{K}\langle S\rangle$ is called *the free vector space over* $S$.

Occasionally, a function $f : S \to \mathbb{K}$ non-vanishing only on a finite subset of $S$ will be said to have "compact support".

*Remark 2.4.* Let $v \in S$ be fixed. We denote by

$$\chi(v) : S \to \mathbb{K}, \quad \chi(v)(s) := \begin{cases} 1, & \text{if } s = v \\ 0, & \text{if } s \neq v \end{cases} \tag{2.2}$$

the characteristic function of $\{v\}$ on $S$. With this notation at hand, it is easily seen that one has

$$\mathbb{K}\langle S\rangle = \mathrm{span}\big\{\chi(v) \,|\, v \in S\big\}, \tag{2.3}$$

so that the generic element of $\mathbb{K}\langle S\rangle$ is of the form

$$\sum_{j=1}^{n} \lambda_j\, \chi(v_j), \qquad \text{where } n \in \mathbb{N},\ \lambda_1, \dots, \lambda_n \in \mathbb{K},\ v_1, \dots, v_n \in S.$$

In the sequel, when there is no possibility of confusion, we shall identify $v \in S$ with $\chi(v) \in \mathbb{K}\langle S\rangle$, so that the generic element of $\mathbb{K}\langle S\rangle$ is of the form $\sum_{j=1}^{n} \lambda_j\, v_j$ (with $n$, $\lambda_j$ and $v_j$ as above), that is, $\mathbb{K}\langle S\rangle$ can be thought of as the set of the "formal linear combinations" of elements of $S$. Thus $S$ can be viewed as a subset (actually, a basis) of $\mathbb{K}\langle S\rangle$. Occasionally, we shall also write an element $f$ of $\mathbb{K}\langle S\rangle$ as

$$f = \sum_{s\in S} f(s)\,\chi(s) \qquad \Big(\text{or } \ f = \sum_{s\in S} f_s\,\chi(s)\Big), \tag{2.4}$$

the sum being finite, for $f : S \to \mathbb{K}$ has compact support.

*Remark 2.5.* With the above notation, *the set* $\chi(S) := \{\chi(v)\,|\, v \in S\}$ *is a linear basis of* $\mathbb{K}\langle S\rangle$. Indeed, let $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ and let $v_1, \dots, v_n$ be pairwise distinct elements of $S$ and suppose $\sum_{j=1}^{n} \lambda_j\, \chi(v_j) = 0$ in $\mathbb{K}\langle S\rangle$. For any fixed $i \in \{1, \dots, n\}$ we then have[2]

$$0 = \big(\textstyle\sum_{j=1}^{n} \lambda_j\, \chi(v_j)\big)(v_i) = \sum_{j=1}^{n} \lambda_j\, \chi(v_j)\, \delta_{i,j} = \lambda_i\, 1,$$

whence $\chi(v_1), \dots, \chi(v_n)$ are linearly independent. Moreover (2.3) proves that $\chi(S)$ generates $\mathbb{K}\langle S\rangle$.

We remark that the linear independence of the set $\chi(S)$ implies in particular that $\chi : S \to \mathbb{K}\langle S\rangle$ *is an injective map*.

As a consequence, $\mathbb{K}\langle S\rangle$ is finite dimensional iff $S$ is finite. In this case, if $S = \{v_1, \dots, v_N\}$, we also use the brief notation $\mathbb{K}\langle v_1, \dots, v_N\rangle := \mathbb{K}\langle S\rangle$.

In the rest of this Book, the following result will be used many times. This is the first of a series of *universal properties* of algebraic objects, which we shall encounter frequently.

**Theorem 2.6 (Universal Property of the Free Vector Space).**

(i) *Let $S$ be any set. Then for every vector space $X$ and every map $F : S \to X$ there exists a unique linear map $F^\chi : \mathbb{K}\langle S\rangle \to X$ such that*

$$F^\chi(\chi(v)) = F(v) \quad \textit{for every } v \in S, \tag{2.5}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[d, "\chi"'] & X \\ \mathbb{K}\langle S\rangle \arrow[ur, "F^\chi"'] & \end{tikzcd}$$

[2] Here and throughout, $\delta_{i,j}$ represents as usual the Kronecker symbol, i.e., $\delta_{i,i} = 1$, $\delta_{i,j} = 0$ if $i \neq j$.

(ii) *Vice versa, suppose $V, \varphi$ are respectively a vector space and a map $\varphi : S \to V$ with the following property: For every vector space $X$ and every map $F{:}S{\to}X$ there exists a unique linear map $F^\varphi : V \to X$ such that*

$$F^\varphi(\varphi(v)) = F(v) \quad \textit{for every } v \in S, \tag{2.6}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[d, "\varphi"'] & X \\ V \arrow[ur, "F^\varphi"'] & \end{tikzcd}$$

*Then $V$ is canonically isomorphic to $\mathbb{K}\langle S\rangle$, the isomorphism being $\varphi^\chi : \mathbb{K}\langle S\rangle \to V$ and its inverse being $\chi^\varphi : V \to \mathbb{K}\langle S\rangle$. Furthermore $\varphi$ is injective and the set $\varphi(S)$ is a basis of $V$. Actually, it holds that $\varphi = \varphi^\chi \circ \chi$.*

When the identification $S \ni v \equiv \chi(v) \in \mathbb{K}\langle S\rangle$ applies, the above map $\chi$ is the associated inclusion $\iota : S \hookrightarrow \mathbb{K}\langle S\rangle$, so that we may think of $F^\chi$ as a "prolongation" of $F$.

*Proof.* See page 393 in Chap. 7. □

We recall the definitions of (external) direct sum and of product of a family of vector spaces. Let $\{V_i\}_{i\in\mathcal{I}}$ be a family of vector spaces (indexed over a set $\mathcal{I}$, finite, denumerable or not). We set

$$\prod_{i\in\mathcal{I}} V_i := \left\{ (v_i)_{i\in\mathcal{I}} \,\middle|\, v_i \in V_i \text{ for every } i \in \mathcal{I} \right\},$$

$$\bigoplus_{i\in\mathcal{I}} V_i := \left\{ (v_i)_{i\in\mathcal{I}} \,\middle|\, v_i \in V_i \text{ for every } i \in \mathcal{I} \text{ and } v_i \neq 0 \text{ for finitely many } i \right\}.$$

The former is called the *product space* of the vector spaces $V_i$, the latter is called the *(external) direct sum* of the spaces $V_i$. More precisely, we use a "sequence-style" notation $(v_i)_{i\in\mathcal{I}}$ to mean a *function* $v : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} V_i$, $v(i) =: v_i$ with $v_i \in V_i$ for every $i \in \mathcal{I}$. In other words

$$(v_i)_{i\in\mathcal{I}} = (v'_i)_{i\in\mathcal{I}} \quad \Longleftrightarrow \quad \Big(\text{for all } i \in \mathcal{I}, v_i, v'_i \in V_i \text{ and } v_i = v'_i\Big). \tag{2.7}$$

Occasionally, *when $\mathcal{I}$ is at most denumerable* we may also use the notation $\sum_{i\in\mathcal{I}} v_i$ instead of $(v_i)_{i\in\mathcal{I}}$. For example, according to this notation when

$\mathfrak{I} = \mathbb{N}$, the generic element of $\bigoplus_{n\in\mathbb{N}} V_n$ is of the form $v_1 + \ldots + v_p$ where $p \in \mathbb{N}$ and $v_n \in V_n$ for every $n = 1, \ldots, p$.

This notation is justified by the fact that the product space and the external direct sum of the spaces $V_i$ are naturally endowed with a vector space structure (simply by defining the vector space operations *componentwise*). Obviously $\bigoplus_{i\in\mathfrak{I}} V_i$ is a subspace of $\prod_{i\in\mathfrak{I}} V_i$.

*Remark 2.7.* With the above notation, for any fixed $j \in \mathfrak{I}$ let

$$\widetilde{V}_j := \prod_{i\in\mathfrak{I}} V_i' \quad \text{where} \quad \begin{cases} V_i' := V_j, & \text{for } i = j, \\ V_i' := \{0\}, & \text{for } i \neq j. \end{cases}$$

Note that, for every $j \in \mathfrak{I}$, $\widetilde{V}_j$ is a vector *subspace* of $\bigoplus_{i\in\mathfrak{I}} V_i$ (hence of $\prod_{i\in\mathfrak{I}} V_i$). We now leave to the Reader the simple verification that the spaces $\widetilde{V}_j$ have the following property: Any $v \in \bigoplus_{i\in\mathfrak{I}} V_i$ can be written in a *unique* way as a (finite) sum $\sum_{i\in\mathfrak{I}} v_i$ with $v_i \in \widetilde{V}_i$ for every $i \in \mathfrak{I}$. Consequently, $\bigoplus_{i\in\mathfrak{I}} V_i$ is the (usual) *direct sum of its subspaces* $\{\widetilde{V}_i\}_{i\in\mathfrak{I}}$ (and the name "external direct sum" is thus well justified).

If, for every fixed $j \in \mathfrak{I}$, we consider the linear map

$$\iota_j : V_j \to \bigoplus_{i\in\mathfrak{I}} V_i, \quad V_j \ni v \overset{\iota_j}{\mapsto} (v_i')_{i\in\mathfrak{I}} \quad \text{where} \quad \begin{cases} v_i' := v, \text{ for } i = j, \\ v_i' := 0, \text{ for } i \neq j, \end{cases}$$

it is easily seen that $\iota_j(V_j) = \widetilde{V}_j$. Moreover, $\iota_j$ is an isomorphism of $V_j$ onto its image $\widetilde{V}_j$, so that $\widetilde{V}_j \simeq V_j$ for every $j \in \mathfrak{I}$. As claimed above, using also (2.7), for any $v = (v_i)_{i\in\mathfrak{I}} \in \bigoplus_{i\in\mathfrak{I}} V_i$ we have the decomposition

$$v = \textstyle\sum_{i\in\mathfrak{I}} \iota_i(v_i) \qquad \Big(\text{with } \iota_i(v_i) \in \widetilde{V}_i \text{ for all } i \in \mathfrak{I}\Big). \tag{2.8}$$

Hence, throughout the sequel *we shall always identify any $V_j$ as a subspace of $\bigoplus_{i\in\mathfrak{I}} V_i$ (or of $\prod_{i\in\mathfrak{I}} V_i$) by the canonical identification $V_j \simeq \widetilde{V}_j$ via $\iota_j$.*

The following simple fact holds:

**Theorem 2.8 (Universal Property of the External Direct Sum).**

(i) *Let $\{V_i\}_{i\in\mathfrak{I}}$ be an indexed family of vector spaces. Then, for every vector space $X$, and every family of linear maps $\{F_i\}_{i\in\mathfrak{I}}$ (also indexed over $\mathfrak{I}$) with $F_i : V_i \to X$ (for every $i \in \mathfrak{I}$) there exists a unique linear map $F_\Sigma : \bigoplus_{i\in\mathfrak{I}} V_i \to X$ prolonging $F_i$, for every $i \in \mathfrak{I}$. More precisely it holds that*

$$F_\Sigma(\iota_i(v)) = F_i(v) \quad \textit{for every } i \in \mathfrak{I} \textit{ and every } v \in V_i, \tag{2.9}$$

*thus making the following a family (over $i \in \mathfrak{I}$) of commutative diagrams:*

$$\begin{tikzcd} V_i \arrow[r, "F_i"] \arrow[d, "\iota_i"'] & X \\ \bigoplus_{i\in\mathfrak{I}} V_i \arrow[ur, "F_\Sigma"'] & \end{tikzcd}$$

*(The notation $\oplus_{i\in\mathfrak{I}} F_i$ for $F_\Sigma$ will also be allowed.)*

(ii) *Conversely, suppose $V$, $\{\varphi_i\}_{i\in\mathfrak{I}}$ are respectively a vector space and a family of linear maps $\varphi_i : V_i \to V$ with the following property: For every vector space $X$ and every family of linear maps $\{F_i\}_{i\in\mathfrak{I}}$ with $F_i : V_i \to X$ (for every $i \in \mathfrak{I}$) there exists a unique linear map $F_\varphi : V \to X$ such that*

$$F_\varphi(\varphi_i(v)) = F_i(v) \quad \textit{for every } i \in \mathfrak{I} \textit{ and every } v \in V_i, \tag{2.10}$$

*thus making the following a family (over $i \in \mathfrak{I}$) of commutative diagrams:*

$$\begin{tikzcd} V_i \arrow[r, "F_i"] \arrow[d, "\varphi_i"'] & X \\ V \arrow[ur, "F_\varphi"'] & \end{tikzcd}$$

*(The notation $\Phi_{i\in\mathfrak{I}} F_i$ for $F_\varphi$ will also be allowed.) Then $V$ is canonically isomorphic to $\bigoplus_{i\in\mathfrak{I}} V_i$, the isomorphism being $\oplus_{i\in\mathfrak{I}}\varphi_i : \bigoplus_{i\in\mathfrak{I}} V_i \to V$ and its inverse being $\Phi_{i\in\mathfrak{I}}\iota_i : V \to \bigoplus_{i\in\mathfrak{I}} V_i$. Furthermore any $\varphi_i$ is injective and $V = \bigoplus_{i\in\mathfrak{I}} \varphi_i(V_i)$ (direct sum of subspaces of $V$). Actually, it holds that $\varphi_i \equiv (\oplus_{i\in\mathfrak{I}}\varphi_i) \circ \iota_i$.*

*Proof.* (i) follows from (2.8), by setting (here $v_i \in V_i$ for all $i$)

$$F_\Sigma : \bigoplus_{i\in\mathfrak{I}} V_i \to X, \quad F_\Sigma\big(\sum_{i\in\mathfrak{I}} \iota_i(v_i)\big) := \sum_{i\in\mathfrak{I}} F_i(v_i).$$

A simple verification shows that this map is linear and obviously it is the unique linear map satisfying (2.9).

(ii) follows by arguing as in the proof of Theorem 2.6 (see page 393). The fact that $V = \bigoplus_{i\in\mathfrak{I}} \varphi_i(V_i)$ derives from the following ingredients:

- The decomposition of $\bigoplus\limits_{i\in\mathfrak{I}} V_i$ into the direct sum of its subspaces $\widetilde{V}_i = \iota_i(V_i)$.
- The isomorphism $\oplus_{i\in\mathfrak{I}}\varphi_i : \bigoplus_{i\in\mathfrak{I}} V_i \to V$.
- The set equality $\big(\oplus_{i\in\mathfrak{I}} \varphi_i\big)(\iota_i(V_i)) = \varphi_i(V_i)$. $\square$

The following is easily seen to hold.

**Proposition 2.9.** *Let $\{V_i\}_{i\in\mathcal{I}}$ be a family of vector spaces. For every $i\in\mathcal{I}$, let $\mathcal{B}_i$ be a basis of $V_i$. Then the following is a basis for the external direct sum $\bigoplus_{i\in\mathcal{I}} V_i$:*

$$\Big\{(w_i)_{i\in\mathcal{I}}\,\Big|\, w_i\in\mathcal{B}_i \text{ for every } i\in\mathcal{I} \text{ and } \exists!\ i_0\in\mathcal{I} \text{ such that } w_{i_0}\neq 0\Big\}.$$

## 2.1.2 *Magmas, Algebras and (Unital) Associative Algebras*

### 2.1.2.1 Some Structures and Their Morphisms

Since there are no universal agreements for names, we make explicit our convention to say that a set $A$ is:

1. *A magma*, if on $A$ there is given a binary operation $A\times A\to A$, $(a,a')\mapsto a*a'$.
2. *A monoid*, if $(A,*)$ is a magma, $*$ is associative and endowed with a unit element.
3. *An algebra*, if $(A,*)$ is a magma, $A$ is a vector space and $*$ is bilinear.
4. *An associative algebra*, if $(A,*)$ is an algebra and $*$ is associative.
5. *A unital associative algebra* (UA algebra, for brevity), if $(A,*)$ is an associative algebra and $*$ is endowed with a unit element.
6. *A Lie algebra*, if $(A,*)$ is an algebra, $*$ is skew-symmetric and the following *Jacobi identity* holds

$$a*(b*c)+b*(c*a)+c*(a*b)=0,\quad \text{for all } a,b,c\in A.$$

As usual, in the context of Lie algebras, the associated operation will be denoted by $(a,a')\mapsto[a,a']$ (occasionally, $[a,a']_A$) and it will be called *the Lie bracket* (or simply, *bracket* or, sometimes, *commutator*[3]) of $A$.

Other structures (which we shall use less frequently) are recalled in the following (self-explanatory) table:

| $(A,*)$ | $*$ Binary | $*$ Associative | $*$ Has a unit | $*$ Bilinear ($A$ vector space) |
|---|---|---|---|---|
| Magma | $\surd$ | | | |
| Unital magma | $\surd$ | | $\surd$ | |
| Semigroup | $\surd$ | $\surd$ | | |
| Monoid | $\surd$ | $\surd$ | $\surd$ | |
| Algebra | $\surd$ | | | $\surd$ |
| Associative algebra | $\surd$ | $\surd$ | | $\surd$ |
| UA algebra | $\surd$ | $\surd$ | $\surd$ | $\surd$ |

[3] In the literature, the term "commutator" is commonly used as a synonym of "bracket". In this Book we shall use the term commutator only for a special kind of bracket: that obtained from an underlying associative algebra structure.

If $(A, \circledast)$, $(B, \circledcirc)$ are two magmas (respectively, two monoids, two algebras, two unital associative algebras, two Lie algebras), we say that a given map $\varphi : A \to B$ is:

1. *A magma morphism*, if $\varphi(a \circledast a') = \varphi(a) \circledcirc \varphi(a')$, for every $a, a' \in A$.
2. *A monoid morphism*, if $\varphi$ is a magma morphism mapping the unit of $A$ into the unit of $B$.
3. *An algebra morphism*, if $\varphi$ is a linear magma morphism.
4. *A morphism of unital associative algebras* (UAA morphism, in short), if $\varphi$ is a linear monoid morphism, or equivalently, if $\varphi$ is an algebra morphism mapping the unit of $A$ into the unit of $B$.
5. *A Lie algebra morphism* (LA morphism, in short), if $\varphi$ is an algebra morphism, i.e. (with the alternative notation for the algebra operation)

$$\varphi([a, a']_A) = [\varphi(a), \varphi(a')]_B, \quad \text{for every } a, a' \in A.$$

The prefix "iso" applies to any of the above notions of morphism $\varphi$, when $\varphi$ is also a bijection. Plenty of examples of the above algebraic structures will be given in the next sections. The following definitions will also be used in the sequel:

1. Let $(M, *)$ be a magma (possibly, a monoid) and let $U \subseteq M$; we say that $U$ is a *set of magma-generators* for $M$ (or that *$U$ generates $M$ as a magma*) if every element of $M$ can be written as an iterated $*$-product (with any coherent insertion of parentheses) of finitely many elements of $U$. In the presence of associativity, this amounts to saying that every element of $M$ can be written in the form $u_1 * \cdots * u_k$, for some $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in U$. When $M$ is a monoid, the locution *$U$ generates $M$ as a monoid* will also apply.
2. Let $(A, *)$ be an algebra (associative or not, unital or not) and let $U \subseteq A$; we say that $U$ is a *set of algebra-generators* for $A$ (or that *$U$ generates $A$ as an algebra*) if every element of $A$ can be written as a *finite linear combination* of iterated $*$-products (with coherent insertions of parentheses) of finitely many elements of $U$.
3. When $(A, [\cdot,\cdot])$ is a Lie algebra, in case (2) we say that $U$ is a *set of Lie-generators* for $A$ (or that *$U$ Lie-generates $A$*). In this case (see Theorem 2.15 at the end of the section), this is equivalent to saying that every element of $A$ can be written as a *finite linear combination* of nested elements of the form $[u_1 \cdots [u_{k-1}, u_k] \cdots]$, for $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in U$.

**Definition 2.10 (Derivation of an Algebra).** If $(A, *)$ is an algebra, we say that a map $D : A \to A$ is a *derivation* of $A$ if $D$ is linear and it holds that

$$D(a * b) = (Da) * b + a * (Db), \quad \text{for every } a, b \in A.$$

When $A$ is a Lie algebra, this can be rewritten

$$D[a,b] = [Da,b] + [a,Db], \quad \text{for every } a,b \in A.$$

Here is another definition that will play a central rôle.

**Definition 2.11 (Graded and Filtered Algebras).**

**Graded Algebra:** We say that an algebra $(A,*)$ is a *graded algebra* if it admits a decomposition of the form $A = \bigoplus_{j=1}^{\infty} A_j$, where the $A_j$ are vector subspaces of $A$ such that $A_i * A_j \subseteq A_{i+j}$ for every $i,j \geq 1$. In this case, the family $\{A_j\}_{j\geq 1}$ will be called a *grading* of $A$.

**Filtered Algebra:** We say that an algebra $(A,*)$ is a *filtered algebra* if $A = \bigcup_{j=1}^{\infty} F_j$, where the sets $F_j$ are vector subspaces of $A$ such that $F_i * F_j \subseteq F_{i+j}$ for every $i,j \geq 1$ and

$$F_j \subseteq F_{j+1}, \quad \text{for every } j \in \mathbb{N}.$$

In this case, the family $\{F_j\}_{j\geq 1}$ will be called a *filtration* of $A$.

For example, in the case of Lie algebras, a graded Lie algebra $A = \bigoplus_{j=1}^{\infty} A_j$ fulfils $[A_i, A_j] \subseteq A_{i+j}$, for every $i,j \geq 1$. Note that if $\{A_j\}_{j\geq 1}$ is a grading of $A$ then $A$ admits the filtration $\{F_j\}_{j\geq 1}$, where $F_j := \bigoplus_{i=1}^{j} A_j$.

The following simple result will be applied frequently in this Book.

**Proposition 2.12 (Quotient Algebra).** *Let $(A,*)$ be an algebra and let $I \subseteq A$ be a two-sided ideal*[4] *of $A$. Then the quotient vector space $A/I$ is an algebra (called* quotient algebra *of $A$ modulo $I$), when equipped with the operation*

$$\circledast : A/I \times A/I \to A/I, \qquad [a]_I \circledast [b]_I := [a*b]_I, \quad \forall\, a,b \in A.$$

*Moreover, the associated projection $\pi : A \to A/I$ (i.e., $\pi(a) := [a]_I$ for every $a \in A$) is an algebra morphism. Finally, if $(A,*)$ is associative (respectively, unital), then the same is true of $(A/I, \circledast)$ (and respectively, its unit is $[1_A]_I$).*

The proof is simple and we only remark that the well-posedness of $\circledast$ follows by this argument: if $[a]_I = [a']_I$ and $[b]_I = [b']_I$ then $a' = a + x$ and $b' = b + y$ with $x, y \in I$ so that

$$a' * b' = a * b + \underbrace{a * y + x * b + x * y}_{\in I}, \quad \text{whence } [a' * b']_I = [a * b]_I.$$

[4] We recall that this means that $I$ is a vector subspace of $A$ and that $a * i, i * a \in I$ for every $i \in I$ and every $a \in A$.

#### 2.1.2.2 Some Notation on Lie Algebras

In this section, $(A, [\cdot,\cdot])$ denotes a Lie algebra. If $U, V \subseteq A$ we set

$$[U,V] := \operatorname{span}\{[u,v] \,|\, u \in U,\ v \in V\}.$$

Note that (unlike some customary notation) $[U,V]$ is not the set of brackets $[u,v]$ with $u \in U$, $v \in V$, but the *span* of these.

Let $U \subseteq A$. We say that the elements of $U$ are brackets of length $1$ of $U$. Inductively, once brackets of length $1,\dots,k-1$ have been defined, we say that $[u,v]$ is a bracket of length $k$ of $U$, if $u,v$ are, respectively, brackets of lengths $i,j$ of $U$ and $i+j=k$. As synonyms for "length", we shall also use *height* or *order*. For example, if $u_1,\dots,u_7 \in U$, then

$$[[u_1,u_2],[[[u_3,[u_4,u_5]],u_6],u_7]], \qquad [[[u_1,[[u_2,u_3],u_4]],u_5],[u_6,u_7]]$$

are brackets of length 7 of $U$. Note that an element of a Lie algebra may have more than one length (or even infinitely many!). For example, if $A$ is the Lie algebra of the smooth vector fields on $\mathbb{R}^1$ and $X=\partial_x$, $Y = x\,\partial_x$, then

$$X = [\cdots[X,\underbrace{Y]\cdots Y}_{k \text{ times}}], \quad \forall\, k \in \mathbb{N},$$

so that $X$ is a bracket of length $k$ of $U=\{X,Y\}$, for every $k \in \mathbb{N}$.

When $u_1,\dots,u_k \in U$, brackets of the form

$$[u_1,[u_2\cdots[u_{k-1},u_k]\cdots]], \quad [[\cdots[u_1,u_2]\cdots u_{k-1}],u_k]$$

are called *nested* (respectively, *right-nested* and *left-nested*). The following result shows that the right-nested brackets span the brackets of any order. First we give a definition.

**Definition 2.13 (Lie Subalgebra Generated by a Set).** Let $A$ be a Lie algebra and let $U \subseteq A$. We denote by $\mathrm{Lie}\{U\}$ the smallest Lie subalgebra of $A$ containing $U$ and we call it *the Lie algebra generated by $U$ in $A$*. More precisely, $\mathrm{Lie}\{U\} = \bigcap \mathfrak{h}$, where the spaces $\mathfrak{h}$ run over the set of subalgebras of $A$ containing $U$.

*Remark 2.14.* With the above notation, it is easily seen that $\mathrm{Lie}\{U\}$ coincides with the span of the brackets of $U$ of any order. More precisely, if $W_k$ denotes the span of the brackets of $U$ of order $k$, it holds that $\mathrm{Lie}\{U\} = \biguplus_{k\in\mathbb{N}} W_k$, where $\biguplus$ denotes the sum of vector subspaces of $A$. Equivalently,

$$\begin{aligned}\mathrm{Lie}\{U\} &= \operatorname{span}\{W_k \,|\, k \in \mathbb{N}\}\\ &= \operatorname{span}\{w|\, w \text{ is a bracket of order } k \text{ of } U, \text{ with } k \in \mathbb{N}\}.\end{aligned}$$

**Theorem 2.15 (Nested Brackets).** *Let $A$ be a Lie algebra and $U \subseteq A$. Set*

$$U_1 := \mathrm{span}\{U\}, \qquad U_n := [U, U_{n-1}], \quad n \geq 2.$$

*Then we have* $\mathrm{Lie}\{U\} = \mathrm{span}\{U_n \mid n \in \mathbb{N}\}$. *Moreover, it holds that*

$$[U_i, U_j] \subseteq U_{i+j}, \quad \textit{for every } i, j \in \mathbb{N}. \tag{2.11}$$

We remark that, from the definition of $U_n$, the elements of $U_n$ are linear combination of right-nested brackets of length $n$ of $U$. The above theorem states that *every element of* $\mathrm{Lie}\{U\}$ *is in fact a linear combination of right-nested brackets* (an analogous statement holding for the left case).

To show the idea behind the proof (which is a consequence of the Jacobi identity and the skew-symmetry of the bracket), let us take $u_1, u_2$, $v_1, v_2 \in U$ and prove that $[[u_1, u_2], [v_1, v_2]]$ is a linear combination of right-nested brackets of length 4. By the Jacobi identity $[X, [Y, Z]] = -[Y, [Z, X]] - [Z, [X, Y]]$ one has

$$\begin{aligned}
[\underbrace{[u_1, u_2]}_{X}, [\underbrace{v_1}_{Y}, \underbrace{v_2}_{Z}]] &= -[v_1, [v_2, [u_1, u_2]]] - [v_2, [[u_1, u_2], v_1]] \\
&= -[v_1, [v_2, [u_1, u_2]]] + [v_2, [v_1, [u_1, u_2]]] \in U_4.
\end{aligned}$$

*Proof (of Theorem 2.15).* We set $U^* := \mathrm{span}\{U_n \mid n \in \mathbb{N}\}$. Obviously, $U^*$ contains $U$ and is contained in any Lie subalgebra of $A$ which contains $U$. Hence, we are left to prove that $U^*$ is closed under the bracket operation. Obviously, it is enough to show that, for any $i$, $j \in \mathbb{N}$ and for any $u_1, \ldots, u_i$, $v_1, \ldots, v_j \in U$ we have

$$\Big[[u_1[u_2[\cdots[u_{i-1}, u_i]\cdots]]]; [v_1[v_2[\cdots[v_{j-1}, v_j]\cdots]]]\Big] \in U_{i+j}.$$

We argue by induction on $k := i + j \geq 2$. For $k = 2$ and 3 the assertion is obvious whilst for $k = 4$ we proved it after the statement of this theorem. Let us now suppose that the result holds for every $i + j \leq k$, with $k \geq 4$, and prove it then holds when $i + j = k + 1$. We can assume, by skew-symmetry, that $j \geq 3$. Exploiting repeatedly the induction hypothesis, the Jacobi identity and skew-symmetry, we have

$$\begin{aligned}
&\Big[u; [v_1[v_2[\cdots[v_{j-1}, v_j]\cdots]]]\Big] \\
&\quad = -[v_1, \underbrace{[[v_2, [v_3, \cdots]], u]}_{\text{length } k}] - [[v_2, [v_3, \cdots]], [u, v_1]] \\
&\quad = \{\text{element of } U_{k+1}\} - [[v_1, u], [v_2, [v_3, \cdots]]]
\end{aligned}$$

$$
\begin{aligned}
&= \{\text{element of } U_{k+1}\} + [v_2, \underbrace{[[v_3, \cdots], [v_1, u]]}_{\text{length } k}] + [[v_3, \cdots], [[v_1, u]v_2]] \\
&= \{\text{element of } U_{k+1}\} + [[v_2, [v_1, u]], [v_3, \cdots]]
\end{aligned}
$$

(after finitely many steps)

$$
\begin{aligned}
&= \{\text{element of } U_{k+1}\} + (-1)^{j-1}[[v_{j-i}, [v_{j-2}, \cdots [v_1, u]]], v_j] \\
&= \{\text{element of } U_{k+1}\} + (-1)^{j}[v_j, [v_{j-i}, [v_{j-2}, \cdots [v_1, u]]]] \\
&\in U_{k+1}.
\end{aligned}
$$

This ends the proof. □

The previous proof shows something more: An arbitrary bracket $u$ of length $k$ of $\{u_1, \ldots, u_k\}$ (the minimal set of elements appearing in $u$) is a linear combination (with coefficients in $\{-1, 1\}$) of right-nested brackets of length $k$ of the same set $\{u_1, \ldots, u_k\}$ and in any such summand there appear all the $u_i$ for $i = 1, \ldots, k$. (An analogous result also holds for left-nested brackets.)

**Definition 2.16.** Let $(A, *)$ be an associative algebra. Let us set

$$
[a, b]_* := a * b - b * a, \quad \text{for every } a, b \in A. \tag{2.12}
$$

Then $(A, [\cdot,\cdot]_*)$ is a Lie algebra, called *the Lie algebra related to* $A$.

The Lie bracket defined in (2.12) will be referred to as the *commutator related to* $A$ *(or the* $*$-commutator*)* and the Lie algebra $(A, [\cdot,\cdot]_*)$ will also be called *the commutator-algebra related to* $A$. The notation $[\cdot,\cdot]_A$ will occasionally apply instead of $[\cdot,\cdot]_*$ when confusion may not arise.

Even if authors often use the term "commutator" as a synonym for "bracket", we shall reserve it for brackets obtained from an associative multiplication as in (2.12).

Due to the massive use of commutators throughout the Book, we exhibit here the proof of the Jacobi identity (anti-symmetry and bilinearity being trivial):

$$
\begin{aligned}
&[a, [b, c]_*]_* + [b, [c, a]_*]_* + [c, [a, b]_*]_* \\
&\quad = \underline{a * b * c} - \underline{\underline{a * c * b}} - \underline{\underline{\underline{b * c * a}}} + \overline{c * b * a} + \underline{\underline{\underline{b * c * a}}} - \overline{\overline{b * a * c}} + \\
&\qquad - \overline{\overline{\overline{c * a * b}}} + \underline{\underline{a * c * b}} + \overline{\overline{\overline{c * a * b}}} - \overline{c * b * a} - \underline{a * b * c} + \overline{\overline{b * a * c}} \\
&\quad = 0 \qquad \text{(summands canceling as over-/under-lined.)}
\end{aligned}
$$

It will be via the Poincaré-Birkhoff-Witt Theorem (a highly nontrivial result) that we shall be able to prove that (roughly speaking) *every* Lie bracket can be realized as a suitable commutator (see Sect. 2.4).

**Convention.** *Let $(A,*)$ be an associative algebra. When a Lie algebra structure on $A$ is invoked, unless otherwise stated, we refer to the Lie algebra on $A$ which is induced by the associated $*$-commutator.* So, for example, if $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a Lie algebra, $(A,*)$ is an associative algebra and $\varphi : \mathfrak{g} \to A$ is a map, when we say that "$\varphi$ is a Lie algebra morphism", we mean that $\varphi$ is linear and that it satisfies $\varphi([a,b]_{\mathfrak{g}}) = \varphi(a) * \varphi(b) - \varphi(b) * \varphi(a)$, for every $a, b \in \mathfrak{g}$.

*Remark 2.17. Let $(A, \circledast)$, $(B, \circledcirc)$ be associative algebras and let $\varphi : A \to B$ be an algebra morphism. Then $\varphi$ is also a Lie algebra morphism of the associated commutator-algebras.* Indeed, for every $a, a' \in A$ one has

$$\begin{aligned} \varphi([a,a']_{\circledast}) &= \varphi(a \circledast a' - a' \circledast a) = \varphi(a) \circledcirc \varphi(a') - \varphi(a') \circledcirc \varphi(a) \\ &= [\varphi(a), \varphi(a')]_{\circledcirc}. \end{aligned}$$

*Remark 2.18. Let $(A,*)$ be an associative algebra and let $D : A \to A$ be a derivation of $A$. Then $D$ is also a derivation of the commutator-algebra related to $A$.* Indeed, for every $a, a' \in A$ one has

$$\begin{aligned} D([a,a']_*) &= D(a * a' - a' * a) \\ &= D(a) * a' + a * D(a') - D(a') * a - a' * D(a) \\ &= \big(D(a) * a' - a' * D(a)\big) + \big(a * D(a') - D(a') * a\big) \\ &= [D(a), a']_* + [a, D(a')]_*. \end{aligned}$$

#### 2.1.2.3 Free Magma and Free Monoid

The remainder of this section is devoted to the construction of the free magma, the free monoid and the free algebra (associative or not) generated by a set. These structures will turn out to be of fundamental importance when we shall be dealing with the construction of free Lie algebras, without the use of the Poincaré-Birkhoff-Witt Theorem (see Sect. 2.2).

We begin with the construction of a free magma generated by a set. We follow the construction in [26, I, §7, n.1]. Henceforth, $X$ will denote a fixed set.

To begin with, we inductively set $M_1(X) := X$, and (if $\coprod$ denotes disjoint union[5] of sets)

[5]We recall the relevant definition: let $\{A_i\}_{i \in \mathcal{I}}$ be an indexed family of sets ($\mathcal{I}$ may be finite, denumerable or not). By $\coprod_{i \in \mathcal{I}} A_i$ we mean the set of the *ordered couples* $(i, a)$ where $i \in \mathcal{I}$ and $a \in A_i$, and we call it *the disjoint union of (the indexed family of) sets* $\{A_i\}_{i \in \mathcal{I}}$. As a common habit, the first entry of the couple is dropped, but care must be paid since the same element $a$ possibly belonging to $A_i$ and $A_j$ with $i \neq j$ gives rise to *distinct* elements in $\coprod_i A_i$.

$$M_2(X) := X \times X, \quad M_3(X) := \big(M_2(X) \times M_1(X)\big) \coprod \big(M_1(X) \times M_2(X)\big),$$

$$M_n(X) := \coprod_{p \in \{1,\dots,n-1\}} M_{n-p}(X) \times M_p(X), \quad \text{for every } n \geq 2; \tag{2.13}$$

$$M(X) := \coprod_{n \in \mathbb{N}} M_n(X). \tag{2.14}$$

Equivalently, we can drop the sign of disjoint union and replace it with standard set-union, *provided we consider as distinct the Cartesian products*

$$\underbrace{\big(X \times \cdots \times X\big)}_{n \text{ times}} \times \underbrace{\big(X \times \cdots \times X\big)}_{m \text{ times}} \neq \underbrace{X \times \cdots \times X}_{n+m \text{ times}}.$$

Hence, we have

$$\begin{aligned}
&M_1(X) = X, \quad M_2(X) = X \times X,\\
&M_3(X) = ((X \times X) \times X) \cup (X \times (X \times X)),\\
&M_4(X) = (((X \times X) \times X) \times X) \cup ((X \times (X \times X)) \times X) \cup\\
&\cup ((X \times X) \times (X \times X)) \cup (X \times ((X \times X) \times X)) \cup (X \times (X \times (X \times X))),\\
&\vdots\\
&M_n(X) := \bigcup_{p \in \{1,\dots,n-1\}} M_{n-p}(X) \times M_p(X), \quad \text{for every } n \geq 2,
\end{aligned}$$

and $M(X) := \bigcup_{n \in \mathbb{N}} M_n(X)$.

Roughly, $M(X)$ is the set of *non-commutative and non-associative* words on the letters of $X$, where parentheses are inserted in any coherent way (different parentheses defining different words). For brevity, we set $M_n := M_n(X)$. For example, if $x \in X$, the following are distinct elements of $M_7$:

$$\Bigg((x,x), \bigg(\Big(\big(x,(x,x)\big), x\Big), x\bigg)\Bigg), \qquad \bigg(\Big(\big(x,((x,x),x)\big), x\Big), (x,x)\bigg)$$

Via the natural injection $X \equiv M_1 \subset M(X)$, we consider $X$ as a subset of $M(X)$ (and the same is done for every $M_n$). For every $w \in M(X)$ there exists a unique $n \in \mathbb{N}$ such that $w \in M_n$, which is denoted by $n = \ell(w)$ and called the *length* of $w$. Note that any $w \in M(X)$ with $\ell(w) \geq 2$ is of the form $w = (w', w'')$ for unique $w', w'' \in M(X)$ satisfying $\ell(w') + \ell(w'') = \ell(w)$. For any $w, w' \in M(X)$ with $w \in M_n$ and $w' \in M_{n'}$, we denote by $w.w'$ the (unique) element of $M_{n+n'}$ corresponding to $(w, w')$ in the canonical injections $M_n \times M_{n'} \subset M_{n+n'} \subset M(X)$. The binary operation $(w, w') \mapsto w.w'$ endows $M(X)$ with the structure of a magma, called the *free magma over* $X$.

*Remark 2.19.* Obviously, $X$ is a set of magma-generators for $M(X)$. Moreover, we have a sort of "grading" on $M(X)$ ($M(X)$ has no vector space structure though), for it holds that $M(X) = \bigcup_{n\in\mathbb{N}} M_n(X)$ and $M_i(X).M_j(X) \subseteq M_{i+j}(X)$, for every $i, j \geq 1$.

**Lemma 2.20 (Universal Property of the Free Magma).** *Let $X$ be any set.*

(i) *For every magma $M$ and every function $f : X \to M$, there exists a unique magma morphism $\overline{f} : M(X) \to M$ prolonging $f$, thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\iota"'] & M \\ M(X) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

(ii) *Vice versa, suppose $N, \varphi$ are respectively a magma and a function $\varphi : X \to N$ with the following property: For every magma $M$ and every function $f : X \to M$, there exists a unique magma morphism $f^\varphi : N \to M$ such that*

$$f^\varphi(\varphi(x)) = f(x), \quad \textit{for every } x \in X,$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & M \\ N \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then $N$ is canonically magma-isomorphic to $M(X)$, the magma isomorphism being (see the notation in part* (i) *above) $\overline{\varphi} : M(X) \to N$ and its inverse being $\iota^\varphi : N \to M(X)$. Furthermore $\varphi$ is injective and $N$ is generated, as a magma, by $\varphi(X)$. Actually, it holds that $\varphi = \overline{\varphi} \circ \iota$. Finally, we have $N \simeq M(\varphi(X))$.*

*Proof.* (i) The map $\overline{f}$ is defined as follows: Let $*$ be the operation on $M$ and let us consider the maps $f_n$ defined by

$$f_1 : M_1 \to M, \;\; f_1(x) := f(x), \;\; \forall\, x \in X,$$

$$f_2 : M_2 \to M, \;\; f_2(x_1.x_2) := f(x_1) * f(x_2), \;\; \forall\, x_1, x_2 \in X,$$

$$f_3 : M_3 \to M, \;\; \begin{cases} f_3((x_1.x_2).x_3) := (f(x_1) * f(x_2)) * f(x_3) \\ f_3(x_1.(x_2.x_3)) := f(x_1) * (f(x_2) * f(x_3)) \end{cases} \;\; \forall\, x_1, x_2, x_3 \in X,$$

and, inductively, $f_n : M_n \to M$ is defined by setting $f_n(w.w') := f_{n-p}(w) * f_p(w')$, for every $p \in \{1, \dots, n-1\}$ and every $(w, w') \in M_{n-p} \times M_p$. Finally, let $\overline{f} : M(X) \to M$ be the unique map such that $\overline{f}|_{M_n}$ coincides with $f_n$. It is easily seen that $\overline{f}$ is a magma morphism and that it is the only morphism fulfilling (i).

(ii) follows by arguing as in the proof of Theorem 2.6 (see page 393). We recall the scheme of the proof. We have the commutative diagrams

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, hook, "\iota"'] & N \\
M(X) \arrow[ur, "\overline{\varphi}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[d, "\varphi"'] & M(X) \\
N \arrow[ur, "\iota^{\varphi}"'] &
\end{tikzcd}
$$

Obviously, the following are commutative diagrams too

$$
\begin{tikzcd}
X \arrow[r, hook, "\iota"] \arrow[d, hook, "\iota"'] & M(X) \\
M(X) \arrow[ur, "\mathrm{id}_{M(X)}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & N \\
N \arrow[ur, "\mathrm{id}_N"'] &
\end{tikzcd}
$$

The maps $\iota^{\varphi} \circ \overline{\varphi} : M(X) \to M(X)$, $\overline{\varphi} \circ \iota^{\varphi} : N \to N$ are magma morphisms such that

$$
(\iota^{\varphi} \circ \overline{\varphi})(\iota(x)) = \iota(x) \quad \forall\, x \in X, \qquad (\overline{\varphi} \circ \iota^{\varphi})(\varphi(x)) = \varphi(x) \quad \forall\, x \in X.
$$

Hence, by the uniqueness of the morphisms represented by the "diagonal" arrows in the last couples of commutative diagrams above, we have

$$
\iota^{\varphi} \circ \overline{\varphi} \equiv \mathrm{id}_{M(X)}, \qquad \overline{\varphi} \circ \iota^{\varphi} \equiv \mathrm{id}_N.
$$

The rest of the proof is straightforward. □

We next construct the free monoid over $X$. We could realize it as a quotient of the free magma $M(X)$ by identifying any two elements in $M_n$ which are obtained by inserting parentheses to the same ordered $n$-tuple of elements of $X$. Alternatively, we proceed as follows (which allows us to introduce in a rigorous way the important notion of a *word over a set*).

Let $X$ be any fixed set. Any ordered $n$-tuple $w = (x_1, \dots, x_n)$ of elements of $X$ is called a *word on* $X$ and $n =: \ell(w)$ is called its *length*. By convention, the empty set is called the *empty word*, it is denoted by $e$ and its length is taken to be $0$. The set of all words of length $n$ is denoted by $W_n$ and we set

$$
\mathrm{Mo}(X) := \bigcup_{n \geq 0} W_n.
$$

Obviously, $X$ is identified with the set of words in $\mathrm{Mo}(X)$ whose length is 1. If $w = (x_1, \dots, x_n)$ and $w' = (x'_1, \dots, x'_{n'})$ are two words on $X$, we define a new word $w'' = (x''_1, \dots, x''_{n+n'})$ (by juxtaposition of $w$ and $w'$) by setting

$$x''_j := \begin{cases} x_j, & \text{for } j = 1, \dots, n, \\ x'_{j-n}, & \text{for } j = n+1, \dots, n+n'. \end{cases}$$

With the above definition, we set $w.w' := w''$. It then holds $\ell(w.w') = \ell(w) + \ell(w')$ so that $W_n.W_{n'} = W_{n+n'}$ for every $n, n' \geq 0$. Any word $w = (x_1, \dots, x_n)$ (with $x_1, \dots, x_n \in X$) is written in a unique way as $w = x_1.x_2.\cdots.x_n$, so that

$$W_0 = \{e\}, \qquad W_n = \big\{x_1.x_2.\cdots.x_n \,\big|\, x_1, \dots, x_n \in X\big\}, \quad n \in \mathbb{N}. \tag{2.15}$$

Obviously, one has $e.w = w.e = w$ for every $w \in \mathrm{Mo}(X)$.

If $w, w', w'' \in \mathrm{Mo}(X)$, then $(w.w').w''$ and $w.(w'.w'')$ are both equal to the word $w''' = (x'''_1, \dots, x'''_h)$ where $h = \ell(w) + \ell(w') + \ell(w'')$ and

$$x'''_j := \begin{cases} x_j, & j = 1, \dots, \ell(w), \\ x'_{j-\ell(w)}, & j = \ell(w)+1, \dots, \ell(w)+\ell(w'), \\ x''_{j-\ell(w)-\ell(w')}, & j = \ell(w)+\ell(w')+1, \dots, \ell(w)+\ell(w')+\ell(w''). \end{cases}$$

As a result, $(\mathrm{Mo}(X), .)$ is a monoid, called *the free monoid over $X$*.

*Remark 2.21.* Obviously, $\{e\} \cup X$ is a set of generators for $\mathrm{Mo}(X)$ as a monoid. Note that $\mathrm{Mo}(X) \setminus \{e\}$ *is a semigroup, i.e., an associative magma* (which is not unital, though) and that $X$ is a set of magma-generators for $\mathrm{Mo}(X) \setminus \{e\}$ (i.e., every element of $\mathrm{Mo}(X) \setminus \{e\}$ can be written as a finite – nonempty – product of elements of $X$).

Moreover, we have a sort of "grading" on $\mathrm{Mo}(X)$ (though $\mathrm{Mo}(X)$ is not a vector space), for it holds that $\mathrm{Mo}(X) = \bigcup_{n \geq 0} W_n$ and $W_i.W_j \subseteq W_{i+j}$, for every $i, j \geq 0$.

The adjective "free" is justified by the following universal property, whose proof is completely analogous to that of Lemma 2.20.

**Lemma 2.22 (Universal Property of the Free Monoid).** *Let $X$ be any set.*

(i) *For every monoid $M$ and every function $f : X \to M$, there exists a unique monoid morphism $\overline{f} : \mathrm{Mo}(X) \to M$ prolonging $f$, thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\iota"'] & M \\ \mathrm{Mo}(X) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

(ii) *Conversely, suppose* $N, \varphi$ *are respectively a monoid and a function* $\varphi : X \to N$ *with the following property: For every monoid* $M$ *and every function* $f : X \to M$*, there exists a unique monoid morphism* $f^\varphi : N \to M$ *such that*

$$f^\varphi(\varphi(x)) = f(x), \quad \textit{for every } x \in X,$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & M \\ N \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then* $N$ *is canonically monoid-isomorphic to* $\mathrm{Mo}(X)$*, the monoid isomorphism being (see the notation in part* (i) *above)* $\overline{\varphi} : \mathrm{Mo}(X) \to N$ *and its inverse being* $\iota^\varphi : N \to \mathrm{Mo}(X)$*. Furthermore* $\varphi$ *is injective and* $N$ *is generated, as a monoid, by* $\varphi(X)$*. Actually, it holds that* $\varphi = \overline{\varphi} \circ \iota$*. Finally, we have* $N \simeq \mathrm{Mo}(\varphi(X))$*.*

#### 2.1.2.4 Free Associative and Non-associative Algebras

We now associate to each of $M(X), \mathrm{Mo}(X)$ of the previous section an algebra (over $\mathbb{K}$). Let, in general, $(M, .)$ be a magma. Let $M_{\mathrm{alg}}$ be the free vector space over $M$ (see Definition 2.3), i.e.,

$$M_{\mathrm{alg}} := \mathbb{K}\langle M \rangle.$$

With reference to the map $\chi$ in Remark 2.4, we know from Remark 2.5 that $\{\chi(m) | \, m \in M\}$ is a basis for $M_{\mathrm{alg}}$. We now define on $M_{\mathrm{alg}}$ an algebra structure, compatible with the underlying structure $(M, .)$. With this aim we set

$$\Big( \sum_{i=1}^{p} \lambda_i \, \chi(m_i) \Big) * \Big( \sum_{i'=1}^{p'} \lambda'_{i'} \, \chi(m'_{i'}) \Big) := \sum_{1 \le i \le p, \; 1 \le i' \le p'} \lambda_i \lambda'_{i'} \, \chi(m_i . m'_{i'}),$$

for any arbitrary $p, p' \in \mathbb{N}$, $\lambda_1, \dots, \lambda_p \in \mathbb{K}$, $\lambda'_1, \dots, \lambda'_{p'} \in \mathbb{K}$, $m_1, \dots, m_p \in M$, $m'_1, \dots, m'_{p'} \in M$. Following the notation in (2.4), the $*$ operation can be rewritten (w.r.t. the basis $\chi(M)$) as

$$f * f' = \sum_{m \in M} \Big( \sum_{a, a' \in M: \; a.a' = m} f(a) f'(a') \Big) \chi(m), \qquad \forall \, f, f' \in M_{\mathrm{alg}}$$

(having set $f = \sum_{a \in M} f(a) \, \chi(a)$, $f' = \sum_{a' \in M} f'(a') \, \chi(a')$).

It is easy to prove that $(M_{\text{alg}}, *)$ is an algebra (when $M$ is a magma), an associative algebra (when $M$ is a semigroup) and a UA algebra (when $M$ is a monoid) with unit $\chi(e)$ ($e$ being the unit of $M$), called *the algebra of* $M$. Clearly $m * m' = m.m'$ for every $m, m' \in M$ (by identifying $m \equiv \chi(m)$, $m' \equiv \chi(m')$) so that $*$ can be viewed as a prolongation of the former . operation.

*Remark 2.23.* If $(M, .)$ is a magma (resp., a monoid), then the *injective* map

$$\chi : (M, .) \to (M_{\text{alg}}, *)$$

is a magma morphism (resp., a monoid morphism). Indeed, one has $\chi(m) * \chi(m') = \chi(m.m')$, for every $m, m' \in M$ by the definition of $*$ (together with the fact that $\chi(e)$ is the unit of $M_{\text{alg}}$ when $e$ is the unit of the monoid $M$).

The passage from $M$ to the corresponding $M_{\text{alg}}$ has a universal property:

**Lemma 2.24 (Universal Property of the Algebra of a Magma, of a Monoid).** *Let $M$ be a magma.*

(i) *For every algebra $A$ and every magma morphism $f : M \to A$ (here $A$ is equipped only with its magma structure), there exists a unique algebra morphism $f^\chi : M_{\text{alg}} \to A$ with the following property*

$$f^\chi(\chi(m)) = f(m), \quad \textit{for every } m \in M, \tag{2.16}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\chi"'] & A \\ M_{\text{alg}} \arrow[ur, "f^\chi"'] & \end{tikzcd}$$

(ii) *Vice versa, suppose $N, \varphi$ are respectively an algebra and a magma morphism $\varphi : M \to N$ with the following property: For every algebra $A$ and every magma morphism $f : M \to A$, there exists a unique algebra morphism $f^\varphi : N \to A$ such that*

$$f^\varphi(\varphi(m)) = f(m), \quad \textit{for every } m \in M,$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi"'] & A \\ N \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then $N$ is canonically algebra-isomorphic to $M_{\mathrm{alg}}$, the algebra isomorphism being (see the notation in part* (i) *above) $\varphi^\chi : M_{\mathrm{alg}} \to N$ and its inverse being $\chi^\varphi : N \to M_{\mathrm{alg}}$. Furthermore $\varphi$ is injective and $\varphi(M)$ is a linear basis for $N$. Actually, it holds that $\varphi = \varphi^\chi \circ \chi$. Finally, it also holds $N \simeq (\varphi(M))_{\mathrm{alg}} = \mathbb{K}\langle\varphi(M)\rangle$, the algebra of the magma $\varphi(M)$ or, equivalently, the free vector space over the set $\varphi(M)$.*

(iii) *Statements analogous to (i) and (ii) hold when $M$ is a monoid, by replacing, respectively, the above algebras $A, N$, the magma morphisms $f, \varphi$ and the algebra morphisms $f^\chi, f^\varphi$ by, respectively, UA algebras $A, N$, monoid morphisms $f, \varphi$ and UA algebra morphisms $f^\chi, f^\varphi$.*

*Proof.* See page 396 in Chap. 7. □

In the particular case when $M = M(X)$ is the free magma over the set $X$, we set $\mathrm{Lib}(X) := (M(X))_{\mathrm{alg}}$ and we call it *the free (non-associative) algebra over* $X$. Moreover, when $M = \mathrm{Mo}(X)$ is the free monoid over $X$, we set $\mathrm{Libas}(X) := (\mathrm{Mo}(X))_{\mathrm{alg}}$ and we call it *the free UA algebra over* $X$.

More explicitly, we have

$$\mathrm{Lib}(X) := \mathbb{K}\langle M(X)\rangle, \qquad \mathrm{Libas}(X) := \mathbb{K}\langle \mathrm{Mo}(X)\rangle, \tag{2.17}$$

i.e., *the free (non-associative) algebra over $X$ is the free vector space related to the free magma over $X$ and the free UA algebra over $X$ is the free vector space related to the free monoid over $X$*, both endowed with the associated algebra structure introduced at the beginning of this section.

It is customary to identify $M(X)$ (resp., $\mathrm{Mo}(X)$) with a subset of $\mathrm{Lib}(X)$ (resp., of $\mathrm{Libas}(X)$) via the associated map $\chi$, and we shall do this when confusion does not arise. Hence, it is customary to consider $X$ as a subset of $\mathrm{Lib}(X)$ and of $\mathrm{Libas}(X)$. (But within special commutative diagrams we shall often preserve the map $\chi$.)

*Remark 2.25.* By an abuse of notation, we shall use the same symbol $\chi|_X$ in the following statements, whose proof is straightforward:

1. The map $\chi|_X : X \to \mathrm{Lib}(X)$ obtained by composing the maps $X \hookrightarrow M(X) \xrightarrow{\chi} \mathbb{K}\langle M(X)\rangle = \mathrm{Lib}(X)$ is injective and $\chi(X)$ generates $\mathrm{Lib}(X)$ as an algebra (in the non-associative case).
2. The map $\chi|_X : X \to \mathrm{Libas}(X)$ obtained by composing the maps $X \hookrightarrow \mathrm{Mo}(X) \xrightarrow{\chi} \mathbb{K}\langle \mathrm{Mo}(X)\rangle = \mathrm{Libas}(X)$ is injective and $\{\chi(e)\} \cup \chi(X)$ generates $\mathrm{Libas}(X)$ as an algebra (in the associative case).

*Remark 2.26.* 1. *The set $\chi(X)$ is a set of generators for* $\mathrm{Lib}(X)$*, as an algebra* (this follows from Remark 2.19). Identifying $M(X)$ with $\chi(M(X))$, we shall also say that $X$ is a set of generators for $\mathrm{Lib}(X)$, as an algebra. If we set ($M_n$ being defined in (2.13))

$$\mathrm{Lib}_n(X) := \mathrm{span}\{\chi(M_n(X))\}, \quad n \in \mathbb{N}, \tag{2.18}$$

then $\mathrm{Lib}(X)$ is a graded algebra, for it holds that

$$\mathrm{Lib}(X) = \bigoplus_{n \geq 1} \mathrm{Lib}_n(X), \qquad \mathrm{Lib}_i(X) * \mathrm{Lib}_j(X) \subseteq \mathrm{Lib}_{i+j}(X), \quad i, j \geq 1, \tag{2.19}$$

where $*$ here denotes the algebra structure on $\mathrm{Lib}(X)$ induced by the magma $(M(X), .)$.

2. Let $e$ denote the empty word, i.e., the unit of $\mathrm{Mo}(X)$. Then *the set* $\{\chi(e)\} \cup \chi(X)$ *is a set of generators for* $\mathrm{Libas}(X)$*, as an algebra* (this follows from Remark 2.21). With the identification of $\mathrm{Mo}(X)$ with $\chi(\mathrm{Mo}(X))$, we shall also say that $\{e\} \cup X$ is a set of generators for $\mathrm{Libas}(X)$, as an algebra.
If we set ($W_n$ being defined in (2.15))

$$\mathrm{Libas}_n(X) := \mathrm{span}\{\chi(W_n)\}, \quad n \geq 0, \tag{2.20}$$

then $\mathrm{Libas}(X)$ is a graded algebra, for it holds that

$$\begin{aligned} &\mathrm{Libas}(X) = \textstyle\bigoplus_{n \geq 0} \mathrm{Libas}_n(X), \\ &\mathrm{Libas}_i(X) * \mathrm{Libas}_j(X) \subseteq \mathrm{Libas}_{i+j}(X), \quad i, j \geq 0, \end{aligned} \tag{2.21}$$

where $*$ here denotes the algebra structure on $\mathrm{Libas}(X)$ induced by the monoid $(\mathrm{Mo}(X), .)$.

The above Lemma 2.24 produces the following results, which we explicitly state for the sake of future reference.

**Theorem 2.27 (Universal Property of the Free Algebra).** *Let $X$ be a set.*

(i) *For every algebra $A$ and every function $f : X \to A$, there exists a unique algebra morphism $f^\chi : \mathrm{Lib}(X) \to A$ with the following property*

$$f^\chi(\chi(x)) = f(x), \quad \textit{for every } x \in X, \tag{2.22}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi|_X"'] & A \\ \mathrm{Lib}(X) \arrow[ur, "f^\chi"'] & \end{tikzcd}$$

*(Here $\chi|_X : X \to \mathrm{Lib}(X)$ is the composition of maps $X \hookrightarrow M(X) \xrightarrow{\chi} \mathbb{K}\langle M(X)\rangle = \mathrm{Lib}(X)$.)*

(ii) *Conversely, suppose* $N, \varphi$ *are respectively an algebra and a map* $\varphi : X \to N$ *with the following property: For every algebra* $A$ *and every function* $f : X \to A$*, there exists a unique algebra morphism* $f^\varphi : N \to A$ *such that*

$$f^\varphi(\varphi(x)) = f(x), \quad \textit{for every } x \in M,$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & A \\ N \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then* $N$ *is canonically algebra-isomorphic to* $\mathrm{Lib}(X)$*, the algebra isomorphism being (see the notation in part* (i) *above)* $\varphi^\chi : \mathrm{Lib}(X) \to N$ *and its inverse being* $(\chi|_X)^\varphi : N \to \mathrm{Lib}(X)$*. Furthermore* $\varphi$ *is injective and* $\varphi(X)$ *generates* $N$ *as an algebra. Actually, it holds that* $\varphi = \varphi^\chi \circ (\chi|_X)$*. Finally, it also holds* $N \simeq \mathrm{Lib}(\varphi(X))$*, the free non-associative algebra over the set* $\varphi(X)$*.*

*Proof.* (i): From Lemma 2.20-(i), there exists a magma morphism $\overline{f} : M(X) \to A$ prolonging $f$. From Lemma 2.24-(i), there exists an algebra morphism

$$\overline{f}^\chi : (M(X))_{\mathrm{alg}} = \mathrm{Lib}(X) \to A$$

such that $\overline{f}^\chi(\chi(m)) = \overline{f}(m)$ for every $m \in M(X)$. The choice $f^\chi := \overline{f}^\chi$ does the job. The uniqueness part of the thesis derives from the fact that $\chi(X)$ generates $\mathrm{Lib}(X)$ as an algebra.

Part (ii) is standard (it makes use of Remark 2.25-1). $\square$

**Theorem 2.28 (Universal Property of the Free UA Algebra).** *Let* $X$ *be a set.*

(i) *For every UA algebra* $A$ *and every function* $f : X \to A$*, there exists a unique UAA morphism* $f^\chi : \mathrm{Libas}(X) \to A$ *with the following property*

$$f^\chi(\chi(x)) = f(x), \quad \textit{for every } x \in X, \tag{2.23}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi|_X"'] & A \\ \mathrm{Libas}(X) \arrow[ur, "f^\chi"'] & \end{tikzcd}$$

*(Here* $\chi|_X : X \to \mathrm{Libas}(X)$ *is the composition of maps* $X \hookrightarrow \mathrm{Mo}(X) \xrightarrow{\chi} \mathbb{K}\langle \mathrm{Mo}(X)\rangle = \mathrm{Libas}(X)$*.)*

(ii) *Vice versa, suppose* $N, \varphi$ *are respectively a UA algebra and a map* $\varphi : X \to N$ *with the following property: For every UA algebra* $A$ *and every function* $f : X \to A$*, there exists a unique UAA morphism* $f^\varphi : N \to A$ *such that*

$$f^\varphi(\varphi(x)) = f(x), \quad \textit{for every } x \in X,$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & A \\ N \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then* $N$ *is canonically isomorphic to* $\mathrm{Libas}(X)$ *as UA algebras, the UAA isomorphism being (see the notation in part* (i) *above)* $\varphi^\chi : \mathrm{Libas}(X) \to N$ *and its inverse being* $(\chi|_X)^\varphi : N \to \mathrm{Libas}(X)$*. Furthermore* $\varphi$ *is injective and* $\{e_N\} \cup \varphi(X)$ *generates* $N$ *as an algebra (*$e_N$ *denoting the unit of* $N$*). Actually, it holds that* $\varphi = \varphi^\chi \circ (\chi|_X)$*. Finally, it holds that* $N \simeq \mathrm{Libas}(\varphi(X))$*, the free UA algebra over* $\varphi(X)$*.*

*Proof.* The proof is analogous to that of Theorem 2.27, making use of Lemma 2.22-(i), Lemma 2.24-(iii) and Remark 2.25-2. □

### 2.1.3 *Tensor Product and Tensor Algebra*

Let $n \in \mathbb{N}$, $n \geq 2$ and $V_1, \dots, V_n$ be vector spaces. Let us consider the Cartesian product $V_1 \times \cdots \times V_n$ (which we *do not* endow with a vector space structure!) and the corresponding free vector space $\mathbb{K}\langle V_1 \times \cdots \times V_n\rangle$ (see Definition 2.3). The notation $\chi(v_1, \dots, v_n)$ agrees with the one given in Remark 2.4.

Let us consider the subspace of $\mathbb{K}\langle V_1 \times \cdots \times V_n\rangle$, say $W$, spanned by the elements of the following form

$$\begin{aligned} &\chi(v_1, \dots, a\, v_i, \dots, v_n) - a\, \chi(v_1, \dots, v_i, \dots, v_n), \\ &\chi(v_1, \dots, v_i + v_i', \dots, v_n) - \chi(v_1, \dots, v_i, \dots, v_n) - \chi(v_1, \dots, v_i', \dots, v_n), \end{aligned} \tag{2.24}$$

where $a \in \mathbb{K}$, $i \in \{1, \ldots, n\}$, $v_j, v'_j \in V_j$ for every $j \in \{1, \ldots, n\}$. The main definition of this section is the following one:

$$V_1 \otimes \cdots \otimes V_n := \mathbb{K}\langle V_1 \times \cdots \times V_n \rangle / W.$$

We say that $V_1 \otimes \cdots \otimes V_n$ is the *tensor product* of the (ordered) vector spaces $V_1, \ldots, V_n$ (orderly). Moreover, if $\pi : \mathbb{K}\langle V_1 \times \cdots \times V_n \rangle \to V_1 \otimes \cdots \otimes V_n$ is the associated projection, we also set

$$v_1 \otimes \cdots \otimes v_n := \pi(\chi(v_1, \ldots, v_n)), \quad \forall\, v_1 \in V_1, \ldots, \forall\, v_n \in V_n.$$

The element $v_1 \otimes \cdots \otimes v_n$ of the tensor product $V_1 \otimes \cdots \otimes V_n$ is called an *elementary tensor* of $V_1 \otimes \cdots \otimes V_n$. Not every element of $V_1 \otimes \cdots \otimes V_n$ is elementary, but every element of $V_1 \otimes \cdots \otimes V_n$ is a linear combination of elementary tensors. Finally we introduce the notation

$$\psi : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n, \quad \psi(v_1, \ldots, v_n) := v_1 \otimes \cdots \otimes v_n.$$

In other words $\psi = \pi \circ \chi$.

*Remark 2.29.* With the above notation, *$\psi$ is $n$-linear and $\psi(V_1 \times \cdots \times V_n)$ generates $V_1 \otimes \cdots \otimes V_n$*. The last statement is obvious, whilst the former follows from the computation:

$$\begin{aligned}
&\psi(v_1, \ldots, a\,v_i + a'\,v'_i, \ldots, v_n) = \big[\chi(v_1, \ldots, a\,v_i + a'\,v'_i, \ldots, v_n)\big]_W \\
&\quad = \Big[\chi(v_1, \ldots, a\,v_i + a'\,v'_i, \ldots, v_n) \\
&\qquad - \chi(v_1, \ldots, a\,v_i, \ldots, v_n) - \chi(v_1, \ldots, a'\,v'_i, \ldots, v_n)\Big]_W \\
&\qquad + \big[\chi(v_1, \ldots, a\,v_i, \ldots, v_n) + \chi(v_1, \ldots, a'\,v'_i, \ldots, v_n)\big]_W \\
&\quad = 0 + \big[\chi(v_1, \ldots, a\,v_i, \ldots, v_n) - a\,\chi(v_1, \ldots, v_i, \ldots, v_n)\big]_W \\
&\qquad + \big[\chi(v_1, \ldots, a'\,v'_i, \ldots, v_n) - a'\,\chi(v_1, \ldots, v'_i, \ldots, v_n)\big]_W \\
&\qquad + a\,[\chi(v_1, \ldots, v_i, \ldots, v_n)]_W + a'\,[\chi(v_1, \ldots, v'_i, \ldots, v_n)]_W \\
&\quad = 0 + 0 + 0 + a\,\psi(v_1, \ldots, v_i, \ldots, v_n) + a'\,\psi(v_1, \ldots, v'_i, \ldots, v_n).
\end{aligned}$$

Using the "$\otimes$" notation instead of $\psi$, the previous remark takes the form

$$\begin{aligned}
v_1 \otimes \cdots \otimes (a\,v_i + a'\,v'_i) \otimes \cdots \otimes v_n &= a\,\big(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n\big) \\
&\quad + a'\,\big(v_1 \otimes \cdots \otimes v'_i \otimes \cdots \otimes v_n\big),
\end{aligned}$$

for every $a, a' \in \mathbb{K}$, every $i \in \{1, \ldots, n\}$ and every $v_j, v'_j \in V_j$ for $j = 1, \ldots, n$.

We are ready for another universal-property theorem of major importance.

**Theorem 2.30 (Universal Property of the Tensor Product).**

(i) *Let $n \in \mathbb{N}$, $n \geq 2$ and let $V_1, \ldots, V_n$ be vector spaces. Then, for every vector space $X$ and every $n$-linear map $F : V_1 \times \cdots \times V_n \to X$, there exists a unique linear map $F^\psi : V_1 \otimes \cdots \otimes V_n \to X$ such that*

$$F^\psi(\psi(v)) = F(v) \quad \textit{for every } v \in V_1 \times \cdots \times V_n, \tag{2.25}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "F"] \arrow[d, "\psi"'] & X \\ V_1 \otimes \cdots \otimes V_n \arrow[ur, "F^\psi"'] & \end{tikzcd}$$

(ii) *Conversely, suppose that $V, \varphi$ are respectively a vector space and an $n$-linear map $\varphi : V_1 \times \cdots \times V_n \to V$ with the following property: for every vector space $X$ and every $n$-linear map $F : V_1 \times \cdots \times V_n \to X$, there exists a unique linear map $F^\varphi : V \to X$ such that*

$$F^\varphi(\varphi(v)) = F(v) \quad \textit{for every } v \in V_1 \times \cdots \times V_n, \tag{2.26}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} V_1 \times \cdots \times V_n \arrow[r, "F"] \arrow[d, "\varphi"'] & X \\ V \arrow[ur, "F^\varphi"'] & \end{tikzcd}$$

*Then $V$ is canonically isomorphic to $V_1 \otimes \cdots \otimes V_n$, the isomorphism in one direction being $\varphi^\psi : V_1 \otimes \cdots \otimes V_n \to V$ with its inverse being $\psi^\varphi : V \to V_1 \otimes \cdots \otimes V_n$. Furthermore the set $\varphi(S)$ is a set of generators for $V$.*

*Proof.* See page 396 in Chap. 7. □

Some natural properties of tensor products are now in order.

**Theorem 2.31 (Basis of the Tensor Product).** *Let $V, W$ be vector spaces with bases $\{v_i\}_{i \in \mathcal{I}}$ and $\{w_k\}_{k \in \mathcal{K}}$ respectively. Then*

$$\left\{v_i \otimes w_k\right\}_{(i,k) \in \mathcal{I} \times \mathcal{K}}$$

*is a basis of $V \otimes W$.*

*Proof.* The proof of this expected result is unexpectedly delicate: See page 399 in Chap. 7. □

**Proposition 2.32 ("Associativity" of $\otimes$).** *Let $n, m \in \mathbb{N}$, $n, m \geq 2$ and let $V_1, \dots, V_n$, $W_1, \dots, W_m$ be vector spaces. Then we have the isomorphism (of vector spaces)*

$$(V_1 \otimes \cdots \otimes V_n) \otimes (W_1 \otimes \cdots \otimes W_m) \simeq V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m.$$

*To this end, we can consider the* canonical *isomorphism mapping*

$$(v_1 \otimes \cdots \otimes v_n) \otimes (w_1 \otimes \cdots \otimes w_m)$$

*into* $v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$.

*Proof.* See page 403 in Chap. 7. □

If $V$ is a vector space and $k \in \mathbb{N}$, we set

$$\mathscr{T}_k(V) := \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}.$$

Thus, the generic element of $\mathscr{T}_k(V)$ is a finite linear combination of tensors of the form $v_1 \otimes \cdots \otimes v_k$, with $v_1, \dots, v_k \in V$. We also set $\mathscr{T}_0(V) := \mathbb{K}$. The elements of $\mathscr{T}_k(V)$ are referred to as being tensors of *degree* (or *order*, or *length*) $k$ on $V$. We are in a position to introduce a fundamental definition.

**Definition 2.33 (Tensor Algebra of a Vector Space).** Let $V$ be a vector space. We set $\mathscr{T}(V) := \bigoplus_{k \in \mathbb{N} \cup \{0\}} \mathscr{T}_k(V)$ (in the sense of external direct sums).
On $\mathscr{T}(V)$ we consider the operation defined by

$$(v_i)_{i \geq 0} \cdot (w_j)_{j \geq 0} := \Big( \sum_{j=0}^{k} v_{k-j} \otimes w_j \Big)_{k \geq 0}, \tag{2.27}$$

where $v_i, w_i \in \mathscr{T}_i(V)$ for every $i \geq 0$. Here, we identify any tensor product

$$\mathscr{T}_{k-j}(V) \otimes \mathscr{T}_j(V)$$

with $\mathscr{T}_k(V)$, for every $k \in \mathbb{N} \cup \{0\}$ and every $j = 0, \dots, k$ (thanks to Proposition 2.32). We call $\mathscr{T}(V)$ (equipped with this operation) *the tensor algebra of* $V$.

Throughout the Book, we consider any $\mathscr{T}_k(V)$ as a subset of $\mathscr{T}(V)$ as described in Remark 2.7. Moreover, we make the identification $V \equiv \mathscr{T}_1(V)$ so that $V$ is considered as a subset of its tensor algebra. When there is no possibility of confusion, we denote $\mathscr{T}_k(V)$ and $\mathscr{T}(V)$ simply by $\mathscr{T}_k$ and $\mathscr{T}$.

If $v = (v_i)_{i\geq 0} \in \mathscr{T}(V)$ (being $v_i \in \mathscr{T}_i(V)$ for every $i \geq 0$), we say that $v_i$ is the *homogeneous component* of $v$ of degree (or order, or length) $i$. Moreover, in writing $v = (v_i)_{i\geq 0}$ for an element $v \in \mathscr{T}(V)$ we tacitly mean that $v_i \in \mathscr{T}_i(V)$ for every $i \in \mathbb{N} \cup \{0\}$. The notation $\sum_{i\geq 0} v_i$ for $(v_i)_{i\geq 0}$ will sometimes apply.

*Remark 2.34.* We have the following remarks.

1. The operation $\cdot$ on $\mathscr{T}(V)$ is the only bilinear operation on $\mathscr{T}(V)$ whose restriction to $\mathscr{T}_{k-j} \times \mathscr{T}_j$ coincides with the map
$$\mathscr{T}_{k-j}(V) \times \mathscr{T}_j(V) \ni (v_{k-j}, w_j) \mapsto v_{k-j} \otimes w_j \in \mathscr{T}_k(V),$$
for every $k \in \mathbb{N} \cup \{0\}$ and every $j = 0, \ldots, k$. Equivalently, it holds that $v \cdot w = v \otimes w$, whenever $v \in \mathscr{T}_i(V)$ and $\mathscr{T}_j(V)$ for some $i, j \geq 0$. Note that $\mathscr{T}(V)$ is generated, as an algebra, by the elements of $V$ (or of a basis of $V$) through iterated $\otimes$ operations (or equivalently, iterated $\cdot$ operations).
2. The name "tensor algebra" is motivated by the fact that $(\mathscr{T}(V), \cdot)$ *is a unital associative algebra*. The unit is $1_{\mathbb{K}} \in \mathscr{T}_0(V)$. As for the other axioms of UA algebra, we leave them all to the Reader, apart from the associativity of $\cdot$, which we prove explicitly as follows:
$$(u_i)_{i\geq 0} \cdot \Big((v_i)_{i\geq 0} \cdot (w_i)_{i\geq 0}\Big) = (u_i)_{i\geq 0} \cdot \Big(\sum_{j=0}^{i} v_{i-j} \otimes w_j\Big)_{i\geq 0}$$
$$= \left(\sum_{h=0}^{i} u_{i-h} \otimes \Big(\sum_{j=0}^{h} v_{h-j} \otimes w_j\Big)\right)_{i\geq 0} = \left(\sum_{h=0}^{i}\sum_{j=0}^{h} u_{i-h} \otimes v_{h-j} \otimes w_j\right)_{i\geq 0}$$
(we interchange the sums and then rename the dummy index $h - j =: k$)
$$= \left(\sum_{j=0}^{i}\sum_{h=j}^{i} \cdots\right)_{i\geq 0} = \left(\sum_{j=0}^{i}\sum_{k=0}^{i-j} u_{i-j-k} \otimes v_k \otimes w_j\right)_{i\geq 0}$$
$$= \left(\sum_{j=0}^{i} \Big((u_i)_{i\geq 0} \cdot (v_i)_{i\geq 0}\Big)_{i-j} \otimes w_j\right)_{i\geq 0} = \Big((u_i)_{i\geq 0} \cdot (v_i)_{i\geq 0}\Big) \cdot (w_i)_{i\geq 0}.$$
3. By the very definition of $\mathscr{T}(V)$, we have
$$\mathscr{T}(V) = \bigoplus_{i\geq 0} \mathscr{T}_i(V), \quad \text{and } \mathscr{T}_i(V) \cdot \mathscr{T}_j(V) \subseteq \mathscr{T}_{i+j}(V) \text{ for every } i, j \geq 0. \tag{2.28}$$
In particular, $\mathscr{T}(V)$ *is a graded algebra.* We next introduce a notation which will be used repeatedly in the sequel: for $k \in \mathbb{N} \cup \{0\}$ we set
$$U_k(V) := \bigoplus_{i\geq k} \mathscr{T}_i(V), \quad \mathscr{T}_+(V) := U_1(V) = \bigoplus_{i\geq 1} \mathscr{T}_i(V). \tag{2.29}$$

The notation $U_k, \mathscr{T}_+$ will also apply. We have the following properties:

a. Every $U_k$ is an ideal in $\mathscr{T}(V)$ containing $\mathscr{T}_k(V)$.
b. $\mathscr{T}(V) = U_0(V) \supset U_1(V) \supset \cdots U_k(V) \supset U_{k+1}(V) \supset \cdots$.
c. $U_i(V) \cdot U_j(V) \subseteq U_{i+j}(V)$, for every $i, j \geq 0$.
d. $\bigcap_{i \geq 0} U_i(V) = \{0\}$ and, more generally, $\bigcap_{i \geq k} U_i(V) = \{0\}$ for every $k \in \mathbb{N} \cup \{0\}$.

Note that $\mathscr{T}_+(V)$ (also, any $U_k(V)$ with $k \geq 1$) is an associative algebra with the operation $\cdot$, but *it is not a unital associative algebra*.

**Proposition 2.35 (Basis of the Tensor Algebra).** *Let $V$ be a vector space and let $\mathcal{B} = \{e_i\}_{i \in \mathcal{I}}$ be a basis of $V$. Then the following facts hold:*

1. *For every fixed $k \in \mathbb{N}$, the system $\mathcal{B}_k := \big\{e_{i_1} \otimes \cdots \otimes e_{i_k} \,\big|\, i_1, \ldots, i_k \in \mathcal{I}\big\}$ is a basis of $\mathscr{T}_k(V)$ (which we call* induced by $\mathcal{B}$*).*
2. *The system*

$$\{1_{\mathbb{K}}\} \cup \bigcup_{k \in \mathbb{N}} \mathcal{B}_k = \Big\{1_K, \; e_{i_1} \otimes \cdots \otimes e_{i_k} \,\Big|\, k \in \mathbb{N}, \;\; i_1, \ldots, i_k \in \mathcal{I}\Big\}$$

*is a basis of $\mathscr{T}(V)$ (which we call* induced by $\mathcal{B}$*).*

*Proof.* (1) follows from Theorem 2.31, whilst (2) follows from (1) together with Proposition 2.9. □

*Remark 2.36.* Also, the following are systems of generators for $\mathscr{T}(V)$:

$$\{1_K\} \cup \Big\{v_1^n \otimes \cdots \otimes v_n^n \;\text{ where }\; n \in \mathbb{N}, \;\; v_i^n \in V \text{ for } i \leq n\Big\};$$

and $$\Big\{\Big(1_{\mathbb{K}}, v_1^1, v_1^2 \otimes v_2^2, v_1^3 \otimes v_2^3 \otimes v_3^3, \ldots, v_1^n \otimes \cdots \otimes v_n^n, 0, \ldots\Big),$$
$$\text{where} \quad n \in \mathbb{N}, \;\; v_i^j \in V \text{ for every } j \leq n \text{ and } i \leq j\Big\}.$$

*Remark 2.37.* The previous remark shows that $V$ *generates $\mathscr{T}_+(V)$ as an algebra* and that *the set $\{1_{\mathbb{K}}\} \cup V$ generates $\mathscr{T}(V)$ as an algebra.* (Indeed, if $v_1, \ldots, v_n \in V$ we have $v_1 \cdot \ldots \cdot v_n = v_1 \otimes \cdots \otimes v_n$.)

Together with the fact that $\chi(X)$ generates $\mathbb{K}\langle X\rangle$ as a vector space, we get that $\{1_{\mathbb{K}}\} \cup \chi(X)$ generates $\mathscr{T}(\mathbb{K}\langle X\rangle)$ as an algebra (and $\chi(X)$ generates $\mathscr{T}_+(\mathbb{K}\langle X\rangle)$). By identifying $X$ and $\chi(X)$, this last fact amounts simply to saying that the letters of $X$ and the unit $1_{\mathbb{K}}$ generate $\mathscr{T}(\mathbb{K}\langle X\rangle)$, the *free UA algebra of the words on $X$*.

The following result will be used again and again in this Book.

**Theorem 2.38 (Universal Property of the Tensor Algebra).**
*Let $V$ be a vector space.*

(i) *For every associative algebra $A$ and every linear map $f : V \to A$, there exists a unique algebra morphism $\overline{f} : \mathscr{T}_+(V) \to A$ prolonging $f$, thus making the following a commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook, "\iota"'] & A \\ \mathscr{T}_+(V) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

(ii) *For every unital associative algebra $A$ and every linear map $f : V \to A$, there exists a unique UAA morphism $\overline{f} : \mathscr{T}(V) \to A$ prolonging $f$, thus making the following a commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook, "\iota"'] & A \\ \mathscr{T}(V) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

(iii) *Vice versa, suppose $W, \varphi$ are respectively a UA algebra and a linear map $\varphi : V \to W$ with the following property: For every UA algebra $A$ and every linear map $f : V \to A$, there exists a unique UAA morphism $f^\varphi : W \to A$ such that*

$$f^\varphi(\varphi(v)) = f(v) \quad \textit{for every } v \in V, \tag{2.30}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & A \\ W \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then $W$ is canonically isomorphic, as UA algebra, to $\mathscr{T}(V)$, the isomorphism being (see the notation in* (ii) *above) $\overline{\varphi} : \mathscr{T}(V) \to W$ and its inverse being $\iota^\varphi : W \to \mathscr{T}(V)$. Furthermore, $\varphi$ is injective and $W$ is generated, as an algebra, by the set $\{1_W\} \cup \varphi(V)$. Actually it holds that $\varphi = \overline{\varphi} \circ \iota$. Finally we have $W \simeq \mathscr{T}(\varphi(V))$, canonically.*

*Proof.* Explicitly, if $\star$ is the operation on $A$, $\overline{f}$ in (i) is the unique linear map such that

$$\overline{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1) \star \cdots \star f(v_k), \tag{2.31}$$

for every $k \in \mathbb{N}$ and every $v_1, \ldots, v_k \in V$. Also, if $e_A$ is the unit of the UA algebra $A$, the map $\overline{f}$ in (ii) is the unique linear map such that

$$\overline{f}(1_K) = e_A, \quad \overline{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1) \star \cdots \star f(v_k), \tag{2.32}$$

for every $k \in \mathbb{N}$ and every $v_1, \ldots, v_k \in V$. For the proof of this theorem, see page 404 in Chap. 7. $\square$

*Remark 2.39.* Let $V, W$ be two isomorphic vector spaces with isomorphism $\Psi : V \to W$. Then $\mathscr{T}(V)$ and $\mathscr{T}(W)$ are isomorphic as UA algebras, via the UAA isomorphism $\widetilde{\Psi} : \mathscr{T}(V) \to \mathscr{T}(W)$ such that $\widetilde{\Psi}(v) = \Psi(v)$ for every $v \in V$ and

$$\widetilde{\Psi}(v_1 \otimes \cdots \otimes v_k) = \Psi(v_1) \otimes \cdots \otimes \Psi(v_k), \tag{2.33}$$

for every $k \in \mathbb{N}$ and every $v_1, \ldots, v_k \in V$. [Indeed, the above map $\widetilde{\Psi}$ is the unique UAA morphism prolonging the linear map $V \xrightarrow{\Psi} W \hookrightarrow \mathscr{T}(W)$; $\widetilde{\Psi}$ is an isomorphism, for its inverse is the unique UAA morphism from $\mathscr{T}(W)$ to $\mathscr{T}(V)$ prolonging the linear map $W \xrightarrow{\Psi^{-1}} V \hookrightarrow \mathscr{T}(V)$.]

The following theorem describes one of the distinguished properties of $\mathscr{T}(V)$ as the "container" of several of our universal objects (we shall see later that it contains the free Lie algebra of $V$ and the symmetric algebra of $V$, too).

**Theorem 2.40 ($\mathscr{T}(\mathbb{K}\langle X\rangle)$ is isomorphic to Libas$(X)$).** *Let $X$ be any set and $\mathbb{K}$ a field.*

(1). *The tensor algebra $\mathscr{T}(\mathbb{K}\langle X\rangle)$ of the free vector space $\mathbb{K}\langle X\rangle$ is isomorphic, as UA algebra, to* $\mathrm{Libas}(X)$*, the free unital associative algebra over $X$. As a (canonical) UAA isomorphism, we can consider the linear map $\Psi : \mathscr{T}(\mathbb{K}\langle X\rangle) \to$* $\mathrm{Libas}(X)$ *such that*[6]

$$\Psi(x_1 \otimes \cdots \otimes x_k) = x_1 . \cdots . x_k, \quad \text{for every } k \in \mathbb{N} \text{ and every } x_1, \ldots, x_k \in X,$$

*and such that $\Psi(1_{\mathbb{K}}) = e$, $e$ being the unit of* $\mathrm{Libas}(X)$.

(2). *More precisely, the couple $(\mathscr{T}(\mathbb{K}\langle X\rangle), \varphi)$ satisfies the universal property of the free UA algebra over $X$, where $\varphi : X \to \mathscr{T}(\mathbb{K}\langle X\rangle)$ denotes the canonical injection*

$$X \xrightarrow{\chi} \mathbb{K}\langle X\rangle \xhookrightarrow{\iota} \mathscr{T}(\mathbb{K}\langle X\rangle).$$

[6] Here we are thinking of $X$ (respectively, $\mathrm{Mo}(X)$) as a subset of $\mathbb{K}\langle X\rangle \hookrightarrow \mathscr{T}(\mathbb{K}\langle X\rangle)$ (of $\mathbb{K}\langle \mathrm{Mo}(X)\rangle = \mathrm{Libas}(X)$, respectively).

*This means that, for every UA algebra $A$ and every function $f : X \to A$, there exists a unique UAA morphism $f^\varphi : \mathscr{T}(\mathbb{K}\langle X\rangle) \to A$ with the following property*

$$f^\varphi(\varphi(x)) = f(x), \quad \textit{for every } x \in X, \tag{2.34}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & A \\ \mathscr{T}(\mathbb{K}\langle X\rangle) \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Proof.* In view of Theorem 2.28-(ii), it is enough to show that the above couple $(\mathscr{T}(\mathbb{K}\langle X\rangle), \varphi)$ satisfies the universal property of the free UA algebra over $X$. With this aim, let $A$ be a UA algebra and let $f : X \to A$ be any map. By Theorem 2.6-(i), there exists a linear map $f^\chi : \mathbb{K}\langle X\rangle \to A$ such that $f^\chi(\chi(x)) = f(x)$ for every $x \in X$. Then, by Theorem 2.38-(ii), there exists a UAA morphism $\overline{f^\chi} : \mathscr{T}(\mathbb{K}\langle X\rangle) \to A$ such that $\overline{f^\chi}(\iota(v)) = f^\chi(v)$, for every $v \in \mathbb{K}\langle X\rangle$. Setting $f^\varphi := \overline{f^\chi}$, one obviously has

$$f^\varphi(\varphi(x)) = \overline{f^\chi}((\iota \circ \chi)(x)) = f^\chi(\chi(x)) = f(x), \quad \forall\, x \in X.$$

Moreover $f^\varphi$ is the unique UAA morphism such that $f^\varphi(\varphi(x)) = f(x)$, for every $x \in X$, since $\varphi(X) = \chi(X)$ and $\{1\} \cup \chi(X)$ generates $\mathscr{T}(\mathbb{K}\langle X\rangle)$, as an algebra.

By Theorem 2.28-(ii), we thus have $\mathscr{T}(\mathbb{K}\langle X\rangle) \simeq \mathrm{Libas}(X)$ via the (unique) UAA isomorphism $\varPsi : \mathscr{T}(\mathbb{K}\langle X\rangle) \to \mathrm{Libas}(X)$ mapping $x \equiv \chi(x) \in \mathbb{K}\langle X\rangle$ into $x \equiv \chi(x) \in \mathbb{K}\langle \mathrm{Mo}(X)\rangle$. The theorem is proved. $\square$

#### 2.1.3.1 Tensor Product of Algebras

Let $(A, \circledast)$ and $(B, \circledcirc)$ be two UA algebras (over $\mathbb{K}$). We describe a natural way to equip $A \otimes B$ with a UA algebra structure. Consider the Cartesian product $A \times B \times A \times B$ and the map

$$F : A \times B \times A \times B \to A \otimes B, \quad (a_1, b_1, a_2, b_2) \mapsto (a_1 \circledast a_2) \otimes (b_1 \circledcirc b_2).$$

We fix $(a_2, b_2) \in A \times B$ and we consider the restriction of $F$ defined by $A \times B \ni (a_1, b_1) \mapsto F(a_1, b_1, a_2, b_2)$. This map is clearly bilinear. Hence, by the universal property of the tensor product in Theorem 2.30, there exists a unique *linear* map $G_{a_2,b_2} : A \otimes B \to A \otimes B$ such that

$$G_{a_2,b_2}(a \otimes b) = F(a, b, a_2, b_2) = (a \circledast a_2) \otimes (b \circledcirc b_2), \qquad \forall\, (a, b) \in A \times B. \tag{2.35}$$

Then we fix $c_1 \in A \otimes B$ and we consider the map

$$\alpha_{c_1} : A \times B \to A \otimes B, \quad (a_2, b_2) \mapsto G_{a_2,b_2}(c_1).$$

It is not difficult to prove that this map is bilinear. Hence, again by Theorem 2.30, there exists a unique *linear* map $\beta_{c_1} : A \otimes B \to A \otimes B$ such that

$$\beta_{c_1}(a \otimes b) = \alpha_{c_1}(a, b) = G_{a,b}(c_1), \qquad \forall\ (a, b) \in A \times B. \tag{2.36}$$

Furthermore, we set

$$H : (A \otimes B) \times (A \otimes B) \to A \otimes B, \quad H(c_1, c_2) := \beta_{c_1}(c_2).$$

By (2.35) and (2.36), we have $H(a_1 \otimes b_1, a_2 \otimes b_2) = (a_1 \circledast a_2) \otimes (b_1 \circledcirc b_2)$. Finally, we define a composition $\bullet$ on $A \otimes B$ as follows:

$$c_1 \bullet c_2 := H(c_1, c_2), \quad \forall\ c_1, c_2 \in A \otimes B.$$

With the above definitions, we have the following fact:

$$(A \otimes B, \bullet) \textit{ is a unital associative algebra.}$$

The (tedious) proof of this fact is omitted: the Reader will certainly have no problem in deriving it. Hence, the following result follows:

**Proposition 2.41.** *Let $(A, \circledast)$ and $(B, \circledcirc)$ be two UA algebras (over $\mathbb{K}$). Then $A \otimes B$ can be equipped with a UA algebra structure by an operation $\bullet$ which is characterized (in a unique way) by its action on elementary tensors as follows:*

$$(a_1 \otimes b_1) \bullet (a_2 \otimes b_2) = (a_1 \circledast a_2) \otimes (b_1 \circledcirc b_2), \quad \forall\ (a_1 \otimes b_1), (a_2 \otimes b_2) \in A \otimes B. \tag{2.37}$$

#### 2.1.3.2 The Algebra $\mathscr{T}(V) \otimes \mathscr{T}(V)$

Let $V$ be a vector space. Following the above section, the tensor product $\mathscr{T}(V) \otimes \mathscr{T}(V)$ can be equipped with a UA algebra structure by means of the operation $\bullet$ such that

$$(a \otimes b) \bullet (a' \otimes b') = (a \cdot a') \otimes (b \cdot b'), \qquad (a, b), (a', b') \in \mathscr{T}(V) \otimes \mathscr{T}(V), \tag{2.38}$$

where $\cdot$ is as in (2.27). Obviously, extended by bilinearity to $\mathscr{T}(V) \otimes \mathscr{T}(V)$, (2.38) characterizes $\bullet$. For any $i, j \in \mathbb{N} \cup \{0\}$, we set[7]

$$\mathscr{T}_{i,j}(V) := \mathscr{T}_i(V) \otimes \mathscr{T}_j(V) \qquad \text{(as a subset of } \mathscr{T}(V) \otimes \mathscr{T}(V)\text{)}. \tag{2.39}$$

[7]The Reader will have care, this time, not to identify

$$\mathscr{T}_i(V) \otimes \mathscr{T}_j(V) = \underbrace{V \otimes \cdots \otimes V}_{i \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{j \text{ times}} \quad \text{with } \underbrace{V \otimes \cdots \otimes V}_{i+j \text{ times}},$$

as we had to do in Definition 2.33.

Given $\mathscr{T} = \bigoplus_{i\geq 0} \mathscr{T}_i$, one obviously has

$$\mathscr{T}(V) \otimes \mathscr{T}(V) = \bigoplus_{i,j\geq 0} \mathscr{T}_{i,j}(V). \tag{2.40}$$

Thanks to the definition of $\bullet$ in (2.38), it holds that

$$\mathscr{T}_{i,j}(V) \bullet \mathscr{T}_{i',j'}(V) \subseteq \mathscr{T}_{i+i',j+j'}(V), \qquad \forall\ i,j,i',j' \geq 0. \tag{2.41}$$

Occasionally, we will also invoke the following direct-sum decomposition:

$$\mathscr{T}(V) \otimes \mathscr{T}(V) = \bigoplus_{k\geq 0} K_k(V), \quad \text{where } K_k(V) := \bigoplus_{i+j=k} \mathscr{T}_{i,j}(V). \tag{2.42}$$

More explicitly,

$$\begin{aligned}
&\mathscr{T}(V) \otimes \mathscr{T}(V) = \\
&= \underbrace{\mathscr{T}_0 \otimes \mathscr{T}_0}_{\mathscr{T}_{0,0}} \oplus \underbrace{\mathscr{T}_1 \otimes \mathscr{T}_0}_{\mathscr{T}_{1,0}} \oplus \underbrace{\mathscr{T}_0 \otimes \mathscr{T}_1}_{\mathscr{T}_{0,1}} \oplus \underbrace{\mathscr{T}_2 \otimes \mathscr{T}_0}_{\mathscr{T}_{2,0}} \oplus \underbrace{\mathscr{T}_1 \otimes \mathscr{T}_1}_{\mathscr{T}_{1,1}} \oplus \underbrace{\mathscr{T}_0 \otimes \mathscr{T}_2}_{\mathscr{T}_{0,2}} \oplus \cdots \\
&= \underbrace{\mathscr{T}_0 \otimes \mathscr{T}_0}_{K_0} \oplus \underbrace{(\mathscr{T}_1 \otimes \mathscr{T}_0 \oplus \mathscr{T}_0 \otimes \mathscr{T}_1)}_{K_1} \oplus \underbrace{(\mathscr{T}_2 \otimes \mathscr{T}_0 \oplus \mathscr{T}_1 \otimes \mathscr{T}_1 \oplus \mathscr{T}_0 \otimes \mathscr{T}_2)}_{K_2} \oplus \cdots
\end{aligned}$$

In particular, with the decomposition (2.42), $\mathscr{T}(V) \otimes \mathscr{T}(V)$ *is a graded algebra*: Indeed, (2.41) proves that

$$K_k(V) \bullet K_{k'}(V) \subseteq K_{k+k'}(V) \qquad \text{for every } k,k' \geq 0. \tag{2.43}$$

We next introduce a notation analogous to (2.29), which will be used repeatedly in the sequel: for $k \in \mathbb{N} \cup \{0\}$ we set

$$W_k(V) := \bigoplus_{i+j\geq k} \mathscr{T}_{i,j}(V), \quad (\mathscr{T} \otimes \mathscr{T})_+(V) := W_1(V) = \bigoplus_{i+j\geq 1} \mathscr{T}_{i,j}(V). \tag{2.44}$$

Note that, with reference to $K_k$ in (2.42), we have $W_k(V) = \bigoplus_{i\geq k} K_i(V)$, for every $k \geq 0$. The notation $W_k$, $(\mathscr{T} \otimes \mathscr{T})_+$ will also apply.

*Remark 2.42.* The following facts are easily seen to hold true:

1. Every $W_k$ is an ideal in $\mathscr{T}(V) \otimes \mathscr{T}(V)$ containing $\mathscr{T}_{i,j}(V)$ for $i + j = k$.
2. $\mathscr{T} \otimes \mathscr{T} = W_0(V) \supset W_1(V) \supset \cdots W_k(V) \supset W_{k+1}(V) \supset \cdots$.
3. $W_i(V) \bullet W_j(V) \subseteq W_{i+j}(V)$, for every $i,j \geq 0$.
4. $\bigcap_{i\geq 0} W_i(V) = \{0\}$ and, more generally, $\bigcap_{i\geq k} W_i(V) = \{0\}$ for every $k \in \mathbb{N} \cup \{0\}$.

To avoid confusion in the notation (as it appears from the note at page 81), we decided to apply the following conventional notation:

**Convention.** *When the tensor products in the sets $\mathscr{T}_i(V) = V \otimes \cdots \otimes V$ ($i$ times) and the tensor product of $\mathscr{T}(V) \otimes \mathscr{T}(V)$ simultaneously arise, with the consequent risk of confusion, we use the larger symbol "$\bigotimes$" for the latter.*

For example, if $u, v, w \in V$ then

$$(u \otimes v) \bigotimes w \in \mathscr{T}_{2,1}(V), \qquad \text{whereas} \quad u \bigotimes (v \otimes w) \in \mathscr{T}_{1,2}(V),$$

and the above tensors are distinct in $\mathscr{T}(V) \otimes \mathscr{T}(V)$. Instead, $(u \otimes v) \otimes w$ and $u \otimes (v \otimes w)$ denote the same element $u \otimes v \otimes w \in \mathscr{T}_3(V)$ in $\mathscr{T}(V)$.

**Proposition 2.43 (Basis of $\mathscr{T}(V) \otimes \mathscr{T}(V)$).** *Let $V$ be a vector space and let $\mathcal{B} = \{e_h\}_{h \in \mathcal{I}}$ be a basis of $V$. Then the following facts hold*[8]*:*

1. *For every fixed $i, j \geq 0$, the system*

$$\mathcal{B}_{i,j} := \Big\{ \big(e_{h_1} \otimes \cdots \otimes e_{h_i}\big) \bigotimes \big(e_{k_1} \otimes \cdots \otimes e_{k_j}\big) \,\Big|\, h_1, \ldots, h_i,\ k_1, \ldots, k_j \in \mathcal{I} \Big\}$$

*is a basis of $\mathscr{T}_{i,j}(V)$ (which we call* induced by $\mathcal{B}$*).*

2. *The system $\bigcup_{i,j \geq 0} \mathcal{B}_{i,j}$, i.e.,*

$$\Big\{ \big(e_{h_1} \otimes \cdots \otimes e_{h_i}\big) \bigotimes \big(e_{k_1} \otimes \cdots \otimes e_{k_j}\big) \,\Big|\, i, j \geq 0,\ \ h_1, \ldots, h_i,\ k_1, \ldots, k_j \in \mathcal{I} \Big\}$$

*is a basis of $\mathscr{T}(V) \otimes \mathscr{T}(V)$ (which we call* induced by $\mathcal{B}$*).*

*Proof.* It follows from Theorem 2.31, and Propositions 2.9 and 2.35. □

*Remark 2.44.* Thanks to Remark 2.36, the following is a system of generators for $\mathscr{T}(V) \otimes \mathscr{T}(V)$:

$$\Big\{ \big(u_1 \otimes \cdots \otimes u_i\big) \bigotimes \big(v_1 \otimes \cdots \otimes v_j\big) \,\Big|\, i, j \geq 0,\ \ u_1, \ldots, u_i,\ v_1, \ldots, v_j \in V \Big\},$$

where the convention $u_1 \otimes \cdots \otimes u_i = 1_{\mathbb{K}} = v_1 \otimes \cdots \otimes v_j$ applies when $i, j = 0$.

Following the decomposition in (2.40) (and the notation we used in direct sums), an element of $\mathscr{T} \otimes \mathscr{T}$ will be also denoted with a double-sequence styled notation:

$$(t_{i,j})_{i,j \geq 0}, \quad \text{where } t_{i,j} \in \mathscr{T}_{i,j}(V) \text{ for every } i, j \geq 0.$$

[8] When $i = 0$, the term $e_{h_1} \otimes \cdots \otimes e_{h_i}$ has to be read as $1_{\mathbb{K}}$.

The notation $(t_{i,j})_{i,j}$ will equally apply (and there will be no need to specify that $t_{i,j} \in \mathscr{T}_{i,j}(V)$). Then the $\bullet$ operation in (2.38) is recast in Cauchy form as follows (thanks to (2.41)):

$$(t_{i,j})_{i,j} \bullet (\widetilde{t}_{i,j})_{i,j} = \Big( \sum_{r+\widetilde{r}=i,\ s+\widetilde{s}=j} t_{r,s} \bullet \widetilde{t}_{\widetilde{r},\widetilde{s}} \Big)_{i,j \geq 0}. \tag{2.45}$$

We now introduce a selected subspace of $\mathscr{T} \otimes \mathscr{T}$, which will play a central rôle in Chap. 3: we set

$$K := \big\{ v \otimes 1 + 1 \otimes w \,\big|\, v, w \in V \big\} \subset \mathscr{T}(V) \otimes \mathscr{T}(V). \tag{2.46}$$

Here and henceforth, $1$ will denote the unit in $\mathbb{K}$, which is also the identity element of the algebra $\mathscr{T}(V)$. By the bilinearity of $\otimes$, we have

$$K = \mathscr{T}_{1,0}(V) \oplus \mathscr{T}_{0,1}(V) = K_1(V). \tag{2.47}$$

The following computations are simple consequences of (2.38):

$$\begin{aligned} (u_1 \otimes 1) \bullet \cdots \bullet (u_i \otimes 1) &= (u_1 \otimes \cdots \otimes u_i) \bigotimes 1, \\ (1 \otimes v_1) \bullet \cdots \bullet (1 \otimes v_j) &= 1 \bigotimes (v_1 \otimes \cdots \otimes v_j), \end{aligned} \tag{2.48a}$$

$$\begin{aligned} &(u_1 \otimes 1) \bullet \cdots \bullet (u_i \otimes 1) \bullet (1 \otimes v_1) \bullet \cdots \bullet (1 \otimes v_j) \\ &\quad = (u_1 \otimes \cdots \otimes u_i) \bigotimes (v_1 \otimes \cdots \otimes v_j), \end{aligned} \tag{2.48b}$$

for every $i, j \in \mathbb{N}$, and every $u_1, \ldots, u_i, v_1, \ldots, v_j \in V$. From (2.48b) and Remark 2.44, we derive the next proposition:

**Proposition 2.45.** *The following is a system of generators for $\mathscr{T}(V) \otimes \mathscr{T}(V)$:*

$$\Big\{ (u_1 \otimes 1) \bullet \cdots \bullet (u_i \otimes 1) \bullet (1 \otimes v_1) \bullet \cdots \bullet (1 \otimes v_j) \Big| i, j \geq 0, u_1, \ldots, u_i, v_1, \ldots, v_j \in V \Big\},$$

*where the convention $u_1 \otimes \cdots \otimes u_i = 1_{\mathbb{K}} = v_1 \otimes \cdots \otimes v_j$ applies, when $i, j = 0$. Moreover, if $\mathcal{B} = \{e_h\}_{h \in \mathcal{I}}$ is a basis of $V$, the following is a basis of $\mathscr{T}(V) \otimes \mathscr{T}(V)$:*

$$\begin{aligned} \Big\{ 1 \otimes 1, \quad &(e_{h_1} \otimes 1) \bullet \cdots \bullet (e_{h_i} \otimes 1), \quad (1 \otimes e_{k_1}) \bullet \cdots \bullet (1 \otimes e_{k_j}), \\ &(e_{\alpha_1} \otimes 1) \bullet \cdots \bullet (e_{\alpha_a} \otimes 1) \bullet (1 \otimes e_{\beta_1}) \bullet \cdots \bullet (1 \otimes e_{\beta_b}), \\ &\textit{where } i, j, a, b \in \mathbb{N} \textit{ and } h_1, \ldots, h_i, k_1, \ldots, k_j, \alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b \in \mathcal{I} \Big\}. \end{aligned}$$

#### 2.1.3.3 The Lie Algebra $\mathcal{L}(V)$

The aim of this section is to describe another distinguished subset of $\mathscr{T}(V)$ having important features in Lie Algebra Theory.

First the relevant definition.

**Definition 2.46 (Free Lie Algebra Generated by a Vector Space).** Let $V$ be a vector space and consider its tensor algebra $(\mathscr{T}(V), \cdot)$. We equip $\mathscr{T}(V)$ with the Lie algebra structure related to the corresponding commutator (see Definition 2.16).

We denote by $\mathcal{L}(V)$ the Lie algebra generated by the set $V$ in $\mathscr{T}(V)$ (according to Definition 2.13) and we call it *the free Lie algebra generated by* $V$. Namely, $\mathcal{L}(V)$ is the smallest Lie subalgebra of the (commutator-) Lie algebra $\mathscr{T}(V)$ containing $V$.

The above adjective "free" will be soon justified in Theorem 2.49 below (though its proof requires a lot of work and will be deferred to Sect. 2.2). We straightaway remark that we are not using the phrasing "free Lie algebra over $V$" (which, according to previous similar expressions in this Book, would – and will – mean a free object *over the set* $V$). All will be clarified in Sect. 2.2.

**Convention.** To avoid the (proper) odd notation $[u, v]_{\boldsymbol{\cdot}}$ for the commutator related to $(\mathscr{T}(V), \cdot)$, we shall occasionally make use of the abuse of notation $[u, v]_\otimes$ for $u \cdot v - v \cdot u$ (when $u, v \in \mathscr{T}(V)$). This notation becomes particularly suggestive when applied to elementary tensors $u, v$ of the form $w_1 \otimes \cdots \otimes w_n$, for in this case the $\cdot$ product coincides with $\otimes$.

**Proposition 2.47.** *Let $V$ be a vector space and let the notation in Definition 2.46 apply. We set $\mathcal{L}_1(V) := V$ and, for every $n \in \mathbb{N}$, $n \geq 2$,*

$$\mathcal{L}_n(V) := \underbrace{[V \cdots [V, V] \cdots]}_{n \text{ times}} = \operatorname{span}\Big\{ [v_1 \cdots [v_{n-1}, v_n] \cdots] \,\Big|\, v_1, \ldots, v_n \in V \Big\}. \tag{2.49}$$

*Then $\mathcal{L}_n(V) \subseteq \mathscr{T}_n(V)$ for every $n \in \mathbb{N}$, and we have the direct sum decomposition*

$$\mathcal{L}(V) = \bigoplus_{n \geq 1} \mathcal{L}_n(V). \tag{2.50}$$

*In particular, the set $V$ Lie generates $\mathcal{L}(V)$. Moreover, $\mathcal{L}(V)$ is a graded Lie algebra, for it holds that*

$$[\mathcal{L}_i(V), \mathcal{L}_j(V)] \subseteq \mathcal{L}_{i+j}(V), \quad \textit{for every } i, j \geq 1. \tag{2.51}$$

*Proof.* From Theorem 2.15, we deduce that $\bigcup_n \mathcal{L}_n(V)$ spans $\mathcal{L}(V)$ and that (2.51) holds (see (2.11)). Finally, (2.50) follows from $\mathcal{L}_n(V) \subseteq \mathscr{T}_n(V)$, which can be proved by an inductive argument, starting from:

$$[v_1, v_2] = v_1 \cdot v_2 - v_2 \cdot v_1 = v_1 \otimes v_2 - v_2 \otimes v_1 \in \mathscr{T}_2(V),$$

holding for every $v_1, v_2 \in V$, and using (2.28). This ends the proof. $\square$

*Remark 2.48.* Let $V, W$ be isomorphic vector spaces and let $\Psi : V \to W$ be an isomorphism. Let $\widetilde{\Psi} : \mathscr{T}(V) \to \mathscr{T}(W)$ be the UAA isomorphism constructed in Remark 2.39. We claim that

$$\widetilde{\Psi}_{\mathcal{L}} := \widetilde{\Psi}|_{\mathcal{L}(V)} : \mathcal{L}(V) \to \mathcal{L}(W) \textit{ is a Lie algebra isomorphism.}$$

Indeed, since $\widetilde{\Psi}$ is a UAA isomorphism, it is also a Lie algebra isomorphism, when $\mathscr{T}(V)$ and $\mathscr{T}(W)$ are equipped with the associated commutator-algebra structures (see Remark 2.17). As a consequence, the restriction of $\widetilde{\Psi}$ to $\mathcal{L}(V)$ is a Lie algebra isomorphism onto $\widetilde{\Psi}(\mathcal{L}(V))$ (recall that $\mathcal{L}(V)$ is a Lie subalgebra of the commutator-algebra of $\mathscr{T}(V)$). To complete the claim, we have to show that $\widetilde{\Psi}(\mathcal{L}(V)) = \mathcal{L}(W)$. To prove this, we begin by noticing that (in view of (2.33) in Remark 2.39) $\widetilde{\Psi}_{\mathcal{L}}(v) = \Psi(v)$ for every $v \in V$ and

$$\widetilde{\Psi}_{\mathcal{L}}\Big([v_1 \cdots [v_{k-1}, v_k] \cdots]_{\mathscr{T}(V)}\Big) = [\Psi(v_1) \cdots [\Psi(v_{k-1}), \Psi(v_k)] \cdots]_{\mathscr{T}(W)}, \tag{2.52}$$

for every $k \in \mathbb{K}$ and every $v_1, \ldots, v_k \in V$. Here we have denoted by $[\cdot,\cdot]_{\mathscr{T}(V)}$ the commutator related to the associative algebra $\mathscr{T}(V)$ (and analogously for $[\cdot,\cdot]_{\mathscr{T}(W)}$). Now, (2.52) shows that $\widetilde{\Psi}(\mathcal{L}(V)) \subseteq \mathcal{L}(W)$ (recall Proposition 2.47). To prove that "$=$" holds instead of "$\subseteq$", it suffices to recognize that the arbitrary element $[w_1 \cdots [w_{k-1}, w_k] \cdots]_{\mathscr{T}(W)}$ of $\mathcal{L}(W)$ (where $k \in \mathbb{K}$ and $w_1, \ldots, w_k \in V$) is the image via $\widetilde{\Psi}$ of

$$[\Psi^{-1}(w_1) \cdots [\Psi^{-1}(w_{k-1}), \Psi^{-1}(w_k)] \cdots]_{\mathscr{T}(V)}.$$

**Theorem 2.49 (Universal Property of $\mathcal{L}(V)$).** *Let $V$ be a vector space.*

(i) *For every Lie algebra $\mathfrak{g}$ and every linear map $f : V \to \mathfrak{g}$, there exists a unique Lie algebra morphism $\overline{f} : \mathcal{L}(V) \to \mathfrak{g}$ prolonging $f$, thus making the following a commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook, "\iota"'] & \mathfrak{g} \\ \mathcal{L}(V) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

(ii) *Conversely, suppose $L, \varphi$ are respectively a Lie algebra and a linear map $\varphi : V \to L$ with the following property: For every Lie algebra $\mathfrak{g}$ and every linear* *map $f : V \to \mathfrak{g}$, there exists a unique Lie algebra morphism $f^\varphi : L \to \mathfrak{g}$ such that*

$$f^\varphi(\varphi(v)) = f(v) \quad \textit{for every } v \in V, \tag{2.53}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathfrak{g} \\ L \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then $L$ is canonically isomorphic, as a Lie algebra, to $\mathcal{L}(V)$, the isomorphism being (see the notation in* (i) *above) $\overline{\varphi} : \mathcal{L}(V) \to L$ and its inverse being $\iota^\varphi : L \to \mathcal{L}(V)$. Furthermore, $\varphi$ is injective and $L$ is Lie-generated by the set $\varphi(V)$. Actually it holds that $\varphi = \overline{\varphi} \circ \iota$. Finally we have $L \simeq \mathcal{L}(\varphi(V))$, canonically.*

*Proof.* Explicitly, if $[\cdot,\cdot]_\mathfrak{g}$ is the Lie bracket of $\mathfrak{g}$, $\overline{f}$ in (i) is the unique linear map such that

$$\overline{f}\big([v_1 \cdots [v_{k-1}, v_k]_\otimes \cdots]_\otimes\big) = [f(v_1) \cdots [f(v_{k-1}), f(v_k)]_\mathfrak{g} \cdots]_\mathfrak{g},$$

for every $k \in \mathbb{N}$ and every $v_1, \ldots, v_k \in V$.

Unfortunately, the proof of this theorem requires the results of Sect. 2.2, on the existence of $\mathrm{Lie}(X)$, the free Lie algebra related to a set $X$ (together with the characterization $\mathrm{Lie}(X) \simeq \mathcal{L}(\mathbb{K}\langle X\rangle)$). Alternatively, it can be proved by means of the fact that every Lie algebra $\mathfrak{g}$ can be embedded in its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ (a corollary of the Poincaré-Birkhoff-Witt Theorem, see Sect. 2.4). Hence, we shall furnish two proofs of Theorem 2.49, see pages 92 and 112. □

## 2.2 Free Lie Algebras

The aim of this section is to prove the existence of the so-called free Lie algebra $\mathrm{Lie}(X)$ related to a set $X$. Classically, the existence of $\mathrm{Lie}(X)$ follows as a trivial corollary of a highly nontrivial theorem, the Poincaré-Birkhoff-Witt Theorem. For a reason that will become apparent in later chapters concerning with the CBHD Theorem, our aim here is *to prove the existence of* $\mathrm{Lie}(X)$ *without the aid of the Poincaré-Birkhoff-Witt Theorem*.

Moreover, for the aims of this Book, it is also a central fact to obtain the isomorphism of $\mathrm{Lie}(X)$ with $\mathcal{L}(\mathbb{K}\langle X\rangle)$, the smallest Lie subalgebra of the tensor algebra over the free vector space $\mathbb{K}\langle X\rangle$.

The main reference for the topics of this section is Bourbaki [27, Chapitre II, §2 n.2 and §3 n.1]. Unfortunately, there is a feature in [27] which does not allow us to simply rerun Bourbaki's arguments: Indeed, the isomorphism $\mathrm{Lie}(X) \simeq \mathcal{L}(\mathbb{K}\langle X\rangle)$ is proved in [27, Chapitre II, §3 n.1] as a consequence[9] of the Poincaré-Birkhoff-Witt Theorem. So we are forced to present a new argument, which bypasses this inconvenience.

To avoid confusion between the notion of free Lie algebra *generated by a vector space* (see Definition 2.46) and the new notion – we are giving here – of free Lie algebra *related to a set*, we introduce dedicated notations.

**Definition 2.50 (Free Lie Algebra Related to a Set).** Let $X$ be any set. We say that the couple $(L, \varphi)$ is a *free Lie algebra related to* $X$, if the following facts hold: $L$ is a Lie algebra and $\varphi : X \to L$ is a map such that, for every Lie algebra $\mathfrak{g}$ and every map $f : X \to \mathfrak{g}$, there exists a unique Lie algebra morphism $f^\varphi : L \to \mathfrak{g}$, such that the following fact holds

$$f^\varphi(\varphi(x)) = f(x) \quad \text{for every } x \in X, \tag{2.54}$$

thus making the following a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathfrak{g} \\ L \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

If, in the above definition, $X \subset L$ (set-theoretically) and $\varphi = \iota$ is the set inclusion, we say that $(L, \iota)$ is a *free Lie algebra over* $X$.

By abuse, if $(L, \varphi)$ (respectively, $(L, \iota)$) is as above, we shall also say that $L$ itself is a free Lie algebra related to $X$ (respectively, a free Lie algebra over $X$). It is easily seen that any two free Lie algebras related to $X$ are canonically isomorphic. More precisely, the following facts hold.

**Proposition 2.51.** *Let $X$ be a nonempty set.*

1. *If $(L_1, \varphi_1)$, $(L_2, \varphi_2)$ are two free Lie algebras related to the same set $X$, then $L_1, L_2$ are isomorphic Lie algebras via the isomorphisms (inverse to each other) $\varphi_2^{\varphi_1} : L_1 \to L_2$, $\varphi_1^{\varphi_2} : L_2 \to L_1$ and $\varphi_2 \equiv \varphi_2^{\varphi_1} \circ \varphi_1$ (analogously, $\varphi_1 \equiv \varphi_1^{\varphi_2} \circ \varphi_2$).*

[9]See [27, Chapitre II, §3, n.1, Théorème 1] where it is employed [25, Chapitre I, §2, n.7, Corollaire 3 du Théorème 1] which is the Poincaré-Birkhoff-Witt Theorem.

2. *If $(L_1, \varphi_1)$ is a free Lie algebra related to $X$, if $L_2$ is a Lie algebra isomorphic to $L_1$ and $\psi : L_1 \to L_2$ is a Lie algebra isomorphism, then $(L_2, \varphi_2)$ is another free Lie algebra related to $X$, where $\varphi_2 := \psi \circ \varphi_1$.*

*Proof.* (1). As usual, it suffices to consider the commutative diagrams

$$
\begin{tikzcd}
X \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & L_2 \\
L_1 \arrow[ur, "\varphi_2^{\varphi_1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & L_1 \\
L_2 \arrow[ur, "\varphi_1^{\varphi_2}"'] &
\end{tikzcd}
$$

and to show that the diagonal arrows in the following commutative diagrams are respectively "closed" by the maps $\varphi_1^{\varphi_2} \circ \varphi_2^{\varphi_1}$ and $\varphi_2^{\varphi_1} \circ \varphi_1^{\varphi_2}$:

$$
\begin{tikzcd}
X \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1"'] & L_1 \\
L_1 \arrow[ur, "\mathrm{id}_{L_1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\varphi_2"] \arrow[d, "\varphi_2"'] & L_2 \\
L_2 \arrow[ur, "\mathrm{id}_{L_2}"'] &
\end{tikzcd}
$$

We conclude by the uniqueness of the "closing" morphism, as stated in the definition of free Lie algebra related to $X$. Part (2) of the proposition is a simple verification. $\square$

We next turn to the actual *construction* of a free Lie algebra related to the set $X$. First we need some preliminary results.

Whereas ideals are usually defined in an associative setting, we need the following (non-standard) definition.

**Definition 2.52 (Magma Ideal).** Let $(M, *)$ be an algebra (not necessarily associative).

1. We say that $S \subseteq M$ is a *magma ideal in $M$*, if $S$ is a subspace of the vector space $M$ such that $s * m$ and $m * s$ belong to $S$, for every $s \in S$ and $m \in M$.
2. Let $A$ be any subset of $M$. The smallest magma ideal in $M$ containing $A$ is called *the magma ideal generated by $A$*.

With the above definition, it is evident that the magma ideal generated by $A$ coincides with $\bigcap S$, where the intersection runs over the magma ideals $S$ in $M$ containing $A$.

Up to the end of this section, $X$ will denote a fixed set. Let us now consider $\mathrm{Lib}(X)$, i.e., the free non-associative algebra over $X$, introduced in Sect. 2.1.2 (see (2.17)). We shall denote its operation by $*$, recalling that this is the bilinear map extending the operation of the free magma $(M(X), .)$

(and that Lib$(X)$ is the free vector space of the formal linear combinations of elements of $M(X)$). Let us introduce the subset of Lib$(X)$ defined as

$$\begin{gathered} A := \big\{Q(a),\ J(a,b,c) \,\big|\, a,b,c \in \mathrm{Lib}(X)\big\}, \quad \text{where} \\ Q(a) := a*a,\ \ J(a,b,c) := a*(b*c)+b*(c*a)+c*(a*b). \end{gathered} \tag{2.55}$$

We henceforth denote by $\mathfrak{a}$ the magma ideal in Lib$(X)$ generated by $A$, according to Definition 2.52. We next consider the quotient vector space

$$\mathrm{Lie}(X) := \mathrm{Lib}(X)/\mathfrak{a}, \tag{2.56}$$

and the associated natural projection

$$\pi : \mathrm{Lib}(X) \to \mathrm{Lie}(X), \quad \pi(t) := [t]_{\mathfrak{a}}. \tag{2.57}$$

Then the following fact holds:

**Proposition 2.53.** *With all the above notation, the map* $[\cdot,\cdot] : \mathrm{Lie}(X) \to \mathrm{Lie}(X)$ *defined by*

$$[\pi(a), \pi(b)] := \pi(a*b) \quad \textit{for every } a,b \in \mathrm{Lib}(X), \tag{2.58}$$

*is well posed and it endows* $\mathrm{Lie}(X)$ *with a* Lie algebra *structure. Moreover, the map* $\pi$ *in* (2.57) *is an algebra morphism (when we consider* $\mathrm{Lie}(X)$ *as an algebra with the binary bilinear operation* $\mathrm{Lie}(X) \times \mathrm{Lie}(X) \ni (\ell,\ell') \mapsto [\ell,\ell'] \in \mathrm{Lie}(X)$*).*

*Proof.* The well posedness of $[\cdot,\cdot]$ follows from $\mathfrak{a}$ being a magma ideal[10], while the fact that it endows $\mathrm{Lie}(X)$ with a Lie algebra structure is a simple consequence[11] of the definition of $A$. Finally, $\pi$ is an algebra morphism because it is obviously linear ($\mathrm{Lie}(X)$ is a quotient vector space and $\pi$ is the associated projection!) and it satisfies (2.58). □

The Reader will take care not to confuse $\mathrm{Lie}(X)$ with $\mathrm{Lie}\{X\}$ (the latter being the smallest Lie subalgebra – of some Lie algebra $\mathfrak{g}$ – containing $X$, in case $X$ is a subset of a pre-existing Lie algebra $\mathfrak{g}$). Obviously, there is an expected meaning for the similarity of the notation, which will soon be clarified (see Remark 2.55 below). We are ready to state the important fact that $\mathrm{Lie}(X)$ is a free Lie algebra related to $X$.

---

[10] Indeed, if $\pi(a) = \pi(a')$ and $\pi(b) = \pi(b')$ there exist $\alpha, \beta \in \mathfrak{a}$ such that $a' = a + \alpha$, $b' = b + \beta$. Hence $a' * b' = a*b + a*\beta + \alpha*b + \alpha*\beta \in a*b + a*\mathfrak{a} + \mathfrak{a}*b + \mathfrak{a}*\mathfrak{a} \subseteq a*b + \mathfrak{a}$, so that $\pi(a'*b') = \pi(a*b)$.

[11] For example, the Jacobi identity follows from $[\pi(a),[\pi(b),\pi(c)]] = \pi(a*(b*c))$ so that $[\pi(a),[\pi(b),\pi(c)]] + [\pi(b),[\pi(c),\pi(a)]] + [\pi(c),[\pi(a),\pi(b)]] = \pi(J(a,b,c)) = 0$.

**Theorem 2.54 (**$\mathrm{Lie}(X)$ **is a Free Lie Algebra Related to** $X$**).** *Let* $X$ *be any set and, with the notation in* (2.56) *and* (2.57)*, let us consider the map*

$$\varphi : X \to \mathrm{Lie}(X), \quad x \mapsto \pi(x), \tag{2.59}$$

*i.e.,*[12]*,* $\varphi \equiv \pi|_X$*. Then the following facts hold:*

1. *The couple* $(\mathrm{Lie}(X), \varphi)$ *is a free Lie algebra related to* $X$ *(see Definition 2.50).*
2. *The set* $\{\varphi(x)\}_{x\in X}$ *is independent in* $\mathrm{Lie}(X)$*, whence* $\varphi$ *is injective.*
3. *The set* $\varphi(X)$ *Lie-generates* $\mathrm{Lie}(X)$*, that is, the smallest Lie subalgebra of* $\mathrm{Lie}(X)$ *containing* $\varphi(X)$ *coincides with* $\mathrm{Lie}(X)$*.*

*Proof.* See Sect. 8.1 (page 459) in Chap. 8. □

*Remark 2.55.* Part 3 of the statement of Theorem 2.54 says that $\mathrm{Lie}\{\varphi(X)\} = \mathrm{Lie}(X)$, the former being meant as the smallest subalgebra – of the latter – containing $X$ (see Definition 2.13). This fact, together with the identification $X \equiv \varphi(X)$ (this is possible due to part 2 of Theorem 2.54) says that

$$\mathrm{Lie}\{X\} \equiv \mathrm{Lie}(X)$$

(which is extremely convenient given the abundance of notation for free Lie algebras generated by a set!).

Here is another (very!) desirable result concerning free Lie algebras.

**Theorem 2.56 (The Isomorphism** $\mathcal{L}(\mathbb{K}\langle X\rangle) \simeq \mathrm{Lie}(X)$**).** *Let* $X$ *be any set and consider the free vector space* $\mathbb{K}\langle X\rangle$ *over* $X$*. Consider also* $\mathcal{L}(\mathbb{K}\langle X\rangle)$*, the smallest Lie subalgebra of* $\mathscr{T}(\mathbb{K}\langle X\rangle)$ *containing* $X$*.*

*Then* $\mathcal{L}(\mathbb{K}\langle X\rangle)$ *and* $\mathrm{Lie}(X)$ *are isomorphic, as Lie algebras. More precisely, the pair* $(\mathcal{L}(\mathbb{K}\langle X\rangle), \chi)$ *is a free Lie algebra related to* $X$*.*

When, occasionally, we shall allow ourselves to identify $X$ with the subset $\chi(X)$ of $\mathbb{K}\langle X\rangle$ (via the injective map $\chi$), the map $\chi : X \to \mathcal{L}(\mathbb{K}\langle X\rangle)$ becomes the map of set inclusion, whence Theorem 2.56 will permit us to say that $\mathcal{L}(\mathbb{K}\langle X\rangle)$ is a free Lie algebra *over* $X$.

*Proof.* If $\varphi$ is as in (2.59), we know from Theorem 2.54 that $(\mathrm{Lie}(X), \varphi)$ is a free Lie algebra related to $X$. Hence, considering the map $X \ni x \mapsto \chi(x) \in \mathcal{L}(\mathbb{K}\langle X\rangle)$, there exists a unique Lie algebra morphism (see the notation in Definition 2.50) $\chi^{\varphi}$, say $f$ for short, such that

$$f : \mathrm{Lie}(X) \to \mathcal{L}(\mathbb{K}\langle X\rangle) \quad \text{and } f(\varphi(x)) = \chi(x), \text{ for every } x \in X. \tag{2.60}$$

We claim that $f$ *is a Lie algebra isomorphism*. This claim is proved in Sect. 8.1 in Chap. 8 (precisely in Corollary 8.6, page 469). Hence, by Proposition 2.51-2,

[12]More precisely, the map $\varphi$ is the composition

$$X \xrightarrow{\iota} M(X) \xrightarrow{\chi} \mathrm{Lib}(X) \xrightarrow{\pi} \mathrm{Lie}(X).$$

Via the identification $X \equiv \chi(X) \xrightarrow{\iota} \mathrm{Lib}(X)$ we can write $\varphi \equiv \pi|_X$.

$(\mathcal{L}(\mathbb{K}\langle X\rangle), \varphi_2)$ is a free Lie algebra related to $X$, with $\varphi_2 = f \circ \varphi \equiv \chi$ on $X$ (where we also used (2.60)). $\square$

Collecting together Theorems 2.54 and 2.56 (and Definition 2.50), we can deduce that, if $X$ is any set, $\mathfrak{g}$ any Lie algebra and $f : X \to \mathfrak{g}$ any map, there exist Lie algebra morphisms

$$f^{\varphi} : \mathrm{Lie}(X) \to \mathfrak{g}, \qquad f^{\chi} : \mathcal{L}(\mathbb{K}\langle X\rangle) \to \mathfrak{g}$$

such that

$$f^{\varphi}(\varphi(x)) = f(x) = f^{\chi}(\chi(x)), \qquad \forall\, x \in X$$

and, more explicitly, these morphisms act – on typical elements of their respective domains – as follows:

$$\begin{aligned} & f^{\varphi}\Big([\varphi(x_1) \cdots [\varphi(x_{k-1}), \varphi(x_k)]_{\mathrm{Lie}(X)} \cdots ]_{\mathrm{Lie}(X)}\Big) \\ & \quad = f^{\chi}\big([\chi(x_1) \cdots [\chi(x_{k-1}), \chi(x_k)]_{\otimes} \cdots ]_{\otimes}\big) \\ & \quad = [f(x_1) \cdots [f(x_{k-1}), f(x_k)]_{\mathfrak{g}} \cdots ]_{\mathfrak{g}}, \end{aligned}$$

for every $x_1, \ldots, x_k \in X$ and every $k \in \mathbb{N}$. Here

$$[\cdot,\cdot]_{\mathrm{Lie}(X)}, \quad [\cdot,\cdot]_{\otimes}, \quad [\cdot,\cdot]_{\mathfrak{g}}$$

are, respectively, the Lie brackets of $\mathrm{Lie}(X)$, of $\mathcal{L}(\mathbb{K}\langle X\rangle)$ (with Lie bracket inherited from the commutator on $\mathscr{T}(\mathbb{K}\langle X\rangle)$) and of $\mathfrak{g}$.

With Theorem 2.56 at hand, we are ready to provide the following:

*Proof (of Theorem 2.49, page 86).* Since (ii) is standard, we restrict our attention to the proof of (i). Let $\mathfrak{g}$ be a Lie algebra and let $f : V \to \mathfrak{g}$ be any linear map. We have to prove that there exists a unique Lie algebra morphism $\overline{f} : \mathcal{L}(V) \to \mathfrak{g}$ prolonging $f$. Since $\mathcal{L}(V)$ is Lie-generated by $V$ (see e.g., Proposition 2.47) the uniqueness of $\overline{f}$ will follow from its existence. To prove this latter fact, we make use of a basis of $V$ (the "non-canonical" nature of this argument being completely immaterial). See also the diagram below:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[d, hook] & & \mathfrak{g} \\ \mathcal{L}(V) \arrow[urr, "\overline{f}"] \arrow[d, "\widetilde{\Psi}_{\mathcal{L}}"'] & & \\ \mathcal{L}(\mathbb{K}\langle \mathcal{B}\rangle) \arrow[uurr, "(f|_{\mathcal{B}})^{\chi}"'] & & \end{tikzcd}$$

With this aim, let $\mathcal{B} = \{b_i\}_{i\in\mathcal{I}}$ be a basis of $V$. Then $V$ is isomorphic (as a vector space) to the free vector space $\mathbb{K}\langle\mathcal{B}\rangle$, via the (unique) linear map $\Psi : V \to \mathbb{K}\langle\mathcal{B}\rangle$ mapping $b_i \in V$ into $\chi(b_i) \in \mathbb{K}\langle\mathcal{B}\rangle$ for every $i \in \mathcal{I}$ (recall the notation in (2.2)): more explicitly

$$\Psi\big(\textstyle\sum_{i\in\mathcal{I}'} \lambda_i\, b_i\big) = \textstyle\sum_{i\in\mathcal{I}'} \lambda_i\, \chi(b_i), \tag{2.61}$$

where $\mathcal{I}'$ is any finite subset of $\mathcal{I}$ and the coefficients $\lambda_i$ are arbitrary scalars. Since $\Psi : V \to \mathbb{K}\langle\mathcal{B}\rangle$ is an isomorphism, by Remark 2.48 we can deduce that $\mathcal{L}(V)$ and $\mathcal{L}(\mathbb{K}\langle\mathcal{B}\rangle)$ are isomorphic via the unique LA isomorphism $\widetilde{\Psi}_{\mathcal{L}} : \mathcal{L}(V) \to \mathcal{L}(\mathbb{K}\langle\mathcal{B}\rangle)$ such that

$$\widetilde{\Psi}_{\mathcal{L}}(v) = \Psi(v), \quad \text{for every } v \in V. \tag{2.62}$$

Since the pair $(\mathcal{L}(\mathbb{K}\langle\mathcal{B}\rangle), \chi)$ is a free Lie algebra related to $\mathcal{B}$ (see Theorem 2.56), considering the map $f|_{\mathcal{B}} : \mathcal{B} \to \mathfrak{g}$, there exists an LA morphism $(f|_{\mathcal{B}})^{\chi} : \mathcal{L}(\mathbb{K}\langle\mathcal{B}\rangle) \to \mathfrak{g}$ such that

$$(f|_{\mathcal{B}})^{\chi}(\chi(b_i)) = f(b_i), \quad \forall\ i \in \mathcal{I}. \tag{2.63}$$

We claim that $\overline{f} := (f|_{\mathcal{B}})^{\chi} \circ \widetilde{\Psi}_{\mathcal{L}} : \mathcal{L}(V) \to \mathfrak{g}$ prolongs $f$ (see the diagram above). Indeed, if $v \in V$, say $v = \sum_{i\in\mathcal{I}'} \lambda_i\, b_i$, we have

$$\begin{aligned}\overline{f}(v) &= (f|_{\mathcal{B}})^{\chi}(\widetilde{\Psi}_{\mathcal{L}}(v)) \overset{(2.62)}{=} (f|_{\mathcal{B}})^{\chi}(\Psi(v)) \overset{(2.61)}{=} (f|_{\mathcal{B}})^{\chi}\big(\textstyle\sum_{i\in\mathcal{I}'} \lambda_i\, \chi(b_i)\big)\\ &\overset{(2.63)}{=} \textstyle\sum_{i\in\mathcal{I}'} \lambda_i\, f(b_i) = f\big(\textstyle\sum_{i\in\mathcal{I}'} \lambda_i\, b_i\big) = f(v).\end{aligned}$$

This ends the proof. $\square$

## 2.3 Completions of Graded Topological Algebras

The aim of this section is to equip a certain class of algebras $A$ with a topology endowing $A$ with the structure of a topological algebra. It will turn out that a structure of metric space will also be available in this setting. Then we shall describe the general process of completion of a metric space. Finally, we shall focus on graded algebras and the concept of formal power series will be closely investigated. All these topics will be of relevance when we shall deal with the CBHD Formula (and convergence aspects concerned with it).

### 2.3.1 Topology on Some Classes of Algebras

**Definition 2.57.** Let $(A, *)$ be an associative algebra. We say that $\{\Omega_k\}_{k\in\mathbb{N}}$ *is a topologically admissible family in* $A$ if the sets $\Omega_k$ are subsets of $A$ satisfying the properties:

*(H1.)* $\Omega_k$ is an ideal of $A$, for every $k \in \mathbb{N}$.
*(H2.)* $\Omega_1 = A$ and $\Omega_k \supseteq \Omega_{k+1}$, for every $k \in \mathbb{N}$.
*(H3.)* $\Omega_h * \Omega_k \subseteq \Omega_{h+k}$, for every $h, k \in \mathbb{N}$.
*(H4.)* $\bigcap_{k\in\mathbb{N}} \Omega_k = \{0\}$.

The main aim of this section is to prove the following theorem.

**Theorem 2.58.** *Let $(A, *)$ be an associative algebra and suppose that $\{\Omega_k\}_{k\in\mathbb{N}}$ is a topologically admissible family of subsets of $A$. Then the family*

$$\emptyset \cup \Big\{a + \Omega_k\Big\}_{a\in A,\ k\in\mathbb{N}} \tag{2.64}$$

*is a basis for a topology $\Omega$ on $A$ endowing $A$ with the structure of a topological algebra.*[13] *Even more, the topology $\Omega$ is induced by the metric $d : A \times A \to [0, \infty)$ defined as follows* ($\exp(-\infty) := 0$ *applies)*

$$d(x, y) := \exp(-\nu(x - y)), \quad \textit{for all } x, y \in A, \tag{2.65}$$

*where $\nu : A \to \mathbb{N} \cup \{0, \infty\}$ is defined by $\nu(z) := \sup\big\{n \geq 1 \,\big|\, z \in \Omega_n\big\}$, or more precisely*

$$\nu(z) := \begin{cases} \textit{if } z \neq 0, & \max\big\{n \geq 1 \,\big|\, z \in \Omega_n\big\} \\ \textit{if } z = 0, & \infty. \end{cases} \tag{2.66}$$

*The triangle inequality for $d$ holds in the stronger form*[14]*:*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \quad \textit{for every } x, y, z \in A. \tag{2.67}$$

*Proof.* See page 407 in Chap. 7. □

---

[13]We recall that a topological algebra is a pair $(A, \Omega)$ where $(A, +, *)$ is an algebra and $\Omega$ is a topology on $A$ such that the maps

$$A \times A \ni (a, b) \mapsto a + b, a * b \in A, \quad \mathbb{K} \times A \ni (k, a) \mapsto k\, a \in A$$

are continuous (with the associated product topologies, $\mathbb{K}$ being equipped with the discrete topology) and such that $(A, \Omega)$ is a Hausdorff topological space.

[14]A metric space $(A, d)$ whose distance satisfies (2.67) (called the *strong triangle inequality* or *ultrametric inequality*) is usually referred to as an *ultrametric space*. Hence, a topologically admissible family of subsets of an algebra $A$ endows $A$ with the structure of an ultrametric space.

*Remark 2.59.* In the notation of the previous theorem, (2.67) easily implies the following peculiar fact: *A sequence $\{a_n\}_n$ in $A$ is a Cauchy sequence in $(A, d)$ if and only if* $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$.

Indeed, given a sequence $\{a_n\}_n$ in $A$, as a consequence of (2.67) the following telescopic estimate applies, for every $n, p \in \mathbb{N}$:

$$\begin{aligned}
d(a_n, a_{n+p}) &\le \max\{d(a_n, a_{n+1}), d(a_{n+1}, a_{n+p})\} \\
&\le \max\big\{d(a_n, a_{n+1}), \max\{d(a_{n+1}, a_{n+2}), d(a_{n+2}, a_{n+p})\}\big\} \\
&= \max\big\{d(a_n, a_{n+1}), d(a_{n+1}, a_{n+2}), d(a_{n+2}, a_{n+p})\big\} \\
&\quad \text{(after finitely many steps)} \\
&\le \max\big\{d(a_n, a_{n+1}), d(a_{n+1}, a_{n+2}), \ldots, d(a_{n+p-1}, a_{n+p})\big\}.
\end{aligned}$$

This shows that $\{a_n\}_n$ is a Cauchy sequence in $(A, d)$ if and only if

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0.$$

**Definition 2.60.** If $A$ is an associative algebra and if $\{\Omega_k\}_{k\in\mathbb{N}}$ is a topologically admissible family of subsets of $A$, the topology $\Omega$ (respectively, the metric $d$) in Theorem 2.58 will be called *the topology on $A$ induced by* $\{\Omega_k\}_{k\in\mathbb{N}}$ (respectively, *the metric on $A$ induced by* $\{\Omega_k\}_{k\in\mathbb{N}}$).

*Remark 2.61.* When $A$ is an associative algebra and $d$ is the metric on $A$ induced by a topologically admissible family $\{\Omega_k\}_{k\in\mathbb{N}}$, we have the following *algebraic properties* of the metric $d$ in (2.65) (proved in due course within Chap. 7, see page 415):

1. $d(x, y) = d(x + z, y + z)$, for every $x, y, z \in A$.
2. $d(k\,x, k\,y) = d(x, y)$, for every $k \in \mathbb{K} \setminus \{0\}$ and every $x, y \in A$.
3. $d(x * y, \xi * \eta) \le \max\{d(x, \xi), d(y, \eta)\}$, for every $x, y, \xi, \eta \in A$.

*Remark 2.62.* In the notation of Theorem 2.58, we have the following fact: *A sequence $\{a_n\}_n$ in $A$ is a Cauchy sequence in the metric space $(A, d)$ sequence if and only if* $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$.

Consequently, *a series $\sum_{n=1}^{\infty} a_n$ consisting of elements in $A$ is a Cauchy sequence in $(A, d)$ if and only if* $\lim_{n\to\infty} a_n = 0$ *in $(A, d)$.*

Indeed, by Remark 2.59 $\{a_n\}_n$ is Cauchy in $(A, d)$ iff $\lim_{n\to\infty} d(a_n, a_{n+1}) = 0$. In its turn, by (1) in Remark 2.61, we see that this latter fact coincides with $\lim_{n\to\infty} d(0, a_{n+1} - a_n) = 0$. Finally, this is the definition of $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ in $(A, d)$.

The above remark shows how different are the metrics in Theorem 2.58 (indeed, all ultrametrics), if they are compared to the usual Euclidean metric in $\mathbb{R}^n$, where the above facts are false (as shown by the trivial example $\sum_{n=1}^{\infty} 1/n = \infty$ in the usual Euclidean space $\mathbb{R}$).

*Remark 2.63.* With the notation of Theorem 2.58, by unraveling the definition of $d$ we have that, for two points $x, y \in A$ and a positive real number $\varepsilon$, the condition $d(x,y) < \varepsilon$ is equivalent to $\sup\{n \geq 1 \,|\, z \in \Omega_n\} > \ln(1/\varepsilon)$, that is,

$$\big(x, y \in A \quad d(x,y) < \varepsilon\big) \quad \Longleftrightarrow \quad \begin{pmatrix} \text{there exists } n \in \mathbb{N}\text{, with } n > \ln(1/\varepsilon) \\ \text{such that } x - y \in \Omega_n \end{pmatrix}. \tag{2.68}$$

*Example 2.64.* Before proceeding, we make explicit some examples of topologically admissible families, useful for the sequel.

1. Let $(A, *)$ be an associative algebra and let $I \subseteq A$ be an ideal. Let us set $\Omega_0 := A$ and, for $k \in \mathbb{N}$, let

$$\begin{aligned} \Omega_k &:= \text{ideal generated by } \{I * \cdots * I \ (k \text{ times})\} \\ &= \left\{ \begin{array}{c} \text{set of the finite sums of elements of the form } r * i_1 * \cdots * i_k * \rho \\ \text{where } r, \rho \in A \text{ and } i_1, \ldots, i_k \in I \end{array} \right\}. \end{aligned}$$

   Then it is easily seen that the family $\{\Omega_k\}_{k\geq 0}$ fulfils hypotheses (H1), (H2) and (H3) in Definition 2.57. Hence, whenever $\{\Omega_k\}_{k\geq 0}$ fulfils also hypothesis (H4), it is a topologically admissible family in $A$.
2. Suppose $(A, *)$ is an associative algebra which is also *graded* (see Definition 2.11). We set $A_+ := \bigoplus_{j=1}^\infty A_j$. Also, let $\Omega_0 := A$ and, for $k \in \mathbb{N}$,

$$\Omega_k := \mathrm{span}\big\{a_1 * \cdots * a_k \,\big|\, a_1, \ldots, a_k \in A_+\big\}.$$

   It is not difficult to show that $\{\Omega_k\}_{k\geq 0}$ is a topologically admissible family in $A$. For example, we prove (H4): First note that $\Omega_k \subseteq \bigoplus_{j=k}^\infty A_j$ for any $k \in \mathbb{N} \cup \{0\}$ (indeed equality holds); hence we have

$$\{0\} \subseteq \textstyle\bigcap_{k\geq 0} \Omega_k \subseteq \bigcap_{k\geq 0} \bigoplus_{j=k}^\infty A_j = \{0\}.$$

   The last equality is proved as follows: if $a \in A$ and $a \neq 0$, then we can write $a = \sum_{i=1}^N a_i$ with $a_i \in A_i$ and $a_N \neq 0$; in this case $a \notin \bigoplus_{j=N+1}^\infty A_j$ and the assertion follows. Note that also

$$\Omega_k = \textstyle\bigoplus_{j\geq k} A_j, \qquad \forall\, k \in \mathbb{N} \cup \{0\}. \tag{2.69}$$

   As for hypotheses (H1)-(H2)-(H3), they follow by the previous Example 1, since it can be easily seen that, for all $k \in \mathbb{N}$,

$$\Omega_k = \text{ideal generated by } \{A_+ * \cdots * A_+ \ (k \text{ times})\}. \tag{2.70}$$

3. Let $V$ be a vector space and let $A = \mathscr{T}(V)$. We can construct the family $\{\Omega_k\}_{k\in\mathbb{N}}$ according to the previous example, with respect to the usual

grading $\mathscr{T} = \bigoplus_{j=0}^{\infty} \mathscr{T}_j$. By (2.69), we have $\Omega_k = U_k$ for every $k \in \mathbb{N}$, where $U_k$ has been defined in (2.29). Hence $\{U_k\}_{k\in\mathbb{N}}$ *is a topologically admissible family in* $\mathscr{T}(V)$*, thus equipping* $\mathscr{T}(V)$ *with both a metric space and a topological algebra structure.* (This same fact also follows from the results in Remark 2.34-(3).) As stated in Example 2, we can view this example as a particular case of Example 1 above, since $U_k$ coincides with the ideal generated by the $k$-products of $I = \mathscr{T}_+(V)$ (or equivalently, of $I = V$).

4. Let $V$ be a vector space, and on the tensor product $\mathscr{T}(V) \otimes \mathscr{T}(V)$ let us consider the family of subsets $\{W_k\}_{k\in\mathbb{N}}$ introduced in (2.44). Then, by the results in Remark 2.42, $\{W_k\}_{k\in\mathbb{N}}$ *is a topologically admissible family in* $\mathscr{T}(V) \otimes \mathscr{T}(V)$*, thus equipping* $\mathscr{T}(V) \otimes \mathscr{T}(V)$ *with both a metric space and a topological algebra structure.* Analogously, the fact that $\{W_k\}_{k\in\mathbb{N}}$ is a topologically admissible family can be proved by Example 2 above, by considering the grading $\mathscr{T} \otimes \mathscr{T} = \bigoplus_{k\geq 0} K_k$ as in (2.42). Indeed, $W_k = \mathrm{span}\{a_1 \bullet \cdots \bullet a_k \mid a_1, \ldots, a_k \in \bigoplus_{j\geq 1} K_j = (\mathscr{T} \otimes \mathscr{T})_+\}$. As stated in Example 2, we can also view this example as a particular case of Example 1 above, since $W_k$ coincides with the ideal generated by the $k$-products of $I = (\mathscr{T} \otimes \mathscr{T})_+$ (or equivalently, of $I := K$ introduced in (2.46)).

Since we are mainly interested in graded algebras, for the sake of future reference, we collect many of the aforementioned results in the following proposition, and we take the opportunity to prove some further facts.

**Proposition 2.65 (Metric Related to a Graded Algebra).** *Let* $(A, *)$ *be an associative graded algebra with grading* $\{A_j\}_{j\geq 0}$*. For every* $k \in \mathbb{N} \cup \{0\}$*, set* $\Omega_k := \bigoplus_{j\geq k} A_j$*.*

(a) *Then* $\{\Omega_k\}_{k\geq 0}$ *is a topologically admissible family of* $A$*, thus endowing* $A$ *with both a metric space and a topological algebra structure (both structures are referred to as "related to the grading* $\{A_j\}_{j\geq 0}$*").*

(b) *The induced metric* $d$ *has the algebraic (and ultrametric) properties*

$$\begin{aligned} d(x,y) &= d(x+z, y+z), \qquad d(k\,x, k\,y) = d(x,y) \\ d(x*y, \xi*\eta) &\leq \max\{d(x,\xi), d(y,\eta)\}, \end{aligned} \tag{2.71}$$

*for every* $x, y, z, \xi, \eta \in A$ *and every* $k \in \mathbb{K} \setminus \{0\}$*.*

(c) *A sequence* $\{a_n\}_n$ *of elements of* $A$ *is a Cauchy sequence in* $(A, d)$ *if and only if* $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ *in* $(A, d)$*. Moreover a series* $\sum_{n=1}^{\infty} a_n$ *of elements of* $A$ *is a Cauchy sequence in* $(A, d)$ *if and only if* $\lim_{n\to\infty} a_n = 0$ *in* $(A, d)$*.*

(d) *For every* $z = (z_j)_{j\geq 0} \in A$*, we have*

$$\begin{aligned} d(z) &= \begin{cases} \exp\big(-\min\{j \geq 0 \,:\, z_j \neq 0\}\big), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0 \end{cases} \\ &= \begin{cases} \max\{e^{-j} \,:\, z_j \neq 0\}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases} \end{aligned} \tag{2.72}$$

(e) *Let* $\{b_n\}_{n\in\mathbb{N}}$ *be a sequence of elements in* $A$ *and let* $\beta \in A$*; let us write*

$$b_n = (a_j^{(n)})_{j\geq 0} \quad \textit{and} \quad \beta = (a_j)_{j\geq 0},$$

*with* $a_j^{(n)}, a_j \in A_j$ *for every* $j \geq 0$ *and every* $n \in \mathbb{N}$.
*Then we have* $\lim_{n\to\infty} b_n = \beta$ *in* $(A, d)$ *if and only if*

$$\forall\, J \geq 0\ \exists\, N_J \in \mathbb{N}: \quad n \geq N_J \textit{ implies } a_j^{(n)} = a_j \quad \textit{for } 0 \leq j \leq J. \tag{2.73}$$

*Proof.* See page 416 in Chap. 7. □

## 2.3.2 Completions of Graded Topological Algebras

We begin by recalling some classical results of Analysis concerning metric spaces. As usual, the associated proofs are postponed to Chap. 7.

**Definition 2.66.** Let $(X, d)$ be a metric space. We say that $(Y, \delta)$ is an *isometric completion* of $(X, d)$, if the following facts hold:

1. $(Y, \delta)$ is a complete metric space.
2. There exists a metric subspace $X_0$ of $Y$ which is dense in $Y$ and such that $(X_0, \delta)$ is isometric (in the sense of metric spaces[15]) to $(X, d)$.

The following simple fact holds, highlighting the fact that the notion of isometric completion is unique, up to isomorphism.

**Proposition 2.67.** *Let* $(X, d)$ *be a metric space. If* $(Y_1, \delta_1)$ *and* $(Y_2, \delta_2)$ *are isometric completions of* $(X, d)$*, then they are (canonically) isomorphic.*

*Proof.* See page 417 in Chap. 7. □

The following remarkable result states that every metric space always admits an isometric completion.

**Theorem 2.68 (Completion of a Metric Space).** *Let* $(X, d)$ *be a metric space. Then there exists an isometric completion* $(\widetilde{X}, \widetilde{d})$ *of* $(X, d)$*, which can be constructed as follows. We first consider the set* $\mathcal{C}$ *of all the Cauchy sequences* $\widetilde{x} = (x_n)_n$ *in* $(X, d)$*. We introduce in* $\mathcal{C}$ *an equivalence relation by setting*

$$(x_n)_n \sim (x'_n)_n \quad \textit{iff} \quad \lim_{n\to\infty} d(x_n, x'_n) = 0. \tag{2.74}$$

---

[15] We recall that, given two metric spaces $(Y_1, d_1), (Y_2, d_2)$, a map $\Phi : Y_1 \to Y_2$ is called an isomorphism of metric spaces if $\Phi$ is bijective and such that $d_2(\Phi(y), \Phi(y')) = d_1(y, y')$ for every $y, y' \in Y_1$ (note that this last condition implicitly contains the injectivity of $\Phi$ together with the fact that $\Phi$ is a homeomorphism of the associated topological spaces).

*We take as $\widetilde{X}$ the quotient set $\mathcal{C}/_\sim$, with the metric defined by*

$$\widetilde{d}\Big(\big[(x_n)_n\big]_\sim, \big[(y_n)_n\big]_\sim\Big) := \lim_{n\to\infty} d(x_n, y_n). \tag{2.75}$$

*Furthermore (according to the notation in Definition 2.66), we take as $X_0$ (say, the isometric copy of $X$ inside $\widetilde{X}$) the quotient set of the constant sequences $(x)_n$ with $x \in X$ and the associated isometry is the map*

$$\alpha : X \to X_0, \quad x \mapsto [(x_n)_n]_\sim \quad \textit{with } x_n = x \textit{ for every } n \in \mathbb{N}. \tag{2.76}$$

*Proof.* See page 418 in Chap. 7. □

In the sequel, when dealing with isometric completions of a given metric space $X$, we shall reserve the notation $\widetilde{X}$ for the metric space introduced in Theorem 2.68. The following result states that the passage to the isometric completion preserves many of the underlying algebraic structures, in a very natural way.

**Theorem 2.69 (Algebraic Structure on the Isometric Completion of a UAA).** *Let $(A, +, *)$ be a UA algebra. Suppose $\{\Omega_k\}_{k\in\mathbb{N}}$ is a topologically admissible family in $A$ and let $d$ be the metric on $A$ induced by $\{\Omega_k\}_{k\in\mathbb{N}}$. Finally, consider the isometric completion $\widetilde{A}$ of $(A, d)$ as in Theorem 2.68 and let $A_0 \subseteq \widetilde{A}$ be the set containing the equivalence classes of the constant sequences.*

*Then $\widetilde{A}$ can be equipped with a structure of a UA algebra $(\widetilde{A}, \widetilde{+}, \widetilde{*})$, which is also a topological algebra containing $A_0$ as a (dense) subalgebra isomorphic to $A$. More precisely, the map $\alpha$ in (2.76) is an isomorphism of metric spaces and of UA algebras. The relevant operations on $\widetilde{A}$ are defined as follows:*

$$\begin{aligned}
\big[(x_n)_n\big]_\sim \,\widetilde{+}\, \big[(y_n)_n\big]_\sim &:= \big[(x_n + y_n)_n\big]_\sim, \\
\big[(x_n)_n\big]_\sim \,\widetilde{*}\, \big[(y_n)_n\big]_\sim &:= \big[(x_n * y_n)_n\big]_\sim, \\
k\,\big[(x_n)_n\big]_\sim &:= \big[(k\, x_n)_n\big]_\sim, \quad k \in \mathbb{K}, \\
1_{\widetilde{A}} &:= [(1_A)_n]_\sim.
\end{aligned} \tag{2.77}$$

*Proof.* See page 422 in Chap. 7. □

*Remark 2.70.* Let $(A, *), \{\Omega_k\}_k, d, \widetilde{A}$ be as in Theorem 2.69. Suppose $B$ is equipped with a UAA structure by the operation $\star$, that it is equipped with a metric space structure by the metric $\delta$ and that the following properties hold:

1. $A$ is a *subset* of $B$.
2. $\star$ coincides with $*$ on $A \times A$.
3. $\delta$ coincides with $d$ on $A \times A$.
4. $A$ is dense in $B$.
5. $B$ is a complete metric space.

Then $B$ and $\widetilde{A}$ are not only isomorphic as metric spaces (according to Proposition 2.67) but also as UA algebras (via the same isomorphism). (See the proof in Chap. 7, page 425.)

By collecting together some results obtained so far (and derived within the proofs of some of the previous results), we obtain the following further characterization of the isometric completion $\widetilde{A}$ of $A$.

**Theorem 2.71 (Characterizations of the Isometric Completion of a UAA).** *Let $(A, +, *)$ be a UA algebra. Suppose $\{\Omega_k\}_{k\in\mathbb{N}}$ is a topologically admissible family in $A$ and let $d$ be the metric on $A$ induced by $\{\Omega_k\}_{k\in\mathbb{N}}$. Finally, consider the isometric completion $\widetilde{A}$ of $(A, d)$ as in Theorem 2.68.*

*If $\alpha \in \widetilde{A}$ is represented by the (Cauchy) sequence $(a_n)_n$ in $A$ (that is, $\alpha = [(a_n)_n]_\sim$), we have*

$$\alpha = \lim_{n\to\infty} a_n \quad \text{in } (\widetilde{A}, \widetilde{d}),$$

*where each $a_n \in A$ is identified with an element of $\widetilde{A}$ via the map $\alpha$ in (2.76). Hence, roughly, $\widetilde{A}$ can be thought of as the set of the "limits" of the Cauchy sequences in $A$, more precisely*

$$\widetilde{A} = \Big\{ \lim_{j\to\infty} \big[(a_j, a_j, \cdots)\big]_\sim \,\Big|\, (a_n)_n \text{ is a Cauchy sequence in } A \Big\}. \tag{2.78a}$$

*Equivalently (see also Proposition 2.65-(c)), $\widetilde{A}$ can also be thought of as the set of the $A$-valued series associated to a vanishing sequence, more precisely*

$$\widetilde{A} = \Big\{ \sum_{j=1}^{\infty} \big[(b_j, b_j, \cdots)\big]_\sim \,\Big|\, (b_n)_n \text{ is a sequence converging to zero in } (A, d) \Big\}. \tag{2.78b}$$

Here is a very natural result on the relation $A \mapsto \widetilde{A}$.

**Lemma 2.72.** *Let $A, B$ be two isomorphic UA algebras. Suppose $\varphi : A \to B$ is a UAA isomorphism and suppose that $\{\Omega_k\}_{k\in\mathbb{N}}$ is a topologically admissible family in $A$. Set $\widetilde{\Omega}_k := \varphi(\Omega_k)$, for every $k \in \mathbb{N}$.*

*Then the family $\{\widetilde{\Omega}_k\}_{k\in\mathbb{N}}$ is a topologically admissible family in $B$. Moreover the metric spaces induced on $A$ and on $B$ respectively by the families $\{\Omega_k\}_{k\in\mathbb{N}}$ and $\{\widetilde{\Omega}_k\}_{k\in\mathbb{N}}$ are isomorphic metric spaces and $\varphi$ can be uniquely prolonged to a continuous map $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$ which is both a metric isomorphism and a UAA isomorphism.*

*Proof.* As claimed, as an isomorphism of metric spaces we can take the map

$$\widetilde{\varphi} : \widetilde{A} \to \widetilde{B} \text{ such that } \quad \widetilde{\varphi}([(a_n)_n]_\sim) := [(\varphi(a_n))_n]_\sim. \tag{2.79}$$

Here, we used the following notation: Let $(A, d)$ denote the metric induced on $A$ by the family $\{\Omega_k\}_{k\in\mathbb{N}}$, and let $(B, \delta)$ denote the metric induced on $B$ by the family $\{\widetilde{\Omega}_k\}_{k\in\mathbb{N}}$; in (2.79), $(a_n)_n$ is a Cauchy sequence in $(A, d)$ while the two classes $[\cdot]_\sim$ from left to right in (2.79) are the equivalence classes as in (2.74) related respectively to the equivalence relations induced by the metrics $d$ and $\delta$. See page 426 in Chap. 7 for the complete proof. □

### 2.3.3 *Formal Power Series*

Throughout this section, $(A, *)$ is a UA *graded* algebra with a fixed grading $\{A_j\}_{j\geq 0}$. Following the notation in the previous section, for $k \geq 0$ we set

$$\Omega_k := \bigoplus\nolimits_{j\geq k} A_j. \tag{2.80}$$

We know from Proposition 2.65 that $\{\Omega_k\}_{k\in\mathbb{N}}$ is a topologically admissible family of $A$, thus endowing $A$ with both a metric space and a topological algebra structure. We aim to give a very explicit realization of an isometric completion of $(A, d)$, as the set of the so-called formal power series on $A$ (w.r.t. the grading $\{A_j\}_j$). We begin with the relevant definitions.

**Definition 2.73 (Formal Power Series on $A$).** Let $A = \bigoplus_{j\geq 0} A_j$ be a UA graded algebra. We set

$$\widehat{A} := \prod\nolimits_{j\geq 0} A_j, \tag{2.81}$$

and we call $\widehat{A}$ the space of *formal power series on $A$ (w.r.t. the grading $\{A_j\}_j$)*. On $\widehat{A}$ we consider the operation $\widehat{*}$ defined by

$$(a_j)_j \;\widehat{*}\; (b_j)_j := \Big(\sum_{k=0}^{j} a_{j-k} * b_k\Big)_{j\geq 0} \qquad (a_j, b_j \in A_j,\ \ \forall\, j \geq 0). \tag{2.82}$$

Then $(\widehat{A}, \widehat{*})$ is a UA algebra, called *the algebra of the formal power series on $A$*.

*Remark 2.74.* Note that (2.82) is well posed thanks to the fact that $a_{j-k} * b_k \in A_{j-k} * A_k \subseteq A_j$ for every $j \geq 0$ and every $k = 0, \ldots, j$. Obviously, $A$ is a subset of $\widehat{A}$ and it is trivially seen that

$$a, b \in A \quad \Longrightarrow \quad a \,\widehat{*}\, b = a * b. \tag{2.83}$$

We now introduce on $\widehat{A}$ a distinguished topology, by introducing a suitable topologically admissible family. To this aim, we set

$$\widehat{\Omega}_k := \prod\nolimits_{j\geq k} A_j, \quad k \in \mathbb{N} \cup \{0\}, \tag{2.84}$$

naturally considered as subspaces of $\widehat{A}$. The following facts hold:

1. Every $\widehat{\Omega}_k$ is an ideal in $\widehat{A}$.
2. $\widehat{A} = \widehat{\Omega}_0 \supseteq \widehat{\Omega}_1 \supseteq \cdots \widehat{\Omega}_k \supseteq \widehat{\Omega}_{k+1} \supseteq \cdots$.
3. $\widehat{\Omega}_i \mathbin{\widehat{*}} \widehat{\Omega}_j \subseteq \widehat{\Omega}_{i+j}$, for every $i, j \geq 0$.
4. $\bigcap_{i \geq 0} \widehat{\Omega}_i = \{0\}$.

As a consequence, $\{\widehat{\Omega}_k\}_{k \geq 0}$ is a topologically admissible family of $\widehat{A}$. By means of Theorem 2.58 we can deduce that $\{\widehat{\Omega}_k\}_{k \geq 0}$ endows $\widehat{A}$ with a topology $\widehat{\Omega}$ (more, with the structures of a topological algebra and of a metric space) and we call $(\widehat{A}, \widehat{\Omega})$ the *topological space of the formal power series* (related to the given grading). Note that

$$\Omega_k = A \cap \widehat{\Omega}_k, \quad \forall\, k \in \mathbb{N} \cup \{0\}, \tag{2.85}$$

whence the inclusion $A \hookrightarrow \widehat{A}$ is continuous (here $A$ has the topology induced by $\{\Omega_k\}_k$ and $\widehat{A}$ has the topology induced by $\{\widehat{\Omega}_k\}_k$). We have the following important result.

**Theorem 2.75 (The Isometry $\widehat{A} \simeq \widetilde{A}$).** *Let $(A, *)$ be a UA graded algebra with grading $\{A_j\}_{j \geq 0}$. Let $\Omega_k$ and $\widehat{\Omega}_k$ be defined, respectively, as in (2.80) and (2.84).*

*Then the space $\widehat{A}$ (with the metric induced by $\{\widehat{\Omega}_k\}_{k \geq 0}$) is a complete metric space and it is an isometric completion of $A$ (with the metric induced by $\{\Omega_k\}_{k \geq 0}$). The natural inclusion $A \hookrightarrow \widehat{A}$ is both an isometry and a UAA isomorphism, and $A$ is dense in $\widehat{A}$.*

*In particular, denoting by $d$ (resp. by $\widehat{d}$) the metric on $A$ (resp. on $\widehat{A}$) induced by the family $\{\Omega_k\}_{k \geq 0}$ (resp. by the family $\{\widehat{\Omega}_k\}_{k \geq 0}$) we have that the restriction of $\widehat{d}$ to $A \times A$ coincides with $d$.*

*Proof.* See page 428 in Chap. 7. $\square$

*Remark 2.76.* We have the following results.

1. By Proposition 2.65-(d), we get:
   *A sequence $w_k = (u_0^k, u_1^k, \ldots)$ in $\widehat{A}$ converges to $w = (u_0, u_1, \ldots)$ in $\widehat{A}$ if and only if for every $N \in \mathbb{N}$ there exists $k(N) \in \mathbb{N}$ such that, for all $k \geq k(N)$, it holds that*
   $$w_k = \Big(u_0, u_1, u_2, u_3, \ldots, u_N,\; u_{N+1}^k, u_{N+2}^k, u_{N+3}^k, \ldots\Big).$$
2. With all the above notation, if $a = (a_j)_j \in \widehat{A} = \prod_{j \geq 0} A_j$ (with $a_j \in A_j$ for every $j \geq 0$) then we have the limit
   $$A \ni \sum_{j=0}^{N} a_j \equiv (a_0, a_1, \ldots, a_N, 0, 0, \ldots) \xrightarrow[N \to \infty]{} a, \tag{2.86}$$

the limit being taken in the metric space $\widehat{A}$. We can thus represent the elements of $\widehat{A}$ as series $\sum_{j=0}^{\infty} a_j$ (with $a_j \in A_j$ for every $j \geq 0$).

3. Furthermore, any series $\sum_{n=1}^{\infty} b_n$ of elements of $A$ converges in $\widehat{A}$ if and only if it is Cauchy (completeness of $\widehat{A}$), which is equivalent to $\lim_{n\to\infty} b_n = 0$ in $\widehat{A}$ (see Remark 2.62), which, in its turn, is equivalent to $\lim_{n\to\infty} b_n = 0$ in $A$ (see (2.85)). For example, if $a_n \in A_n$ for every $n \geq 0$, the series $\sum_{n=1}^{\infty} a_n$ is convergent in $\widehat{A}$.

   Analogously, *any series* $\sum_{n=1}^{\infty} b_n$ *of elements of* $\widehat{A}$ *converges in* $\widehat{A}$ *if and only if* $\lim_{n\to\infty} b_n = 0$ *in* $\widehat{A}$ (again by an application of Remark 2.62).
4. Any set $\widehat{\Omega}_k$ ($k \in \mathbb{N} \cup \{0\}$) is both open and closed in $\widehat{A}$. Thus, by (2.85), the same is true of any $\Omega_k$ in $A$. More generally, see Proposition 2.77 below.

**Proposition 2.77.** *Let $J$ be any fixed subset of $\mathbb{N} \cup \{0\}$. Then the set*

$$H := \Big\{ (u_j)_j \in \widehat{A} \,\Big|\, u_j \in A_j \text{ for every } j \geq 0 \text{ and } u_j = 0 \text{ for every } j \in J \Big\}$$

*is closed in the topological space $\widehat{A}$.*

*Proof.* Suppose $\{w_k\}_k$ is a sequence in $H$ converging to $w$ in $\widehat{A}$. We use for $w_k$ and $w$ the notation in Remark 2.76-(1). Let $j_0 \in J$ be fixed. By the cited remark, there exists $k(j_0) \in \mathbb{N}$ such that, for all $k \geq k(j_0)$,

$$w_k = \big(u_0, u_1, u_2, \dots, u_{j_0},\ u^k_{j_0+1}, u^k_{j_0+2}, \dots\big). \tag{2.87}$$

Since $w_k \in H$ for every $k$, its $j_0$-component is null. By (2.87), this $j_0$-component equals the $j_0$-component of $w$. Since $j_0$ is arbitrary in $H$, this proves that $w \in H$. □

*Remark 2.78.* For example, we can apply Proposition 2.77 in the cases when $J = \{0, 1, \dots, k-1\}$, or $J = \{0\}$ or $J = (\mathbb{N} \cup \{0\}) \setminus \{k\}$, in which cases we obtain respectively the closed sets $\widehat{\Omega}_k$, $\widehat{A}_+ := \prod_{j\geq 1} A_j$, and $A_k$.

The following lemma will be used frequently in the sequel.

**Lemma 2.79 (Prolongation Lemma).** *Suppose $A = \bigoplus_{j\geq 0} A_j$ and $B = \bigoplus_{j\geq 0} B_j$ are graded UA algebras and let $\widehat{A}, \widehat{B}$ be the corresponding topological spaces of their formal power series.*

*Following (2.80), we use the notation $\Omega^A_k := \bigoplus_{j\geq k} A_j$ and $\Omega^B_k := \bigoplus_{j\geq k} B_j$. Suppose $\varphi : A \to B$ is a linear map with the following property:*

$$\begin{aligned} &\textit{There exists a sequence } \{k_n\}_n \textit{ in } \mathbb{N} \textit{ such that } \lim_{n\to\infty} k_n = \infty \textit{ and} \\ &\varphi\big(\Omega^A_n\big) \subseteq \Omega^B_{k_n} \quad \textit{for every } n \in \mathbb{N}. \end{aligned} \tag{2.88}$$

*Then $\varphi$ is uniformly continuous (considering $A$, $B$ as subspaces of the metric spaces $\widehat{A}$, $\widehat{B}$, respectively[16]). Hence, $\varphi$ can be extended in a unique way to a continuous linear map $\widehat{\varphi}:\widehat{A}\to\widehat{B}$. Moreover, if $\varphi$ is a UAA morphism, the same is true of $\widehat{\varphi}$.*

*Proof.* See page 430 in Chap. 7. □

*Remark 2.80.* Theorem 2.75 can be applied to the graded algebras

$$\mathscr{T}(V)=\bigoplus_{j\geq 0}\mathscr{T}_j(V)\quad\text{and}\quad \mathscr{T}(V)\otimes\mathscr{T}(V)=\bigoplus_{j\geq 0}K_j(V)$$

(see (2.28) and (2.42), respectively). Thus, on the algebras $\mathscr{T}$ and $\mathscr{T}\otimes\mathscr{T}$ we are given metric space structures induced respectively by the topologically admissible families $\{U_k\}_k$ and $\{W_k\}_k$, where

$$U_k=\bigoplus_{j\geq k}\mathscr{T}_j(V),\qquad W_k=\bigoplus_{i+j\geq k}\mathscr{T}_{i,j}(V).$$

The formal power series related to the graded algebras $\mathscr{T}$ and $\mathscr{T}\otimes\mathscr{T}$, denoted henceforth by $\widehat{\mathscr{T}}(V)$ and $\widehat{\mathscr{T}\otimes\mathscr{T}}(V)$ (or, shortly, by $\widehat{\mathscr{T}}$ and $\widehat{\mathscr{T}\otimes\mathscr{T}}$), are the algebras

$$\widehat{\mathscr{T}}(V)=\prod_{j\geq 0}\mathscr{T}_j(V),\qquad \widehat{\mathscr{T}\otimes\mathscr{T}}(V)=\prod_{i+j\geq 0}\mathscr{T}_{i,j}(V),\tag{2.89}$$

with operations as in (2.82) (respectively inherited from the operations on $(\mathscr{T},\cdot)$ and on $(\mathscr{T}\otimes\mathscr{T},\bullet)$), respectively equipped with the metric space structures induced by the topologically admissible families $\{\widehat{U}_k\}_k$ and $\{\widehat{W}_k\}_k$, where

$$\widehat{U}_k:=\prod_{j\geq k}\mathscr{T}_j(V),\qquad \widehat{W}_k:=\prod_{i+j\geq k}\mathscr{T}_{i,j}(V).\tag{2.90}$$

**Convention.** In order to avoid heavy notation, the operations $\widehat{\cdot}$ and $\widehat{\bullet}$ (see the notation in (2.82)) will usually appear without the "$\widehat{\ }$" sign. This slight abuse of notation is in accordance with (2.83).

[16]Which is the same as considering $A,B$ as metric spaces with metrics induced by the families $\{\Omega_k^A\}_k$ and $\{\Omega_k^B\}_k$, respectively (see (2.85)).

As a consequence, we have

$$\cdot : \widehat{\mathscr{T}} \times \widehat{\mathscr{T}} \to \widehat{\mathscr{T}}, \quad (a_j)_j \cdot (b_j)_j = \Bigg( \sum_{k=0}^{j} a_{j-k} \cdot b_k \Bigg)_{j \geq 0} \tag{2.91}$$

where $a_j, b_j \in \mathscr{T}_j(V)$ for all $j \geq 0$;

$$\bullet : \widehat{\mathscr{T}\otimes\mathscr{T}} \times \widehat{\mathscr{T}\otimes\mathscr{T}} \to \widehat{\mathscr{T}\otimes\mathscr{T}}, \quad (a_j)_j \bullet (b_j)_j = \Bigg( \sum_{k=0}^{j} a_{j-k} \bullet b_k \Bigg)_{j \geq 0} \tag{2.92}$$

where $a_j, b_j \in \bigoplus_{h+k=j} \mathscr{T}_{h,k}(V)$ for all $j \geq 0$.

The operation $\bullet$ on $\widehat{\mathscr{T}\otimes\mathscr{T}}$ can also be rewritten by using the double-sequenced notation $(u_{i,j})_{i,j}$ for the elements of $\widehat{\mathscr{T}\otimes\mathscr{T}}$ (this means that $u_{i,j} \in \mathscr{T}_{i,j}(V) = \mathscr{T}_i(V) \otimes \mathscr{T}_j(V)$ for every $i, j \geq 0$): indeed, following (2.45), we have

$$\bullet : \widehat{\mathscr{T}\otimes\mathscr{T}} \times \widehat{\mathscr{T}\otimes\mathscr{T}} \to \widehat{\mathscr{T}\otimes\mathscr{T}}$$

$$(t_{i,j})_{i,j} \bullet (\widetilde{t}_{i,j})_{i,j} = \Bigg( \sum_{r+\widetilde{r}=i,\ s+\widetilde{s}=j} t_{r,s} \bullet \widetilde{t}_{\widetilde{r},\widetilde{s}} \Bigg)_{i,j \geq 0}, \tag{2.93}$$

where $t_{i,j}, \widetilde{t}_{i,j} \in \mathscr{T}_i(V) \otimes \mathscr{T}_j(V)$ for all $i, j \geq 0$.

*Remark 2.81.* When expressed in coordinate form on the product space $\widehat{\mathscr{T}} = \prod_{j \geq 0} \mathscr{T}_j$, the Lie bracket operation takes a particularly easy form: Indeed, if $u, v \in \widehat{\mathscr{T}}$ and $u = (u_j)_j$ and $v = (v_j)_j$, with $u_j, v_j \in \mathscr{T}_j(V)$ (for every $j \in \mathbb{N}$), we have

$$[u, v] = \big[(u_j)_j, (v_j)_j\big] = \Big( \textstyle\sum_{h+k=j} [u_h, v_k] \Big)_{j \geq 0}. \tag{2.94}$$

Indeed, the following computation holds

$$\begin{aligned}
\big[(u_j)_j, (v_j)_j\big] &= (u_j)_j \cdot (v_j)_j - (v_j)_j \cdot (u_j)_j \\
&\overset{(2.91)}{=} \Bigg( \sum_{k=0}^{j} u_{j-k} \cdot v_k \Bigg)_{j \geq 0} - \Bigg( \sum_{k=0}^{j} v_{j-k} \cdot u_k \Bigg)_{j \geq 0}
\end{aligned}$$

(change the dummy index in the second sum)

$$= \Bigg( \sum_{k=0}^{j} \big( u_{j-k} \cdot v_k - v_k \cdot u_{j-k} \big) \Bigg)_{j \geq 0} = \Bigg( \sum_{k=0}^{j} [u_{j-k}, v_k] \Bigg)_{j \geq 0}.$$

Now note that the last term in the above chain of equalities is indeed the coordinate expression of $[u, v]$, since (as $u_{j-k} \in \mathscr{T}_{j-k}$, $v_k \in \mathscr{T}_k$) one has $[u_{j-k}, v_k] \in [\mathscr{T}_{j-k}, \mathscr{T}_k] \subseteq \mathscr{T}_{j-k} \otimes \mathscr{T}_k = \mathscr{T}_j$. □

In Chap. 3, we will have occasion to apply the following result.

**Proposition 2.82 ($\widehat{\mathscr{T}} \otimes \widehat{\mathscr{T}}$ as a Subalgebra of $\widehat{\mathscr{T}\otimes\mathscr{T}}$).** *Let $V$ be a vector space. With the notation of this section, the tensor product $\widehat{\mathscr{T}}(V) \otimes \widehat{\mathscr{T}}(V)$ can be identified with a subalgebra of $\widehat{\mathscr{T}\otimes\mathscr{T}}(V)$.*

*Indeed, we can identify the element $(u_i)_i \otimes (v_j)_j$ of $\widehat{\mathscr{T}} \otimes \widehat{\mathscr{T}}$ (where $u_i, v_i \in \mathscr{T}_i(V)$ for every $i \geq 0$) with the element $(u_i \otimes v_j)_{i,j}$ of $\widehat{\mathscr{T}\otimes\mathscr{T}}$, this identification being a UAA morphism. Here, $\widehat{\mathscr{T}}(V) \otimes \widehat{\mathscr{T}}(V)$ is equipped with the UA algebra structure obtained, as in Proposition 2.41, from the UAA structure of $(\widehat{\mathscr{T}}(V), \cdot)$. Hereafter, when writing $\widehat{\mathscr{T}}(V) \otimes \widehat{\mathscr{T}}(V) \hookrightarrow \widehat{\mathscr{T}\otimes\mathscr{T}}(V)$, we shall understand the previously mentioned immersion:*

$$\widehat{\mathscr{T}}(V) \otimes \widehat{\mathscr{T}}(V) \ni (u_i)_i \otimes (v_j)_j \mapsto \big(u_i \otimes v_j\big)_{i,j} \in \widehat{\mathscr{T}\otimes\mathscr{T}}(V). \tag{2.95}$$

*Proof.* See page 433 in Chap. 7. □

*Remark 2.83. Let $\{\alpha_k\}_k$ and $\{\beta_k\}_k$ be two sequences of elements in $\widehat{\mathscr{T}}(V)$ such that $\lim_{k\to\infty} \alpha_k = \alpha$ and $\lim_{k\to\infty} \beta_k = \beta$ in $\widehat{\mathscr{T}}(V)$. Then*

$$\lim_{k\to\infty} \alpha_k \otimes \beta_k = \alpha \otimes \beta \quad \textit{in } \widehat{\mathscr{T}\otimes\mathscr{T}}(V),$$

*where we consider $\alpha \otimes \beta$ and any $\alpha_k \otimes \beta_k$ as elements of $\widehat{\mathscr{T}\otimes\mathscr{T}}(V)$ (according to* (2.95) *in Proposition 2.82).* For the proof, see page 434 in Chap. 7.

*Remark 2.84.* Following the notation in (2.91) and (2.92) and by using the immersion $\widehat{\mathscr{T}}(V) \otimes \widehat{\mathscr{T}}(V) \hookrightarrow \widehat{\mathscr{T}\otimes\mathscr{T}}(V)$ in (2.95), it is not difficult to prove that

$$(a \otimes b) \bullet (\alpha \otimes \beta) = (a \cdot b) \otimes (b \cdot \beta), \qquad \text{for every } a, b, \alpha, \beta \in \widehat{\mathscr{T}}(V), \tag{2.96}$$

where this is meant as an equality of elements of $\widehat{\mathscr{T}\otimes\mathscr{T}}(V)$.

### 2.3.4 *Some More Notation on Formal Power Series*

Let $n \in \mathbb{N}$ and let $S = \{x_1, \dots, x_n\}$ be a set of cardinality $n$. The free vector space $\mathbb{K}\langle S\rangle$ will be denoted by

$$\mathbb{K}\langle x_1, \dots, x_n\rangle.$$

The algebras $\mathscr{T}(\mathbb{K}\langle x_1, \dots, x_n\rangle)$ and $\widehat{\mathscr{T}}(\mathbb{K}\langle x_1, \dots, x_n\rangle)$ can be thought of as, respectively, the *algebra of polynomials in the $n$ non-commuting indeterminates* $x_1, \dots, x_n$ and the *algebra of formal power series in the $n$ non-commuting indeterminates* $x_1, \dots, x_n$.

Recall that $\mathscr{T}(\mathbb{K}\langle x_1,\dots,x_n\rangle)$ is isomorphic to $\mathrm{Libas}(\{x_1,\dots,x_n\})$, the free UAA over $\{x_1,\dots,x_n\}$ (see Theorem 2.40). Analogously, the Lie algebra $\mathcal{L}(\mathbb{K}\langle x_1,\dots,x_n\rangle)$ can be thought of as the *Lie algebra of the Lie-polynomials in the $n$ non-commuting indeterminates* $x_1,\dots,x_n$. Recall that $\mathcal{L}(\mathbb{K}\langle x_1,\dots,x_n\rangle)$ is a free Lie algebra related to the set $\{x_1,\dots,x_n\}$, being isomorphic to $\mathrm{Lie}(\{x_1,\dots,x_n\})$ (see Theorem 2.56).

When $n=1$, it is customary to write

$$\mathbb{K}[x] := \mathscr{T}(\mathbb{K}\langle x\rangle) \quad \text{and} \quad \mathbb{K}[[x]] := \widehat{\mathscr{T}}(\mathbb{K}\langle x\rangle).$$

(Note that $\mathscr{T}(\mathbb{K}\langle x\rangle)$ and $\widehat{\mathscr{T}}(\mathbb{K}\langle x\rangle)$ are commutative algebras!) Some very important features of $\mathbb{K}[[x]]$ will be stated in Sect. 4.3 of Chap. 4 (and proved in Chap. 9).

Finally, when writing expressions like

$$\mathscr{T}(\mathbb{K}\langle x,y\rangle),\quad \widehat{\mathscr{T}}(\mathbb{K}\langle x,y\rangle),\quad \mathscr{T}(\mathbb{K}\langle x,y,z\rangle),\quad \widehat{\mathscr{T}}(\mathbb{K}\langle x,y,z\rangle),$$

we shall always mean (possibly without the need to say it explicitly) that the sets $\{x,y\}$ and $\{x,y,z\}$ have cardinality, respectively, two and three.

For the sake of future reference, we explicitly state the contents of Theorem 2.40 and 2.56 in the cases of two $\{x,y\}$ and three $\{x,y,z\}$ non-commuting indeterminates. We also seize the opportunity to introduce a new notation $\Phi_{a,b}$. In what follows, by an abuse of notation, we identify the canonical injection $\varphi: X\to\mathscr{T}(\mathbb{K}\langle X\rangle)$ defined by

$$X \xrightarrow{\chi} \mathbb{K}\langle X\rangle \xhookrightarrow{\iota} \mathscr{T}(\mathbb{K}\langle X\rangle),$$

with the set inclusion $X\hookrightarrow\mathscr{T}(\mathbb{K}\langle X\rangle)$.

**Theorem 2.85.** *The following universal properties are satisfied.*

(1a). *For every UA algebra $A$ and every pair of elements $a,b\in A$, there exists a unique UAA morphism $\Phi_{a,b}:\mathscr{T}(\mathbb{K}\langle x,y\rangle)\to A$ such that*

$$\Phi_{a,b}(x)=a \quad \textit{and} \quad \Phi_{a,b}(y)=b. \tag{2.97}$$

(1b). *For every Lie algebra $\mathfrak{g}$ and every pair of elements $a,b\in\mathfrak{g}$, there exists a unique LA morphism $\Phi_{a,b}:\mathcal{L}(\mathbb{K}\langle x,y\rangle)\to\mathfrak{g}$ such that* (2.97) *holds.*

(2a). *For every UA algebra $A$ and every triple of elements $a,b,c\in A$, there exists a unique UAA morphism $\Phi_{a,b,c}:\mathscr{T}(\mathbb{K}\langle x,y,z\rangle)\to A$ such that*

$$\Phi_{a,b,c}(x)=a,\quad \Phi_{a,b,c}(y)=b,\quad \textit{and} \quad \Phi_{a,b,c}(z)=c. \tag{2.98}$$

(2b). *For every Lie algebra $\mathfrak{g}$ and every triple of elements $a,b,c\in\mathfrak{g}$, there exists a unique LA morphism $\Phi_{a,b,c}:\mathcal{L}(\mathbb{K}\langle x,y,z\rangle)\to\mathfrak{g}$ such that* (2.98) *holds.*

## 2.4 The Universal Enveloping Algebra

The aim of this section is to introduce the so-called universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ and to collect some useful related results. In particular, we will present the remarkable Poincaré-Birkhoff-Witt Theorem.

Throughout this section, $\mathfrak{g}$ will denote a fixed Lie algebra and its Lie bracket is denoted by $[\cdot,\cdot]_{\mathfrak{g}}$ (or simply by $[\cdot,\cdot]$). As usual, $(\mathscr{T}(\mathfrak{g}),\cdot)$ is the tensor algebra of (the vector space of) $\mathfrak{g}$. We denote by $\mathscr{J}(\mathfrak{g})$ (sometimes $\mathscr{J}$ for short) the two-sided ideal in $\mathscr{T}(\mathfrak{g})$ generated by the set

$$\{x\otimes y - y\otimes x - [x,y]_{\mathfrak{g}} \;:\; x,y\in\mathfrak{g}\}.$$

More explicitly, we have

$$\mathscr{J}(\mathfrak{g}) = \operatorname{span}\Big\{t\cdot\big(x\otimes y - y\otimes x - [x,y]_{\mathfrak{g}}\big)\cdot t' \;\Big|\; x,y\in\mathfrak{g},\; t,t'\in\mathscr{T}(\mathfrak{g})\Big\}. \tag{2.99}$$

*Remark 2.86.* We remark that *the ideal $\mathscr{J}(\mathfrak{g})$ is not homogeneous (in the natural grading of $\mathscr{T}(\mathfrak{g})$)*. Indeed, in the sequence-style notation $(t_k)_{k\geq 0}$ for the elements of $\mathscr{T}(\mathfrak{g}) = \bigoplus_{k\geq 0}\mathscr{T}_k(\mathfrak{g})$, the element $x\otimes y - y\otimes x - [x,y]_{\mathfrak{g}}$ is rewritten as

$$\big(0,-[x,y]_{\mathfrak{g}}, x\otimes y - y\otimes x, 0, 0\cdots\big)\in\mathscr{T}_1\oplus\mathscr{T}_2. \tag{2.100}$$

**Definition 2.87 (Universal Enveloping Algebra).** With all the above notation, we consider the quotient space

$$\mathscr{U}(\mathfrak{g}) := \mathscr{T}(\mathfrak{g})/\mathscr{J}(\mathfrak{g})$$

and we call it *the universal enveloping algebra of* $\mathfrak{g}$. We denote by

$$\pi:\mathscr{T}(\mathfrak{g})\to\mathscr{U}(\mathfrak{g}),\qquad \pi(t):=[t]_{\mathscr{J}(\mathfrak{g})},\;\; t\in\mathscr{T}(\mathfrak{g}) \tag{2.101}$$

the associated projection. The natural operation[17] on $\mathscr{U}(\mathfrak{g})$

$$\mathscr{U}(\mathfrak{g})\times\mathscr{U}(\mathfrak{g})\ni(\pi(t),\pi(t'))\mapsto\pi(t\cdot t'),\qquad (t,t'\in\mathscr{T}(\mathfrak{g})),$$

which equips $\mathscr{U}(\mathfrak{g})$ with the structure of a UA algebra (see Proposition 2.12 on page 58), will be simply denoted by juxtaposition.

The natural injection $\mathfrak{g}\hookrightarrow\mathscr{T}(\mathfrak{g})$ induces a linear map

$$\mu:\mathfrak{g}\to\mathscr{U}(\mathfrak{g}),\qquad \mu(x):=[x]_{\mathfrak{g}}\quad(x\in\mathfrak{g}), \tag{2.102}$$

[17]This operation is well-posed because $\mathscr{J}(\mathfrak{g})$ is an ideal of $\mathscr{T}(\mathfrak{g})$.

that is, $\mu = \pi|_{\mathfrak{g}}$. The following important proposition proves that the Lie bracket of $\mathfrak{g}$ is turned by $\mu$ into the commutator of $\mathscr{U}(\mathfrak{g})$. As soon as we will know that $\mu$ is injective (a corollary of the Poincaré-Birkhoff-Witt Theorem), this will prove that (up to an identification) *every Lie bracket is a commutator* (in the very meaning used in this Book).

As usual, if $\mathscr{U}(\mathfrak{g})$ is involved as a Lie algebra, it is understood to be equipped with the associated commutator, which we denote by $[\cdot,\cdot]_{\mathscr{U}}$.

*Remark 2.88.* By its very definition, *the map* $\pi : \mathscr{T}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$ *is a UAA morphism, whence it is a Lie algebra morphism*, when $\mathscr{T}(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g})$ are equipped with their appropriate commutators (see Remark 2.17). Note that this does *not* prove (yet) that $\mu$ is a Lie algebra morphism, since $\mathfrak{g}$ (equipped with its *intrinsic* Lie bracket) is not a Lie subalgebra of $\mathscr{T}(\mathfrak{g})$ (equipped with its commutator).

*Remark 2.89.* The set $\{\pi(1)\} \cup \mu(\mathfrak{g})$ generates $\mathscr{U}(\mathfrak{g})$ as an algebra. (This follows from the fact that $\{1\} \cup \mathfrak{g}$ generates $\mathscr{T}(\mathfrak{g})$ as an algebra, together with the fact that $\pi$ is a UAA morphism.)

**Proposition 2.90.** *With the above notation, the map $\mu$ in (2.102) is a Lie algebra morphism, i.e.,*

$$\mu([x,y]_{\mathfrak{g}}) = \mu(x)\mu(y) - \mu(y)\mu(x), \qquad \textit{for every } x,y \in \mathfrak{g}. \tag{2.103}$$

*In particular, $\mu(\mathfrak{g})$ is a Lie subalgebra of $\mathscr{U}(\mathfrak{g})$, equipped with the associated commutator-algebra structure.*

Note that (2.103) can be rewritten as

$$\mu([x,y]_{\mathfrak{g}}) = [\mu(x),\mu(y)]_{\mathscr{U}}, \quad \text{for every } x,y \in \mathfrak{g}. \tag{2.104}$$

*Proof.* First we remark that (2.103) is equivalent to $\pi([x,y]_{\mathfrak{g}}) = \pi(x \otimes y - y \otimes x)$, which in its turn is equivalent to $x \otimes y - y \otimes x - [x,y]_{\mathfrak{g}} \in \mathscr{J}(\mathfrak{g})$. This is true (for any $x, y \in \mathfrak{g}$) by the definition of $\mathscr{J}(\mathfrak{g})$. □

*Remark 2.91.* Via the map $\pi$, the grading $\mathscr{T}(\mathfrak{g}) = \bigoplus_{k\geq 0} \mathscr{T}_k(\mathfrak{g})$ turns into $\mathscr{U}(\mathfrak{g}) = \biguplus_{k\geq 0} \pi(\mathscr{T}_k(\mathfrak{g}))$ (in the sense of sum of vector subspaces) but the family of vector spaces $\{\pi(\mathscr{T}_k(\mathfrak{g}))\}_{k\geq 0}$ does not furnish a *direct sum* decomposition of $\mathscr{U}(\mathfrak{g})$. Indeed, if $x, y \in \mathfrak{g}$ we have

$$\underbrace{\pi([x,y]_{\mathfrak{g}})}_{\in \pi(\mathscr{T}_1(\mathfrak{g}))} = \underbrace{\pi(x \otimes y - y \otimes x)}_{\in \pi(\mathscr{T}_2(\mathfrak{g}))}$$

(and we shall see explicit examples where this does not vanish). This is obviously due to the non-homogeneity of $\mathscr{J}(\mathfrak{g})$.

As expected, $\mathscr{U}(\mathfrak{g})$ has a universal property:

**Theorem 2.92 (Universal Property of the Universal Enveloping Algebra).** *Let $\mathfrak{g}$ be a Lie algebra and let $\mathscr{U}(\mathfrak{g})$ be its universal enveloping algebra.*

(i) *For every UA algebra $(A, *)$ and for every Lie algebra morphism $f : \mathfrak{g} \to A$, there exists a unique UAA morphism $f^\mu : \mathscr{U}(\mathfrak{g}) \to A$ such that*

$$f^\mu(\mu(x)) = f(x) \quad \textit{for every } x \in \mathfrak{g}, \tag{2.105}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "\mu"'] & A \\ \mathscr{U}(\mathfrak{g}) \arrow[ur, "f^\mu"'] & \end{tikzcd}$$

(ii) *Vice versa, suppose $U, \varphi$ are respectively a UA algebra and a Lie algebra morphism $\varphi : \mathfrak{g} \to U$ with the following property: For every UA algebra $(A, *)$ and for every Lie algebra morphism $f : \mathfrak{g} \to A$, there exists a unique UAA morphism $f^\varphi : U \to A$ such that*

$$f^\varphi(\varphi(x)) = f(x) \quad \textit{for every } x \in \mathfrak{g}, \tag{2.106}$$

*thus making the following a commutative diagram:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "\varphi"'] & A \\ U \arrow[ur, "f^\varphi"'] & \end{tikzcd}$$

*Then $U$ is canonically isomorphic to $\mathscr{U}(\mathfrak{g})$, the isomorphism being $\varphi^\mu : \mathscr{U}(\mathfrak{g}) \to U$ and its inverse being $\mu^\varphi : U \to \mathscr{U}(\mathfrak{g})$. Moreover, $\varphi = \varphi^\mu \circ \mu$. Furthermore (if $1_U$ denotes the unit of $U$) the set $\{1_U\} \cup \varphi(\mathfrak{g})$ is a set of algebra generators for $U$ and $U \simeq \mathscr{U}(\varphi(\mathfrak{g}))$, canonically as UA algebras.*

*Proof.* Explicitly, the map $f^\mu$ is defined by

$$f^\mu : \mathscr{U}(\mathfrak{g}) \to A, \qquad \pi(t) \mapsto \overline{f}(t) \quad (t \in \mathscr{T}(\mathfrak{g})), \tag{2.107}$$

where $\overline{f} : \mathscr{T}(\mathfrak{g}) \to A$ is the unique UAA morphism extending $f : \mathfrak{g} \to A$.
For the rest of the proof, see page 435 in Chap. 7. $\square$

We are in a position to prove a useful result on the enveloping algebra of the *free* Lie algebra generated by a vector space.

**Proposition 2.93.** *Let $X$ be any set. Let $V := \mathbb{K}\langle X\rangle$ denote the free vector space over $X$. Let $\mathcal{L}(V)$ be the free Lie algebra generated by the vector space $V$ (i.e., $\mathcal{L}(V)$ is the smallest Lie subalgebra of $\mathscr{T}(V)$ containing $V$).*

*Then $\mathscr{U}(\mathcal{L}(V))$ and $\mathscr{T}(V)$ are isomorphic (as unital associative algebras).*

*Proof.* More explicitly, we can take as isomorphism $j : \mathscr{U}(\mathcal{L}(V)) \to \mathscr{T}(V)$ the only UAA morphism such that

$$j(\pi(t)) = \iota(t), \quad \text{for every } t \in \mathcal{L}(V). \tag{2.108}$$

See page 437 in Chap. 7 for the proof. We remark that in that proof we will not use explicitly the fact that $\mathcal{L}(\mathbb{K}\langle X\rangle)$ is a free Lie algebra related to $X$ (proved in Theorem 2.56). □

Here we have the fundamental result on the universal enveloping algebra.

**Theorem 2.94 (Poincaré-Birkhoff-Witt).** *Let $\mathfrak{g}$ be a Lie algebra and let $\mathscr{U}(\mathfrak{g})$ be its universal enveloping algebra. Let $1$ denote the unit of $\mathscr{U}(\mathfrak{g})$ and let $\mu$ be the map in (2.102). Suppose $\mathfrak{g}$ is endowed with an indexed (linear) basis $\{x_i\}_{i\in\mathfrak{I}}$, where $\mathfrak{I}$ is totally ordered by the relation $\preccurlyeq$. Set $X_i := \mu(x_i)$, for $i \in \mathfrak{I}$.*

*Then the following elements form a linear basis of $\mathscr{U}(\mathfrak{g})$:*

$$1, \quad X_{i_1}\cdots X_{i_n}, \qquad \text{where} \quad n \in \mathbb{N},\ i_1,\dots,i_n \in \mathfrak{I},\ i_1 \preccurlyeq \ldots \preccurlyeq i_n. \tag{2.109}$$

*Proof.* The (laborious) proof of this key result is given in Chap. 7 (starting from page 438). For other proofs, the Reader is referred for example to [25, 85, 95, 99, 159, 171]. □

In the sequel, the Poincaré-Birkhoff-Witt Theorem will be referred to as PBW for short. Apparently until 1956, the theorem was only referred to as the "Birkhoff-Witt Theorem": see Schmid [153], Grivel [74], Ton-That, Tran [168] for a historical overview on this topic and for a description of (the long forgotten) contribution of Poincaré to this theorem, dated back to 1900.

**Corollary 2.95.** *Let $\mathfrak{g}$ be a Lie algebra and let $\mathscr{U}(\mathfrak{g})$ be its universal enveloping algebra. Then the map $\mu$ in (2.102) is injective, so that $\mu : \mathfrak{g} \to \mu(\mathfrak{g})$ is a Lie algebra isomorphism.*

*As a consequence, every Lie algebra can be identified with a Lie subalgebra of a UA algebra (endowed with the commutator), in the following way:*

$$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \equiv (\mu(\mathfrak{g}), [\cdot,\cdot]_{\mathscr{U}}) \hookrightarrow \mathscr{U}(\mathfrak{g}) \quad \begin{pmatrix} \textit{both a UA algebra} \\ \textit{and a commutator-algebra} \end{pmatrix}.$$

*Proof.* Let $x \in \mathfrak{g}$ be such that $\mu(x) = 0$. With the notation of Theorem 2.94, we have $x = \sum_{i \in \mathcal{I}'} \lambda_i\, x_i$, where $\mathcal{I}' \subseteq \mathcal{I}$ is finite and the $\lambda_i$ are scalars. Thus $0 = \mu(\sum_{i \in \mathcal{I}'} \lambda_i\, x_i) = \sum_{i \in \mathcal{I}'} \lambda_i\, X_i$, which is possible iff $\lambda_i = 0$ for every $i \in \mathcal{I}'$, since the vectors $X_i$ appear in the basis (2.109) of $\mathscr{U}(\mathfrak{g})$, i.e., $x = 0$.

Hence, the map $\mu : \mathfrak{g} \to \mu(\mathfrak{g})$ is a bijection and it is also a Lie algebra morphism, in view of Proposition 2.90, when $\mu(\mathfrak{g})$ is equipped with the commutator from the UA algebra $\mathscr{U}(\mathfrak{g})$. □

By means of the PBW Theorem, we are able to give a short proof of the existence of free Lie algebras generated by a vector space.

*Proof (of Theorem 2.49, page 86).* Let $V$ be a vector space and let $f : V \to \mathfrak{g}$ be a linear map, $\mathfrak{g}$ being a Lie algebra. We need to prove that there exists a unique LA morphism $\overline{f} : \mathcal{L}(V) \to \mathfrak{g}$ prolonging $f$. The uniqueness is trivial, once existence is proved. To this end, let us consider the LA morphism $\mu : \mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ in (2.102). Since the map $\mu \circ f : V \to \mathscr{U}(\mathfrak{g})$ is linear and $\mathscr{U}(\mathfrak{g})$ is a UA algebra, by Theorem 2.38 there exists a UAA morphism $\overline{\mu \circ f} : \mathscr{T}(V) \to \mathscr{U}(\mathfrak{g})$ prolonging $\mu \circ f$. Now we restrict $\overline{\mu \circ f}$ both in domain and codomain, by considering the map

$$\widehat{f} : \mathcal{L}(V) \to \mu(\mathfrak{g}), \qquad \widehat{f}(t) := \overline{\mu \circ f}(t) \quad (t \in \mathcal{L}(V)).$$

To prove that $\widehat{f}$ is well posed, we need to show that

$$\overline{\mu \circ f}(t) \in \mu(\mathfrak{g}) \quad \text{for every } t \in \mathcal{L}(V). \tag{2.110}$$

Since $\mathcal{L}(V)$ is Lie-generated by $V$ (see Proposition 2.47) it suffices to prove (2.110) when $t = [v_1 \cdots [v_{n-1}, v_n] \cdots]$, for any $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in V$. To this end (denoting by $[\cdot,\cdot]_{\mathscr{U}}$ the commutator of $\mathscr{U}(\mathfrak{g})$), we argue as follows:

$$\begin{aligned} \overline{\mu \circ f}(t) &= [\overline{\mu \circ f}(v_1) \cdots [\overline{\mu \circ f}(v_{n-1}), \overline{\mu \circ f}(v_n)]_{\mathscr{U}} \cdots]_{\mathscr{U}} \\ &= [\mu(f(v_1)) \cdots [\mu(f(v_{n-1})), \mu(f(v_n))]_{\mathscr{U}} \cdots]_{\mathscr{U}} \end{aligned}$$

In the first equality we applied the fact that $\overline{\mu \circ f}$ is a UAA morphism and in the second equality the fact that $\overline{\mu \circ f}$ coincides with $\mu \circ f$ on $V$. Now the above right-hand side is an element of $\mu(\mathfrak{g})$ since $f(v_i) \in \mathfrak{g}$ for every $i = 1, \ldots, n$ and $\mu(\mathfrak{g})$ is a Lie subalgebra of $\mathscr{U}(\mathfrak{g})$ (see Proposition 2.90). This proves (2.110). We now remark that $\widehat{f}$ is an LA morphism (of the associated commutator-algebras) since it is the restriction of $\overline{\mu \circ f}$, which is an LA morphism (being a UAA morphism).

Since $\mu : \mathfrak{g} \to \mu(\mathfrak{g})$ is a Lie algebra isomorphism (thanks to Corollary 2.95), the map

$$\mu^{-1} \circ \widehat{f} : \mathcal{L}(V) \xrightarrow{\widehat{f}} \mu(\mathfrak{g}) \xrightarrow{\mu^{-1}} \mathfrak{g}$$

is an LA morphism, since both $\mu^{-1}$ and $\widehat{f}$ are. We set $\overline{f} := \mu^{-1} \circ \widehat{f}$. It remains to show that $\overline{f}$ prolongs $f$. This follows immediately from

$$\overline{f}(v) = \mu^{-1}(\overline{\mu \circ f}(v)) = (\mu^{-1} \circ \mu \circ f)(v) = f(v), \quad \forall\, v \in V.$$

This ends the proof. □

The following diagram describes the maps in the above argument:

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, hook] & \mathfrak{g} \arrow[r, shift left, "\mu"] & \mu(\mathfrak{g}) \arrow[l, shift left, "\mu^{-1}"] \arrow[r, hook] & \mathscr{U}(\mathfrak{g}) \\
\mathcal{L}(V) \arrow[ur, "\overline{f}"] \arrow[urr, "\widehat{f}"] \arrow[d, hook] & & & \\
\mathscr{T}(V) \arrow[uurrr, "\overline{\mu \circ f}"'] & & &
\end{tikzcd}$$

We end the section with an example of how the injective map $\mu : \mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ can be used to perform computations involving the Lie bracket of a Lie algebra (without "explicit knowledge" of the Lie bracket on $\mathfrak{g}$).

*Example 2.96.* We prove that, for every Lie algebra $\mathfrak{g}$, one has

$$[a, [b, [a, b]]]_{\mathfrak{g}} = -[b, [a, [b, a]]]_{\mathfrak{g}}, \quad \text{for every } a, b \in \mathfrak{g}. \tag{2.111}$$

Obviously, this computation can be a consequence only of the skew-symmetry and the Jacobi identity, but it may not at first be obvious how to perform the computation.[18]

Let us use, instead, the injection $\mu : \mathfrak{g} \to \mathscr{U}(\mathfrak{g})$. Given arbitrary $a, b \in \mathfrak{g}$, we set $A := \mu(a)$ and $B := \mu(b)$. We begin by showing that

$$[A, [B, [A, B]]]_{\mathscr{U}} = -[B, [A, [B, A]]]_{\mathscr{U}}, \quad \text{in } \mathscr{U}(\mathfrak{g}). \tag{2.112}$$

[18]Indeed, (2.111) follows from the following argument: Set $x := [a, b]$, $y := a$, $z := b$ and write the Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$; the first summand is $[[a, b], [a, b]]$ which is null by skew-symmetry. Hence we get $[a, [b, [a, b]]] + [b, [[a, b], a]] = 0$ which leads directly to (2.111), again by skew-symmetry.

Indeed, unraveling the commutators (and dropping the subscript $\mathscr{U}$)

$$\begin{aligned}[A,[B,[A,B]]] &= [A,[B,AB-BA]] = [A, BAB - B^2A - AB^2 + BAB] \\ &= ABAB - AB^2A - A^2B^2 + ABAB + \\ &\quad - BABA + B^2A^2 + AB^2A - BABA \\ &= 2\,ABAB - 2\,BABA + B^2A^2 - A^2B^2.\end{aligned}$$

Hence, by interchanging $A$ and $B$ we get

$$[B,[A,[B,A]]] = 2\,BABA - 2\,ABAB + A^2B^2 - B^2A^2,$$

which proves (2.112). By exploiting (2.104), we thus get

$$\begin{aligned}\mu([a,[b,[a,b]]]_{\mathfrak{g}}) &\overset{(2.104)}{=} [\mu(a),[\mu(b),[\mu(a),\mu(b)]]]_{\mathscr{U}} = [A,[B,[A,B]]]_{\mathscr{U}} \\ &\overset{(2.112)}{=} -[B,[A,[B,A]]]_{\mathscr{U}} = -[\mu(b),[\mu(a),[\mu(b),\mu(a)]]]_{\mathscr{U}} \\ &\overset{(2.104)}{=} -\mu([b,[a,[b,a]]]_{\mathfrak{g}}).\end{aligned}$$

This yields the identity

$$\mu([a,[b,[a,b]]]_{\mathfrak{g}}) = \mu(-[b,[a,[b,a]]]_{\mathfrak{g}}).$$

The injectivity of $\mu$ now gives the claimed formula in (2.111). $\square$

http://www.springer.com/978-3-642-22596-3

Topics in Noncommutative Algebra
The Theorem of Campbell, Baker, Hausdorff and Dynkin
Bonfiglioli, A.; Fulci, R.
2012, XXII, 539 p. 5 illus., Softcover
ISBN: 978-3-642-22596-3