Chapter 1
Overview of Mathematical Methods in Partial Differential Equations

> Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.
>
> Bertrand Russell (1872–1970)

In this chapter we collect some results in Nonlinear Analysis that will be frequently used in the book. The first part of this chapter deals with comparison principles for second order differential operators and enables us to obtain an ordered structure of the solution set and, in most of the cases, the uniqueness of the solution. In the second part of this chapter we review the celebrated method of moving planes that allows us to deduce the radial symmetry of the solution. The third part of this chapter is concerned with variational methods. The final section contains some results in degree theory that will be mostly used to derive existence and nonexistence of a stationary solution to some reaction-diffusion systems.

1.1 Comparison Principles

We start this section with the following result which is due to Lou and Ni (see [139] or [140]).

Theorem 1.1 *Let* $g \in C^1(\overline{\Omega} \times \mathbb{R})$.

(i) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\Delta w + g(x,w) \geq 0 \text{ in } \Omega, \qquad \frac{\partial w}{\partial n} \leq 0 \text{ on } \partial\Omega, \tag{1.1}$$

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and $w(x_0) = \max_{\overline{\Omega}} w$*, then* $g(x_0, w(x_0)) \geq 0$.

(ii) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\Delta w + g(x,w) \leq 0 \ \ in\ \ \Omega, \qquad \frac{\partial w}{\partial n} \geq 0 \ \ on\ \ \partial\Omega,$$

and $w(x_0) = \min_{\overline{\Omega}} w$*, then* $g(x_0, w(x_0)) \leq 0$.

Proof. We shall prove only part (i) as (ii) can be established in a similar way. There are two possibilities for our consideration.

Case 1: $x_0 \in \Omega$. Since $w(x_0) = \max_{\overline{\Omega}} w$ we have $\Delta w(x_0) \leq 0$ and now from the first inequality in (1.1) we obtain $g(x_0, w(x_0)) \leq 0$.

Case 2: $x_0 \in \partial\Omega$. Assume by contradiction that $g(x_0, w(x_0)) < 0$. By the continuity of g and w, there exists a ball $B \subset \overline{\Omega}$ with $\partial B \cap \partial\Omega = \{x_0\}$ such that

$$g(x, w(x)) < 0 \qquad \text{for all } x \in B.$$

Thus, from (1.1) we find $\Delta w > 0$ in B. Since $w(x_0) = \max_{\overline{B}} w$, it follows from the Hopf boundary lemma that $\partial w/\partial n(x_0) > 0$ which contradicts the boundary condition in (1.1). This completes the proof of Theorem 1.1. □

Basic to our purposes in this book we state and prove the following result which is suitable for singular nonlinearities.

Theorem 1.2 *Let* $\Psi : \Omega \times (0,\infty) \to \mathbb{R}$ *be a Hölder continuous function such that the mapping* $(0,\infty) \ni t \longmapsto \Psi(x,t)/t$ *is decreasing for each* $x \in \Omega$*. Assume that there exist* v_1*,* $v_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ *such that*

(a) $\Delta v_1 + \Psi(x, v_1) \leq 0 \leq \Delta v_2 + \Psi(x, v_2)$ *in* Ω;

(b) $v_1, v_2 > 0$ *in* Ω *and* $v_1 \geq v_2$ *on* $\partial\Omega$;

(c) $\Delta v_1 \in L^1(\Omega)$ *or* $\Delta v_2 \in L^1(\Omega)$.

Then $v_1 \geq v_2$ *in* Ω.

Proof. Suppose by contradiction that $v \leq w$ is not true in Ω. Then, we can find ε_0, $\delta_0 > 0$ and a ball $B \subset\subset \Omega$ such that

$$v - w \geq \varepsilon_0 \qquad \text{in } B, \tag{1.2}$$

$$\int_B vw \left(\frac{\Phi(x,w)}{w} - \frac{\Phi(x,v)}{v} \right) dx \geq \delta_0. \tag{1.3}$$

Let us assume that $\Delta w \in L^1(\Omega)$ and set

$$M = \max\{1, \|\Delta w\|_{L^1(\Omega)}\},\ \varepsilon = \min\left\{1, \varepsilon_0, \frac{\delta_0}{4M}\right\}.$$

Consider $\theta \in C^1(\mathbb{R})$ a nondecreasing function such that $0 \le \theta \le 1$, $\theta(t) = 0$, if $t \le 1/2$ and $\theta(t) = 1$ for all $t \ge 1$. Define

$$\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

Because $w \ge v$ on $\partial\Omega$, we can find a smooth subdomain $\Omega^* \subset\subset \Omega$ such that

$$B \subset \Omega^* \quad \text{and } v - w < \frac{\varepsilon}{2} \text{ in } \Omega \setminus \Omega^*.$$

Using hypotheses (i) and (ii) we deduce

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \ge \int_{\Omega^*} vw\left(\frac{\Phi(x,w)}{w} - \frac{\Phi(x,v)}{v}\right)\theta_\varepsilon(v-w)dx. \quad (1.4)$$

By relation (1.3), we have

$$\begin{aligned}
\int_{\Omega^*} vw&\left(\frac{\Phi(x,w)}{w} - \frac{\Phi(x,v)}{v}\right)\theta_\varepsilon(v-w)dx \\
&\ge \int_B vw\left(\frac{\Phi(x,w)}{w} - \frac{\Phi(x,v)}{v}\right)\theta_\varepsilon(v-w)dx \\
&= \int_B vw\left(\frac{\Phi(x,w)}{w} - \frac{\Phi(x,v)}{v}\right)dx \\
&\ge \delta_0.
\end{aligned}$$

To raise a contradiction, we need only to prove that the left-hand side in (1.4) is smaller than δ_0. For this purpose, define

$$\Theta_\varepsilon(t) := \int_0^t s\theta'_\varepsilon(s)ds, \quad t \in \mathbb{R}.$$

It is easy to see that

$$\Theta_\varepsilon(t) = 0, \text{ if } t < \frac{\varepsilon}{2} \quad \text{and } 0 \le \Theta_\varepsilon(t) \le 2\varepsilon, \text{ for all } t \in \mathbb{R}. \quad (1.5)$$

Now, using Green's first formula, we evaluate the left side of (1.4):

$$\begin{aligned}
&\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \\
&\quad = \int_{\partial\Omega^*} w\theta_\varepsilon(v-w)\frac{\partial v}{\partial n}d\sigma(x) - \int_{\Omega^*}(\nabla w\cdot\nabla v)\theta_\varepsilon(v-w)dx \\
&\qquad - \int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla v\cdot\nabla(v-w)dx - \int_{\partial\Omega^*} v\theta_\varepsilon(v-w)\frac{\partial w}{\partial n}d\sigma(x) \\
&\qquad + \int_{\Omega^*}(\nabla w\cdot\nabla v)\theta_\varepsilon(v-w)dx + \int_{\Omega^*} v\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)dx \\
&\quad = \int_{\Omega^*}\theta_\varepsilon'(v-w)(v\nabla w - w\nabla v)\cdot\nabla(v-w)dx.
\end{aligned}$$

The previous relation can be rewritten as

$$\begin{aligned}
\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx = &\int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla(w-v)\cdot\nabla(v-w)dx \\
&+ \int_{\Omega^*}(v-w)\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)dx.
\end{aligned}$$

Because $\int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla(w-v)\cdot\nabla(v-w)dx \le 0$, the last equality yields

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \le \int_{\Omega^*}(v-w)\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)dx.$$

Therefore,

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \le \int_{\Omega^*}\nabla w\cdot\nabla(\Theta_\varepsilon(v-w))dx.$$

Again by Green's first formula, and by (1.5), we have

$$\begin{aligned}
\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \le &\int_{\partial\Omega^*}\Theta_\varepsilon(v-w)\frac{\partial w}{\partial n}d\sigma(x) \\
&- \int_{\Omega^*}\Theta_\varepsilon(v-w)\Delta w dx \\
\le &-\int_{\Omega^*}\Theta_\varepsilon(v-w)\Delta w dx \le 2\varepsilon\int_{\Omega^*}|\Delta w|dx \\
\le &\, 2\varepsilon M < \frac{\delta_0}{2}.
\end{aligned}$$

Thus, we have obtained a contradiction. Hence $v \le w$ in Ω, which completes the proof. □

A direct consequence of Theorem 1.2 is the result below.

Corollary 1.3 *Let $k \in C(0,\infty)$ be a positive decreasing function and $a_1, a_2 \in C(\Omega)$ with $0 < a_2 \le a_1$ in Ω. Assume that there exist $\beta \ge 0$, $v_1, v_2 \in C^2(\Omega)\cap C(\overline{\Omega})$ such* *that* $v_1, v_2 > 0$ *in* Ω*,* $v_1 \geq v_2$ *on* $\partial\Omega$ *and*

$$\Delta v_1 - \beta v_1 + a_1(x)k(v_1) \leq 0 \leq \Delta v_2 - \beta v_2 + a_2(x)k(v_2) \quad \text{in } \Omega.$$

Then $v_1 \geq v_2$ *in* Ω*.*

Proof. We simply apply Theorem 1.2 in the particular case

$$\Phi(x,t) = -\beta t + a_1(x)k(t), \quad (x,t) \in \Omega \times (0,\infty).$$

□

Let us now consider the more general elliptic operator in divergence form

$$\mathscr{L}u := \operatorname{div}[A(|\nabla u|)\nabla u],$$

where $A \in C(0,\infty)$ is positive such that the mapping $t \mapsto tA(t)$ is increasing.

Theorem 1.4 *Let* Ω *be a bounded and smooth domain in* $\mathbb{R}^N$ *(*$N \geq 1$*),* $\rho \in C(\overline{\Omega})$ *and* $f \in C(\mathbb{R})$*. Assume that* $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfy*

(i) $\mathscr{L}u - \rho(x)f(u) \geq 0 \geq \mathscr{L}v - \rho(x)f(v)$ *in* Ω*;*

(ii) $u \leq v$ *on* $\partial\Omega$*.*

Then $u \leq v$ *in* Ω*.*

Proof. Let $\phi : \mathbb{R} \to [0,\infty)$ be a C^1-function such that $\phi = 0$ on $(-\infty, 0]$ and ϕ is strictly increasing on $[0,\infty)$. We first multiply by $\phi(u-v)$ in (i) and obtain

$$(\mathscr{L}u - \mathscr{L}v)\phi(u-v) \geq \rho(x)(f(u) - f(v))\phi(u-v) \quad \text{in } \Omega.$$

Integrating over Ω, by the divergence theorem we find

$$-\int_\Omega \Big[A(|\nabla u|)\nabla u - A(|\nabla v|)\nabla v\Big] \cdot \nabla(u-v)\phi(u-v)dx \geq \int_\Omega \rho(x)(f(u) - f(v))\phi(u-v)dx \geq 0.$$

Hence

$$\int_\Omega \Big[A(|\nabla u|)\nabla u - A(|\nabla v|)\nabla v\Big] \cdot \nabla(u-v)\phi(u-v)dx \leq 0. \tag{1.6}$$

On the other hand,

$$\begin{aligned}\Big[A(|\nabla u|)\nabla u - A(|\nabla v|)\nabla v\Big]\cdot\nabla(u-v) \\ &= \Big[A(|\nabla u|)|\nabla u| - A(|\nabla v|)|\nabla v|\Big](|\nabla u| - |\nabla v|) \\ &\quad + \Big[A(|\nabla u|) + A(|\nabla v|)\Big](|\nabla u||\nabla v| - \nabla u\cdot\nabla v),\end{aligned}$$

so that

$$\Big[A(|\nabla u|)\nabla u - A(|\nabla v|)\nabla v\Big]\cdot\nabla(u-v) \geq 0 \quad \text{in } \Omega,$$

with equality if and only if $\nabla u = \nabla v$. Using this fact in (1.6) it follows that $u \leq v$ in Ω. This finishes the proof of our result. □

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a smooth bounded domain, $T > 0$, and*

$$\mathscr{L}u := \partial_t u - a(x,t,u)\Delta u + f(x,t,u),$$

where $a, f : \overline{\Omega} \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ are continuous functions such that $a \geq 0$ in $\overline{\Omega} \times [0,\infty)$. Assume that there exist $u_1, u_2 \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T])$ such that:

(i) *$\mathscr{L}u_1 \leq \mathscr{L}u_2$ in $\Omega \times (0,T)$.*
(ii) *$u_1 \leq u_2$ on $\Sigma_T := (\partial\Omega \times (0,T)) \cup (\Omega \times \{0\})$.*
(iii) *at least for one $i \in \{1,2\}$ we have $|D^2 u_i| \in L^\infty(\overline{\Omega} \times [0,T])$ and the functions a and f are Lipschitz with respect to the u variable in the neighborhood of $K := u_i(\overline{\Omega} \times [0,T])$.*

Then $u_1 \leq u_2$ in $\Omega \times [0,T]$.

1.2 Radial Symmetry of Solutions to Semilinear Elliptic Equations

An important tool in establishing the radial symmetry of a solution to elliptic PDEs is the so-called *moving plane method* that goes back to A.D. Alexandroff and J. Serrin. It was then refined by Gidas, Ni and Nirenberg in the celebrated paper [97]. The requirements on the regularity of the domain were further simplified by Berestycki and Nirenberg [16]. We follow here the line in [16] and [25] to provide the reader with a simple and instructive proof of the radial symmetry of solutions to semilinear elliptic PDEs in bounded and convex domains Ω that vanish on $\partial\Omega$.

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^N$ be a convex domain which is symmetric about the x_1 axis. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and $\rho : [0,\infty) \to \mathbb{R}$ is a decreasing function. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u = \rho(|x|) f(u), u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

then u is symmetric with respect to the x_1 axis.

Proof. We first need a version of the maximum principle for small domains as stated below.

Lemma 1.7 *Let $a \in C(\overline{\Omega})$ and $w \in C^2(\omega) \cap C(\overline{\omega})$ be such that*

$$\begin{cases} -\Delta w + a(x) w \geq 0 & \text{in } \omega, \\ w \geq 0 & \text{on } \partial\omega. \end{cases} \tag{1.8}$$

If

$$||a^-||_{L^{N/2}(\omega)} \geq S_N, \tag{1.9}$$

where S_N is the best Sobolev constant in ω, then $w \geq 0$ in ω. In particular, if

$$\|a^-\|_{L^\infty(\omega)} |\omega|^{N/2} \leq S_N,$$

that is, if ω is small, then $w \geq 0$ in ω.

Proof. We multiply the first inequality in (1.8) by $w^- = \max\{-w, 0\}$. Integrating over ω we obtain

$$\int_\omega |\nabla w^-|^2 dx + \int_\omega a(x) |w^-|^2 dx \leq 0.$$

This also yields

$$\int_\omega |\nabla w^-|^2 dx \leq \int_\omega a^-(x) |w^-|^2 dx.$$

On the other hand, by Sobolev and Hölder inequalities we find

$$S_N \|w^-\|_{L^{2N/(N-2)}(\omega)} \leq \int_\omega a^-(x) |w^-|^2 dx \leq \|a^-\|_{L^{N/2}(\omega)} \|w^-\|_{L^{2N/(N-2)}}(\omega).$$

Using (1.9), the above inequality implies $\|w^-\|_{L^{2N/(N-2)}(\omega)} = 0$, so $w \geq 0$ in ω. $\square$

Let us now come back to the proof of Theorem 1.6. For any $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ we write $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$. Let

$$\lambda_0 = \max\{x_1 \,:\, (x_1, x') \in \overline{\Omega}\}.$$

We claim that

$$u(x_1, x') < u(y_1, x'), \tag{1.10}$$

for all $(x_1, x') \in \Omega$ with $x_1 > 0$ and all $y_1 \in \mathbb{R}$ with $|y_1| < x_1$.

Then (1.10) implies $u(x_1, x') \le u(x_1, -x')$ and similarly $u(x_1, x') \ge u(x_1, -x')$. so $u(x_1, x') = u(x_1, -x')$, that is, u is symmetric about the x_1 axis. For any $0 < \lambda < \lambda_0$ define

$$\Sigma_\lambda = \{x = (x_1, x') \in \Omega \,:\, x_1 > \lambda\}$$

and

$$w_\lambda(x) = u_\lambda(x) - u(x), \quad x \in \Sigma_\lambda,$$

where $u_\lambda(x) = u(2\lambda - x_1, x')$. Note that w_λ is well defined in Σ_λ since Ω is convex and symmetric about the hyperplane $x_1 = 0$. Let us further remark that (1.10) is equivalent to

$$w_\lambda > 0 \quad \text{in } \Sigma_\lambda, \quad \text{for all } 0 < \lambda < \lambda_0. \tag{1.11}$$

From (1.7) we have

$$-\Delta w_\lambda + \rho(|x|)\frac{f(u) - f(u_\lambda)}{u_\lambda - u} w_\lambda + \rho(|x|) - \rho(|x_\lambda|) w_\lambda = 0 \quad \text{in } \Sigma_\lambda,$$

where $x_\lambda = (2\lambda - x_1, x')$. This yields

$$-\Delta w_\lambda + a(x) w_\lambda \ge 0 \quad \text{in } \Sigma_\lambda,$$

where

$$a(x) = \begin{cases} \rho(|x|)\dfrac{f(u) - f(u_\lambda)}{u_\lambda - u} & \text{if } w_\lambda(x) \ne 0, \\ 0 & \text{if } w_\lambda(x) = 0. \end{cases}$$

Remark that $a \in L^\infty(\Omega)$ and $\|a\|_{L^\infty(\Omega)} \le L\|\rho\|_{L^\infty(\Omega)}$, where L is the Lipschitz constant of f on the interval $[-\|u\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}]$. Furthermore, we have

$$w_\lambda > 0 \quad \text{on } \partial\Sigma_\lambda \cap \partial\Omega, \quad w_\lambda = 0 \quad \text{on } \partial\Sigma \cap \Omega.$$

Thus, by taking λ close to λ_0, by Lemma 1.7 we obtain $w_\lambda \ge 0$ in Σ_λ. Let

$$\mathscr{A} = \{0 < \lambda < \lambda_0 \,:\, w_\lambda \ge 0 \text{ in } \Sigma_\lambda\}.$$

It is easily seen that $\mathscr{A}$ is closed. We next prove that $\mathscr{A}$ is open. To this aim, let $\lambda \in \mathscr{A}$. By the strong maximum principle, $w_\lambda > 0$ in Σ. Let $K \subset \Sigma_\lambda$ be a compact set such that $|\Sigma_\mu \setminus K|$ is small, for μ in a neighborhood of λ. Also, there exists $c > 0$ such that $w_\lambda \geq c > 0$ in K, so by continuity arguments we have $w_\mu > 0$ in K for μ near λ. This yields $w_\mu \geq 0$ on $\partial(\Sigma_\mu \setminus K)$, so by Lemma 1.7 we find $w_\mu \geq 0$ in $\Sigma_\mu \setminus K$, so $w_\mu \geq 0$ in Σ_μ. This proves that $\mathscr{A}$ is open, so $\mathscr{A} = (0, \lambda_0)$. This implies that (1.11) holds, that is, u is symmetric with respect to the x_1 axis. □

1.3 Variational Methods

1.3.1 Ekeland's Variational Principle

Ekeland's variational principle [67] was established in 1974, with its main feature of how to use the norm completeness and a partial ordering to obtain a point where a linear functional achieves its supremum on a closed bounded convex set. In its original form, Ekeland's variational principle can be stated as follows.

Theorem 1.8 (Ekeland's Variational Principle) *Let (M,d) be a complete metric space and assume that $\Phi : M \to (-\infty, \infty]$, $\Phi \not\equiv \infty$, is a lower semicontinuous functional that is bounded from below.*

Then, for every $\varepsilon > 0$ and for any $z_0 \in M$, there exists $z \in M$ such that

(i) $\Phi(z) \leq \Phi(z_0) - \varepsilon d(z, z_0)$;
(ii) $\Phi(x) \geq \Phi(z) - \varepsilon d(x,z)$, *for any $x \in M$.*

Proof. We may assume without loss of generality that $\varepsilon = 1$. Define the following binary relation on M:

$$y \leq x \qquad \text{if and only if} \qquad \Phi(y) - \Phi(x) + d(x,y) \leq 0\,.$$

Then "$\leq$" is a partial order relation—that is,
(a) $x \leq x$, for any $x \in M$;
(b) if $x \leq y$ and $y \leq x$ then $x = y$;
(c) if $x \leq y$ and $y \leq z$ then $x \leq z$.

For arbitrary $x \in M$, set

$$S(x) := \{y \in M : y \leq x\}.$$

Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ and fix $z_0 \in M$. For any $n \geq 0$, let $z_{n+1} \in S(z_n)$ be such that

$$\Phi(z_{n+1}) \leq \inf_{S(z_n)} \Phi + \varepsilon_{n+1}.$$

The existence of z_{n+1} follows from the definition of $S(x)$. We prove that the sequence $\{z_n\}$ converges to some element z, which satisfies (i) and (ii).

Let us first remark that $S(y) \subset S(x)$, provided that $y \leq x$. Hence, $S(z_{n+1}) \subset S(z_n)$. It follows that for any $n \geq 0$,

$$\Phi(z_{n+1}) - \Phi(z_n) + d(z_n, z_{n+1}) \leq 0,$$

which implies $\Phi(z_{n+1}) \leq \Phi(z_n)$. Because Φ is bounded from below, we deduce that the sequence $\{\Phi(z_n)\}$ converges.

We prove in what follows that $\{z_n\}$ is a Cauchy sequence. Indeed, for any n and p we have

$$\Phi(z_{n+p}) - \Phi(z_n) + d(z_{n+p}, z_n) \leq 0. \tag{1.12}$$

Therefore,

$$d(z_{n+p}, z_n) \leq \Phi(z_n) - \Phi(z_{n+p}) \to 0 \quad \text{as } n \to \infty,$$

which shows that $\{z_n\}$ is a Cauchy sequence, so it converges to some $z \in M$. Now, taking $n = 0$ in (1.12), we find

$$\Phi(z_p) - \Phi(z_0) + d(z_p, z_0) \leq 0.$$

So, as $p \to \infty$, we find (i).

To prove (ii), let us choose arbitrarily $x \in M$. We distinguish the following situations.

Case 1: $x \in S(z_n)$, for any $n \geq 0$. It follows that $\Phi(z_{n+1}) \leq \Phi(x) + \varepsilon_{n+1}$, which implies that $\Phi(z) \leq \Phi(x)$.

Case 2: There exists an integer $N \geq 1$ such that $x \notin S(z_n)$, for any $n \geq N$ or, equivalently,

$$\Phi(x) - \Phi(z_n) + d(x, z_n) > 0 \quad \text{for every } n \geq N.$$

Passing to the limit in this inequality as $n \to \infty$ we find (ii). $\square$

A major consequence of Ekeland's variational principle is that even if it is not always possible to minimize a nonnegative C^1 functional Φ on a Banach space; however, there is always a minimizing sequence $(u_n)_{n\geq 1}$ such that $\Phi'(u_n) \to 0$ as $n \to \infty$. More precisely we have

Corollary 1.9 *Let E be a Banach space and let $\Phi : E \to \mathbb{R}$ be a C^1 functional that is bounded from below. Then, for any $\varepsilon > 0$, there exists $z \in E$ such that*

$$\Phi(z) \leq \inf_E \Phi + \varepsilon \qquad \text{and} \qquad \|\Phi'(z)\|_{E^\star} \leq \varepsilon.$$

Proof. The first part of the conclusion follows directly from Theorem 1.8. For the second part we have

$$\|\Phi'(z)\|_{E^\star} = \sup_{\|u\|=1} \langle \Phi'(z), u \rangle.$$

But,

$$\langle \Phi'(z), u \rangle = \lim_{\delta \to 0} \frac{\Phi(z+\delta u) - \Phi(z)}{\delta \|u\|}.$$

So, by Ekeland's variational principle,

$$\langle \Phi'(z), u \rangle \geq -\varepsilon.$$

Replacing now u with $-u$ we find

$$\langle \Phi'(z), u \rangle \leq \varepsilon,$$

which concludes our proof. □

1.3.2 Mountain Pass Theorem

The mountain pass theorem was established by Ambrosetti and Rabinowitz in [7]. It is a powerful tool for proving the existence of critical points of energy functionals, hence of weak solutions to wide classes of nonlinear problems. We first recall the following definition.

Definition 1.10 (Palais–Smale condition) *Let E be a real Banach space. A functional $J : E \to \mathbb{R}$ of class C^1 satisfies the Palais–Smale condition if any sequence $\{u_n\}$ in E is relatively compact, provided*

$$\{J(u_n)\} \quad \textit{is bounded and} \quad \|J'(u_n)\|_{E^\star} \to 0 \ \ \textit{as} \ \ n \to \infty. \tag{1.13}$$

By means of Ekeland's variational principle, one deduces the following result.

Proposition 1.11 *Let E be a real Banach space and assume that $\Phi : E \to \mathbb{R}$ is a functional of class C^1 that is bounded from below, and satisfies the Palais–Smale condition. Then the following properties hold true:*

(i) *Φ is coercive.*
(ii) *Any minimizing sequence of Φ has a convergent subsequence.*

We are now in position to state the mountain pass theorem.

Theorem 1.12 (Mountain Pass Theorem) *Let E be a real Banach space and assume that $J : E \to \mathbb{R}$ is a C^1 functional that satisfies the following conditions: There exist positive constants α and R such that*

(i) *$J(0) = 0$ and $J(v) \geq \alpha$ for all $v \in E$ with $\|v\| = R$;*
(ii) *$J(v_0) \leq 0$, for some $v_0 \in E$ with $\|v_0\| > R$.*

Set

$$\Gamma := \{p \in C([0,1];E) : p(0) = 0 \text{ and } p(1) = v_0\}$$

and

$$c := \inf_{p \in \Gamma} \max_{t \in [0,1]} J(p(t)).$$

Then there exists a sequence $\{u_n\}$ in E such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. Moreover, if J satisfies the Palais–Smale condition, then c is a nontrivial critical value of J, that is, there exists $u \in E$ such that $J(u) = c$ and $J'(u) = 0$.

1.3.3 Around the Palais–Smale Condition for Even Functionals

In this section we recall some notions and results from critical point theory for even functionals. These results are due to Tanaka [195]. Let X be an infinite dimensional separable Hilbert space and let $J : X \to \mathbb{R}$ be a C^2 functional such that

(A_1) J is even and $J(0) = 0$;

(A_2) For any finite dimensional subspace W of X there exists $R = R(W) > 0$ such that $J(u) < 0$ for all $u \in W$ with $\|u\| \geq R$;

(A_3) The Fréchet derivative $J' : X \to X$ satisfies

$$J'(u) = u + K(u) \quad \text{for all } \; u \in X\,,$$

where $K : X \to X$ is a compact operator. Let $\{X_k\}$ be a sequence of subspaces of X such that

$$\dim X_k = k \quad \text{and} \quad X = \overline{\bigcup_{k=1}^{\infty} X_k}.$$

For every $k \geq 1$ let $R_k = R(X_k) > 0$ from the hypothesis (A_2) and set $D_k = X_k \cap \overline{B(0,R_k)}$. Let also

$$\mathscr{C}_k = \left\{ \gamma \in C(D_k, X) \; : \gamma \text{ is odd and } \gamma\Big|_{X_k \cap \partial B(0,R_k)} = Id \right\}$$

and

$$b_k = \inf_{\gamma \in \mathscr{C}_k} \sup_{u \in D_k} J(\gamma(u)).$$

Definition 1.13 *We say that*

(i) J satisfies the $(PS)_k$ condition if every sequence $\{u_n\}$ in X_k such that $\{J(u_n)\}$ is bounded and

$$\left\| \left(J\big|_{X_k} \right)' (u_n) \right\|_{X_k} \to 0 \quad \text{as } n \to \infty$$

admits a convergent subsequence in X_k.

(ii) J satisfies the $(PS)_*$ condition if every sequence $\{u_k\}$ in X with $u_k \in X_k$ and such that $\{J(u_k)\}$ is bounded and

$$\left\| \left(J\big|_{X_k} \right)' (u_k) \right\|_{X_k} \to 0 \quad \text{as } k \to \infty$$

admits a convergent subsequence in X.

Definition 1.14 *Let u be a critical point of $J : X \to \mathbb{R}$. The large Morse index of J at u, denoted by $m^*(J,u)$ is the infimum of the codimensions of all subspaces of X on which the quadratic form $J''(u)$ is positive definite.*

Theorem 1.15 (see [195]) *Assume that J satisfies $(A_1) - (A_3)$, (PS), $(PS)_k$ and $(PS)_*$. Then, for each $k \geq 1$ there exists a critical point $u_k \in X$ such that*

$$J(u_k) \leq k \quad \text{and} \quad m^*(J, u_k) \geq k.$$

1.3.4 Bolle's Variational Method for Broken Symmetries

In the following we recall some notions and results from critical point theory for functionals with broken symmetry in the spirit of Bolle [22]. Let X be an infinite dimensional separable Hilbert space and let $J : [0,1] \times X \to \mathbb{R}$ be a C^2 functional. We set $J_\theta = J(\theta, \cdot)$ and denote by $J_\theta : X \to X$ the Fréchet derivative of J_θ.

Consider $\{e_k\}$ an orthonormal system of X and for any $k \geq 1$ set $X_k = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$.

(B_1) J satisfies the Palais–Smale condition on $[0,1] \times X$;

(B_2) For any $b > 0$ there exists a positive constant $C = C(b) > 0$ such that

$$\left| \frac{\partial J}{\partial \theta}(\theta, u) \right| \leq C(1 + \|J'_\theta(u)\|_X)(1 + \|u\|_K),$$

for all $(\theta, u) \in [0,1] \times X$ satisfying $|J_\theta(u)| \leq b$.

Assume that J is even and $J(0) = 0$;

(B_3) There exist two flows $\eta_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $\eta_i(\theta, \cdot)$ are Lipschitz continuous for all $\theta \in [0,1]$ and

$$\eta_1(\theta, J_\theta(u)) \leq \frac{\partial J}{\partial \theta}(\theta, u) \leq \eta_2(\theta, J_\theta(u))$$

at each critical point u of J_θ.

(B_4) J is even and for any finite dimensional subspace W of X we have

$$\lim_{\substack{u \in W \\ \|u\|_X \to \infty}} \sup_{\theta \in [0,1]} J_\theta(u) = -\infty.$$

Denote by $\psi_i : [0,1] \times X \to X$ the solutions of the problem

$$\begin{cases} \dfrac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)), \\ \psi_i(0, s) = s. \end{cases}$$

Remark that $\psi_i(\theta, \cdot)$ are continuous, nondecreasing and $\psi_1 \leq \psi_2$. Define

$$\bar{\eta}_i(s) = \sup_{\theta \in [0,1]} \eta_i(\theta, s).$$

Let

$$\mathscr{C} = \{\zeta \in C(X, X) : \zeta \text{ is odd and } \zeta(u) = u \text{ if } \|u\|_X \geq R\},$$

and

$$c_k = \inf_{\zeta \in \mathscr{C}} \sup_{u \in X_k} J_0(\zeta(u)).$$

The main result of this section is due to Bolle [22].

Theorem 1.16 (see [22]) *Assume that the sequence*

$$\left\{ \frac{c_{k+1} - c_k}{\bar{\eta}_1(c_{k+1}) + \bar{\eta}_2(c_k) + 1} \right\} \quad \textit{is unbounded.}$$

Then, the functional J_1 admits a sequence of critical values $\{d_k\}$ such that

$$\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \le d_k \quad \textit{for all } k \ge 1.$$

1.4 Degree Theory

1.4.1 Brouwer Degree

We start by recalling some basic facts about Brouwer degree.

Definition 1.17 *Let Ω be an open set in $\mathbb{R}^N$, $N \ge 1$, and $F \in C^1(\overline{\Omega}; \mathbb{R}^N)$.*

(i) We say that $x_0 \in \Omega$ is a regular point if the Jacobian matrix $J_F(x_0) = (\partial F_i / \partial x_j)$ has rank N. If x_0 is not a regular point, we say that x_0 is a critical point of F.
(ii) We say that $y_0 \in \mathbb{R}^N$ is a regular value of F if the preimage $F^{-1}(y_0)$ does not contain any critical point; otherwise we say that y_0 is a critical value.

A first characterization of the set of critical values is given by the following result.

Theorem 1.18 *(Sard Lemma) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $F \in C^2(\overline{\Omega}; \mathbb{R}^N)$. Then the set of critical values of F has zero Lebesgue measure.*

Definition 1.19 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $F \in C^2(\overline{\Omega}; \mathbb{R}^N)$, $p \in \mathbb{R}^N \setminus F(\partial\Omega)$.*

(i) If p is a regular value of F then

$$\deg(F, \Omega, p) = \sum_{x \in F^{-1}(p)} \mathrm{sign}(\det J_F(x)).$$

(ii) If p is a critical value of F then

$$\deg(F, \Omega, p) = \deg(F, \Omega, p_1),$$

where p_1 is a regular value of F such that $\|p - p_1\| < \operatorname{dist}(p, F(\partial\Omega))$.

It can be proved that $\deg(F, \Omega, p)$ is independent of the choice of p_1.

Definition 1.20 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $F \in C(\overline{\Omega}; \mathbb{R}^N)$, $p \in \mathbb{R}^N \setminus F(\partial\Omega)$. The Brouwer degree of F in Ω at point p is defined as*

$$\deg(F, \Omega, p) = \deg(G, \Omega, p),$$

where $g \in C^2(\overline{\Omega}; \mathbb{R}^N)$ is an arbitrary function such that $\|F - G\|_{L^\infty} < \operatorname{dist}(p, \partial\Omega)$.

It can be proved that $\deg(F, \Omega, p)$ is independent of the choice of G.

Some basic properties of Brouwer degree theory are stated below.

Theorem 1.21 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set.*

(i) (Normality)

$$\deg(F, \Omega, p) = \begin{cases} 1 & p \in \Omega, \\ 0 & p \notin \overline{\Omega}. \end{cases}$$

(ii) (Domain Additivity) Let $F \in C(\overline{\Omega}; \mathbb{R}^N)$. If Ω_1, Ω_2 are two open subsets of Ω with $\Omega_1 \cap \Omega_2 = \emptyset$ and $p \notin F(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ then

$$\deg(F, \Omega, p) = \deg(F, \Omega_1, p) + \deg(F, \Omega_2, p).$$

(iii) (Invariance of Homotopy) Let $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^N$ be a continuous mapping. Assume that $p : [0,1] \to \mathbb{R}^N$ satisfies $p(t) \neq H(x,t)$ for all $(x,t) \in \partial\Omega \times [0,1]$. Then $\deg(H(\cdot,t), \Omega, p(t))$ is independent of t.

1.4.2 Leray–Schauder Degree

Let X be a Banach space and let Ω be a bounded open set in X. If $T : \overline{\Omega} \to X$ is a compact operator, then there exists a sequence of finite rank operators $\{T_\varepsilon\}$ such that $\|T - T_\varepsilon\| \to 0$ as $\varepsilon \to \infty$.

Let now $p \in X \setminus F(\partial\Omega)$. If $0 < \varepsilon < \operatorname{dist}(p, \partial\Omega)$, there exists a finite rank operator T_ε such that $\|T - T_\varepsilon\| < \varepsilon$. Letting $F_\varepsilon = I - T_\varepsilon$, then $p \notin F_\varepsilon(\partial\Omega)$ so that considering $X_\varepsilon = T_\varepsilon(\overline{\Omega})$ it follows that

$$F_\varepsilon\big|_{X_\varepsilon \cap \overline{\Omega}} : X_\varepsilon \cap \overline{\Omega} \to X_\varepsilon,$$

the Brouwer degree of F_ε on $X_\varepsilon \cap \overline{\Omega}$ at point p is well defined.

Definition 1.22 *(Leray–Schauder Degree) Let $F = I - T$ where T is a compact operator. The Leray–Schauder degree of F in Ω at point $p \in X \setminus F(\partial\Omega)$ is defined as*

$$\deg(F, \Omega, p) = \deg(F_\varepsilon, X_\varepsilon \cap \overline{\Omega}, p).$$

It can be proved that $\deg(F, \Omega, p)$ is independent of the choice of ε. As a consequence, the results in Theorem 1.21 (i)–(ii) concerning Brouwer degree of maps in finite dimensional space transfer to Leray–Schauder degree for $F = I - T$ by applying these results to the finite dimensional approximation F_ε. As regards the invariance to the homotopy, we state here the counterpart result for Theorem 1.21(iii).

Theorem 1.23 *(Invariance to Homotopy) Let $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^N$ be a compact operator and let $p : [0,1] \to X$ be a continuous function such that*

$$p(t) \neq x - H(x,t) \quad \text{for all } (x,t) \in \partial\Omega \times [0,1].$$

Then $\deg(H(\cdot,t), \Omega, p(t))$ *is independent of t.*

1.4.3 Leray–Schauder Degree for Isolated Solutions

As before, let Ω be a bounded open set of a Banach space and $F : \overline{\Omega} \to X$ such that $0 \notin F(\partial\Omega)$ and $T = I - F$ is compact. We assume that $x_0 \in \Omega$ is an isolated solution of $F(x) = 0$ and that $F'(x_0) = I - T'(x_0)$ is invertible. By the implicit function theorem, there exists a ball $B_r(x_0) \subset \Omega$ such that $F(x) \neq 0$ for all $x \in B_r(x_0)$, $x \neq x_0$.

Definition 1.24 *The index of F at x_0 is given by*

$$\operatorname{index}(F, x_0) = \deg(F, B_r(x_0), 0).$$

It can be shown that $\operatorname{index}(F, x_0)$ is independent of r.

Theorem 1.25 *Under the above conditions,*

$$\operatorname{index}(F, x_0) = (-1)^\beta, \quad \beta = \sum_{\substack{\lambda \in \sigma(T'(x_0)) \\ \lambda > 1}} n_\lambda,$$

where

$$n_\lambda = \dim\Big[\bigcup_{p\geq 1} \mathrm{Ker}(\lambda I - T'(x_0))^p\Big].$$

Proof. Without loss of generality we may assume $x_0 = 0$. For $0 \leq t \leq 1$ and $x \in \overline{\Omega}$ let

$$H(x,t) = \begin{cases} x - \dfrac{1}{t}T(tx) & 0 < t \leq 1\,, \\ x - T'(0)x & t = 0\,. \end{cases}$$

Then, by the invariance of the compact homotopy we have

$$\begin{aligned} \mathrm{index}(F,x_0) &= \deg(F,B_r,0) = \deg(H(1,\cdot),B_r,0) \\ &= \deg(H(1,\cdot),B_r,0) = \deg(I - T'(0),B_r,0). \end{aligned}$$

We next decompose $X = X_1 \bigoplus X_2$ where

$$X_1 = \mathrm{span}\Big\{ \bigcup_{\substack{\lambda\in\sigma(T'(x_0)) \\ \lambda>1}} \bigcup_{p\geq 1} \mathrm{Ker}(\lambda I - T'(x_0))^p \Big\}.$$

Then

$$\deg(I - T'(0),B_r,0) = \deg((I - T'(0))\big|_{X_1}, B_r \cap X_1, 0)\cdot \deg((I - T'(0))\big|_{X_2}, B_r \cap X_2, 0).$$

Further, if $\Gamma(t,\cdot) = I - tT'(0)$, $0 \leq t \leq 1$, then

$$0 \notin \Gamma(x,t) \qquad \text{for all } (x,t) \in [0,1]\times \partial(B_r \cap X_2)$$

so

$$\deg((I - T'(0))\big|_{X_2}, B_r \cap X_2, 0) = \deg(\Gamma(0,\cdot)\big|_{X_2}, B_r \cap X_2, 0) = 1.$$

Thus, from the above equalities we find

$$\begin{aligned} \deg(I - T'(0),B_r,0) &= \deg((I - T'(0))\big|_{X_1}, B_r \cap X_1, 0) \\ &= \mathrm{sgn}\big(\det(I - T'(0))\big) = (-1)^\beta. \end{aligned}$$

This finishes the proof. $\square$

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