Chapter 2
Basic Algebraic and Geometric Features

We gather in this chapter useful algebraic and geometric features of the Schrödinger–Virasoro algebra that have not been derived in the previous chapter because they are not directly related to Newton-Cartan structures. The unifying concept here is that of *graduations*. Recall that a *graduation* of a Lie algebra $\mathfrak{g}$ is a decomposition of $\mathfrak{g}$ into a direct sum $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ such that $[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$. If $X \in \mathfrak{g}_n$ for some n, then X is said to be an element of degree n. Allowing for $\mathbb{Z}^k$-valued graduations, a particularly interesting case is that of the graduation given by the root system if $\mathfrak{g}$ is semi-simple.

Graduations – and more generally, root diagrams – play a prominent rôle for Lie algebras, both from a classification and a representation point of view; think for instance about highest-weight modules, or (in the infinite-dimensional case) about O. Mathieu's theorem [86] on the classification of simple graded Lie algebras with polynomial growth. In this respect, it is interesting to note the two following facts:

- The Schrödinger–Virasoro Lie algebra enjoys two independent graduations (both of which are equally interesting from a representation theoretic point of view, as we shall see in Chaps. 4–6), one of which is given by the indices of the generators L_n, Y_m, M_p (see Sect. 2.1). The Lie algebra $\mathfrak{sv}$ also appears naturally as a subalgebra of an *extended Poisson algebra* on the torus, quotiented out by the ideal generated by the elements with negative graduation (see Sect. 2.3). The latter idea is developed in full generality in Chap. 11 when considering supersymmetric extensions;
- The finite-dimensional Lie algebra $\mathfrak{sch} \subset \mathfrak{sv}$ may be embedded into the complexified *conformal Lie algebra* $\mathfrak{conf}(3)_{\mathbb{C}}$ in three dimensions (see Sect. 2.2). More generally, $\mathfrak{sch}(d)$ may be embedded into $\mathfrak{conf}(d+2)_{\mathbb{C}}$. This simple statement has already been made in the Preface (see 8.); it comes from the fact that a Laplace transformation $\mathscr{M} \rightsquigarrow \partial_\zeta$ formally intertwines the free Schrödinger operators in $(d+1)$-dimensions with the Klein-Gordon operator in $(d+2)$-dimensions. We give here explicit formulas for the embedding.

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A natural question arises then: does there exist a Lie algebra containing both $\mathfrak{conf}(3)$ and $\mathfrak{sv}$, such that both natural embeddings $\mathfrak{sch} \subset \mathfrak{conf}(3)$, $\mathfrak{sch} \subset \mathfrak{sv}$ are preserved? It turns out that there is no natural answer to this problem. In analogy with the well-known result in gauge theory stating that there does not exist a non-trivial extension containing both the Poincaré group and the external gauge group, we shall call this a *'no-go' theorem* (see Sect. 2.3).

Finally, we recall in Sect. 2.4 classical results concerning Schrödinger and conformal *tensor invariants*. These tensor invariants are called *covariant n-point functions* in the context of statistical physics. The general idea is that symmetries constrain n-point functions, which in turn yields predictions for the correlators of physical quantities. This idea will be developed in Chap. 6, and also in a supersymmetric setting in Chap. 11.

The results of this chapter are taken from [106].

2.1 On Graduations and Some Deformations of the Lie Algebra $\mathfrak{sv}$

Let us emphasize the following statement from Sect. 1.1.2.

Proposition 2.1. *As a* $\mathrm{Vect}(S^1)$*-module, the Schrödinger–Virasoro algebra is isomorphic to the semi-direct product* $\mathrm{Vect}(S^1) \ltimes (\mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_0)$.

Recall (see footnote in Sect. 1.1.2) that a field X in $\mathcal{F}_\lambda$ gives rise in the formalism of conformal field theory to a (quantum) *primary field* $\hat{X}$ with *conformal weight* $\lambda + 1$, i.e. such that $[L_n, \hat{X}_m] = (\lambda n - m)\hat{X}_{n+m}$ for the operator bracket, so that (depending on the terminology) the Y, resp. M field behaves as a $(-\frac{1}{2})$-density, resp. 0-density or, in other words, Y, resp. M has *conformal weight* $\frac{3}{2}$, resp. 1. We shall spend some time explaining the basics of conformal field theory in Chap. 6; for the moment, it is sufficient to know that it is based upon the explicit construction out of creation and annihilation operators of quantum fields acting on Fock spaces, whose Fourier components satisfy commutation relations of a certain type, including in particular those of the Virasoro algebra and those between the Virasoro field and a primary field. Conformal field theory is thus a very fruitful way of constructing infinite-dimensional algebras and representations thereof. Let us now remark that quantum fields with *half-integer weight* (such as Y) are usually *fermionic*. As a matter of fact, the celebrated spin-statistics theorem *in dimension* ≥ 3 [110] states that bosonic, resp. fermionic fields should have an integer-, resp. half-integer-valued spin. Fermionic fields produce *odd* generators in *superalgebras*. The most celebrated superalgebra containing the Virasoro field *and* a field with conformal weight $3/2$ is the so-called *Neveu-Schwarz superalgebra* $\mathfrak{ns}$ (see [66]). It is generated by the Virasoro field $(L_n)_{n\in\mathbb{Z}}$ and a fermionic current (G_m) with $m \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$, with supplementary anti-commutation relations $\{G_n, G_m\} = 2L_{n+m}$. Contrary to $\mathfrak{sv}$, it is a *simple* superalgebra. The liberty of choosing either integer of half-integer indices for the G-generators leas to subtle differences when considering its representations in conformal field theory.[1]

Turning back to the Schrödinger–Virasoro algebra, and trying to play with its definition, one may note the following facts:

- A theorem due to O. Mathieu [86], already alluded to in the introduction to this chapter, gives a classification of all *simple* $\mathbb{Z}$-graded algebras $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ with polynomial growth, i.e. such that $\dim \mathfrak{g}_n \le C|n|^k$ for some $C > 0$ and $k \in \mathbb{N}$. They come in four series: (1) *finite-dimensional simple Lie algebras*; (2) *Kac-Moody Lie algebras*, or in other words current Lie algebras on the circle; (3) the so-called *Cartan-type Lie algebras*, which are Lie algebras of formal vector fields in n variables;[2] (4) the two Virasoro-type algebras, $\mathrm{Vect}(S^1)$ and $\mathfrak{vir}$. From this theorem one sees clearly the impossibility of deforming the bracket of $\mathfrak{sv}$ so as to obtain a new *simple* algebra. On the other hand, the Neveu-Schwarz algebra is a counterexample to this statement in the category of *superalgebras*.
- The same ambiguity that exists in the range of indices for the generators of the Neveu-Schwarz algebra $\mathfrak{ns}$ is also found for $\mathfrak{sv}$, yielding a competition between two algebras, the Schrödinger–Virasoro algebra $\mathfrak{sv}$ and the *twisted Schrödinger–Virasoro algebra* $\mathfrak{sv}(0)$, to which we shall come back in Chaps. 7 and 9;
- Contrary to the case of the Neveu-Schwarz algebra, *changing continuously the conformal weight* of the Y and M generators yields two one-parameter families of algebras, $\mathfrak{sv}_\varepsilon$ and $\mathfrak{sv}_\varepsilon(0)$, $\varepsilon \in \mathbb{R}$, with $\mathfrak{sv}_0 = \mathfrak{sv}$ and $\mathfrak{sv}_0(0) = \mathfrak{sv}(0)$. We shall actually show by cohomological methods in Chap.7 that $\mathfrak{sv}$ and $\mathfrak{sv}(0)$ admit *three* independent families of deformations.

Definition 2.2. Let $\mathfrak{sv}_\varepsilon,\ \varepsilon \in \mathbb{R}$ (resp. $\mathfrak{sv}_\varepsilon(0)$) be the Lie algebra generated by $L_n, Y_m, M_p,\ n, p \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}$ (resp. $m \in \mathbb{Z}$), with relations

$$[L_n, L_{n'}] = (n - n')L_{n+n'},\ [L_n, Y_m] = \left(\frac{(1+\varepsilon)n}{2} - m\right) Y_{n+m},\ \left[L_n, M_p\right] = (\varepsilon n - p)M_{n+p}$$

$$[Y_m, Y_{m'}] = (m - m')M_{m+m'},\ [Y_m, M_p] = 0,\ [M_p, M_{p'}] = 0, \tag{2.1}$$

The Y, resp. M field is a $-(1+\varepsilon)n/2$, resp. $-\varepsilon$ density, or in other words, has primary weight $(3+\varepsilon)/2$, resp. $1+\varepsilon$.

In close connection to all this, let us now turn to graduations. Any graduation δ on a Lie algebra $\mathfrak{g}$ may be seen alternatively as a *derivation* $\bar{\delta} : \mathfrak{g} \to \mathfrak{g}$ by setting $\bar{\delta}(X_n) = nX_n$ if $\delta(X_n) = n$, i.e. if X_n has degree n. Recall that a *derivation* of $\mathfrak{g}$ is a

[1] Vertex representations with integer, resp. half-integer valued G-component indices correspond to the so-called *Ramond*, resp. *Neveu-Schwarz* sector.

[2] There are four series indexed by n: the Lie algebra $\mathrm{Vect}(n)$ of all formal vector fields, and the subalgebras of $\mathrm{Vect}(n)$ made up of symplectic, unimodular or contact vector fields.

linear map $\bar{\delta} : \mathfrak{g} \to \mathfrak{g}$ such that $\bar{\delta}[X, Y] = [\bar{\delta}(X), Y] - [\bar{\delta}(Y), X]$. Typical examples are *inner derivations* $\bar{\delta} := [X, .]$, given by bracketing against an element $X \in \mathfrak{g}$. *Outer derivations* are then simply non-inner derivations.

It turns out that there exist *two* linearly independent graduations on $\mathfrak{sv}_\varepsilon$ or $\mathfrak{sv}_\varepsilon(0)$.

Definition 2.3. Let δ_1, resp. δ_2, be the graduations on $\mathfrak{sv}_\varepsilon$ or $\mathfrak{sv}_\varepsilon(0)$ defined by

$$\delta_1(L_n) = n,\ \delta_1(Y_m) = m, \delta_1(M_p) = p \tag{2.2}$$

$$\delta_2(L_n) = n,\ \delta_2(Y_m) = m - \frac{1}{2},\ \delta_2(M_p) = p - 1 \tag{2.3}$$

with $n, p \in \mathbb{Z}$ and $m \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.

One immediately checks that both δ_1 and δ_2 define graduations and that they are linearly independent.

Proposition 2.4. *The graduation δ_1, defined either on $\mathfrak{sv}_\varepsilon$ or on $\mathfrak{sv}_\varepsilon(0)$, is given by the inner derivation $\delta_1 = \mathrm{ad}(-L_0)$, while δ_2 is an outer derivation.*

Remark 2.5. As we shall see in Chap. 7, the space $H^1(\mathfrak{sv}, \mathfrak{sv})$ or $H^1(\mathfrak{sv}(0), \mathfrak{sv}(0))$ of outer derivations modulo inner derivations (see introduction to Chap. 7) is three-dimensional, but only δ_2 defines a graduation on the basis (L_n, Y_m, M_p).

Proof. The only non-trivial point is to prove that δ_2 is not an inner derivation. Suppose, to the contrary, that $\delta_2 = \mathrm{ad}X$, $X \in \mathfrak{sv}$ or $X \in \mathfrak{sv}(0)$ (we treat both cases simultaneously). Then $\delta_2(M_0) = 0$ since M_0 is central in $\mathfrak{sv}$ and in $\mathfrak{sv}(0)$. Hence the contradiction. Note that the graduation δ_2 is given by the Lie action of the Euler vector field $t\partial_t + r\partial_r + \zeta\partial_\zeta$ in the vector-field representation $d\tilde{\pi}$ (see Introduction). □

All these Lie algebras may be extended by using the natural extension of the Virasoro cocycle (1.14) of Sect. 1.1, yielding Lie algebras denoted by $\widetilde{\mathfrak{sv}}$, $\widetilde{\mathfrak{sv}}(0)$, $\widetilde{\mathfrak{sv}}_\varepsilon$, $\widetilde{\mathfrak{sv}}_\varepsilon(0)$.

2.2 The Conformal Embedding

Let $\mathfrak{conf}(d)$ be the *conformal algebra* in d dimensions ($d \geq 3$). By definition, it is the Lie algebra of the Lie group of global conformal transformations of $\mathbb{R}^d$ equipped with its standard Riemannian metric, i.e. the Lie group of C^1-diffeomorphisms $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that the differential $d\phi_x : T_x\mathbb{R}^d \to T_x\mathbb{R}^d$ at the point x preserves the metric tensor $g_{\mu\nu}dx^\mu dx^\nu$ up to a scalar multiplication.

Such a Lie group may also be defined on any Riemannian manifold M of dimension d. As shown for instance in [19], in dimension $d \geq 3$, the local constraints coming from the requirement that a one-parameter subgroup of local diffeomorphisms preserve the metric up to scalar multiplication is already strong enough to make the conformal group finite-dimensional. As mentioned in the Introduction, this is not true for $d = 2$, since local conformal transformations are then local holomorphic or anti-holomorphic transformations.

Computations show that $\mathfrak{conf}(d)$ is $\frac{(d+1)(d+2)}{2}$- dimensional and is isomorphic to $\mathfrak{so}(d+1,1)$. Clearly, rotations and translations preserve the metric, so the (Riemannian) Poincaré algebra $\mathfrak{so}(d) \ltimes \mathbb{R}^d$ is naturally embedded into $\mathfrak{so}(d+1,1)$. The dilatations $x \to \lambda x$ ($\lambda \in \mathbb{R}$) preserve the metric up to a constant. There also exist so-called *special conformal transformations* which preserve the metric only up to a non-trivial scalar (see below).

The idea of embedding $\mathfrak{sch}(d)$ into $\mathfrak{conf}(d+2)_{\mathbb{C}}$ comes naturally when considering the wave equation

$$(2\mathscr{M}\partial_t - \Delta_r)\psi(\mathscr{M};t,r) = 0 \tag{2.4}$$

where $\mathscr{M}$ is viewed no longer as a parameter, but as a coordinate. Then the formal Laplace transform of the wave function with respect to the mass

$$\tilde{\psi}(\zeta;t,r) = \int_{\mathbb{R}} \psi(\mathscr{M};t,r)e^{\mathscr{M}\zeta}\, d\mathscr{M} \tag{2.5}$$

satisfies the equation

$$(2\partial_\zeta\partial_t - \Delta_r)\psi(\zeta;t,r) = 0 \tag{2.6}$$

which is nothing but a zero mass Klein-Gordon equation $(\partial_x^2 - \partial_y^2 - \Delta_r)\psi = 0$ on $(d+2)$-dimensional space-time, put into light-cone coordinates $(\zeta,t) = (\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}})$. In this sense, Schrödinger transformations may be seen as the *subgroup of conformal transformations which preserve the mass.*

This simple idea has been developed in a previous article (see [54]) for $d = 1$. Let us here give an explicit embedding for any dimension d.

We need first to fix some notations. Consider the conformal algebra in its standard representation as infinitesimal conformal transformations on $\mathbb{R}^{d+2}$ with coordinates $(\xi_1,\ldots,\xi_{d+2})$. Then there is a natural basis of $\mathfrak{conf}(d+2)$ made of $(d+2)$ translations P_μ, $\frac{1}{2}(d+1)(d+2)$ rotations $\mathscr{M}_{\mu,\nu}$, $(d+2)$ inversions K_μ and the Euler operator D: in coordinates, one has

$$P_\mu = \partial_{\xi_\mu} \tag{2.7}$$

$$\mathscr{M}_{\mu,\nu} = \xi_\mu\partial_\nu - \xi_\nu\partial_\mu \tag{2.8}$$

$$K_\mu = 2\xi_\mu\left(\sum_{\nu=1}^{d+2}\xi_\nu\partial_\nu\right) - \left(\sum_{\nu=1}^{d+2}\xi_\nu^2\right)\partial_\mu \tag{2.9}$$

$$D = \sum_{\nu=1}^{d+2}\xi_\nu\partial_\nu. \tag{2.10}$$

Proposition 2.6. *The formulas*

$$Y^j_{-\frac{1}{2}} = -2^{\frac{1}{2}} e^{-i\pi/4} P_j \tag{2.11}$$

$$Y^j_{\frac{1}{2}} = -2^{-\frac{1}{2}} e^{i\pi/4} \left(\mathscr{M}_{d+2,j} + i\mathscr{M}_{d+1,j}\right) \tag{2.12}$$

$$R_{j,k} = \mathscr{M}_{j,k} \tag{2.13}$$

$$L_{-1} = i(P_{d+2} - iP_{d+1}) \tag{2.14}$$

$$L_0 = -\frac{D}{2} + \frac{i}{2}\mathscr{M}_{d+2,d+1} \tag{2.15}$$

$$L_1 = -\frac{i}{4}(K_{d+2} + iK_{d+1}) \tag{2.16}$$

give an embedding of $\mathfrak{sch}(d)$ *into* $\mathfrak{conf}(d+2)_{\mathbb{C}}$.

Proof. Put

$$t = \frac{1}{2}(-\xi_{d+1} + i\xi_{d+2}), \zeta = \frac{1}{2}(\xi_{d+1} + i\xi_{d+2}), r_j = 2^{-\frac{1}{2}} e^{i\pi/4}\xi_j \ (j = 1, \ldots, d).$$

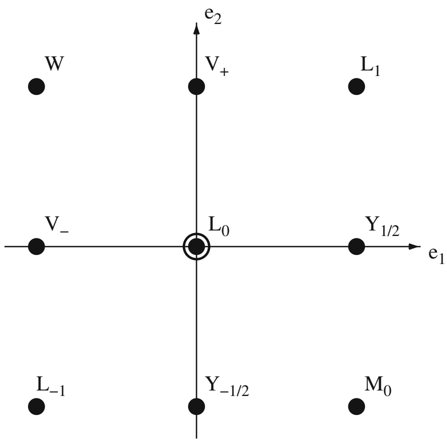
Then the previous definitions yield the representation $d\tilde{\pi}^d_{d/4}$ of $\mathfrak{sch}(d)$ (see (1.11)). □

When $d = 1$, this conformal embedding can be represented by the following root diagram of $\mathfrak{conf}(3)$, of type B_2, as drawn in [54] (see reproduction in Fig. 2.1). The *Schrödinger representation* $d\tilde{\pi}^d_{d/4}$ extends naturally to a representation of $\mathfrak{conf}(3)$. The Cartan subalgebra $\mathfrak{a}$ is two-dimensional, generated by L_0 and N_0, with $d\tilde{\pi}^d_{d/4}(N_0) = -r\partial_r - 2\zeta\partial_\zeta$ (see Sect. 6.1). It is convenient here to introduce instead the linear combinations $D = -L_0 - \frac{N_0}{2}$ and $N = L_0 - \frac{N_0}{2}$, realized in the Schrödinger representation as $t\partial_t + r\partial_r + \zeta\partial_\zeta$ and $-t\partial_t + \zeta\partial_\zeta$ respectively (see Chap. 11). Define the roots e_1, e_2 as $e_i(N) = \delta_{i,1}$, $e_i(D) = \delta_{i,2}$, yielding the coordinates along the horizontal and vertical axes respectively on Fig. 2.1. Then $\Pi := \{-e_2, e_1 + e_2\}$ defines a basis of the root system; the associated set of positive roots, i.e. of roots which are linear combinations with *positive* coefficients of $-e_2$ and $e_1 + e_2$, is $\Delta_+ = \{-e_2, e_1 + e_2, e_1, e_1 - e_2\}$. From Fig. 2.1 one reads directly that the positive Borel subalgebra (or *minimal parabolic subalgebra*) is

$$\mathfrak{a} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha = \langle L_0, N_0, L_1, Y_{\pm\frac{1}{2}}, M_0 \rangle, \tag{2.17}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{conf}(3) \mid \forall H \in \mathfrak{a}, [H, X] = \alpha(H)X\}$ is the one-dimensional root space associated to the root α. Up to the extra generator N_0, this is the *age subalgebra* of the Schrödinger subalgebra introduced in [54], which is a natural symmetry algebra for ageing systems (see Preface). The *extended Schrödinger*

Fig. 2.1 Connected diagram



algebra $\overline{\mathfrak{sch}} = \langle N_0 \rangle \ltimes \mathfrak{sch}$, introduced in Chap. 6, may be understood algebraically as the *maximal parabolic subalgebra* associated to the subset $\Pi_{\overline{\mathfrak{sch}}} = \{e_1 + e_2\} \subset \Pi$; apart from the generators of the positive Borel subalgebra, it also contains $\mathfrak{g}_{-(e_1+e_2)} = \mathbb{C}L_{-1}$, realized as time-translations in the Schrödinger representation. All these notions are standard in the study of the structure and representations of semi-simple Lie groups and algebras [60, 72].

2.3 Relations Between $\mathfrak{sv}$ and the Poisson Algebra on T^*S^1 and 'no-go' Theorem

The relation between the Virasoro algebra and the Poisson algebra on T^*S^1 has been investigated in [93]. We shall consider more precisely the Lie algebra $\mathscr{A}(S^1)$ of smooth functions on $\dot{T}^*S^1 = T^*S^1 \setminus S^1$, the total space of the cotangent bundle with zero section removed, which are Laurent series on the fibers.[3] So $\mathscr{A}(S^1) = C^\infty(S^1) \otimes \mathbb{R}[\partial, \partial^{-1}]]$ and $F \in \mathscr{A}(S^1)$ is of the following form:

$$F(t,\partial) = \sum_{k\in\mathbb{Z}} f_k(t)\partial^k,$$

with $f_k = 0$ for large enough k. The Poisson bracket is defined as usual, following:

$$\{F,G\} = \frac{\partial F}{\partial \partial}\frac{\partial G}{\partial t} - \frac{\partial G}{\partial \partial}\frac{\partial F}{\partial t}.$$

[3]One might also consider the subalgebra of $\mathscr{A}(S^1)$ defined as $\mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\partial, \partial^{-1}]$; it gives the usual description of the Poisson algebra on the torus $\mathbb{T}^2$, sometimes denoted $SU(\infty)$ [59].

(The reader should not be afraid of the notation $\partial\partial$!). In terms of tensor-density modules for $\mathrm{Vect}(S^1)$ (see Chap. 1), one has the natural decomposition: $\mathscr{A}(S^1) = \bigoplus_{k>0} \mathscr{F}_k \bigoplus \left(\prod_{k\leq 0} \mathscr{F}_k \right)$. The Poisson bracket turns out to be homogeneous with respect to that decomposition: $\{\mathscr{F}_k, \mathscr{F}_l\} \subset \mathscr{F}_{k+l-1}$; more explicitly $\{f(x)\partial^k, g(x)\partial^l\} = (kfg' - lf'g)\partial^{k+l-1}$. One recovers the formulae of Chap. 1 for the Lie bracket on $\mathscr{F}_1 = \mathrm{Vect}(S^1)$ and its representations on modules of densities; one has as well the embedding of the semi-direct product $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1) = \mathscr{F}_1 \ltimes \mathscr{F}_0$ as a Lie subalgebra of $\mathscr{A}(S^1)$, representing differential operators of order ≤ 1.

We shall need to extend the above Poisson algebra by considering *half-densities*, whose coefficients are Laurent series in $\sqrt{z}$. Geometrically speaking, half-densities can be described as spinors: let E be a vector bundle over S^1, square root of T^*S^1; in other words one has $E \otimes E = T^*S^1$. Then the space of Laurent polynomials on the fibers of E (with the zero-section removed) is exactly the Poisson algebra $\widetilde{\mathscr{A}}(S^1) = C^\infty(S^1) \otimes \mathbb{C}[\partial^{1/2}, \partial^{-1/2}]]$. We shall also need to consider half-integer power series or polynomials in $\sqrt{z}$ as coefficients of the Laurent series in ∂; one may obtain the corresponding algebra globally by using the pull-back though the application $S^1 \longrightarrow S^1$ defined as $z \mapsto z^2$.

Summarizing, one has obtained the subalgebra $\widehat{\mathscr{A}}(S^1) \supset \widetilde{\mathscr{A}}(S^1)$ generated by terms $z^m\partial^n$ where m and n are either integers or half-integers. One may represent such generators as the points with coordinates (m, n) in the plane $\mathbb{R}^2$. In particular, $\mathfrak{sv} = \mathrm{Vect}(S^1) \ltimes \mathfrak{h}$, with $\mathfrak{h} \simeq \mathscr{F}_{1/2} \ltimes \mathscr{F}_0$ as a $\mathrm{Vect}(S^1)$-module, can be naturally embedded into $\widehat{\mathscr{A}}(S^1)$ as follows:

$$\begin{array}{ccccccccccccc}
\cdots & & L_{-2} & & L_{-1} & & L_0 & & L_1 & & L_2 & & \cdots \\
& \cdots & & Y_{-\frac{3}{2}} & & Y_{-\frac{1}{2}} & & Y_{\frac{1}{2}} & & Y_{\frac{3}{2}} & & \cdots & \\
\cdots & & M_{-2} & & M_{-1} & & M_0 & & M_1 & & M_2 & & \cdots
\end{array} \tag{2.18}$$

while the twisted Schrödinger–Virasoro algebra $\mathfrak{sv}(0)$ (with integer-valued indices for the Y-field) may be represented as follows:

$$\begin{array}{ccccccc}
\cdots & L_{-2} & L_{-1} & L_0 & L_1 & L_2 & \cdots \\
\cdots & Y_{-2} & Y_{-1} & Y_0 & Y_1 & Y_2 & \cdots \\
\cdots & M_{-2} & M_{-1} & M_0 & M_1 & M_2 & \cdots
\end{array}$$

The graduation along the vertical axis is given by the outer derivation $\delta_2 - \delta_1$, and indicates the weights of the associated density modules. One can naturally ask whether this defines a Lie algebra embedding, just as in the case of $\mathrm{Vect}(S^1) \ltimes \mathscr{F}_0$. The answer is *no*:

Proposition 2.7. *The natural vector space embedding $\mathfrak{sv} \hookrightarrow \tilde{\mathscr{A}}(S^1)$ is not a Lie algebra homomorphism.*

Proof. One sees immediately that on the one hand $[Y_m, M_p] = 0$ and $[M_p, M_{p'}] = 0$, while on the other hand $\{\mathscr{F}_{1/2}, \mathscr{F}_0\} \subset \mathscr{F}_{-1/2}$ is non trivial. $\square$

The vanishing of $\{\mathscr{F}_0, \mathscr{F}_0\}$ which makes the embedding of $\mathrm{Vect}(S^1) \ltimes \mathscr{F}_0$ as a Lie subalgebra possible was in some sense an accident. In fact, one can show that, starting from the image of the generators of $\mathfrak{sv}$ and computing successive Poisson brackets, one can generate all the $\mathscr{F}_\lambda$ with $\lambda \leq 0$. In other words,

$$\tilde{\mathscr{A}}(S^1)_{\leq 1} = \mathscr{F}_1 \oplus \mathscr{F}_{1/2} \oplus \mathscr{F}_0 \bigoplus_{\lambda \in \frac{\mathbb{Z}}{2}\ ,\ \lambda < 0} \mathscr{F}_\lambda \tag{2.19}$$

defines the smallest possible Poisson subalgebra of $\tilde{\mathscr{A}}(S^1)$ which contains the image of $\mathfrak{sv}$. Now, let

$$\tilde{\mathscr{A}}(S^1)_{\leq -\frac{1}{2}} = \prod_{\lambda \in \frac{\mathbb{Z}}{2},\ \lambda < 0} \mathscr{F}_\lambda \tag{2.20}$$

be the Poisson subalgebra of $\tilde{\mathscr{A}}(S^1)$ which contains only negative powers of ∂. One easily sees that it is an ideal of $\tilde{\mathscr{A}}(S^1)_{(1)}$. If one considers the quotient $\tilde{\mathscr{A}}(S^1)_{(1)} / \tilde{\mathscr{A}}(S^1)_{(0)}$, then the embedding becomes an isomorphism:

Proposition 2.8 ($\mathfrak{sv}$ as a Poisson subquotient). *There exists a natural Lie algebra isomorphism between $\mathfrak{sv}$ and quotient $\tilde{\mathscr{A}}(S^1)_{\leq 1} / \tilde{\mathscr{A}}(S^1)_{\leq -\frac{1}{2}}$. In other words, $\mathfrak{sv}$ may be seen as a subquotient of the Poisson algebra $\tilde{\mathscr{A}}(S^1)$.*

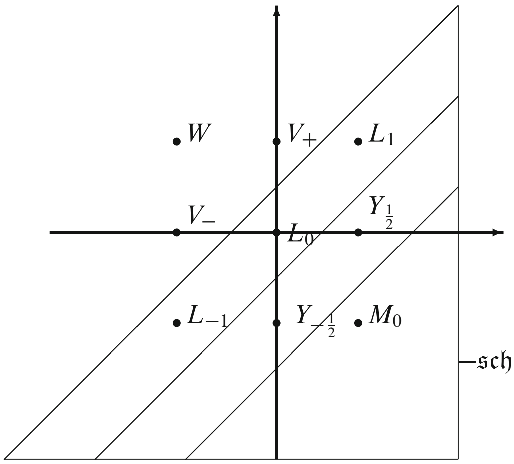
Now a natural question arises: the conformal embedding of Schrödinger algebra described in Sect. 2.2 yields $\mathfrak{sch} \subset \mathfrak{conf}(3)_{\mathbb{C}}$, so one would like to extend the construction of $\mathfrak{sv}$ as generalization of $\mathfrak{sch}$, in order that it contain $\mathfrak{conf}(3)$. In other words, we are looking for an hypothetic Lie algebra $\mathscr{G}$ making the following diagram of embeddings complete:

$$\begin{array}{ccc} \mathfrak{sch} & \hookrightarrow & \mathfrak{conf}(3) \\ \downarrow & & \downarrow \\ \mathfrak{sv} & \hookrightarrow & \mathscr{G} \end{array} \tag{2.21}$$

In the category of abstract Lie algebras, one has an obvious solution to this problem: simply take the amalgamated sum of $\mathfrak{sv}$ and $\mathfrak{conf}(3)$ over $\mathfrak{sch}$. Such a Lie algebra is defined though generators and relations, and is generally intractable. We are looking here for a natural, geometrically defined construction of such a $\mathscr{G}$; we shall give some evidence of its non-existence, a kind of *'no-go' theorem*, analogous to those well-known in gauge theory [4] (see e.g. [67]).

[4]Simply recall that this theorem states that there does not exist a common non-trivial extension containing both the Poincaré group and the external gauge group.

Let us consider once again the root diagram of $\mathfrak{conf}(3)$:



The graduation along the vertical (resp. horizontal) axis is given by the action of δ_2, resp. $2\delta_1 - \delta_2$. Comparing with (2.18), one sees that the successive diagonal strips are contained in $\mathscr{F}_1, \mathscr{F}_{1/2}, \mathscr{F}_0$ respectively. So the first idea might be to try to add $\mathscr{F}_{3/2}$ and $\mathscr{F}_2$, as an infinite-dimensional prolongation of $\langle V_-, V_+, W\rangle$, so that $V_- \longrightarrow t^{-1/2}\partial^{3/2}$, $V_+ \longrightarrow t^{1/2}\partial^{3/2}$, $W \longrightarrow \partial^2$.

Unfortunately, this construction fails at once for two reasons: first, one does not get the right brackets for $\mathfrak{conf}(3)$ with such a choice, and secondly the elements of $\mathscr{F}_\lambda$, $\lambda \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$, taken together with their successive brackets generate the whole Poisson algebra $\tilde{\mathscr{A}}(S^1)$.

Another approach could be the following: take two copies of $\mathfrak{h}$, say $\mathfrak{h}^+$ and $\mathfrak{h}^-$ and consider the semi-direct product $\mathscr{G} = \mathrm{Vect}(S^1) \ltimes (\mathfrak{h}^+ \oplus \mathfrak{h}^-)$, so that $\mathfrak{h}^+$ extends $\langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\rangle$ as in $\mathfrak{sv}$ before, and so that $\mathfrak{h}^-$ extends $\langle V_-, V_+, W\rangle$. Then $\mathscr{G}$ is obtained from density modules. However, it does *not* extend $\mathfrak{conf}(3)$, but only a contraction of it: all the brackets between $\langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\rangle$ on the one hand and $\langle V_-, V_+, W\rangle$ on the other are vanishing. Now, we can try to deform $\mathscr{G}$ in order to obtain the right brackets for $\mathfrak{conf}(3)$. Let Y_m^+, M_m^+ and Y_m^-, M_m^- be the generators of $\mathfrak{h}^+$ and $\mathfrak{h}^-$; we want to find coefficients $a_{p,m}$ such that $[Y_m^+, Y_p^-] = a_{p,m}L_{m+p}$ defines a Lie bracket. So let us check Jacobi identity for (L_n, Y_m^+, Y_p^-). One obtains $(m - \frac{n}{2})a_{p,n+m} + (n - m - p)a_{p,m} + (p - \frac{n}{2})a_{p+n,m} = 0$. If one tries $a_{pm} = \lambda p + \mu m$, one deduces from this relation: $n\lambda(p - \frac{n}{2}) + n\mu(m - \frac{n}{2}) = 0$ for every $n \in \mathbb{Z}$, $p, m \in \frac{1}{2}\mathbb{Z}$, so obviously $\lambda = \mu = 0$.

So our computations show there does not exist a geometrically defined construction of $\mathscr{G}$ satisfying the conditions of diagram (2.16). The two possible extensions of $\mathfrak{sch}$, $\mathfrak{sv}$ and $\mathfrak{conf}(3)$ are shown to be incompatible, and this is our "no-go" theorem.

To finish, note that embeddings into Poisson algebras also appear naturally in a supersymmetric setting, yielding a large class of superizations of $\mathfrak{sv}$ (see Chap. 11, Sect. 11.3).

2.4 Conformal and Schrödinger Tensor Invariants

Consider quite generally a group representation $\rho : G \to Hom(\mathscr{H}, \mathscr{H})$ into some vector space $\mathscr{H}$, and its n-fold tensor product $\rho^{\otimes n} : G \to Hom(\mathscr{H}^{\otimes n}, \mathscr{H}^{\otimes n})$ defined as usual by $\rho^{\otimes n}(g)(v_1 \otimes \ldots v_n) = \rho(g)v_1 \otimes \ldots \otimes \rho(g)v_n$. Then an *n-tensor invariant* of ρ is an element $v \in \mathscr{H}^{\otimes n}$ such that $\rho^{\otimes n}(g) \cdot v = v$ for all $g \in G$. This notion is central in quantum field theory, and more specifically in conformal field theory, as we shall presently see. Let $\Psi_1(x), \ldots, \Psi_n(x)$ be (vector-valued) quantum fields on $\mathbb{R}^d$ (free fields in quantum field theory [115], or *quasi-primary fields* in conformal field theory [19]), acting on some Hilbert space $\mathscr{H}$, usually a Fock space. Assume that some symmetry group G (the Poincaré group for quantum field theory on $\mathbb{R}^{3+1}$, or the group of projective transformations for two-dimensional conformal field theory, resp. the conformal group $\mathrm{Conf}(d)$ in $d \geq 3$ dimensions) has been implemented as unitary operators $U(g) : \mathscr{H} \to \mathscr{H}$ such that

$$U(g)\Psi_i(x)U(g)^{-1} = \sum_i D_{i,j}(g^{-1}) \cdot \Psi_j(g \cdot x), \tag{2.22}$$

where D is a finite-dimensional matrix representation of G acting on the components of the vector Ψ_i, and $g \cdot x$ is the image of x by the natural coordinate transformation induced by $g \in G$.[5] Assume also that the *vacuum state* $|0\rangle \in \mathscr{H}$ is G-invariant. Then the *vacuum state expectation* (or *n-point function*) $\langle 0|\Psi_1(x_1) \ldots \Psi_n(x_n)|0\rangle$ is an n-tensor invariant of ρ.

We consider here two classical examples: conformal and Schrödinger tensor-invariants.

Example 1 (Conformal invariants). Consider the representation $\rho = \rho_\mu$ of the group of homographies, i.e. of holomorphic projective transformations of the sphere: it acts on a *quasi-primary field* of conformal weight μ as

$$\rho_\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Psi_\mu(z) = (ct + d)^{-2\mu} \Psi_\mu \left(\frac{dz - b}{-cz + a} \right). \tag{2.23}$$

In other words, $\rho_\mu(\phi)\Psi_\mu(z) = (\phi'(z))^\mu (\Psi_\mu \circ \phi^{-1})(z)$ for $g \in SL(2, \mathbb{R}) \subset \mathrm{Diff}(S^1)$. The associated classical action is the restriction to $SL(2, \mathbb{R})$ of the action of diffeomorphisms of the circle on μ-densities (see Chap. 1). A *primary field* is by definition transformed in the same way by *all* diffeomorphisms of the circle. However, the constraints induced by this much stronger property on n-point functions are much more subtle because the vacuum state $|0\rangle$ itself is *not* invariant under $\mathrm{Diff}(S^1)$.

[5] In many cases, by replacing the quantum fields $\Psi(x)$ by a function $\psi(x)$, one obtains simply an irreducible unitary representation of G on a one-particle state.

Tensor-invariants (or $SL(2,\mathbb{R})$-covariant n-point functions) are given as follows for $n = 2, 3, 4$:

$$\langle \Psi_{\mu_1}(z_1)\Psi_{\mu_2}(z_2)\rangle = C\delta_{\mu_1,\mu_2}(z_1 - z_2)^{-2\mu_1}; \tag{2.24}$$

$$\langle \Psi_{\mu_1}(z_1)\Psi_{\mu_2}(z_2)\Psi_{\mu_3}(z_3)\rangle = C(z_1 - z_2)^{\mu_3-\mu_1-\mu_2}(z_2 - z_3)^{\mu_1-\mu_2-\mu_3}(z_3 - z_1)^{\mu_2-\mu_3-\mu_1}; \tag{2.25}$$

$$\left\langle \prod_{i=1}^{4} \Psi_{\mu_i}(z_i)\right\rangle = C \prod_{1\le i<j\le 4} (z_i - z_j)^{-2\gamma_{ij}} \cdot f(F(z_1, z_2, z_3, z_4)) \tag{2.26}$$

where f is some arbitrary function (called: *scaling function*) and $F(z_1, z_2, z_3, z_4) := \frac{(z_1-z_3)/(z_1-z_4)}{(z_2-z_3)/(z_2-z_4)}$ is the *cross-ratio* of its arguments, and γ_{ij} are parameters fixed by the relations $\sum_{i\neq j} \gamma_{ij} = \mu_j$.

Example 2 (Schrödinger invariants). Consider the Schrödinger action π_λ on fields $\Psi_{\mathscr{M},\lambda}$ of mass $\mathscr{M}$ and scaling dimension 2λ as in the Preface. Then $\mathrm{Sch}(d)$-covariant (or equivalently $\mathrm{Sch}(d)$-*quasiprimary*) n-point functions are given as follows for $n = 2, 3$ (see Appendix B for references):

$$\langle \Psi_{\mathscr{M}_1,\lambda_1}(t_1, r_1)\Psi_{\mathscr{M}_2,\lambda_2}(t_2, r_2)\rangle = C\delta_{\mathscr{M}_1+\mathscr{M}_2,0}\delta_{\lambda_1,\lambda_2}(t_1-t_2)^{-2\lambda_1}e^{-\mathscr{M}_1\frac{(r_1-r_2)^2}{2(t_1-t_2)}}; \tag{2.27}$$

$$\begin{aligned}\langle \Psi_{\mathscr{M}_1,\lambda_1}&(t_1, r_1)\Psi_{\mathscr{M}_2,\lambda_2}(t_2, r_2)\Psi_{\mathscr{M}_3\lambda_3}(t_3, r_3)\rangle\\ &= C\delta_{\mathscr{M}_1+\mathscr{M}_2+\mathscr{M}_3,0}(t_1 - t_2)^{\lambda_3-\lambda_1-\lambda_2}(t_2 - t_3)^{\lambda_1-\lambda_2-\lambda_3}(t_3 - t_1)^{\lambda_2-\lambda_3-\lambda_1}\\ &\qquad \exp\left[-\frac{\mathscr{M}_1}{2}\frac{(r_1 - r_3)^2}{t_1 - t_3} - \frac{\mathscr{M}_2}{2}\frac{(r_2 - r_3)^2}{t_2 - t_3}\right] f(F(t_1, r_1; t_2, r_2; t_3, r_3)),\end{aligned} \tag{2.28}$$

where f is an arbitrary scaling function, and $F(t_1, r_1; t_2, r_2; t_3, r_3) = \frac{((r_1-r_3)(t_2-t_3)-(r_2-r_3)(t_1-t_3))^2}{(t_1-t_2)(t_2-t_3)(t_1-t_3)}$.

The fact that (contrary to the conformally covariant case) Schrödinger-covariant three-point functions should only be determined up to a scaling function is of course very disturbing for physicists, since (1) it means that Schrödinger-covariant model are less constrained, (2) two-point functions alone give little evidence that a given model should be Schrödinger-covariant. Using the natural embedding $\mathfrak{sch} \subset \mathfrak{conf}(3)_{\mathbb{C}}$ exhibited earlier in this chapter, one can however consider three-point functions which are covariant under the action of the whole conformal group $\mathrm{Conf}(3)$. These are determined up to a constant; formulas (2.25) actually extend to arbitrary dimension. To be explicit,

$$\begin{aligned}\langle \Psi_{\lambda_1}(\zeta_1, t_1, r_1)\Psi_{\lambda_2}(\zeta_2, t_2, r_2)\Psi_{\lambda_3}(\zeta_3, t_3, r_3)\rangle &= C\,|(\zeta_1 - \zeta_2, t_1 - t_2, r_1 - r_2)|^{\lambda_3-\lambda_1-\lambda_2}\\ &|(\zeta_2 - \zeta_3, t_2 - t_3, r_2 - r_3)|^{\lambda_1-\lambda_2-\lambda_3}|(\zeta_3 - \zeta_1, t_3 - t_1, r_3 - r_1)|^{\lambda_2-\lambda_3-\lambda_1},\end{aligned} \tag{2.29}$$

where $|(\zeta, t, r)|^2 := 2\zeta t - r^2$ is the squared Minkowski norm of (ζ, t, r) in light-cone coordinates.

We shall find out in Chap. 6 (see Theorem 6.10) that the three-point functions of the *massive* $\mathfrak{sv}$*-primary field* ψ_{-1} look similar to these up to a time-dependent prefactor. Because of the "no-go" theorem described above, it would have been very surprising to find three-point functions which are conformally covariant *and* transform covariantly under the whole Schrödinger–Virasoro group.

The two appendices at the end of the book give variants and extensions of the above results (in a supersymmetric setting for Appendix B).

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