Chapter 2
Mathematical Prerequisites

Before entering in the field of nonlinear waves on closed contours and surfaces we need to recall some useful mathematical concepts. The *cnoidal waves*, *solitary waves*, and *solitons* are solutions of nonlinear equations that could be partial differential (PDE), integro-differential, finite difference-differential, or even functional equations. They describe the evolution of the wave solutions in space and time. These nonlinear equations are usually coupled with linear or nonlinear boundary conditions (BC), initial conditions, or asymptotic conditions. The properties of solutions are dependent on the topological and geometrical structure of the space on which they are defined. In the following we assume for the reader to be familiar with the general concept of group, Abelian group, quotient group, rank of a group, and group homomorphism.

2.1 Elements of Topology

In this section we introduce some elements of topology related to the idea of boundary [68, 160, 274, 291, 344]. Some working theorems are very important and their generality raises sometimes the question: "how is this possible?" The following few sections try to reveal a little bit of the insights of such properties. When we investigate a space from the topological point of view, the basic questions are: how large, how dense, how tight, or how fuzzy is such a space? In Table 2.1, we present how topology addresses these questions. A topological space (X, τ) is a set X and a family $X, \emptyset \in \tau \subset \mathcal{P}_X$ of *open sets* stable against finite intersection and arbitrary reunions. The complement of any open set is *closed*. To any point $x \in X$ we can associate a family $\mathcal{V}$ of neighborhoods of x, $V(x) \in \mathcal{V}$ defined by the property $V(x) \in \mathcal{V}$ if $\exists A \in \tau, x \in A \subset V(x)$. A family of open sets in (X, τ) is called base if any open set of the topology is a reunion of sets in that family. A point $x \in X$ is called adherent if $\forall V(x), V(x) \cap A \neq \emptyset$.

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Table 2.1 Properties of topological spaces

Question	Topological property (or invariant)
How large?	Compactness
How fuzzy?	Separation
How many pieces?	Connectedness
How complicated?	Separability
How much measurable?	Metric space

A closed set contains all its adherent points. An adherent point is the rudiment of the concept of limit. We need the following definitions:

$$\text{int}\, A = \overset{\circ}{A} = \{x \in A |\, \text{exists}\ D \in \tau, x \in D \subset A\},\ \text{interior of}\ A,$$

$$\overline{A} = A \cup \{x \in X | x\ \text{adherent point to}\ A\},\ \text{closure of}\ A,$$

$$\partial A = \overline{A} - \ \text{int}\, A, \quad \text{boundary of}\ A.$$

The open property of a set is relative to the topology of the space. For example, the real interval (a, b) is open in the usual metric topology on $\mathbb{R}$, but it is neither open nor closed in the plane $\mathbb{R}^2$, while a loop is closed both in $\mathbb{R}^2$ and $\mathbb{R}^3$. A family $B_\alpha \in B \subset \tau$ with the property that $\forall D \in \tau, D = \cup B_\alpha$ is called a *base*. A set $A \subset X$ with the property $\overline{A} = X$ is called *dense* in X. A space with countable base is called *separable*. A topological space which is also a vector space such that the algebraic operations with vectors and scalars are continuous in the topology is a *linear topological space*. The space $C_0[0, 1]$ of continuous real functions defined on $[0, 1]$, for example, is separable because any such function can be the limit of a countable sequence of polynomials. Any harmonic complex function defined on the surface of the unit sphere in $\mathcal{R}_3$ can be expressed as a series of spherical harmonics Y_{lm}, so this space is separable, too.

A function defined on X with values in Y is *continuous* if the inverse image of any open set in Y is an open set in X. A bijective continuous function is called *homeomorphism*. Topological spaces are classified as modulo homeomorphisms and *topological invariants* (properties preserved by homeomorphisms). Topological properties of a space X can be investigated by choosing a test topological space S (known one) like $\mathcal{R}^n$ or $C_0(X)$, and building homeomorphisms $hom : S \to X$. When the image of a topological invariant in S is not anymore a topological invariant in X we know that X moved from a certain homeomorphism class into another [235]. The set of all homeomorphisms between two topological spaces X, Y is denoted by $\text{Hom}(X, Y)$. The property of homeomorphism, like any topological property, can be loosen up by using instead the property of *local homeomorphism*. A function is a local homeomorphisms if for any point of its domain of definition there is an open neighborhood of that point on which the restriction of this function is a homeomorphism onto its image. Obviously, homeomorphism implies local homeomorphism.

Definition 1. A *covering map* from a topological space C to another topological space X is a continuous surjective map $cov : C \to X$ such that $\forall c \in C$ and $\forall U(c)$ an open neighborhood of c we have $cov^{-1}(U) = \bigcup_\alpha V_\alpha$, $V_\alpha \bigcap V_\beta = \emptyset$ a disjoint union of open sets in C, and $cov \in \mathrm{Hom}(V_\alpha, U)$.

The "larger" space C is called the covering space, and the space X is called the base space. Traditional examples of covering maps are projecting a helix to its base circle, or by wrapping a plane around a cylinder.

2.1.1 Separation Axioms

The uniqueness property of solutions of a nonlinear partial differential system is not only important in itself, but it also provides the freedom to build solutions by any available methods. Uniqueness is mainly controlled by two mechanisms. One is related to the boundary, initial, asymptotic, regularity, or normalization conditions. The second is related to the internal constrains of the spaces for variables and parameters. Uniqueness is very strongly related to the topological property of *separation*. In topology there are several more refined definitions for the concept of *separation* [68, 160, 291, 344]. The various forms of separations, i.e., *separation axioms* introduce different types of topological spaces:

– $\mathcal{P}_1$. $x \neq y$. This is the weakest separation criterium.
– $\mathcal{P}_2$. $\mathcal{V}_(x) \neq \mathcal{V}_(y)$, the two points x and y do not have the same families of neighborhoods: they are *topologically distinguishable*.
– $\mathcal{P}_3$. $A \cap \overline{B} = \emptyset$, each set is disjoint from the other's closure; the sets are *separated*.
– $\mathcal{P}_4$. $\exists \mathcal{V}_(x) \cap \mathcal{V}_(y) = \emptyset$, points separated by disjoint neighborhoods. This form of separation is the most used in analysis, since it makes the transition from points to open sets.
– $\mathcal{P}_5$. $\exists \overline{\mathcal{V}_(x)} \cap \overline{\mathcal{V}_(y)} = \emptyset$, points separated by disjoint closed neighborhoods.
– $\mathcal{S}$. $A \cap B = \emptyset$, disjoint sets.
– $\mathcal{PS}$. $x \notin A$, the element does not belong to the set.
– $\mathcal{F}$. $\exists f \in C_0(X)$, $f(\xi_1) \neq f(\xi_2)$. There is a continuous function on X which takes distinct values in two disjoint quantities ξ that can be points and/or sets. This last form of separation is very useful when working with spaces of functions, e.g., in the Weierstrass approximation theorem.

According to the separation axioms there are four types of topological spaces:

1. Regular (R).
   A topological space is Kolmogorov (or T_0) if $\mathcal{P}_1 \to \mathcal{P}_2$, i.e., the space is such that any two distinct points have different families of neighborhoods (are topologically distinguishable). A topological space is *symmetric* (or R_0) if $\mathcal{P}_2 \to \mathcal{P}_3$, i.e., the space is such that any two topologically indistinguishable points have a disjoint neighborhood with respect to the other point (separated) (see Fig. 2.1). A stronger separation axiom defines X as a preregular space (or R_1)

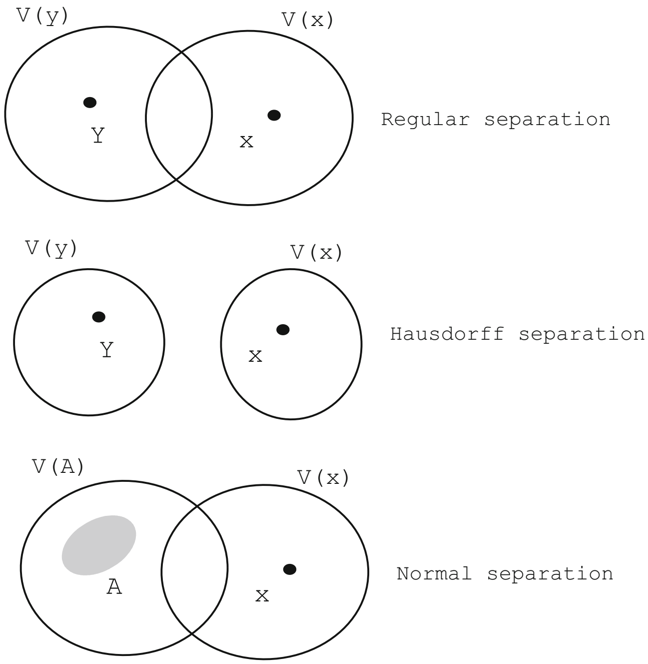

Fig. 2.1 Forms of separation axioms: regular, Hausdorff, and normal. Loops represent neighborhoods

if $\mathcal{P}_2 \to \mathcal{P}_4$, i.e., any two topologically indistinguishable points have disjoint neighborhoods. This axiom can be enhanced even more if we ask that any point x and disjoint closed set C, $x \notin C$ are separated by a continuous function, namely if $\mathcal{PS} \to \mathcal{P}_4$, and the space is called regular. As application, for example, any topological vector space is regular [160].

2. Hausdorff (H).
A topological space is Hausdorff separated (H or T_2) if $\mathcal{P}_1 \to \mathcal{P}_4$, i.e., its distinct points are separated by disjoint neighborhoods (see Fig. 2.1). The Hausdorff separation is the most used one in analysis and operator theory. For example, to build a Banach (commutative) algebra of functions defined on a base space X, we need this space to be Hausdorff (and compact). A very important application of H spaces is related to their property that the intersection of all closed neighborhoods of any point reduces to that point, $\forall x \in X, \cap\overline{\mathcal{V}(x)} = x$. This property is actually the basis of the uniqueness of the limit for the convergent sequences in H spaces. Moreover, this property plays the essential role in the proof of the Cauchy integral representation formula. There is interference between separation and compactness properties: the image of a compact through a continuous function $f : E \to F$ is compact, only if F is Hausdorff. The separation property is requested because we need to label the sets of a finite

covering of E (produced by reciprocal images of an open covering of F) by elements in E. So, if F is not separated, the images of two distinct such elements may belong to the same open set in F, which destroy the construction. As an example, the topology induced by a family of seminorms is in general Hausdorff.

Since the Hausdorff property is so essential to the uniqueness of solutions of equations, we give the following example of a non-Hausdorff space. Let us consider in $\mathbb{R}^2$ the sets $A_1 = \{(x, 0)|x \in \mathbb{R}\}$ and $A_2 = \{(x, 1)|x \in \mathbb{R}\}$, and let us introduce an equivalence relation $\sim$ between the points $(x, y) \in A = A_1 \cup A_2$ defined by $(x, y) \sim (x', y')$ if $x = x'$ and $y = y'$ or $x = x' < 0$ and $y \neq y'$. We organize the quotient set $X = A/\sim$ as a topological space with the canonical interval topology on $\mathbb{R}$. The points $(0, 0)$ and $(0, 1)$ in X are distinct but have no disjoint neighborhoods.

3. Normal (N).
   In a normal topological space, any two disjoint closed sets are separated by neighborhoods, i.e., $\mathcal{S} \to \mathcal{P}_4$, or $\forall \overline{A} \cap \overline{B} = \emptyset, \exists \mathcal{V}(\overline{A}) \cap \mathcal{V}(\overline{B}) = \emptyset$ (see Fig. 2.1). For a Hausdorff space, this request becomes the Tietze–Uryson lemma. A topological space with the topology induced by a metric is normal, and a compact space is also normal [291]. Normal spaces are important in problems related to the partition of unity. Partitions are important in the theory of prolongation of continuous functions.
4. Completely separated (C).
   Here the separation criterium is the function separation. There are already several types of topological spaces completely separated as follows: completely Hausdorff spaces (CH or completely T_2) where $\mathcal{P}_1 \to \mathcal{F}$, completely regular spaces (CR) where $\mathcal{P}_5 \to \mathcal{F}$, and completely normal (CN) where $\mathcal{P}_3 \to \mathcal{P}_4$. We also have perfectly normal spaces (PN) if $\mathcal{S} \to \mathcal{F}$, etc.

In addition to these types of topological spaces, there are other spaces where separation is defined by combining different forms of separation. In Figs. 2.2 and 2.3, we represent some of the interconnections between all these spaces.

2.1.2 Compactness

The compactness property of a topological space (or set) tells if this space is "bounded" in some sense, without having a metric or a distance available. The compactness property is actually more powerful than boundedness, since the latter is not preserved by homeomorphisms. A topological space is a *compact space* if every open covering has a finite subcovering. In metric spaces (see Sect. 2.1.6) compact is equivalent with closed and bounded. Actually, it is easier to understand the concept of noncompact. The real axis is noncompact because if we cover it with the intervals $(n, n + 1)$ and $((2n + 1)/2, (2n + 3)/2)$, n integer, and we eliminate any of them the axis has at least one point uncovered. A compact Hausdorff space

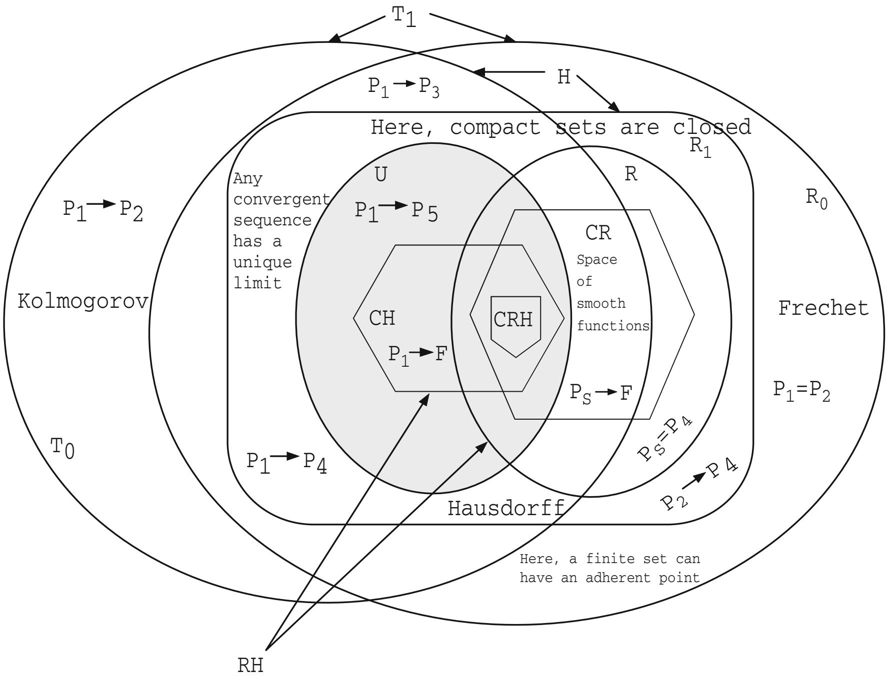

Fig. 2.2 Relationships between separation axioms presented in a Venn diagram. Part 1: the normal spaces are not included here. *Circles* represent classes of spaces fulfilling separation axioms, together with their inclusion and intersection properties. Each space is identified by an abbreviation (H = Hausdorff) and the text shows the corresponding axiom of separation. The *shaded area* represents the regular Hausdorff (T_3) space. The two *inside ovals* represent topological spaces where the separation axioms involve functional separation (definition F)

is usually called a *compact*, and a compact metric space is called *compactum*. An example of a compactum is any finite discrete metric space. A *continuum* is a connected compactum. The image of a compact set through a continuous function into a Hausdorff space is a compact set. As an immediate consequence, a continuous function defined on a compact space is bounded and has a maximum and a minimum.

Although compactness is a global property of a space, it can also be obtained starting from local level. We define a weaker request for compactness, i.e., a *local compact* space as a Hausdorff topological space with the property that any element has at least one compact neighborhood. A local compact space X can always be submerged into a larger topological compact space $\tilde{X}$ such that $X \sqsubset \tilde{X}$ and $\tilde{X} \backslash X = \omega$ (Alexandroff's compactification). The extra element ω is called the point at infinity. In the case of $\mathbb{R}^2 \simeq \mathbb{C}$, $C \cup \omega = \tilde{C}$ is called the extended complex plane. A local compact linear topological space has finite dimension. There are also refinements of the compactness property, like precompact, paracompact,

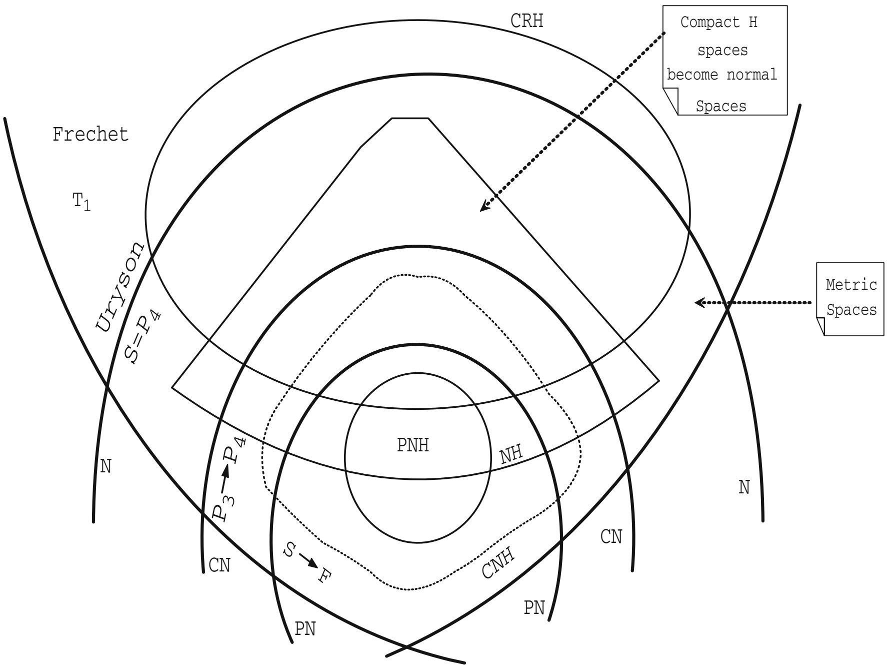

Fig. 2.3 Relationships between separation axioms in a Venn diagram, Part 2: the normal spaces are included. This figure is a zoom in of Fig. 2.2, and the space CRH has the same signification. The *thicker boundaries* represent topological spaces where the separation axioms involve functional separation (definition F)

relatively compact, countable compact, etc., but we do not need these concepts in our book. Basically, they occur whenever we relax one of the three properties defining compactness [68, 160, 291] (see Fig. 2.4).

An *open map* is a function between two topological spaces which maps open sets to open sets. Likewise, a *closed map* is a function which maps closed sets to closed sets. The open or closed maps are not necessarily continuous. A continuous function between topological spaces is called *proper* if inverse images of compact subsets are compact. An *embedding* between two topological spaces is a homeomorphism onto its image.

2.1.3 Weierstrass–Stone Theorem

How is it possible for the Taylor series to exist? That is, how is it possible to know all the values of a continuous function from just knowing a countable sequence of number, the coefficients of the Taylor series. The answer is related to the separation axioms and it is the Weierstrass–Stone theorem. This theorem is also the answer for

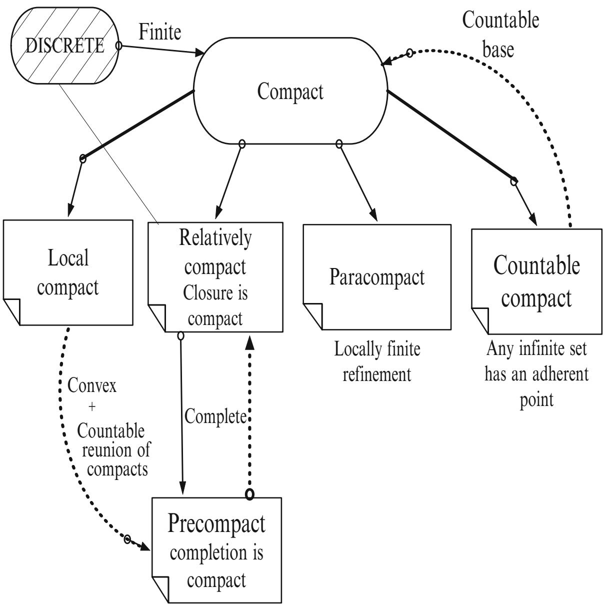

Fig. 2.4 Relation between different categories of compactness and their implications

the questions in Sect. 2.2, namely how is possible to find the values of a function in an n-dimensional domain, knowing only the values of the function in the $(n-1)$-dimensional boundary? Weierstrass proved that a real function defined on $[0, 1]$ is the uniform limit of a series of polynomials. Later on Stone explained that the essential property of the polynomials that allow such a perfect approximation is that they form an algebra.

Theorem 1 (Weierstrass–Stone). *A subalgebra $\mathcal{A}$ of the Banach algebra of $C_0(X)$ continuous real functions defined on a Hausdorff compact space X, is dense in $C_0(X)$ if and only if:*

1. $1 \in \mathcal{A}$.
2. $\forall x \neq y \in X, \exists f \in \mathcal{A}$ *such that* $f(x) \neq f(y)$.

The first condition actually requires $\forall x \in X$, $\exists f \in \mathcal{A}$ such that $f(x) \neq 0$. We meet this condition if we try to generate a Hausdorff linear topological space. The algebraic structure of the functions $\mathcal{A}$ is required to have included in $\mathcal{A}$ the elements $Sup(f, g)$ and $Inf(f, g)$ for $\forall f, g \in \mathcal{A}$. The second condition requires that the algebra $\mathcal{A}$ "separates" points in X, in the sense of the $\mathcal{F}$ form of separation, like in the case for example when X is a completely regular Hausdorff (CRF) space. For details about the proof and Banach algebras one can consult, for example, [160] and references cited therein at page 516. Basically the idea is that any real continuous function defined on a Hausdorff compact X can be infinitely well approximated with other functions selected from a closed subalgebra of $C_0(X)$.

The Weierstrass–Stone theorem tells us that any vector-valued continuous function, no matter how complicated it is, can be infinitely well approximated with simpler functions g_α (where α is a label), as long as these simpler functions form a Banach algebra $\mathcal{A}$, i.e., $\mathcal{A} \ni g_\alpha \to f$. Moreover, if $\mathcal{A}$ is a separable space (to be defined later), then we have a countable basis of continuous functions, $\alpha \simeq n$, and consequently we can express f, for all $x \in X$, by a (maximum) countable set of coefficients associated with f approximating series. Since $\mathcal{A}$ is an abstract Banach algebra which $\mathcal{F}$ separates X, there is freedom to choose its elements, i.e., such a richness of examples: Taylor polynomials, orthogonal polynomials, trigonometric series, etc. The Weierstrass–Stone theorem can be equally applied to complex functions, with an additional request: $\forall g \in \mathcal{A}, \overline{g} \in \mathcal{A}$, where $\overline{g}$ is the complex conjugation.

We have two important corollaries. The space of polynomials defined on a compact $C \in \mathbb{R}^n$ with coefficients in a seminormed vector space $\mathbf{V}$ is dense in the space of continuous bounded functions defined on C with values in $\mathbf{V}$. The second corollary of the Weierstrass–Stone theorem allows us to approximate any complex vector-valued continuous function defined on the unit complex circle $S_1 \subset \mathbb{R}^2$ with trigonometric polynomials [291, Chap. XXII]. This corollary has important consequences for differential systems on closed curves and surfaces. Namely

Lemma 1. *Trigonometric polynomials with coefficients in* $\mathbf{V}$ *are a dense set in* $\{f : \mathbb{R} \to \mathbf{V} | f\,\textit{continuous}, \textit{periodic}\}$.

2.1.4 Connectedness, Connectivity, and Homotopy

A topological space X is *connected* if it is not the disjoint reunion of two or more nonempty open sets. Connected spaces have a very interesting property: the only sets with empty boundary are the total space and the empty set. We can introduce a stronger type of connectedness through the concept of *arc* or *path*. Let $x, y \in X$ be two arbitrary points in a topological space. We have

Definition 2. A path from x to y is a continuous map $\Gamma : [0, 1] \to X$ such that $\Gamma(x) = 0, \Gamma(y) = 1$. An arc from x to y is a path which is also a homeomorphisms onto $\Gamma[0, 1]$.

So, an arc is a path which has also a continuous inverse.

Definition 3. The topological space X is *pathwise-connected* (*or arcwise-connected*) if any two of its points can be joined by a path (by an arc).

Some authors do not make a difference between path and arc in this context, and many references use the term path-connected instead of pathwise-connected, etc. Every path-connected space is connected, but not conversely. A traditional example is the graphics of the real function $\sin(1/x)$ which is in *one-piece* in $\mathbb{R}^2$ but there is no path between the points $(-1/\pi, 0)$ and $(1/\pi, 0)$ of its graphics. Any path-connected Hausdorff space is also arc connected, so again we want to emphasize the importance of axioms of separation. Connectedness is a topological invariant.

Finally, there is third type of criterion for connectedness. If any loop (closed smooth path) in the space is contractible to a point (can be smoothly deformed to a point) the space is called *simply connected* or 1-connected. Such a space is in one piece (connected) and has no "holes." The space is *n-multiply connected* if it is $(n-1)$ multiply connected and if every map from the n-sphere into it extends continuously over the $(n+1)$-disk. By *sphere* we mean here just the boundary of a sphere, for example in an n-dimensional normed space the $(n-1)$-sphere is the set $\{x/\ ||x|| = R\}$. The $(n-1)$-dimensional sphere is the boundary of an n-dimensional disk. The n-connectedness property is a generalization of pathwise connectedness, from paths to higher dimension surfaces.

Let X be a space and a function $f : X \to X$. An element $x_f \in X$ is a *fixed point* for the application f if $f(x) = x$. Also, a set $A \subset X$ is an *invariant set* if $f(A) \subset A$. Any continuous function defined on a real interval $[a, b]$ has at least one fixed point. The fixed point theorems [52] are successfully applied in field theory, biological problems and logistic equations, dynamics of population [327], and in mathematical economics. One of the most important applications is about iterated maps [93, 94]. A theorem due to Tikhonov [160, 312], enounces that compact and convex sets in a Hausdorff local convex space have the fixed point property.

If all the closed smooth curves (loops) in X can be continuously deformed one into another, we call this property *homotopy*. More rigorous

Definition 4. Let $\Phi : [0, 1] \to M$, and $\Psi : [0, 1] \to M$ be piecewise smooth closed paths on a manifold M. A *homotopy* from Φ to Ψ is a continuous function $\gamma : [0, 1]^2 \to M$ such that $\forall t \in [0, 1], \gamma(0, t) = \Phi(t)$, $\gamma(1, t) = \Psi(t)$, and $\forall s \in [0, 1]$, the path $\gamma(s, t)$ parameterized by t is closed and piecewise smooth.

All loops in X belong to the same equivalence class with respect to homotopy equivalence relation, so the group generated by the homotopy classes of X via the composition of curves is trivial identity. We call this group, *homotopy group* of X, and we denote it with $\pi_1(X)$. In algebraic topology one can prove that the groups of homotopy are topological invariants [235, 242].

An interesting result combining some of the concepts we introduced so far is this: any local homeomorphism from a compact space to a connected space is a covering, see Definition 1. The proof of this theorem is based on the fact that the local homeomorphism still preserves the property of being open, and the compactness of C insures that we can always choose a finite sub-cover from any open cover of it. Being finite, we can always choose its neighborhoods small enough to be pairwise disjoint, so all the conditions of being a covering map can be accomplished.

2.1.5 Separability and Basis

A metric space is *separable* if it has a countable dense subset Y, $Y \subset X$, $\bar{Y} = X$, where $\bar{Y}$ is the closure of Y, i.e., Y and all its adherent points (the boundaries).

Usually, the set Y is called *basis*, and if X is separable, members of Y can approximate any $x \in X$ as closely as we like. One of the Weierstrass theorems shows that the set of polynomials is a dense set in $C_0([0, 1])$, so continuous real functions on a compact space can be approximated with polynomials to the best extent.

2.1.6 Metric and Normed Spaces

Metric spaces deal with *completeness* property. A metric topological space (M, τ, d) is a topological space (M, τ) endowed with a positive symmetric function $d : M \times M \rightarrow \mathbb{R}^+$ called *distance*, fulfilling the triangle inequality $\forall x, y, z \in M, d(x, z) \leq d(x, y) + d(y, z)$, and $d(x, y) = 0 \leftrightarrow x = y$. In a metric space M we can define an open ball (or disk) of center $x_0 \in M$ and radius $R \in \mathcal{R}^+$ as $B(x_0; R) = \{x | d(x, x_0) < R\}$. Any metric space is Hausdorff, by inheriting from the common real topology. In a metric space we can define *bounded* sets, if they can be enclosed in a certain ball. A compact metric space is separable. A linear space where we defined a nonnegative real function (a norm) $|| \cdot ||$ which is positively homogenous, subadditive and is zero only in the origin of the linear space is a *normed space*. A normed space is a metric space with the relation $d(x, y) = ||x - y||$, and consequently has all the properties of metric spaces. In a normed space the topology is normed induced and we have convergency in norm (the strong convergency). Any metric space M can be *completed* to $\overline{M}$ by adding to M the limits of all its Cauchy sequences. In a complete metric space all Cauchy sequences are convergent to a certain, unique limit. In a compact metric space any sequence contains a convergent subsequence. A complete normed linear space (where the metric is induced by a norm defined in the linear space) is called a *Banach* space.

A complex bilinear continuous symmetric form defined on a linear vector space $< \cdot, \cdot >: V \rightarrow \mathbb{C}$ is called a *scalar* or *inner* product. A space together with a scalar product, $(X, < \cdot, \cdot >)$ is *Euclidean*. For example on the linear topological space of integrable (in what ever sense integrability is needed) functions defined on a space X we define the scalar product

$$< f, g >= \int_X f(x) g^*(x) dx,$$

with g^* complex conjugated. The scalar product induces a norm, and obviously a distance $||f|| = \sqrt{< f, f >}, d(f, g) = \sqrt{< f - g, f - g >}$. A *Hilbert space* is a complete Euclidean space. The scalar product can measure the property of being *orthogonal* which generalizes the linear independence property in a geometric way. A maximal linear independent set of elements in X is a *basis* in X, and if X is Euclidean and the basis elements are mutually orthogonal and of unit norm, it is called *orthonormal basis*. Special functions, like orthogonal polynomials, spherical harmonics, etc. (Sect. 18.3), form orthonormal bases in spaces where the integral of the square magnitude of the functions are finite, $L_2(X)$.

The key theorem about representation of functions is the following:

Theorem 2. *Every separable Hilbert space H_s has a countable orthonormal basis $B_N \subset H_s$, i.e., $\bar{B}_N = H_s$.*

The following chapters, and all representation formulas theory, are entirely based on this result. It means that on a Hilbert space, any element can be approximated as good as we want with elements from this countable (discrete) basis. As strange as it may look, there are nonseparable Hilbert spaces in physics. For example in canonical quantum gravity, the space of functions defined on connections, A, modulo gauge transformations G, $L_2(A/G)$, is nonseparable [179].

2.2 Elements of Homology

The meaning of homology will become more transparent when we will use it in the Poincaré Lemma, and in compact boundary representation formulas (Sect. 3.1.4). For reference on the topics we suggest the bibliography [112, 235]. An oriented *p-simplex*, $p > 0$ integer, in $\mathbb{R}^n$ is generated by an ordered system of $p+1$ vectors, and it is the p-dimensional manifold

$$\sigma^p = [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_p] = \left\{ \boldsymbol{v} \in \mathbb{R}^n \mid \sum_{i=0}^{p} t_i \boldsymbol{v}_i, \sum_{i=0}^{p} t_i = 1 \right\}.$$

Basically, the generalization of a segment (1-simplex), a triangle (2-simplex), and a tetrahedron (3-simplex) is to higher dimensions. A p-simplex is topologically homeomorphic with a p-ball. The subset $t^i = 0$ is an $(p-1)$-plane, or face, and the end points of the vectors are the vertices. A *simpliceal complex* is a set K of simplexes constructed such that all their faces also belong to K, and any two simplexes in K are either disjoint, or their intersection is a common face of each of them. A topological space homeomorphic to a simpliceal complex is called *triangulated*. In the following we work only on these triangulated spaces. Based on the triangulation K of a given manifold we can construct the Abelian groups $C_p(K)$, $p = 0, \ldots, n$ freely generated by the oriented p-simplexes of K, with integer coefficients, called the *group of chains* (not to be confounded to sets of continuity of order k !). We define the linear *boundary operators* as

$$\partial_p : C_p(K) \to C_{p-1}(K), \tag{2.1}$$

with the action $\partial_p \sigma^p = \sum_{j=0}^{p} (-1)^j [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{j-1}, \boldsymbol{v}_{j+1} \ldots, \boldsymbol{v}_p]$ creating thus a $(p-1)$-simplex. It is easy to verify that the boundary operator is a group homomorphism, $\partial_0 c_p = 0$, and

$$\partial_{p-1} \partial_p = 0, \tag{2.2}$$

which is the central property of homology, and somehow the main philosophy of the compact surfaces, contours, boundaries in general:

The boundary of a boundary is the empty set.
The immediate consequence in cohomology is that
the external derivative of order two is always zero.

Like we mention in Chap. 1, again a pure algebraic property like skew-symmetry of ∂_p provides a deep geometrical result. The kernel of the boundary operator, $Z_p(K) = \mathrm{Kerr}(\partial_p)$, is a subgroup of the group of chains, namely the group of boundary-less chains which are called p-cycles. Also the image of the boundary operator is called the group of the p-boundaries $B_p(K) = \partial_{p+1}(C_{p+1}(K))$. So, basically we have for each p the following succession of (normal) subgroups: $B_p \subset Z_p \subset C_p$. It is easy to notice that we can construct the quotient (factor) groups $C_p(K)/Z_p(K)$, $C_p(K)/B_p(K)$ and $Z_p(K)/B_p(K)$, and we have the group homomorphism $Z_p \sim C_p(K)/B_{p-1}(K)$. The quotient group

$$H_p(K) = Z_p(K)/B_p(K), \tag{2.3}$$

namely the *homology group* of order p of K. This factorization of p-cycles modulo p-boundaries over K introduces an equivalence relation in the group of cycles. In other words, two p-cycles of K are homologous if their difference is a p-boundary. Being Abelian freely generated, all the homology groups are isomorphic with some $\mathbb{Z}^n$ group. The rank of H_p group counts the number of p-dimensional holes of K. The rank of a group is smallest cardinality of its generating set. For example, $H_0(S_n) \sim H_n(S_n) \sim \mathbb{Z}$ and $H_p(S_n) \sim \{0\}$ for $p \neq 0, n$. A $T_2 \subset \mathbb{R}^3$ torus has the homology described by $H_0(T_1) \sim \mathbb{Z}$, $H_2(T_1) \sim \mathbb{Z}^2$, $H_3(T_1) \sim \mathbb{Z}$, and $H_p(T_1) \sim \{0\}$ for the rest of p.

We define the Euler characteristic χ of K the expression

$$\chi(K) = \sum_{p=0}^{n} (-1)^p \ \mathrm{rank}\ H_P(K), \tag{2.4}$$

which is one of the essential topological invariants for the Gauss–Bonnet formula (see Theorem 20) applied to closed Riemannian manifolds and for the Euler–Poincaré formula. For example $\chi(S_1) = 0, \chi(S_2) = 2, \chi(T_1) = 0, \chi(T_2) = -2$, etc. The Euler characteristic defines the genus g of a closed orientable surface by $g = (2 - \chi)/2$, which can be loosely understood as the number of "handles" of the surface.

2.3 Group Action

Let X be a topological space and G a topological group (that is a group which is also topological space and the two structures are reciprocal compatible). We say that G acts on X (from the left) if there is a continuous map $m : G \times X \to X$ such that

1. $m(g, m(h, x)) = m(gh, x)$ for $g, h \in G, x \in X$
2. $m(e, x) = x$, for $x \in X$

The entity (X, G, m) is called a *G-space*. For an efficient introduction in the theory of group actions from the differential geometry point of view we recommend the text [81], while for more technical details and applications we recommend [242]. We have the following definitions. The set $G_x = \{g \in G | m(g, x) = x\}$ is called *isotropy group* of x (or stabilizer subgroup of x). The set $O_x = \{m(g, x) | g \in G\}$ is called the *orbit* of x. The set of all orbits is denoted X/G and it is called orbit space and it is a topological space through the quotient induced topology with respect to the canonic projection $x \to \mathrm{O}_x$.

- The action of G on X is *free* if the isotropy group is trivial for all x.
- The action of G on X is *proper* if the map $\theta : G \times X \to X \times X$ given by $(g, x) \to (x, m(g, x))$ is a proper function.
- The action of G on X is *transitive* if it possesses only a single group orbit, i.e. if all elements are equivalent. The G-space (X, G, m) is a *homogeneous space* if G acts in a transitive way.

The *principal homogeneous space* (or torsor) of G is a homogeneous space X such that the isotropy group of any point is trivial. Equivalently, a principal homogeneous space for a group G is a topological space X on which G acts freely and transitively, so that for any $x, y \in X$ there exists a unique $g \in G$ such that $m(g, x) = y$. If X is a G-space with proper action the quotient space X/G is Hausdorff. All these properties and definitions can be extended if the space X is a differentiable manifold, and G is a Lie group acting on X, case in which the structure (X, G, m) is called a G-manifold. Moreover, if the action of G is proper and free X/G has a differentiable manifold structure and the canonical projection $X \to X/G$ is a submersion.

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