Preface

The object of the present monography is to give an up-to-date, self-contained presentation of a recently discovered mathematical structure: the *Schrödinger–Virasoro algebra*. The study of the structure of this infinite-dimensional Lie algebra containing the Virasoro algebra, and the various contexts in which it appears naturally, will lead us to touch upon such different topics as mechanics, statistical physics, Poisson geometry, integrable systems, supergeometry, representation theory and cohomology of infinite-dimensional Lie algebras, spectral theory of Schrödinger operators.

The original motivation for introducing the Schrödinger–Virasoro algebra was the following (see the Preface for a more physically minded point of view).

There is, in the physical literature of the twentieth century, a deeply rooted belief that physical systems – macroscopic systems for statistical physicists, quantum particles and fields for high energy physicists – could and should be classified according to their symmetries.

Let us just point at two very well-known physical examples : *elementary particles* on the $(3 + 1)$-dimensional Minkowski space-time, and *two-dimensional conformal field theory*.

1. From the point of view of *covariant quantization*, introduced at the time of Wigner back in the 1930s [1], *elementary particles* of relativistic quantum mechanics (of positive mass, say) may be described as irreducible unitary representations of the Poincaré group $P_4 \simeq SO_0(3,1) \ltimes \mathbb{R}^4$, the group of affine isometries of Minkowski space-time, or in other words the semi-direct product of the Lorentz group of rotations and relativistic boosts by space-time translations; the physical states of a particle of mass $m > 0$ and spin $s \in \frac{1}{2}\mathbb{N}$ are in bijection with the states of the Hilbert space corresponding to the associated irreducible representation of P_4.

[1] see e.g. references and a history of the subject in: S. Weinberg. *The quantum theory of fields.*, Cambridge University Press (1996).

This *covariant quantization* was revisited by the school of Souriau in the 1960s and 1970s [2] as a particular case of *geometric quantization*. The latter is a piece of a wide program of geometrization of classical and quantum mechanics. It allows a geometric construction of a mapping from *classical observables* on a symplectic manifold $\mathscr{M}$ (the *phase space* of Hamiltonian mechanics) to operators (that is, *quantum observables*) on L^2-sections of a fiber bundle obtained from $\mathscr{M}$ by polarization. In most physical cases, there is a non-trivial Lie group of symmetries of the symplectic manifold, which is represented in this framework by unitary operators. As for the electromagnetic field, it appears as a perturbation of the underlying symplectic structure; this principle is described by Souriau as *symplectic materialism*. One of the main outcomes of this program is a general method for constructing *wave equations* that are *covariant under the action of a group of symmetries* preserving a certain particular *geometric structure*.

2. *Two-dimensional conformal field theory*, on the other hand, is an attempt at understanding the universal behaviour of two-dimensional statistical systems at *equilibrium* and at the *critical temperature*, where they undergo a second-order phase transition (see the Preface for details and references). Starting from the basic assumption of translational and rotational invariance, together with the fundamental hypothesis (confirmed by the observation of the fractal structure of the systems and the existence of long-range correlations, and made into a cornerstone of renormalization-group theory) that scale invariance holds at criticality, one is [3] naturally led to the idea that invariance under the whole conformal group $\mathrm{Conf}(d)$ should also hold. This group is known to be finite-dimensional as soon as the space dimension d is larger than or equal to three, so physicists became very interested in dimension $d = 2$, where *local conformal transformations* are given by *holomorphic or anti-holomorphic functions*. A systematic investigation of the theory of *representations* of the *Virasoro algebra* (considered as a *central extension* of the algebra of infinitesimal holomorphic transformations) in the 1980s led to introduce a class of physical models (called *unitary minimal models*), corresponding to *degenerate unitary highest weight representations* of the Virasoro algebra with central charge less than one. Miraculously, covariance alone is enough to allow the computation of the statistic correlations – or so-called *n-point functions* – for these highly constrained models.

   Let us emphasize in particular the following well-known facts, to which we shall refer several times in the sequel. Covariance under projective transformations (or homographies) $z \mapsto \frac{az+b}{cz+d}$ fixes up to a constant two- and three-point functions. On the other hand, four-point functions $\langle \phi_1(z_1) \ldots \phi_4(z_4) \rangle$ are fixed only up to a *scaling function* depending on the cross-ratio $\frac{(z_1-z_3)/(z_1-z_4)}{(z_2-z_3)/(z_2-z_4)}$.

[2] J.-M. Souriau. *Structure des systèmes dynamiques*. Maîtrises de mathématiques, Dunod, Paris (1970).

[3] For systems with sufficiently short-ranged interactions (see Preface for details and counterexamples).

* * *

The *Schrödinger–Virasoro algebra* was originally discovered by M. Henkel [4] in 1994 while he was trying to apply the concepts and methods of conformal field theory to models of statistical physics which either undergo a dynamics, whether in or out of equilibrium, or are no longer isotropic. The idea was that, contrary to the case of relativistic physics or conformal field theory, the different coordinates (called for convenience *time* and *space*) should not play the same rôle. Replacing *isotropic scale transformations* $\boldsymbol{r} \mapsto \lambda \boldsymbol{r}$ with *anisotropic dilatations* $(t, \boldsymbol{r}) \mapsto (\lambda^z t, \lambda \boldsymbol{r}), \lambda > 0$ with $z \neq 1$ changes fundamentally the geometry. Then natural questions arise, such as: Is there anything like conformal geometry for $z \neq 1$? Does there exist a notion of local scale invariance in low dimensions as in the case of conformal field theory?

It turns out that the answer to the first question is *positive* for $z = 2$. The geometric theory has been developed by C. Duval, H. Künzle, P. Horváthy and other authors [5]. Lorentzian geometry has to be replaced with the so-called *Newton-Cartan geometry* in $(1 + d)$ dimensions, which is the right geometric framework for Newtonian (instead of relativistic) mechanics, defined by a one-form (locally written dt) representing the time-coordinate, by a metric structure on the fibers $t = \mathrm{Cst}$, and by a (alas non unique) connection preserving these two. Various covariant wave equations may be obtained using the tools of geometric quantization, among which the two simplest ones: the *free Schrödinger equation* $\Delta_0\psi := (-2\mathrm{i}\mathscr{M}\partial_t - \Delta_r)\psi = 0$, and the *Dirac-Lévy Leblond equation*, which is associated to the Schrödinger equation in the same way as the Dirac equation to the Laplace equation $\Delta_r\psi = 0$ in the relativistic setting [6].

Contrary to the conformal case, there are several groups of symmetries associated to the Newton-Cartan geometry, among which (by increasing order with respect to inclusion) the *Galilei group*, the *Schrödinger group*, and also (by weakening the assumptions though) the *Schrödinger–Virasoro group*.

The *Galilei group* $\mathrm{Gal}(d)$ is the group generated by time translations and space rotations, and by motions with constant speed. It is the symmetry group of classical mechanics; the induced changes of frames leave invariant the equations of classical physics. It also leaves invariant the Euler equation for perfect, incompressible fluids without viscosity, obtained from Newton's equation of dynamics by a shift of point of view from Lagrangian to Eulerian mechanics.

The *Schrödinger group* $\mathrm{Sch}(d)$ is the group of *projective Lie symmetries* of the free Schrödinger equation $\Delta_0\psi(t,r) := (-2\mathrm{i}\mathscr{M}\partial_t - \Delta_r)\psi(t,r) = 0$, see Preface, including a subgroup of *time-homographies coupled with space-transformations*

---

[4] M. Henkel. *Schrödinger invariance and strongly anisotropic critical systems*, J. Stat. Phys. **75**, 1023 (1994).

[5] See references to articles by these authors in Chap. 1.

[6] J.-M. Lévy-Leblond. *Nonrelativistic particles and wave equations*, Comm. Math. Phys. **6**, 286 (1967).

that is isomorphic to the group of two-by-two matrices with determinant one, $SL(2,\mathbb{R})$. It contains the *Bargmann group*, which is a central extension of the Galilei group. In physical applications, the value of the central element $M_0 \in \mathfrak{sch}(d) = Lie(\mathrm{Sch}(d))$ is interpreted as the mass of the particle or the total mass of the system. The Schrödinger group also preserves a number of field equations coming either from non-relativistic condensed matter physics (e.g. Landau liquids or Bose-Einstein condensation, see 5. in the Preface) or from a direct application of the principles of geometric quantization (e.g. non-relativistic electromagnetism, see 6. in the Preface).

The *Schrödinger–Virasoro group* $\mathrm{SV}(d)$ is obtained by removing the requirement that symmetries should preserve the particular choice of connection. One then obtains *for any space dimensionality d* an infinite-dimensional Lie group, with Lie algebra $\mathfrak{sv}_d$ given by (8) in the Preface. Details are given in Chap. 1. We concentrate in this book on the particular case $d = 1$. Then $\mathrm{SV} = \mathrm{SV}(1)$ is a *semi-direct product* of $\mathrm{Diff}(S^1)$ – the *group of diffeomorphisms of the circle* – by a rank-2 infinite-dimensional nilpotent group, $\mathrm{SV} \simeq \mathrm{Diff}(S^1) \ltimes H$, with typical element denoted by $(\phi;(\alpha,\beta))$, $\phi \in \mathrm{Diff}(S^1)$, $\alpha,\beta \in C^\infty(S^1)$ [7]. By centrally extending $\mathrm{Diff}(S^1)$ to the Virasoro group Vir, one may also consider the group $\widetilde{\mathrm{SV}} = \mathrm{Vir} \ltimes H$ and its Lie algebra $\widetilde{\mathfrak{sv}} = \mathfrak{vir} \ltimes \mathfrak{h}$, whose center is generated by the mass generator M_0 and by the central generator in $\mathfrak{vir}$.

Schrödinger transformations in $\mathrm{Sch} \subset \mathrm{SV}$ play the same rôle as projective transformations in conformal invariance. Computations show that covariance under Schrödinger transformations only fixes two-point functions, which are given up to a constant by the heat kernel. In three-point functions, contrary to the conformal case (see above), an arbitrary scaling function of some complicated expression in terms of the coordinates appears. The consequences of the postulate of covariance under the Schrödinger group or subgroups of it have been explored quite systematically by M. Henkel and his collaborators [8], and proved analytically or observed numerically on several models, in particular in cases where *physical ageing* sets in. A more detailed account is given in the Preface. While these are interesting, the theory is undeniably not as far-reaching as 2d-conformal field theory because the Schrödinger group is *finite-dimensional*. On the other hand, the *Schrödinger–Virasoro group* may seem at first sight to be the right candidate for local-scale invariance. However, these symmetries have not yet been observed on physical models. Maybe this approach is doomed to fail because it is even difficult to find wave equations invariant under the Schrödinger–Virasoro group [9].

* * *

[7] The elements in $SL(2,\mathbb{R})$ coming from Sch(1) represent the finite projective transformations in the $\mathrm{Diff}(S^1)$-factor.

[8] See M. Henkel, M. Pleimling. *Nonequilibrium phase transitions. Vol. 2, Ageing and dynamical scaling far from equilibrum*, Springer (2010).

[9] R. Cherniha, M. Henkel. *On non-linear partial differential equations with an infinite-dimensional conditional symmetry*, J. Math. Anal. Appl. **298**, 487 (2004).

While the above approach, developed in Chap. 1, has led, for the time being, neither to further developments nor to applications, other fruitful points of view on the Schrödinger–Virasoro group have gradually emerged.

1. The approach closest to the previous one is to see the Schrödinger–Virasoro group not as a symmetry group of a given wave equation, but as a *reparametrization group* for a given class of equations. This way of seeing things is well-known in the case of the Virasoro group: the *group of diffeomorphisms of the circle*, $\mathrm{Diff}(S^1)$, acts on the space of *Hill operators* $\partial_x^2 + u_0(x)$, $x \in \mathbb{R}/2\pi\mathbb{Z}$, otherwise known as periodic Sturm-Liouville operators on the line. This affine action, σ^{Hill}, is a left-and-right action,

$$\phi \mapsto \left(\partial_x^2 + u_0(x) \mapsto \sigma^{Hill}(\phi)(\partial_x^2 + u_0(x)) := \pi_{3/2}^{Hill}(\phi) \circ (\partial_x^2 + u_0(x)) \circ \pi_{-\frac{1}{2}}^{Hill}(\phi)^{-1}\right), \tag{1}$$

   where $\pi_\lambda^{Hill}(\phi)u_0(x) := (\frac{d\phi}{dx})^\lambda (u_0 \circ \phi^{-1})(x)$ is the natural action of $\mathrm{Diff}(S^1)$ on the space of $(-\lambda)$-densities, also called (without specifying the weight) *tensor densities*. The orbits of this action have been classified independently by A.A. Kirillov [10] on the one hand, and by V.F. Lazutkin and T.F. Pankratova [11] on the other. A.A. Kirillov actually deduces a set of *normal forms* for the orbits from a classification up to conjugacy of all possible generators of the symmetry subgroups (also called *isotropy subgroups*) of Sturm-Liouville operators. The main characteristic of a given orbit is its *monodromy*, namely, the Floquet matrix in $SL(2,\mathbb{R})$ relating $\psi(x)$ to $\psi(x+2\pi)$, where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is the general solution of the Hill equation $(\partial_x^2 + u_0)\psi(x) = 0$. The eigenvalues of this matrix give the behaviour at infinity of the solutions. Another important characteristic is the oscillatory character of the solutions [12]. Generic orbits are of *type I* (*elliptic type* [13] with *oscillatory solutions*, or *hyperbolic type* with *non-oscillatory* solutions) or of *type II* (*hyperbolic type*, with *oscillatory solutions*), but there are also non-generic orbits of *type III*, with *unipotent monodromy*.

   The above analysis carries over very nicely to the space of *generalized harmonic oscillators*, which are time-dependent Schrödinger operators in $(1+1)$-

[10] A.A. Kirillov. *Infinite-dimensional Lie groups: their orbits, invariants and representations. The geometry of moments*, Lecture Notes in Mathematics **970**, 101–123 (1982).

[11] V.F. Lazutkin, T.F. Pankratova. *Normal forms and versal deformations for Hill's equations*, Funct. Anal. Appl. **9** (4), 306–311 (1975).

[12] *Oscillatory solutions* have an infinite number of zeroes, *non-oscillatory solutions* have at most one zero.

[13] Two-by two matrices of determinant one with eigenvalues $e^{i\theta}$, $0 < \theta < \pi$, resp. $e^{\pm t}$, $t > 0$, resp. ± 1, are called *elliptic*, resp. *hyperbolic*, resp. *unipotent*.

dimensions of a very special type,

$$\mathscr{S}^{aff}_{\le 2} := \left\{-2\mathrm{i}\mathscr{M}\partial_t - \partial_r^2 + V_0(t) + V_1(t)r + V_2(t)r^2\right\}, \tag{2}$$

with time-dependent periodic quadratic potential. This space is preserved by the following affine action of the Schrödinger–Virasoro group,

$$\sigma_{1/4} : (\phi; (\alpha, \beta)) \mapsto \left(D \mapsto \pi_{5/4}(\phi; (\alpha, \beta)) \circ D \circ \pi_{1/4}(\phi; (\alpha, \beta))^{-1}\right), \tag{3}$$

where the *vector-field representation* π_λ exponentiates the realization $d\pi_\lambda$ of SV found by extrapolation by M. Henkel, as explained in the Preface (formulas are recalled at the end of the Introduction). We summarize here the contents of Chap. 9. In the semi-classical limit, these operators give back the Hill operators, which explains why part of this action is equivalent to σ^{Hill}. The new feature here is the existence of an exact *invariant* I, sometimes called *Ermakov-Lewis invariant* [14] – a time-dependent second-order differential operator in the space coordinates, commuting with the Schrödinger operator –. When I has a discrete spectrum, the solutions of the Schrödinger operator are the eigenfunctions of I, multiplied by an explicit time-dependent *phase* yielding the *monodromy operator* – a bounded operator on $L^2(\mathbb{R})$ this time – ; contrary to the usual *adiabatic scheme* [15], these results are non-perturbative. Let us now come to our results:

- the orbit classification due to A.A. Kirillov carries over to the case of generalized harmonic oscillators for *generic* orbits. There also appear supplementary, *non-generic* orbits of type I or type III, due to a *resonance* between the *quadratic* and the *linear parts of the potential.* A choice of normal forms for these orbits, together with the associated isotropy subgroups, is given in Sect. 9.2.4. The isotropy subgroups are shown to be isomorphic to subgroups of the Schrödinger group or of some covering of it.
- the *invariant* I may be reconstructed from the *orbital data*, thus leading to a *unification of the algebraic, geometric and analytic methods*. One may show that I is conjugate to some simple *model operator* belonging to a finite family of operators [16].
- even when the spectrum of I is non-discrete (so that the standard adiabatic scheme is not valid), the monodromy operator is shown to be essentially conjugate to a *multiplication operator* $L^2(\Sigma) \to L^2(\Sigma),\ f(\lambda) \mapsto e^{\mathrm{i}\lambda T} f(\lambda)$,

[14] H.R. Lewis, W.B. Riesenfeld. *An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field*, J. Math. Phys. **10** (8), 1458–1473 (1969).

[15] See e.g. A. Joye, *Geometric and mathematical aspects of the adiabatic theorem of quantum mechanics*, Ph. D. Thesis, Ecole Polytechnique Fédérale de Lausanne (1992).

[16] Namely, the *free Laplacian*, the *harmonic oscillator*, the *"harmonic repulsor"* and the *Airy operator*.

where $\Sigma \subset \mathbb{R}$, $\Sigma = \frac{1}{2} + \mathbb{N}$, $\mathbb{R}$ or $\mathbb{R}_+$ is the *spectrum* of I, and T is the *period* of the orbit.

2. The above action σ^{Hill} on Hill operators equivalent to the *coadjoint action* of the Virasoro group, and thus Hamiltonian for the *Kirillov-Kostant-Souriau Poisson structure* on the dual of the Virasoro algebra. The latter is the simplest of a family of compatible (i.e. mutually commuting) infinite-dimensional Poisson structures, a so-called *hierarchy*. Such structures usually come up with infinite families of Hamiltonian equations induced by commuting Hamiltonian operators, thus implying their complete integrability. This is the most efficient scheme to construct completely integrable systems. In the infinite-dimensional case, one gets *integrable partial differential equations* (rather than ordinary differential equations), the simplest one in our context being the *Korteweg-De Vries equation* $\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$, which describes an *isospectral deformation* $\partial_x^2 + u_0(x) \rightsquigarrow \partial_x^2 + u(t,x)$ of a Hill operator [17]. The Poisson-Lie group Poisson bracket on the *Volterra group* integrating the *algebra of formal pseudo-differential symbols* of the type $u_{-1}(x)\partial_x^{-1} + u_{-2}(x)\partial_x^{-2} + \ldots$ leads by *symplectic reduction* to the same equation.

   Some but not all of these arguments may be reproduced in the case of the *linear action*

$$\tilde{\sigma}_\mu : (\phi; (\alpha,\beta)) \mapsto \big(D \mapsto \pi_{\mu+2}(\phi;(\alpha,\beta)) \circ D \circ \pi_\mu(\phi;(\alpha,\beta))^{-1}\big) \qquad (4)$$

   on the *linear space* of *time-dependent periodic Schrödinger operators*, $\mathscr{S}^{lin} := \{a(t)(-2\mathrm{i}\mathscr{M}\partial_t - \partial_r^2) + V(t,r)\}$; see Chap. 10. Note that, because the index μ has been shifted by two instead of one, compare with (3), the associated *affine space* $\mathscr{S}^{aff} := \{-2\mathrm{i}\mathscr{M}\partial_t - \partial_r^2 + V(t,r)\}$ is *not* preserved by this action. Contrary to the case of Hill operators, this action is unrelated to the coadjoint action of $\mathfrak{sv}$. On the other hand, it may be retrieved by symplectic reduction from the *current algebra* over a Volterra-type space of formal pseudo-differential symbols. Thus $\mathscr{S}^{lin}$ appears as a coadjoint orbit of a huge looped group integrating this current algebra; the $\tilde{\sigma}_\mu$-actions are in this sense coadjoint actions, and are Hamiltonian for the usual Kirillov-Kostant-Souriau bracket (see Theorem 10.2 in Sect. 10.6). This suggests of course to look for related integrable systems.
3. The above results point out to the importance of the point of view of *Poisson geometry* (see Chaps. 2, 10, 11). They rely on the embedding of the Schrödinger–Virasoro algebra as a *subquotient* of the *extended Poisson algebra* [18] on the two-dimensional torus, $\mathbb{C}[q,q^{-1}][p^{\frac{1}{2}},p^{-\frac{1}{2}}]$ with Poisson bracket $\{F,G\} =$

[17] For a beautiful introduction to the subject, see e.g. P.G. Drazin and R.S. Johnson, *Solitons: an introduction*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge (1989), or the monography by L. Guieu and C. Roger cited in the bibliography.

[18] Unfortunately, the natural embedding of $\mathrm{Vect}(S^1) \subset \mathfrak{sv}$ into the Poisson algebra does not extend to $\mathfrak{sv}$.

$\frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}$, or of its natural quantization, the algebra of *formal pseudo-differential symbols* on the line $\mathbb{C}[\xi,\xi^{-1}][\partial_\xi^{\frac{1}{2}},\partial_\xi^{-\frac{1}{2}}]]$ with Lie bracket induced by the natural associative product of operators. This point of view may be generalized to a supersymmetric setting, yielding a large class of *superizations of the Schrödinger–Virasoro algebra* $\mathfrak{sns}^{(N)}$ called *Schrödinger-Neveu-Schwarz algebras*, containing algebras of *super-contact* vector fields (often called *super-conformal algebras* in the literature), namely, algebras of super-vector fields preserving the kernel of the super-contact form $dq + \sum_{i=1}^N \theta^i d\theta^i$. These arguments will be developed in Chap. 11 (see Definition 11.20 in Sect. 11.3). Similarly to what happened originally with the Schrödinger–Virasoro algebra, its N = 2 supersymmetric generalization, $\mathfrak{sns}^{(2)}$ extending the *Neveu-Schwarz algebra* [19], arises as a natural infinite-dimensional extrapolation of the symmetry generators of a *supersymmetric Schrödinger equation* (see Sect. 11.2.1), obtained from the *supersymmetric model* on super-space-time $\mathbb{R}^{(3|2)}$ with coordinates $(t,\zeta,r;\theta_1,\theta_2)$ by a Fourier transform with respect to ζ, formally $\partial_\zeta \rightsquigarrow \mathrm{i}\mathcal{M}$ (see 8. in the Preface for the analogous construction relating the Klein-Gordon equation in $(d+2)$-dimensions to the Schrödinger equation $(-2\mathrm{i}\mathcal{M}\partial_t - \Delta_r)\psi_{\mathcal{M}}(t,r) = 0$ in $(d+1)$-dimensions).

* * *

Leaving aside the above geometrical and physical aspects, which may be considered as a motivation or maybe as applications, one may also study the *algebraic properties* of this infinite-dimensional Lie algebra for its own sake, which is the subject of Chaps. 3–7.

The Schrödinger–Virasoro algebra $\mathfrak{sv}$ is a semi-direct product of the *centerless Virasoro algebra* $\mathrm{Vect}(S^1) \simeq \langle L_n, n \in \mathbb{Z}\rangle$ – seen as the Lie algebra of *smooth vector fields on the circle*, and naturally identified with the Lie algebra of the diffeomorphism group $\mathrm{Diff}(S^1)$ introduced above – by an infinite-dimensional, rank 2 nilpotent Lie algebra

$$\mathfrak{h} \simeq \left\langle Y_m, M_p, m \in \frac{1}{2}+\mathbb{Z}, p \in \mathbb{Z}\right\rangle \tag{5}$$

In other words, $\mathfrak{sv} \simeq \mathrm{Vect}(S^1) \ltimes \mathfrak{h}$ where $[\mathfrak{h},[\mathfrak{h},\mathfrak{h}]] = 0$. The generic element may be written in terms of three Laurent series [20],

$$\mathcal{L}_f + \mathcal{Y}_g + \mathcal{M}_h := \sum_{n\in\mathbb{Z}} f_n L_n + \sum_{m\in\frac{1}{2}+\mathbb{Z}} g_m Y_m + \sum_{p\in\mathbb{Z}} h_p M_p, \tag{6}$$

where $f(z) = \sum f_n z^{n+1}$, $g(z) = \sum g_m z^{m+\frac{1}{2}}$ and $h(z) = \sum h_p z^p$.

---

[19] A. Neveu and J.H. Schwarz. *Factorizable dual model of pions*, Nucl. Phys. **31**, 86 (1971).

[20] We then use *calligraphic letters* to avoid any confusion with the Laurent components.

Semi-direct products of the type $\mathrm{Vect}(S^1) \ltimes \mathcal{F}_\lambda$, where $\mathcal{F}_\lambda$ is a *tensor density module* of $\mathrm{Vect}(S^1)$ – the semi-classical analogue of a *primary operator* of weight $\lambda + 1$ in the language of conformal field theory –, have been studied in great details [21], in particular from the cohomological point of view. The Lie algebra $\mathfrak{sv}$, as a $\mathrm{Vect}(S^1)$-module, is isomorphic to $\mathrm{Vect}(S^1) \ltimes (\mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_0)$, *but* $\mathfrak{h} \simeq \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_0$ (as a vector space) is *not* abelian. Hence $\mathfrak{sv}$ may be viewed as the next example in the order of increasing complexity. The techniques developed by D.B. Fuks [22] also apply to the case of $\mathfrak{sv}$ and allow a *detailed* and almost exhaustive *cohomological study* (mainly concerning *deformations* and *central extensions*) in Chap. 7. We show in particular (see Theorem 7.2 in Sect. 7.2) that there exist exactly *three independent families of deformations* of the bracket of $\mathfrak{sv}$, among which a family denoted by $\mathfrak{sv}_\epsilon$, which is isomorphic to $\mathrm{Vect}(S^1) \ltimes (\mathcal{F}_{\frac{1}{2}+\epsilon} \oplus \mathcal{F}_{2\epsilon})$ as a $\mathrm{Vect}(S^1)$-module.

Chapters 3–6 are concerned with the study of different classes of representations of $\mathfrak{sv}$, inspired by the well-developed Virasoro theory: *unitary highest-weight modules* (or *induced representations*, or *Verma modules*), *coinduced representations*, and more specifically the *coadjoint representation*. Let us comment on each of these.

1. *Unitary highest-weight modules* (see Chap. 4) are induced from a *character* of the commutative algebra $\langle L_0, M_0 \rangle$, and characterized by their *mass* and by the *conformal weight* of their highest-weight vector. As opposed to the Virasoro case, non-trivial unitary highest-weight modules are all *non-degenerate* (see Theorem 4.2 in Sect. 4.1), so the theory seems to be of little interest.
2. *Coinduced representations*, on the other hand, provide an interesting *class of representations generalizing* the $\mathrm{Vect}(S^1)$-*tensor density modules*; one finds the following explicit formulas in Chap. 5, see Theorem 5.3 in Sect. 5.1,

$$d\tilde{\rho}(\mathcal{L}_f) = \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}f''(t)r^2\partial_\zeta\right) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)d\rho(L_0)$$
$$+\frac{1}{2}f''(t)rd\rho(Y_{\frac{1}{2}}) + \frac{1}{4}f'''(t)r^2 d\rho(M_1); \tag{7}$$

$$d\tilde{\rho}(\mathcal{Y}_f) = \left(-f(t)\partial_r - f'(t)r\partial_\zeta\right) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)d\rho(Y_{\frac{1}{2}}) + f''(t)r\, d\rho(M_1); \tag{8}$$

$$d\tilde{\rho}(\mathcal{M}_f) = -f(t)\partial_\zeta \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)\, d\rho(M_1). \tag{9}$$

[21] T. Tsujishita, *On the continuous cohomology of the Lie algebra of vector fields*, Proc. Japan Acad. Ser. A Math. Sci. **53** (4), 134–138 (1977), and V. Ovsienko, C. Roger. *Generalizations of Virasoro group and Virasoro algebra through extensions by modules of tensor-densities on* S^1, Indag. Math. (N.S.) **9** (2), 277–288 (1998).

[22] D.B. Fuks. *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York (1986).

where $\rho : \mathfrak{g}_0 \to Hom(\mathscr{H}_\rho, \mathscr{H}_\rho)$ is any representation of the solvable Lie subalgebra $\mathfrak{g}_0 := \langle L_0, Y_{\frac{1}{2}}, M_1 \rangle$. The known realizations of $\mathfrak{sv}$ as a *reparametrization group* of *wave equations* – in particular, of *Schrödinger* or *Dirac-Lévy-Leblond operators*, see Chap. 8 – all belong to this family. Such explicit formulas are easily obtained by taking into consideration the *Cartan prolongation structure* of $\mathfrak{sv}$ (see Theorem 5.1 in Sect. 5.1), based on the graduation of $\mathfrak{sv}$ given by the polynomial degree of the vector fields in the vector-field representation.

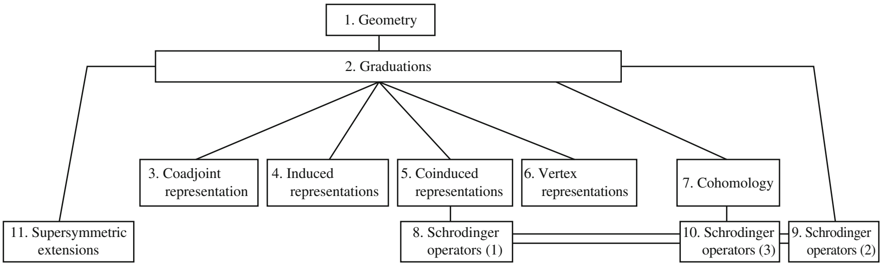
3. *Vertex representations* are defined in Chap. 6. Recall first that *primary operators* for vertex representations of the Virasoro algebra are *quantum fields* $X = \sum_n X_n z^n$ realized on a *Fock space*, whose Laurent components X_n are *quantized tensor densities*, i.e. they satisfy the commutation relations $[L_n, X_m] = (\lambda n - m) X_{n+m}$ for some λ; physicists call $\lambda + 1$ the *conformal weight* of X. Regarding time-and-space dependent operators as Laurent series in the *time coordinate* leads to a natural transposition of these notions to $\mathfrak{sv}$-*primary fields*. The operators defined in Chap. 6 are *primary* with respect to some of the above-defined coinduced representations. The *polynomial fields* which have been successfully constructed (see Theorems 6.6 and 6.7 in Sect. 6.3.2) are unfortunately *degenerate*, in the sense that the action of M_0 is *nilpotent*. Hence M_0 may not be interpreted as a *mass*. We conjecture though, see Sect. 6.5, the existence of *massive fields* – on which M_0 acts as a non-trivial scalar – and compute by a formal analytic extension their *two-* and *three-point functions*, see Theorems 6.9 and 6.10. The latter are different from but closely connected to the *conformally covariant three-point functions* in three dimensions. As shown in Chap. 2, although $\mathfrak{sch} \subset \mathfrak{conf}(3)_{\mathbb{C}}$ on the one hand, and $\mathfrak{sch} \subset \mathfrak{sv}$ on the other, there seems to be no reasonable way to combine conformal *and* Schrödinger–Virasoro symmetries, which makes the above results puzzling.

   Though these preliminary results may probably be extended, it would be reasonable to try first to work out a physical context in which these quantum fields would appear, in order to gain some intuition and to go beyond what may appear for the moment as a somewhat formal exercise.

4. Finally, the *coadjoint representation* (contrary to the Virasoro case) does not belong to the above family of representations, and is studied separately in Chap. 3. As in the case of the coadjoint action of the Virasoro group, the study of the isotropy subalgebras lead to differential equations which are easily solved[23]. The reader acquainted with the classification (obtained independently by O. Mathieu on the one hand and by C. Martin and A. Piard on the other [24]) of the representations of the Virasoro algebra may wonder whether the coadjoint

[23] A.A. Kirillov, op. cit., and the monography by L. Guieu and C. Roger, op. cit.

[24] C. Martin, A. Piard. *Classification of the indecomposable bounded admissible modules over the Virasoro Lie algebra with weight spaces of dimension not exceeding two*, Comm. Math. Phys. **150**, 465–493 (1992); and O. Mathieu, *Classification of Harish-Chandra modules over the Virasoro Lie algebra*, Invent. Math. **107**, 225–234 (1992).



representation belongs to a new class of its own. For the moment, however, it looks isolated in the picture.

Let us give some suggestions for reading. Chapters 1 and 2 are introductive; definitions and results contained in these two chapters are frequently needed, and are quoted throughout the book. The cohomological results contained in Chap. 7 are not required for the other chapters (the deformations and central extensions used elsewhere, in particular in the chapters concerned with representation theory, are all introduced in Chap. 2), but the spectral sequence method used to compute central extensions of semi-direct products, together with Fuks' results on the Virasoro cohomology, are used once more in Chap. 10. Chapters 8–10, devoted to Schrödinger operators, have a strong thematic unity despite the differences in the methods and the language; an introductory discussion is placed at the beginning of Chap. 8, and Sect. 8.1 should be read first. Chapters 3–6 on representation theory also have a strong thematic unity, but may be read separately. Chapter 11 on supergeometry is an extension to the supersymmetric setting of Chap. 2.

The following diagram shows some possible orders of reading.

Almost all the material contained in this monograph has been published elsewhere, save for the introductory parts and Chap. 4. The first steps into Newton-Cartan geometry in Chap. 1 are a short summary of a theory developed by Duval, Horváthy, Künzle,... Chap. 9 includes a detailed account of classical results about Ermakov-Lewis quantum invariants and about Hill operators, in particular the orbit classification due to A.A. Kirillov; the classification of Hill operators by their lifted monodromy (Sect. 9.2.2), and the solution of the Hill equations and the determination of their monodromy in terms of the ξ-invariant in the isotropy subalgebra (Sect. 9.3.2) should be folklore results, but we have not found them in the literature. Apart from these, all other developments are due either to one of the authors or to both – including contributions of the author of the Preface.

The reader is assumed to have a background knowledge on the Virasoro algebra, with its applications to conformal field theory (for Chap. 6 mainly) and to integrable systems. Reading some sections of the frequently cited monograph *L'Algèbre et le Groupe de Virasoro: aspects géométriques et algébriques, généralisations* by L. Guieu and C. Roger [43] may help him find his way around. The two volumes of the (recently appeared) book *Nonequilibrium phase transitions* by M. Henkel, H. Hinrichsen, S. Lübeck and M. Pleimling [57, 58] give in particular an up-to-date overview of Schrödinger invariance in statistical physics – a good complement to this monograph.

* * *

Let us finally write down, for the convenience of the reader, a few formulas concerning the Schrödinger–Virasoro algebra and its *vector-field representations*, which are constantly used in the book.

Generators of the Schrödinger–Virasoro algebra

$$\mathfrak{sv} = \langle L_n,\ n \in \mathbb{Z}\rangle \ltimes \left\langle Y_m, M_p,\ m \in \frac{1}{2} + \mathbb{Z},\ p \in \mathbb{Z}\right\rangle \simeq \mathrm{Vect}(S^1) \ltimes \mathfrak{h} \qquad (10)$$

Commutation relations of the generators

$$[L_n, L_m] = (n-m)L_{n+m}; \quad [L_n, Y_m] = \left(\frac{n}{2} - m\right) Y_{n+m}, \quad [L_n, M_p] = -pM_{n+p} \qquad (11)$$

and

$$[Y_{m_1}, Y_{m_2}] = (m_1 - m_2)M_{m_1+m_2},\ [Y_m, M_p] = [M_{p_1}, M_{p_2}] = 0 \qquad (12)$$

with $n, p, p_1, p_2 \in \mathbb{Z}$ and $m, m_1, m_2 \in \frac{1}{2} + \mathbb{Z}$.

Vector-field representation: massive version $d\pi_\lambda$

One defines

$$d\pi_\lambda(L_n) = -t^{n+1}\partial_t - \lambda(n+1)t^n - \frac{1}{2}(n+1)t^n r\partial_r - \frac{\mathscr{M}}{4}(n+1)nt^{n-1}r^2$$

$$d\pi_\lambda(Y_m) = -t^{m+\frac{1}{2}}\partial_r - \mathscr{M}(m+\frac{1}{2})t^{m-\frac{1}{2}}r \qquad (13)$$

$$d\pi_\lambda(M_n) = -\mathscr{M}t^n \qquad (14)$$

where $\mathscr{M}$ is an arbitrary (non-zero) constant which plays the role of a mass parameter. These formulas agree with the original realization of $\mathfrak{sv}$ found by M. Henkel, see (8) in the Foreword, if one sets $\lambda = x/2$, where x is the *scaling dimension* of the field.

Vector-field representation: Fourier version $d\tilde{\pi}_\lambda$

$$\begin{aligned}
d\tilde{\pi}_\lambda(L_n) &= -t^{n+1}\partial_t - \lambda(n+1)t^n - \frac{1}{2}(n+1)t^n r\partial_r - \frac{1}{4}(n+1)nt^{n-1}r^2\partial_\zeta \\
d\tilde{\pi}_\lambda(Y_m) &= -t^{m+\frac{1}{2}}\partial_r - \left(m+\frac{1}{2}\right)t^{m-\frac{1}{2}}r\partial_\zeta \\
d\tilde{\pi}_\lambda(M_n) &= -t^n\partial_\zeta
\end{aligned} \tag{15}$$

Both families of representations are called *vector field representations* of $\mathfrak{sv}$. As a matter of fact, the representation $d\tilde{\pi}_\lambda$ may be deduced from $d\pi_\lambda$ by a formal Laplace transform with respect to the parameter $\mathscr{M}$.

Sometimes the scaling dimension is unimportant. In such cases we simply write sometimes $d\pi$ or $d\tilde{\pi}$.

Villeurbanne Cedex — *Claude Roger*
Vandoeuvre-les-Nancy — *Jérémie Unterberger*

Springer

http://www.springer.com/978-3-642-22716-5

The Schrödinger-Virasoro Algebra
Mathematical structure and dynamical Schrödinger symmetries
Unterberger, J.; Roger, C.
2012, XLII, 302 p., Hardcover
ISBN: 978-3-642-22716-5