

Hamiltonian Formalisms Applied to Continuum Mechanics: Potential Use for Fracture Mechanics

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Abstract The first part of this chapter deals with several Hamiltonian formalisms in elasticity. The formalisms of Zhong ((1995) Dalian Science & Technology University Press, Liaoning, China) and Bui ((1993) Introduction aux problèmes inverses en mécanique des matériaux, Editions Eyrolles, Paris), which resolve respectively the two-end problem and the Cauchy problem in elasticity, are presented briefly. Then we propose a new Hamiltonian formalism, which resolves simultaneously the two problems mentioned above and shows the link between the two formalisms. The potential use for fracture mechanics purposes is then mentioned. In fact, when traditional theories in fracture mechanics are used, asymptotic analyses are often carried out by using high-order differential equations governing the stress field near the crack tip. The solution of the high-order differential equations becomes difficult when one deals with anisotropic or multilayer media etc. The key of our idea was to introduce the Hamiltonian system, usually studied in rational mechanics, into continuum mechanics. By this way, one can obtain a system of first-order differential equations, instead of the high-order differential equation. This method is very efficient and quite simple to obtain a solution of the governing equations of this class of problems. It allows dealing with a large range of problems, which may be difficult to resolve by using traditional methods. Also, recently we developed another new way to resolve fracture mechanics problems with the use of ordinary differential equations (ODEs) with respect to the circumferential coordinate θ around the crack (or notch) tip. This method presents the opportunity to be coupled with finite element analysis and then allows resolving more complicated geometries.

Keywords Hamiltonian analysis • Stress-singular fields • V-notch • Boundary element method

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1 Introduction

Recently, an important effort has been made in the reform of the classical theory of continuum mechanics in the frame of the Hamiltonian system. In these new approaches, the principle of Hamilton is applied in a special manner, i.e., by considering a dimensional parameter as “time”. In this topic, we can distinguish two formalisms: the formalism of Bui [1] and the formalism of Zhong [2]. By seeking the variations of the couple (displacements, traction forces) on an arbitrary front in a solid when this front virtually moves from an initial position to a neighbor one, a first-order differential equation system governing the mechanical fields was explicitly established. That is the Cauchy problem in elasticity resolved by Bui. On the other hand, the formalism of Zhong looks more classical. In simple words, he established an analogy between quantities in rational mechanics and those in continuum mechanics. For example, a dimensional coordinate in continuum mechanics is considered as time in rational mechanics; the displacement vector as the generalized coordinates; the strain energy density as the Lagrange function and so on. This analogy leads to the canonical equations of Hamilton governing the mechanical fields in elastic bodies. The main advantage of these approaches is that the fundamental equations can directly be resolved. The traditional semi-inverse method is then replaced by a direct, systematic and more structural resolution method.

2 Zhong’s Formalism: The Two-End Problem

Let us consider a solid V described by a coordinate system Z in which z is one chosen coordinate. Let us consider now q the displacements in the Z system associated to neighbor displacements, $q + \delta q$. One notes $\dot{q} = \frac{\partial q}{\partial z}$. If we suppose that the displacements are imposed at $z = z_0$ and $z = z_1$, named the two end points, then we have:

$$\delta q(z = z_0) = \delta q(z = z_1) = 0 \quad (1)$$

Let us write the total potential energy Π of the solid:

$$\Pi = \int_{z_0}^{z_1} \int_S (U_0 - W) dS dz = \int_{z_0}^{z_1} L dz \quad \text{avec} \quad L = \int_S (U_0 - W) dS \quad (2)$$

where U_0 is the strain energy density and W is the work density of the external forces. We define the Lagrange function as the integral over S . If S is constant along z and we neglect the body forces and we just consider a volume element inside the solid, we can write $L = U_0 - W$. In general, L is a function of q and \dot{q} . Following the principle of the minimum of total potential energy, $\delta \Pi = 0$ with respect to δq and using the conditions (1), one obtains the Euler equation in L :

$$\frac{\partial L}{\partial \mathbf{q}} - \frac{\partial}{\partial z} \frac{\partial L}{\partial \dot{\mathbf{q}}} = 0 \quad (3)$$

In rational mechanics, L is named Lagrange's function, and (3) Lagrange's equation. Then we construct the Hamilton function $H(\mathbf{p}, \mathbf{q})$ through the Legendre's transformation:

$$\begin{aligned} \mathbf{p} &= \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \\ H(\mathbf{p}, \mathbf{q}) &= \mathbf{p}^T \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}) \end{aligned} \quad (4)$$

From (3) and (4), one deduces immediately the canonical equations of Hamilton:

$$\frac{\partial H}{\partial \mathbf{q}} = -\frac{\partial L}{\partial \mathbf{q}} = -\dot{\mathbf{p}} \quad \frac{\partial H}{\partial \mathbf{p}} = \dot{\mathbf{q}} \quad (5)$$

\mathbf{q} and \mathbf{p} are dual conjugate variables. Differently from rational mechanics, these two variables represent respectively the displacement vector and the normalized stress vector.

3 Bui's Formalism: Cauchy's Problem in Elasticity

Bui [1] has solved the Cauchy problem in elasticity by seeking the variations of the mechanical quantities (\mathbf{q} as a displacement vector, \mathbf{p} as a traction vector) at an arbitrary front in the solid when it moves from an initial position Γ_t to a neighbour position Γ_{t+dt} , where t defines the movement of the front in the solid. This approach leads to an explicit system of first-order differential equations.

Let us consider a domain divided into two parts Ω and Ω_t by a contour Γ_t . Suppose that mechanical fields are known at the interior of the contour; consequently \mathbf{q} and \mathbf{p} are known at the contour Γ_t . Suppose \mathbf{q}' a virtual compatible displacement. The virtual work principle leads to:

$$\int_{\Omega_t} \nabla \mathbf{q} \cdot \mathbf{A} \cdot \nabla \mathbf{q}' d\Omega = \int_{\Gamma_t} \mathbf{q} \cdot \mathbf{q}' d\Gamma \quad (6)$$

\mathbf{A} is the elastic tensor. Let us consider now an evolution of Γ_t to Ω_t , i.e. at $t + dt$, the contour Γ_t reaches Γ_{t+dt} . It's suitable to consider that Γ_{t+dt} is deduced from Γ_t following the normal to Γ_t with a quantity $\psi \mathbf{n} dt$ where \mathbf{n} is a unit vector normal to the contour and ψ is a positive scalar field describing the velocity of the contour evolution. The derivation of (6) with respect to dt gives:

$$\frac{d}{dt} \int_{\Omega_t} \nabla \mathbf{q} \cdot \mathbf{A} \cdot \nabla \mathbf{q}' d\Omega = \frac{d}{dt} \int_{\Gamma_t} \mathbf{p} \cdot \mathbf{q}' d\Gamma \quad (7)$$

If introducing the following notations of tangential operators:

$$\text{grad}_\Gamma(\cdot) := \nabla(\cdot) - \mathbf{n} \frac{\partial}{\partial n}(\cdot) \quad \text{div}_\Gamma(\cdot) := \text{div}(\cdot) - \mathbf{n} \cdot \frac{\partial}{\partial n}(\cdot) \quad (8)$$

equation (7) leads to:

$$\int_{\Gamma_t} \nabla \mathbf{q} \cdot \mathbf{A} \cdot \nabla \mathbf{q}' \psi d\Gamma - \int_{\Gamma_t} \left\{ \frac{d\mathbf{p}}{dt} + \text{div}_\Gamma(\psi \mathbf{n}) \mathbf{p} \right\} \cdot \mathbf{q}' d\Gamma - \int_{\Gamma_t} \psi \mathbf{p} \cdot \frac{\partial \mathbf{q}'}{\partial n} d\Gamma = 0 \quad (9)$$

After rearrangement and integration by parts, one can deduce the following differential equations:

$$\begin{aligned} \frac{d\mathbf{q}}{dt} &= \mathbf{B}_q(\mathbf{q}, \mathbf{p}, \psi) \\ \frac{d\mathbf{p}}{dt} &= \mathbf{B}_p(\mathbf{q}, \mathbf{p}, \psi) \end{aligned} \quad (10)$$

\mathbf{B}_q and \mathbf{B}_p are expressed as function of quantities defined on the contour Γ_t . Their explicit expressions are given in the [1].

4 Unified Description of the Two Formalisms

Here we describe a formalism unifying the two precedents within the frame of minimization of the total potential energy of the structure [3].

4.1 Hamilton Principle Written as Variation of Total Potential Energy

Following (2) and (4), the total potential energy is written as:

$$\Pi = \int_{z_1}^{z_2} L dz = \int_{z_1}^{z_2} (\mathbf{p} \cdot \dot{\mathbf{q}} - H) dz \quad (11)$$

u is a parameter describing the solid's evolution. The description of a solid between an event a and an event b could be done under parametrical form of six functions in 2D media: two displacements $\mathbf{q}(u)$, three normalised stresses $\mathbf{p}(u)$ and one coordinate $z(u)$. Consider u_1 and u_2 as values of u corresponding to events a and b . For $z_1 = u_1$ and $z_2 = u_2$, the total potential energy is re-written as:

$$\Pi(u) = \int_{u_1}^{u_2} \left(\mathbf{p} \cdot \frac{\partial \mathbf{q}}{\partial u} - H \frac{\partial z}{\partial u} \right) du \quad (12)$$

And its variation becomes:

$$\delta\Pi = \frac{\partial\Pi}{\partial u} \delta u = \left\{ \int_{u_1}^{u_2} \left[\frac{\partial \mathbf{p}}{\partial u} \cdot \frac{\partial \mathbf{q}}{\partial u} - \frac{\partial \mathbf{p}}{\partial u} \cdot \frac{\partial \mathbf{q}}{\partial u} - \frac{\partial H}{\partial u} \cdot \frac{\partial z}{\partial u} + \frac{\partial H}{\partial u} \frac{\partial z}{\partial u} \right] du + \left[\mathbf{p} \cdot \frac{\partial \mathbf{q}}{\partial u} - H \frac{\partial z}{\partial u} \right]_{u_1}^{u_2} \right\} \delta u \quad (13)$$

One notes:

$$\frac{\partial \mathbf{q}}{\partial u} \delta u = \delta \mathbf{q}; \quad \frac{\partial \mathbf{p}}{\partial u} \delta u = \delta \mathbf{p}; \quad \frac{\partial H}{\partial u} \delta u = \delta H; \quad \frac{\partial z}{\partial u} \delta u = \delta z \quad (14)$$

When u represents the coordinate z , (13) is written as follow:

$$\delta\Pi = \int_{z_1}^{z_2} [\dot{\mathbf{q}} \cdot \delta \mathbf{p} - \dot{\mathbf{p}} \cdot \delta \mathbf{q} - \delta H + \dot{H} \delta z] dz + [\mathbf{p} \cdot \delta \mathbf{q} - H \delta z]_{z_1}^{z_2} \quad (15)$$

So we have $\delta\Pi$ divided into two parts, the first one is an integral; the second one is in the square bracket.

4.2 Application to the Two-End Problem

Consider now the variation of \mathbf{q} and z are zero at z_1 and z_2 , $\delta \mathbf{q} = \mathbf{0}$ and $\delta z = 0$. This means we have fixed boundaries and fixed displacement boundary conditions at the two-ends, so we have got the so-called two end point problem. In this case, the quantities in the square bracket of equation (15) vanish. According to the principle of minimum total potential energy, we directly obtain the canonical equations of Hamilton. This is the problem resolved by the formalism of Zhong.

$$\delta\Pi = \int_{z_1}^{z_2} [\dot{\mathbf{q}} \cdot \delta \mathbf{p} - \dot{\mathbf{p}} \cdot \delta \mathbf{q} - \delta H + \dot{H} \delta z] dz = 0 \quad (16)$$

δH being: $\delta H = \frac{\partial H}{\partial \mathbf{q}} \delta \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}} \delta \mathbf{p} + \frac{\partial H}{\partial z} \delta z$, one obtains:

$$\delta\Pi = \int_{z_1}^{z_2} \left[\dot{\mathbf{q}} \cdot \delta \mathbf{p} - \dot{\mathbf{p}} \cdot \delta \mathbf{q} - \frac{\partial H}{\partial \mathbf{q}} \delta \mathbf{q} - \frac{\partial H}{\partial \mathbf{p}} \delta \mathbf{p} - \frac{\partial H}{\partial z} \delta z + \dot{H} \delta z \right] dz = 0 \quad (17)$$

This equation is available for arbitrary $\delta \mathbf{q}$, $\delta \mathbf{p}$ and δz . Consequently, we deduce the Hamilton canonical equations:

$$\frac{\partial H}{\partial z} = \dot{H}; \quad \frac{\partial H}{\partial \mathbf{p}} = \dot{\mathbf{q}}; \quad \frac{\partial H}{\partial \mathbf{q}} = -\dot{\mathbf{p}} \quad (18)$$

4.3 Application to Cauchy's Problem

Now consider a natural evolution of the structure, this means that the Hamilton canonical equations are satisfied everywhere in the structure, but with possible variations of (\mathbf{q}, z) at $z = z_1$ and $z = z_2$. In this case, we have no fixed boundaries neither fixed boundary conditions at the two ends but we have *natural* evolution everywhere, this is the so called Cauchy problem. In this case, the integral in equation (15) vanishes i.e.:

$$\delta\Pi = \mathbf{p}_2 \cdot \delta\mathbf{q}_2 - H_2 \cdot \delta z_2 - \mathbf{p}_1 \cdot \delta\mathbf{q}_1 + H_1 \cdot \delta z_1 \quad (19)$$

For a small displacement of events a and b , the variation of the total potential energy is:

$$\delta\Pi = \delta\mathbf{q}_1 \cdot \frac{\partial\Pi}{\partial\mathbf{q}_1} + \delta z_1 \frac{\partial\Pi}{\partial z_1} + \delta\mathbf{q}_2 \cdot \frac{\partial\Pi}{\partial\mathbf{q}_2} + \delta z_2 \frac{\partial\Pi}{\partial z_2} \quad (20)$$

The variables $\mathbf{q}_1, z_1, \mathbf{q}_2, z_2$ are independent. By identification between (19) and (20), we have got the Hamilton–Jacobi equations:

$$\frac{\partial\Pi}{\partial\mathbf{q}_2} = \mathbf{p}_2 \quad \frac{\partial\Pi}{\partial z_2} = -H_2 \quad \frac{\partial\Pi}{\partial\mathbf{q}_1} = -\mathbf{p}_1 \quad \frac{\partial\Pi}{\partial z_1} = H_1 \quad (21)$$

This is the problem resolved by Bui. We know that the Hamilton canonical equations and the Hamilton–Jacobi equations are equivalent. So we can say the formalism of Zhong and that of Bui are equivalent in the differential point of view, even they look quite different. Now, dealing with Bui's formalism, it's obvious that the virtual work principle (6) could be written as a total potential energy by replacing \mathbf{q}' by virtual displacements $\delta\mathbf{q}$.: (Note that $d\Omega = d\Gamma dt$)

$$\delta \left[\int_t \frac{1}{2} \int_{\Gamma_t} \nabla \mathbf{q} \cdot \mathbf{\Lambda} \cdot \nabla \mathbf{q} d\Gamma dt - \int_{\Gamma_t} \mathbf{p} \cdot \mathbf{q} d\Gamma \right] = \delta\Pi = 0 \quad (22)$$

If we define:

$$L = \frac{1}{2} \int_{\Gamma_t} \nabla \mathbf{q} \cdot \mathbf{\Lambda} \cdot \nabla \mathbf{q} d\Gamma - \frac{d}{dt} \int_{\Gamma_t} \mathbf{p} \cdot \mathbf{q} d\Gamma$$

equation (22) becomes:

$$\delta \int_t L dt = 0 \quad (23)$$

The partial derivation of (6) with respect to t , which represents the variation of virtual works due to virtual displacements during the evolution of the contour is equivalent to equation (23) if we consider a natural evolution.

5 Hamiltonian Formalism Applied to Fracture Mechanics

We can actually write the equations governing the crack tip fields under the form of (5). The main idea [4, 5], is to consider one coordinate in the polar system as “time” and take the total potential energy as the Lagrange function. For example, we can consider the radial coordinate r or the angular coordinate θ as time and take the variational principles established in continuum mechanics as the Hamilton variational principle. Then all the procedures currently used in rational mechanics can be translated into continuum mechanics. In the following, the angular coordinate θ will be substituted to time.

5.1 Governing Equations of the Problem

Consider a notch formed from several elastic materials. We establish a cylindrical coordinate system with their origins at the notch tip and the z -axis representing the notch front. Material 1 occupies domain $[\theta_0, \theta_1]$, named zone 1; Material 2 occupies zone 2, bounded by $[\theta_1, \theta_2]$, and so on. Under remote loading, the stress concentration at the notch tip will take a mixed mode nature due to the anisotropy of the materials.

First, we write the stress components in the polar coordinate system as: $\sigma = \{\sigma_r \quad \sigma_\theta \quad \tau_{r\theta}\}^T$. The corresponding strain components are $\varepsilon = \{\varepsilon_r \quad \varepsilon_\theta \quad \gamma_{r\theta}\}^T$. The linear elastic stress–strain relationship is:

$$\sigma = C \varepsilon. \quad (24)$$

C is the stiffness matrix of the material. All its components are constant.

We write now the fundamental equations of linear elasticity in the polar system:

(a) Equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \quad (25)$$

We perform the following variable changes:

$$\xi = \ln r \quad r = \exp(\xi); \quad (26)$$

and

$$\begin{aligned} S_r = r\sigma_r \quad \sigma_r = S_r/r; \quad S_\theta = r\sigma_\theta \quad \sigma_\theta = S_\theta/r; \\ S_{r\theta} = r\tau_{r\theta} \quad \tau_{r\theta} = S_{r\theta}/r; \dots \text{etc} \end{aligned} \quad (27)$$

Then, by using the notation $(\cdot) = \frac{\partial}{\partial \theta}$, the equilibrium equations (25) can be rewritten as:

$$\dot{S}_{r\theta} = S_\theta - \frac{\partial S_r}{\partial \xi} \quad \dot{S}_\theta = -\frac{\partial S_{r\theta}}{\partial \xi} - S_{r\theta} \quad (28)$$

We define the following variable vectors:

$$\mathbf{p} = \{S_\theta \quad S_{r\theta}\}^T \quad (29)$$

Hence, the equilibrium equations (28) can be rewritten as:

$$\dot{\mathbf{p}} = \mathbf{E}_1 \mathbf{p} + \mathbf{E}_2 \frac{\partial \mathbf{p}}{\partial \xi} \quad (30)$$

where

$$\mathbf{E}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{E}_2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

(b) Displacement-stress relationship:

$$\begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r} & \varepsilon_\theta &= \frac{1}{r} \left(u_r + \frac{\partial u_\theta}{\partial \theta} \right) \\ \gamma_{r\theta} &= \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \end{aligned} \quad (31)$$

By substituting (31) into (24) and by using the variable changes (26) and (27), one obtains:

$$\begin{Bmatrix} S_r \\ S_\theta \\ S_{r\theta} \end{Bmatrix} = \begin{bmatrix} c_{12} & c_{14} \\ c_{22} & c_{24} \\ c_{42} & c_{44} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_\theta}{\partial \theta} \\ \frac{\partial u_r}{\partial \theta} \end{Bmatrix} + \begin{bmatrix} -c_{14} & c_{12} \\ -c_{24} & c_{22} \\ -c_{44} & c_{42} \end{bmatrix} \begin{Bmatrix} u_\theta \\ u_r \end{Bmatrix} + \begin{bmatrix} c_{14} & c_{11} \\ c_{24} & c_{21} \\ c_{44} & c_{41} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_\theta}{\partial \xi} \\ \frac{\partial u_r}{\partial \xi} \end{Bmatrix} \quad (32)$$

Similarly, we define a displacement vector

$$\{\mathbf{q}\} = \{u_\theta \quad u_r\}^T \quad (33)$$

By using the definitions (29) and (33), the relationship (32) can be rewritten as:

$$\mathbf{p} = \mathbf{C}_d \dot{\mathbf{q}} + \mathbf{C}_e \mathbf{q} + \mathbf{C}_f \frac{\partial \mathbf{q}}{\partial \xi} \quad (34)$$

or

$$\dot{\mathbf{q}} = \mathbf{C}_d^{-1} \left(\mathbf{p} - \mathbf{C}_e \mathbf{q} - \mathbf{C}_f \frac{\partial \mathbf{q}}{\partial \xi} \right) \quad (35)$$

with:

$$\mathbf{C}_d = \begin{bmatrix} c_{22} & c_{24} \\ c_{42} & c_{44} \end{bmatrix} \quad \mathbf{C}_e = \begin{bmatrix} -c_{24} & c_{22} \\ -c_{44} & c_{42} \end{bmatrix} \quad \mathbf{C}_f = \begin{bmatrix} c_{24} & c_{21} \\ c_{44} & c_{41} \end{bmatrix} \quad (36)$$

The strain energy in solids is always positive, consequently, \mathbf{C}_d is a positively definite matrix. Therefore, the inversion of the matrix \mathbf{C}_d is permitted

(c) *Governing equations:* By substituting Eq. (34) into the equilibrium equation (30), the variable vector \mathbf{p}_t is eliminated. Then, we obtain, from (30) and (35), the following dual equations that govern the problem:

$$\dot{\mathbf{q}} = \mathbf{H}_{11} \mathbf{q} + \mathbf{H}_{12} \mathbf{p} \quad \dot{\mathbf{p}} = \mathbf{H}_{21} \mathbf{q} + \mathbf{H}_{22} \mathbf{p} \quad (37)$$

with:

$$\begin{aligned} \mathbf{H}_{11} &= \mathbf{E}_1 - \mathbf{C}_d^{-1} \mathbf{C}_f \frac{\partial}{\partial \xi} & \mathbf{H}_{12} &= \mathbf{C}_d^{-1} \\ \mathbf{H}_{21} &= \mathbf{E}_3 (\mathbf{C}_d^{-1} \mathbf{C}_f) \frac{\partial^2}{\partial \xi^2} & \mathbf{H}_{22} &= \mathbf{E}_1 + (\mathbf{E}_2 + \mathbf{E}_3 \mathbf{C}_d^{-1}) \frac{\partial}{\partial \xi} \end{aligned} \quad (38)$$

with:

$$\mathbf{E}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{E}_2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{E}_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

In fact, it is more convenient to define a total vector \mathbf{v} as variables in the state space:

$$\mathbf{v} = \left\{ \mathbf{q}^T \mathbf{p}^T \right\}^T \quad (39)$$

such that the governing equations (37) become:

$$\dot{\mathbf{v}} = \mathbf{H} \mathbf{v} \quad (40)$$

with:

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \quad (41)$$

- (d) *Boundary conditions and continuity conditions:* Referring to Fig. 1, we adopt the superscript ⁽ⁱ⁾ to indicate the quantities in zone i . For example, $\mathbf{v}^{(i)}$, $\mathbf{H}^{(i)}$, etc.

The boundary conditions at the two free surfaces of the notch are:

$$\mathbf{p}^{(1)}(\theta = \theta_0) = \mathbf{0} \quad \mathbf{p}^{(n)}(\theta = \theta_n) = \mathbf{0} \quad (42)$$

The continuity conditions across the interfaces are:

$$\mathbf{v}^{(1)}(\theta = \theta_1) = \mathbf{v}^{(2)}(\theta = \theta_1) \quad \dots \quad \mathbf{v}^{(n-1)}(\theta = \theta_{n-1}) = \mathbf{v}^{(n)}(\theta = \theta_{n-1}) \quad (43)$$

These relations show the advantage of the choice of the dual variables in the present study: the multi-material problem can be dealt with as a single material problem since the variable vector \mathbf{v} is continuous across all the interfaces. This makes much easier the resolution of governing equation (40).

By adapting this new stiffness matrix, all formulations deduced for generalized plane strain can directly be used for plane stress problems.

5.2 Resolution Method

By examining governing equation (40), it is self-evident to try to solve it by using the variable separation method. We suppose that the variable vector $\mathbf{v}(\zeta, \theta)$ can be written under separable form:

$$\mathbf{v}(\zeta, \theta) = \exp(\lambda \zeta) \boldsymbol{\psi}(\theta) \quad (44)$$

where λ is an undetermined eigenvalue and $\boldsymbol{\psi}(\theta)$ is a variable vector depending exclusively on θ . Then, equation (40) becomes:

$$\dot{\boldsymbol{\psi}}(\theta) = \mathbf{H}(\theta) \boldsymbol{\psi}(\theta) \quad (45)$$

In (45), \mathbf{H} is function of θ only,

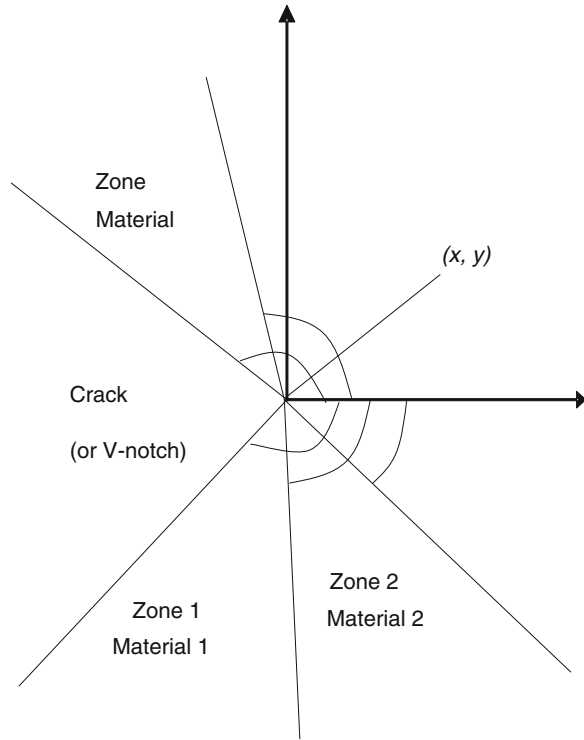
$$\mathbf{H}(\theta) = \begin{vmatrix} \mathbf{E}_1 - \mathbf{C}_d^{-1} \mathbf{C}_f \lambda & \mathbf{C}_d^{-1} \\ \mathbf{E}_3 (\mathbf{C}_d^{-1} \mathbf{C}_f) \lambda^2 & \mathbf{E}_1 + (\mathbf{E}_2 + \mathbf{E}_3 \mathbf{C}_d^{-1}) \lambda \end{vmatrix} \quad (46)$$

The continuity conditions across the interfaces become:

$$\boldsymbol{\psi}^{(1)}(\theta = \theta_1) = \boldsymbol{\psi}^{(2)}(\theta = \theta_1) \quad \dots \quad \boldsymbol{\psi}^{(n-1)}(\theta = \theta_{n-1}) = \boldsymbol{\psi}^{(n)}(\theta = \theta_{n-1}) \quad (47)$$

Any numerical method providing a good accuracy can be used for solving this problem and the eigenvectors $\boldsymbol{\psi}$ can straightforwardly be given with all stress and displacement components.

Fig. 1 V-Notch with various materials



6 Future Extensions

In this paragraph, a new way [6] is proposed in order to determine the orders of singularity for two dimensional V-notch problems. Firstly, on the basis of an asymptotic stress field in terms of radial coordinates at the V-notch tip, the governing equations of the elastic theory are transformed into an eigenvalue problem of ordinary differential equations (ODEs) with respect to the circumferential coordinate θ around the notch tip. Then, the singularity orders of the V-notch problem are determined through solving the corresponding ODEs by means of the interpolating matrix method. Meanwhile, the associated eigenvectors of the displacement and stress fields near the V-notches are also obtained. This method is also available to deal with the plane V-notch problems in bonded orthotropic multi-material.

Firstly, let us consider a V-notch of isotropic material with opening angle $2\pi - \theta_1 - \theta_2$ as shown in Fig. 2.

A polar coordinate system (ρ, θ) is defined taking the notch tip as origin. In the linear elastic analysis, it has been verified that the displacement field in the notch tip region can be expressed as a series expansion with respect to the radial

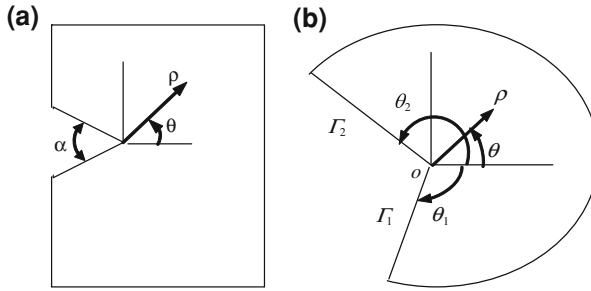


Fig. 2 : Geometry and local field **a** A V-notch with opening angle α **b** Geometry near a V-notch

coordinate ρ originating from the notch tip [7]. One typical term of the series can be written in the following form:

$$u_\rho(\rho, \theta) = \rho^{\lambda+1} \tilde{u}_\rho(\theta) \quad (48a)$$

$$u_\theta(\rho, \theta) = \rho^{\lambda+1} \tilde{u}_\theta(\theta) \quad (48b)$$

where λ , $\tilde{u}_\rho(\theta)$ and $\tilde{u}_\theta(\theta)$ are eigenpairs. Introducing Eqs. (3) into the strain–displacement relations of linear elastic theory yields the strain components as:

$$\varepsilon_{\rho\rho} = (1 + \lambda) \rho^\lambda \tilde{u}_\rho(\theta) \quad (49a)$$

$$\varepsilon_{\theta\theta} = \rho^\lambda \tilde{u}_\rho(\theta) + \rho^\lambda \tilde{u}'_\theta(\theta) \quad (49b)$$

$$\gamma_{\rho\theta} = \rho^\lambda \tilde{u}'_\rho(\theta) + \lambda \rho^\lambda \tilde{u}_\theta(\theta) \quad (49c)$$

where $(\dots)' = d(\dots)/d\theta$. From linear elastic behavior law (Hooke's law) of plane stress problems, the plane stresses are expressed as:

$$\sigma_{\rho\rho} = \frac{E}{1 - \nu^2} \rho^\lambda [(1 + \lambda) \tilde{u}_\rho + \nu \tilde{u}_\rho + \nu \tilde{u}'_\theta] \quad (50a)$$

$$\sigma_{\theta\theta} = \frac{E}{1 - \nu^2} \rho^\lambda [(1 + \lambda) \nu \tilde{u}_\rho + \tilde{u}_\rho + \tilde{u}'_\theta] \quad (50b)$$

$$\sigma_{\rho\theta} = \frac{E}{2(1 + \nu)} \rho^\lambda (\lambda \tilde{u}_\theta + \tilde{u}'_\rho) \quad (50c)$$

where E is the Young's modulus and ν the Poisson's ratio. Neglecting the body forces, the equilibrium equations are:

$$\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\rho\theta}}{\partial \theta} + \frac{\sigma_{\rho\rho} - \sigma_{\theta\theta}}{\rho} = 0 \quad (51a)$$

$$\frac{1}{\rho} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\rho\theta}}{\partial \rho} + \frac{2\sigma_{\rho\theta}}{\rho} = 0 \quad (51b)$$

Substituting Eqs. (50a, 50b, 50c) into Eqs. (51a, 51b) gives:

$$\begin{aligned} \tilde{u}_\rho'' + \left(\frac{1+\nu}{1-\nu} \lambda - 2 \right) \tilde{u}_\theta' + \frac{2}{1-\nu} \lambda (\lambda + 2) \tilde{u}_\rho = 0, \\ \theta \in (\theta_1, \theta_2) \end{aligned} \quad (52a)$$

$$\begin{aligned} \tilde{u}_\theta'' + \left[2 + \frac{1}{2} (1 + \nu) \lambda \right] \tilde{u}_\rho' + \frac{1}{2} (1 - \nu) \lambda (\lambda + 2) \tilde{u}_\theta = 0, \\ \theta \in (\theta_1, \theta_2) \end{aligned} \quad (52b)$$

Assume that all the tractions on the two edges, Γ_1 and Γ_2 , near the notch tip are zero. That is:

$$\left\{ \begin{array}{c} \sigma_{\theta\theta} \\ \sigma_{\rho\theta} \end{array} \right\}_{\theta=\theta_1} = \left\{ \begin{array}{c} \sigma_{\theta\theta} \\ \sigma_{\rho\theta} \end{array} \right\}_{\theta=\theta_2} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \quad (53)$$

Hence, substitution of Eqs. (50a, 50b, 50c) into Eq. (53) yields:

$$\tilde{u}_\theta' + (1 + \nu + \nu \lambda) \tilde{u}_\rho = 0, \quad \theta = \theta_1 \text{ and } \theta_2 \quad (54a)$$

$$\tilde{u}_\rho' + \lambda \tilde{u}_\theta = 0, \quad \theta = \theta_1 \text{ and } \theta_2 \quad (54b)$$

Considering that the appearance of λ^2 in Eqs. (52a, 52b) leads to nonlinear eigenanalysis if Eqs. (52a, 52b) are directly solved, an alternative approach is adopted in this paper to transfer the equation into a linear eigenvalue problem. To this end, two new field variables are introduced as follows:

$$g_\rho(\theta) = \lambda \tilde{u}_\rho(\theta), \quad \theta \in (\theta_1, \theta_2) \quad (55a)$$

$$g_\theta(\theta) = \lambda \tilde{u}_\theta(\theta), \quad \theta \in (\theta_1, \theta_2) \quad (55b)$$

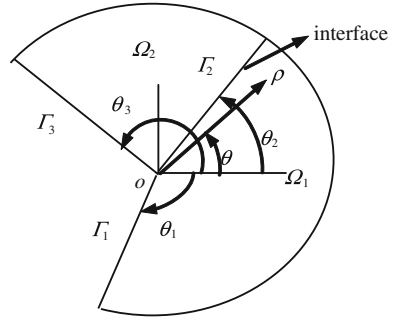
Thus, Eqs. (55a, 55b), Eqs. (52a, 52b) can be rewritten as:

$$\begin{aligned} \tilde{u}_\rho'' + \left(\frac{1+\nu}{1-\nu} \lambda - 2 \right) \tilde{u}_\theta' + \frac{2}{1-\nu} (\lambda + 2) g_\rho = 0, \\ \theta \in (\theta_1, \theta_2) \end{aligned} \quad (56a)$$

$$\begin{aligned} \tilde{u}_\theta'' + \left[2 + \frac{1}{2} (1 + \nu) \lambda \right] \tilde{u}_\rho' + \frac{1}{2} (1 - \nu) (\lambda + 2) g_\theta = 0, \\ \theta \in (\theta_1, \theta_2) \end{aligned} \quad (56b)$$

By following the above procedure, the evaluation of the singularity orders near a V-notch tip is transformed to solving a linear eigenvalue problem of the ODEs governed by Eqs. (55a, 55b), (56a, 56b) subjected to the boundary condition of

Fig. 3 A V-notch of bonded bi-material



Eqs. (54a, 54b). In the solutions, the associated eigenfunctions \tilde{u}_ρ and \tilde{u}_θ can also be obtained and can be used to determine the stresses in the vicinity of the notch tip.

In Fig. 3 we show an example of solution using this method applied in the case of bounded dissimilar linear elastic materials containing a V-notch tip.

Table 1 shows the comparison between the singularity degrees obtained by this method for various mesh levels of the used interpolating matrix method (IMMEI) and those of the literature. Reference [8] gives only one singularity degree λ_1 (one term in Eq. 48). Reference [9] gives two singularity degrees λ_1 and λ_2 as the present method noted (IMMEI) in the table. The value of n in the table indicates the discretization level considered in the IMMEI.

7 Concluding Remarks

In this chapter, we give a new Hamiltonian formalism resolving simultaneously the two-end problem and the problem of Cauchy and as a consequence, showing the relationship between the formalisms of Bui and Zhong which look so different. The key idea is to write the total potential energy of a solid as an integral along a special axis z , then over a section S normal to it. Using integration by part, the variation of the total potential energy can be written as two parts [see Eq. (15)]. The first part is an integral along z , and the second one is an integrated quantity depending on the two ends z_1 and z_2 . For the two end problem, the displacements are imposed at the two ends; so their variations vanish. According to the minimum principle of the total potential energy, the canonical equations of Hamilton are immediately obtained, [see Eqs. (16)–(18)]. On the other hand, for a natural evolution of the structure (i.e., the canonical equations of Hamilton are satisfied everywhere in the solid), but with possible variations of the two ends, the first part in the variation of the total potential energy vanishes [see Eq. (19)]. This corresponds to the Cauchy problem in elasticity. In this case, the equations of Hamilton–Jacobi can be deduced [see Eq. (21)]. Since the canonical equations of

Hamilton and the equations of Hamilton–Jacobi are fundamentally equivalent, we can see that the formalisms of Bui and Zhong are equivalent too.

Zhong’s formalism has been successfully applied to Fracture Mechanics in order to determine the asymptotic mechanical fields near the crack tip [4]. This work has shown that the Hamiltonian approach provides a systematic method in asymptotic analysis near the crack tip. It leads to a first order differential equation system, which is easy to deal with. We insist on the fact that this approach is not only a new formalism other than the traditional methods, but it can be used as a powerful tool in asymptotic analysis of fracture mechanics.

By using this approach, we have resolved various problems. Some of them have been solved previously and some not yet. For example, we can calculate the stress singularities for an interfacial crack between two elastic and isotropic materials. The results are completely identical as those obtained by using the well-known theoretical formula. Similar example is a crack tip normally touching an interface has been resolved see Ref. [4]. For a crack in a generally anisotropic material, we obtained a near tip field identical to theoretical results [5]. The comparison shows no difference between these two stress distributions. Another example consists in finding stress singularities near a notch tip formed from two generally anisotropic materials and stress singularities near an inclined crack tip touching an interface between two generally anisotropic materials [5]. From this work, we see that the present method is particularly efficient for resolving multi-material problems. This is because the selected dual variables are continuous across all the interfaces. So the multi-material problem can be resolved as a single material problem through the construction of the transfer matrix.

We believe that a large domain can be found in applying this new approach into fracture mechanics.

Nevertheless, the connection between the local obtained solution of the stress field and the far field is still a tremendous problem. That is why we investigate a new way transforming the fracture mechanics problem into an eigenvalue problem. That allows us to compute more terms in the stress expansion and then to connect the local field easily to the far field. The far field could be the finite element solution. This way will allow more efficiency to deal with various structural geometries and boundary conditions.

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