

Chapter 2

Basics of Fractional Order Signals and Systems

Abstract This chapter looks at some linear time invariant (LTI) representations of fractional order systems. Since the LTI theory is well developed, casting fractional order systems in state space or transfer function framework helps in the intuitive understanding and also the use of existing tools of LTI systems with some modifications. The discrete versions of fractional order models are also introduced since many practical applications in signal processing work on discrete time system models and discrete signals. Some basic signal processing techniques, like continuous and discrete fractional order realizations, convolution and norms, applied to fractional order signals and systems are also introduced.

Keywords Fractional order (FO) differ-integrators · Constant Phase Element (CPE) · FO realization · FO LTI systems · FO convolution · FO correlation

Modern science has tried to interpret nature from a mathematical framework and has been successful to a large extent in analyzing various phenomena and harnessing control over various physical processes. But on introspection we need to look at the very mathematical tools in our quest for perfection and understanding. Fractional order differential and integral equations are a generalization of our conventional integral and differential equations that extend our notion of modeling the real world around us. The term “fractional” basically implies all non-integer numbers (e.g. fractions, irrational numbers and even complex numbers) and hence a more mathematically correct nomenclature is non-integer order calculus. However, the present focus of the book is real non-integer numbers and the term ‘fractional’ used throughout this book is in this sense.

There are many processes in nature that can be more accurately modeled using fractional differ-integrals. It has been experimentally demonstrated that the charging and discharging of lossy capacitors for example, follows inherently fractional order dynamics. The flow of fluid in a porous media, the conduction of heat in a semi-infinite slab, voltage-current relation in a semi infinite transmission line, non-Fickian

diffusion, etc. are all such examples where the governing equations can be modeled more accurately using fractional order differential or integral operators.

A Fractional order linear time invariant (LTI) system is mathematically equivalent to an infinite dimensional LTI filter. Thus a fractional order system can be approximated using higher order polynomials having integer order differ-integral operators. The concept can be viewed similar to the Taylor series expansion of non-linear functions as a summation of several linear weighted differentials. More involved analysis and review of existing fractional order signals and systems can be found in Magin et al. (2011), Ortigueira (2000a, b, 2003, 2006, 2008, 2010).

2.1 Fractional Order LTI Systems

Fractional order LTI systems can be modeled using the conventional input–output transfer function approach similar to the integer order ordinary differential equations (Monje et al. 2010). Fractional order ordinary differential equations lead to fractional order transfer function models by the well known Laplace transform technique. The conventional notion of state space modeling can also be extended to represent fractional order dynamical system models with additional requirement of defining the state variables as fractional derivative of another.

2.1.1 Transfer Function Representation

Let us consider the following fractional differential equation to represent the dynamics of a system:

$$\begin{aligned} a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \dots + a_0 D^{\alpha_0} y(t) \\ = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \dots + b_0 D^{\beta_0} u(t) \end{aligned} \quad (2.1)$$

In the above fractional differential equation if the order of differentiations be integer multiple of a single base order i.e. $\alpha_k, \beta_k = k\alpha$, $\alpha \in \mathbb{R}_+$, the system will be termed as commensurate order and takes the following form

$$\sum_{k=0}^n a_k D^{k\alpha} y(t) = \sum_{k=0}^m b_k D^{k\alpha} u(t) \quad (2.2)$$

Taking Laplace transform of the above equation and putting zero initial condition, the input–output fractional order transfer function models takes the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \dots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \dots + a_0 s^{\alpha_0}} \quad (2.3)$$

For commensurate fractional order systems the above transfer function takes the following form

$$G(s) = \frac{\sum_{k=0}^m b_k (s^\alpha)^k}{\sum_{k=0}^n a_k (s^\alpha)^k} = \frac{\sum_{k=0}^m b_k \lambda^k}{\sum_{k=0}^n a_k \lambda^k}, \quad \lambda = s^\alpha \quad (2.4)$$

Fractional order transfer function models leads to the concept of fractional poles and zeros in the complex s-plane (Merrikh-Bayat et al. 2009). For the commensurate fractional order system (2.4) with the characteristic equation in terms of the complex variable λ is said to be bounded-input bounded-output (BIBO) stable, if the following condition is satisfied

$$|\arg(\lambda_i)| > \alpha \frac{\pi}{2} \quad (2.5)$$

where λ_i are the roots of the characteristic polynomial in λ .

For $\lambda = 1$, the well known stability condition for integer order transfer functions is obtained which states the real part of the poles should lie in the negative half of the complex s-plane i.e. $|\arg(\lambda_i)| > \frac{\pi}{2}$.

2.1.2 State-Space Representation

A generalized fractional order multiple input multiple output (MIMO) LTI state-space model can be represented as

$$\begin{aligned} D^\alpha x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (2.6)$$

where $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]$ is the commensurate or incommensurate fractional orders. $u \in \mathbb{R}^l$ is the input column vector, $x \in \mathbb{R}^n$ is the state column vector, $y \in \mathbb{R}^p$ is the output column vector, $A \in \mathbb{R}^{n \times n}$ is the state matrix, $B \in \mathbb{R}^{n \times l}$ is the input matrix, $C \in \mathbb{R}^{p \times n}$ is the output matrix, $D \in \mathbb{R}^{p \times l}$ is the direct transmission matrix. In the above fractional order state-space representation, the first equation is called the fractional order state equation and the second one is known as the output equation. The fractional state space model can be converted to the fractional order transfer function form using the following relation

$$G(s) = C(s^\alpha I - A)^{-1} B + D \quad (2.7)$$

Here, I represents the identity matrix of dimension $n \times n$ and $G(s)$ represents the fractional order transfer function matrix of dimension $p \times l$. The fractional order state-space realization in controllable, observable and diagonal canonical forms are similar to the corresponding integer order state-space models and have been detailed in Monje et al. (2010). Accuracy of the numerical approximation of FO state-space has been detailed in Rachid et al. (2011) and Bouafoura et al. (2011).

It is interesting to note that the stability analysis of commensurate fractional order systems is relatively easier (Trigeassou et al. 2011; Trigeassou and Maamri 2011) since the conventional s-plane instability region gets contracted by $\alpha\pi/2$ for $0 < \alpha < 1$. Stability of incommensurate fractional order models are mathematically more involved and have been discussed by Matignon (1998) in a detailed manner.

The discrete time fractional order state space corresponding to the continuous time fractional order state space can be represented as

$$\left. \begin{aligned} x(k+1) &= (AT^\alpha + \alpha I)x(k) + BT^\alpha u(k) && \text{for } k=0 \\ x(k+1) &= (AT^\alpha + \alpha I)x(k) \\ &\quad - \sum_{i=2}^{k+1} (-1)^i \binom{\alpha}{i} x(k+1-i) + BT^\alpha u(k) && \text{for } k \geq 1 \\ y(k) &= Cx(k) + Du(k) \end{aligned} \right\} \quad (2.8)$$

Considering an infinite dimensional memory of a fractional order system, the above discrete time fractional order state space expressed in terms of an expanded state space

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ x(k) \\ x(k-1) \\ \vdots \end{bmatrix} &= \tilde{A} \begin{bmatrix} x(k) \\ x(k-1) \\ x(k-2) \\ \vdots \end{bmatrix} + \tilde{B}u(k) \\ y(k) &= \tilde{C} \begin{bmatrix} x(k) \\ x(k-1) \\ x(k-2) \\ \vdots \end{bmatrix} + Du(k) \end{aligned} \quad (2.9)$$

where

$$\tilde{A} = \begin{bmatrix} (AT^\alpha + \alpha I) - I(-1)^2 \binom{\alpha}{2} - I(-1)^3 \binom{\alpha}{3} \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tilde{B} = \begin{bmatrix} BT^\alpha \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

$$\tilde{C} = [C \ 0 \ 0 \ \cdots]$$

Here, 0 is the null matrix of dimension $n \times n$.

The above discrete time fractional order state space model is asymptotically stable if the following condition $\|\tilde{A}\| < 1$ is satisfied. Here, $\|\cdot\|$ denotes the matrix norm defined as $\max |\lambda_i|$ and λ_i being the i th eigen-value of extended matrix $\tilde{A}$.

2.2 Realization of Fractional Order Differ-integrators

The realization of FO differ-integrators can be done in two ways viz. continuous time and discrete time realization.

2.2.1 Continuous Time Realization

Fractional order elements generally shows a constant phase curve, hence are also known as constant phase elements (CPE). In practice, a fractional order element can be approximated as a higher order system which maintains a constant phase within a chosen frequency band (Krishna 2011; Elwakil 2010; Maundy et al. 2011). The fractional order elements can be rationalized by several iterative techniques namely, Carlson's method (Carlson and Halijak 1964), Oustaloup's method (Oustaloup et al. 2000), Charef's method (Charef et al. 1992), etc. some of which are detailed below.

2.2.1.1 Carlson's Method

Fractional order elements or transfer functions can be recursively approximated using the following formulation known as Carlson's method (Carlson and Halijak 1964).

If $G(s)$ be a rational transfer function and $H(s)$ be a fractional order transfer function such that $H(s) = [G(s)]^q$ where $q = m/p$ is a fractional order of the transfer function, then $H(s)$ can be recursively approximated as

$$H_i(s) = H_{i-1}(s) \frac{(p-m)[H_{i-1}(s)]^2 + (p+m)G(s)}{(p+m)[H_{i-1}(s)]^2 + (p-m)G(s)} \quad (2.10)$$

with an initial guess of $H_0(s) = 1$.

The above recursive formula can be rewritten as

$$H_i(s) = H_{i-1}(s) \frac{G(s) + \alpha [H_{i-1}(s)]^2}{\alpha G(s) + [H_{i-1}(s)]^2} \quad (2.11)$$

where $\alpha := \frac{p-m}{p+m} = \left[\frac{1-\left(\frac{m}{p}\right)}{1+\left(\frac{m}{p}\right)} \right] = \frac{1-q}{1+q}$ and $q := \frac{1-\alpha}{1+\alpha}$

Therefore, for the simplest case i.e. $H(s) = s^q$, it will act like a differentiator for $q > 0$ implying $\alpha < 1$. Also, it will act like an integrator for $q < 0$ implying $\alpha > 1$.

2.2.1.2 Charef's Method

Basic formulation of Charef's approximation technique is detailed in Charef et al. (1992).

Let us take a fractional order transfer function

$$H(s) = \frac{1}{(1 + s/p_T)^m} \quad (2.12)$$

where p_T is the pole of the FO system and m is the fractional order of the system. The above all pole fractional order system can be rationalized by the following recursive formula

$$H(s) = \frac{\prod_{i=0}^{N-1} \left(1 + \frac{s}{z_i}\right)}{\prod_{i=0}^N \left(1 + \frac{s}{p_i}\right)} \quad (2.13)$$

If the maximum allowable discrepancy in estimating the frequency response of the approximated model be y dB, the zeros and poles can be recursively calculated as

$$\begin{aligned} z_{N-1} &= p_{N-1} 10^{[y/10(1-m)]} \\ p_N &= z_{N-1} 10^{[y/10m]} \end{aligned} \quad (2.14)$$

The first approximation of the pole and zero starts with

$$\begin{aligned} p_0 &= p_T 10^{[y/20m]} \\ z_0 &= p_0 10^{[y/10(1-m)]} \end{aligned} \quad (2.15)$$

Now the problem is to determine the value of N so that a specified accuracy of the approximated rational transfer function at the corner frequencies can be obtained.

$$\begin{aligned} \text{Let, } a &= 10^{[y/10(1-m)]} = \frac{z_{N-1}}{p_{N-1}} \text{ and } b = 10^{[y/10m]} = \frac{p_N}{z_{N-1}} \text{ then} \\ ab &= 10^{[y/10m(1-m)]} = \frac{z_{N-1}}{z_{N-2}} = \frac{p_N}{p_{N-1}}. \end{aligned}$$

Now, N can be determined by the following expression

$$N = \text{Integer} \left(\frac{\log \left(\frac{\omega_{\max}}{p_0} \right)}{\log(ab)} \right) + 1 \quad (2.16)$$

2.2.1.3 Oustaloup's Method

Oustaloup's recursive filter gives a very good fitting to the fractional-order elements s^γ within a chosen frequency band (Oustaloup et al. 2000). Let us assume that the expected fitting range is (ω_b, ω_h) . The filter can be written as:

$$G_f(s) = s^\gamma = K \prod_{k=-N}^N \frac{s + \omega'_k}{s + \omega_k} \quad (2.17)$$

where the poles, zeros, and gain of the filter can be evaluated as

$$\omega_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{k+N+\frac{1}{2}(1+\gamma)}{2N+1}}, \quad \omega'_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{k+N+\frac{1}{2}(1-\gamma)}{2N+1}}, \quad K = \omega_h^\gamma$$

2.2.2 Discrete Time Realization

2.2.2.1 Based on Discretization Method

Discrete time realizations are often preferred over continuous time realizations since they can be easily implemented in hardware and updated with the change in the purpose of application. There are mainly two methods of discretization viz. indirect and direct method. The indirect discretization method is accomplished in two steps. Firstly the frequency domain fitting is done in continuous time domain and then the fitted continuous time transfer function is discretized. Direct discretization based methods include the application of power series expansion (PSE), continued fraction expansion (CFE), MacLaurin series expansion, etc. with a suitable generating function. The mapping relation or formula for conversion from continuous time to discrete time operator ($s \leftrightarrow z$) is known as the generating function. Stability of direct discretization has been analyzed in Siami et al. (2011). Among the family of expansion methods, CFE based digital realization has been extensively studied with various types of generating functions like Tustin, Simpson, Al-Alaoui, mixed Tustin-Simpson, mixed Euler-Tustin-Simpson, impulse response based and other higher order generating functions (Gupta et al. 2011; Chen et al. 2004; Visweswaran et al. 2011).

When the continuous time realization like Oustaloup's approximation, etc. with a constant phase is discretized with a suitable generating function the phase of the realized filter may deviate from the original desired one depending on the generating function. However, a direct discretization scheme tries to maintain a constant phase of the FO element directly in the frequency domain. Hence, direct discretization is generally preferred for digital realization over its indirect counterpart.

- *Generating Functions:* The discrete time rational approximation of a simple continuous-time differentiator $s \approx H(z^{-1})$ is as follows

– *Euler (Rectangular Rule):*

$$H_{Euler}(z^{-1}) = \left[\frac{1 - z^{-1}}{T} \right] \quad (2.18)$$

Here, T represents the sampling time and z denotes the discrete time complex frequency. It is clear that the Euler's discretization formula (2.18) is an extension of the backward difference technique of numerical differentiation. Forward or

central differencing schemes is generally not used since it would result in a non causal filter, i.e. future values of a series would be required to obtain the present value.

- *Tustin (Trapezoidal Rule or Bilinear Transform):*

$$H_{Tustin}(z^{-1}) = \left[\frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \right] \quad (2.19)$$

The Tustin's discretization can be obtained from the basic ($s \leftrightarrow z$) mapping relation by expanding the exponential terms with their first order approximations.

$$\begin{aligned} z = e^{sT} &= \frac{e^{sT/2}}{e^{-sT/2}} = \left(1 + \frac{sT}{2}\right) / \left(1 - \frac{sT}{2}\right) = \frac{2 + sT}{2 - sT} \\ \Rightarrow s &= \frac{2}{T} \cdot \left(\frac{z - 1}{z + 1}\right) \end{aligned} \quad (2.20)$$

- *Simpson's Rule:* The well known Simpson's numerical integration formula is given by (in time domain)

$$y(n) = \frac{T}{3} [x(n) + 4x(n-1) + x(n-2)] + y(n-2) \quad (2.21)$$

By applying z transform on (2.21) it is found

$$\frac{Y(z)}{X(z)} = H(z) = \frac{T}{3} \left(\frac{1 + 4z^{-1} + z^{-2}}{1 - z^{-2}} \right) \quad (2.22)$$

The relation given by (2.22) represents a digital integrator and can be inverted to obtain a digital differentiator as

$$H_{Simpson}(z^{-1}) = \left[\frac{3}{T} \cdot \frac{(1 + z^{-1})(1 - z^{-1})}{1 + 4z^{-1} + z^{-2}} \right] \quad (2.23)$$

- *Al-Aloui Operator:* Al-Alaoui has shown that the discretization formula can be improved by interpolating the classical Euler and Tustin's formula as follows

$$\begin{aligned} H_{Al-Alaoui(\alpha)}(z^{-1}) &= \alpha H_{Euler}(z^{-1}) + (1 - \alpha) H_{Tustin}(z^{-1}) \\ &= \frac{\alpha T}{(1 - z^{-1})} + \frac{(1 - \alpha) T}{2} \left(\frac{1 + z^{-1}}{1 - z^{-1}} \right) = \frac{T [(1 + \alpha) + (1 - \alpha) z^{-1}]}{2 (1 - z^{-1})} \\ &= \frac{T (1 + \alpha)}{2} \cdot \frac{\left(1 + \frac{(1 - \alpha)}{(1 + \alpha)} z^{-1}\right)}{(1 - z^{-1})} \end{aligned} \quad (2.24)$$

where $\alpha \in (0, 1)$ is a user-specified weight that balances the impact of the two generating function i.e. Euler (Rectangular) and Tustin (Trapezoidal) and their corresponding accuracies introduced in the discretization. Replacing $\alpha = 3/4$ in (2.24) produces the conventional Al-Alaoui operator as:

$$H_{Al-Alaoui(3/4)}(z^{-1}) = \frac{7T}{8} \cdot \frac{(1 + z^{-1}/7)}{(1 - z^{-1})} \quad (2.25)$$

Generalized Al-Alaoui operator (2.24) shows that the Infinite Impulse Response (IIR) filter has a pole at $z = 1$ and zero varies between $z \in [-1, 0]$ for $\alpha \in [0, 1]$. Thus, the operator (2.24) can be directly inverted to produce a stable IIR realization for a differentiator also.

Early discretization techniques developed by Euler and Tustin are mainly based on the first order polynomial fitting. Simpson's advancement in the discretization technique shows that one can fit higher order polynomial to obtain better accuracy. But this is not a feasible proposition, since expansion with higher order generating function would increase the overall order of the discrete time filter. Also, as the order of the generating function increases, the region of performance in the frequency domain gets shrunk. The generalized Al-Alaoui type generating function (2.24) is ideal for applications where the requirement is to maximize accuracy without going for a higher order realization.

Other higher order generating functions like Chen-Vinagre operator (Chen and Vinagre 2003), Schneider operator, Al-Alaoui-Schneider-Kaneshige-Groutage (Al-Alaoui-SKG) operator, Hsue operator, Barbosa operator, Maione operator, etc. are also used. For details please look in Visweswaran et al. (2011), Gupta et al. (2011), Barbosa et al. (2006), Maione (2008) and the references therein.

2.2.2.2 Series Expansion for FO Element Realization

The generating functions are essentially a $s \leftrightarrow z$ mapping relation. For band-limited rational approximation of fractional order elements, higher order transfer functions in discrete time need to be found out which approximately mimic the constant phase property of the corresponding FO element, within a chosen frequency range. The higher order approximation is done using various recursive formulae for calculating the position of the poles and zeros of the discrete realization. The order of the realized filter is basically a trade-off between the accuracy in frequency domain (obtained by increased number of poles and zeros) and the ease of hardware implementation (obtained by lower number of poles and zeros). Now, the various expansion techniques are listed below.

- *Power Series Expansion (PSE)* Let us now, consider a fractional order differ-integrator

$$G(s) = s^{\pm\gamma}, \gamma \in [0, 1] \subseteq \mathbb{R}_+ \quad (2.26)$$

Using the simple Euler's discretization formula the FO differ-integrator can be approximated via the PSE as

$$T^{\mp\gamma} PSE \left\{ \left(1 - z^{-1}\right)^{\pm\gamma} \right\} \quad (2.27)$$

Performing PSE on the FO differ-integrator gives (Yang Quan et al. 2009)

$$\nabla_T^{\pm\gamma} f(nT) = T^{\pm\gamma} \sum_{k=0}^{\infty} (-1)^k \binom{\mp\gamma}{k} f((n-k)T) \quad (2.28)$$

It is evident that the PSE yields polynomial functions or finite impulse response (FIR) type digital filters.

- *Continued Fraction Expansion (CFE)* In most cases rational approximations which yield both poles and zeros are better than PSE based realization which yields FIR filter structure. These rational approximations converge rapidly and have a wider domain of convergence in the complex plane. A smaller set of coefficients are also required for obtaining a good approximation in case of CFE. The CFE of fractional power of a generating function can be expressed as a rational transfer function

$$\begin{aligned} G(z) &\simeq a_0(z) + \frac{b_1(z)}{a_1(z) + \frac{b_2(z)}{a_2(z) + \frac{b_3(z)}{a_3(z) + \dots}}} \\ &= a_0(z) + \frac{b_1(z)}{a_1(z)} + \frac{b_2(z)}{a_2(z)} + \frac{b_3(z)}{a_3(z)} + \dots \end{aligned} \quad (2.29)$$

where a_i, b_i are either rational functions of the variable or constants.

- *MacLaurin Series Expansion* The MacLaurin series expansion of $(x+a)^v$ is given by

$$\sum_{i=0}^{\infty} a^{v-i} \frac{\Gamma(v+1)}{\Gamma(i+1) \Gamma(v-i+1)} x^i \quad (2.30)$$

For example (Valerio and da Costa 2005),

$$\left(1 - z^{-1}\right)^v = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(v+1)}{\Gamma(i+1) \Gamma(v-i+1)} z^{-i} \quad (2.31)$$

$$\left(1 + z^{-1}\right)^{-v} = \sum_{i=0}^{\infty} \frac{\Gamma(-v+1)}{\Gamma(i+1) \Gamma(-v-i+1)} z^{-i} \quad (2.32)$$

Other expansions like Taylor Series can be used with higher order generating function to get a better approximation (Gupta et al. 2011; Visweswaran et al. 2011).

2.3 Discrete Time Fractional Order Models

2.3.1 ARFIMA Models

Conventional discrete time system models are mostly moving average (Finite Impulse Response or all zero) and AutoRegressive (Infinite Impulse Response or all pole) type. Generalization of these models is the Auto Regressive Moving Average (ARMA) which is a pole-zero (IIR) transfer function.

The Moving Average (MA) model with q terms $MA(q)$ is of the form:

$$y_t = x_t + \theta_1 x_{t-1} + \cdots + \theta_q x_{t-q} \quad (2.33)$$

For the coefficients $\theta_i, i \in \{1, \dots, q\} \subseteq \mathbb{Z}_+$. The Auto Regressive (AR) models with p terms $AR(p)$ is of the form:

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \quad (2.34)$$

for some coefficients $\phi_j, j \in \{1, \dots, p\} \subseteq \mathbb{Z}_+$.

The AutoRegressive Moving Average (ARMA) model with p AutoRegressive terms and q Moving Average terms $ARMA(p, q)$ is given by:

$$y_t = (\phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}) + (x_t + \theta_1 x_{t-1} + \cdots + \theta_q x_{t-q}) \quad (2.35)$$

For a time series X_t and zero mean white Gaussian noise ε_t the ARIMA (p, q) model can be represented as:

$$\left(1 - \sum_{i=1}^p \phi_i L^i\right) X_t = \left(1 + \sum_{i=1}^q \theta_i L^i\right) \varepsilon_t \quad (2.36)$$

where L is the lag operator satisfying the expression $LX_t = X_{t-1}$. The AutoRegressive Integrated Moving Average (ARIMA) model $ARIMA(p, d, q)$ is a generalization of the $ARMA(p, q)$ model with a differencing parameter $d \in \mathbb{Z}_+$. The Eq. 2.36 for the ARIMA model can be extended to the ARFIMA model to obtain:

$$\left(1 - \sum_{i=1}^p \phi_i L^i\right) (1 - L)^d X_t = \left(1 + \sum_{i=1}^q \theta_i L^i\right) \varepsilon_t, \quad (1 - L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-L)^k \quad (2.37)$$

The Auto Regressive Fractional Integrated Moving Average (ARFIMA) model is an extension of the ARIMA model where the degree of differencing d can have fractional values (Hosking 1981). ARFIMA models have found applications in economics, biology, remote sensing, network traffic modeling and similar areas (Sheng and Chen 2011).

An $ARFIMA(p, d, q)$ has the following properties (Chen et al. 2007) depending on various values of d .

1. For $d = 1/2$, the $ARFIMA(p, -1/2, q)$ process is stationary but not invertible.
2. For $-1/2 < d < 0$, the $ARFIMA(p, d, q)$ process has short memory and decay monotonically and hyperbolically to zero.
3. For $d = 0$, the $ARFIMA(p, 0, q)$ process can be white noise.
4. For $0 < d < 1/2$, the $ARFIMA(p, d, q)$ process is a stationary process with long memory. This model is useful in modeling long-range dependence (LRD). The autocorrelation of a LRD time-series decays slowly as a power law function.
5. For $d = 1/2$, the $ARFIMA(p, 1/2, q)$ process is a discrete time $1/f$ noise.

2.3.2 Discrete Time FO Filters

Digital realization of FO filters in FIR and IIR structure is an emerging area of research. Some recent developments include fractional second order filter (Li et al. 2011a), variable order operator (Chan et al. 2010a; Valério and Sá da Costa 2011; Lorenzo and Hartley 2002; Tseng 2006, 2008; Chan et al. 2010b; Charef and Bensouici 2011), distributed order operator (Li et al. 2011b), variable order fractional delay filter (Shyu et al. 2009a, b; Soo-Chang et al. 2010; Tseng 2007b), two-dimensional FO digital differentiator (Chang 2009) and improvement of FO digital differ-integrators (Chien-Cheng 2001, 2006; Tseng 2007a).

2.4 Fractional Order Convolution and Correlation

R_ρ^ϕ is the fractional shift operator and is a generalization of the unitary time shift operator $(T_\tau s)(t) = s(t - \tau)$, R_ρ^ϕ describes a signal shifted along the arbitrary orientation ϕ of the time–frequency plane by a radial distance ρ . R_ρ^ϕ operating on the time domain signals (t) is given by

$$(R_\rho^\phi s)(t) = s(t - \rho \cos \phi) e^{-j2\pi(\rho^2/2) \cos \phi \sin \phi + j2\pi \rho \sin \phi} \quad (2.38)$$

The additivity and inversion properties for R_ρ^ϕ can be stated as

$$R_{\rho_1}^\phi R_{\rho_2}^\phi = R_{\rho_1 + \rho_2}^\phi \quad (2.39)$$

$$(R_\rho^\phi)^{-1} = R_{-\rho}^\phi \quad (2.40)$$

$\mathcal{F}_\phi$ is the Fractional Fourier Transform (FrFT) operator associated with angle ϕ . For $\phi = \pi/2$.

It results in the conventional Fourier Transform, and for $\phi = \pi$.

$\mathcal{F}_\pi$ is the axis-reversal operator $(\mathcal{F}_\pi s)(t) = s(-t)$.

It is actually the application of the conventional Fourier Transform twice on the signal. For further details on the FrFT please see Chap. 4 of this book.

The fractional convolution associated with the angle ϕ is found by computing the inner product of the input signals (t) with the axis-reversed, complex-conjugated, and fractionally shifted version of the function $h(t)$ as

$$\begin{aligned} (s_\phi^* h)(r) &= \langle s, R_r^\phi \mathcal{F}_\pi h^* \rangle = \langle s, R_r^\phi \tilde{h} \rangle \\ &= e^{j2\pi \frac{r^2}{2} \cos \phi \sin \phi} \int s(\beta) h(r \cos \phi - \beta) e^{-j2\pi \beta r \sin \phi} d\beta \end{aligned} \quad (2.41)$$

where $\tilde{h}(t) = h^*(-t)$. This reduces to the integer order convolution for $\phi = 0$

$$(s_0^* h)(t) = \int s(\beta) h(t - \beta) d\beta \quad (2.42)$$

The fractional order cross-correlation between $s(t)$ and $h(t)$ can be defined by replacing T_τ with R_ρ^ϕ in the definition of the LTI cross-correlation equation

$$(s_0^* h)(\tau) = \langle s, T_\tau h \rangle = \int s(\beta) h^*(\beta - \tau) d\beta \quad (2.43)$$

Thus the fractional order cross-correlation can be defined as

$$(s_\phi^* h)(\rho) = e^{j2\pi \frac{\rho^2}{2} \cos \phi \sin \phi} \int s(\beta) h^*(\beta - \rho \cos \phi) e^{-j2\pi \beta \rho \sin \phi} d\beta \quad (2.44)$$

which reduces to the integer order definition of cross-correlation in (2.43) for $\phi = 0$.

The fractional order autocorrelation is defined as

$$(s_\phi^* s)(\rho) = e^{j2\pi \frac{\rho^2}{2} \cos \phi \sin \phi} \int s(\beta) s^*(\beta - \rho \cos \phi) e^{-j2\pi \beta \rho \sin \phi} d\beta \quad (2.45)$$

which reduces to the integer order definition of autocorrelation for $\phi = 0$

$$(s_0^* s)(\tau) = \int s(\beta) s^*(\beta - \tau) d\beta \quad (2.46)$$

The defining property of an LTI system and their associated LTI operations is their covariance to time shifts. The LTI convolution fully reflects the effect of the unitary time shift operator T_k on the signal $s(t)$ as follows,

$$\begin{aligned} (T_k s_0^* h)(t) &= \langle T_k s, T_t \tilde{h} \rangle = \langle s, T_{-k} T_t \tilde{h} \rangle \\ &= \langle s, T_{t-k} \tilde{h} \rangle = (s_0^* h)(t - k) \end{aligned} \quad (2.47)$$

The same covariance property is also valid for the LTI cross-correlation and the LTI autocorrelation.

Similar to time-shift covariance of LTI systems, fractional convolution and correlation operations are covariant to fractional shifts represented by the unitary fractional-shift operator R_ρ^ϕ .

For fractional convolution this can be shown by

$$\begin{aligned} \left(R_k^\phi s_\phi^* h \right) (r) &= \left\langle R_k^\phi s, R_r^\phi \tilde{h} \right\rangle = \left\langle s, R_{-k}^\phi s, R_r^\phi \tilde{h} \right\rangle \\ &= \left\langle s, R_{r-k}^\phi \tilde{h} \right\rangle = \left(s_\phi^* h \right) (r - k) \end{aligned} \quad (2.48)$$

where we used the fact that R_ρ^ϕ is unitary and $\left(R_r^\phi \right)^{-1} = R_{-r}^\phi$ as given by the inversion property in (2.40).

Fractional correlation operators are also covariant to fractional shifts in the same fashion, i.e.

$$\left(R_k^\phi s_\phi^* h \right) (\rho) = \left(s_\phi^* h \right) (\rho - k) \quad (2.49)$$

For further details please see Akay and Boudreaux-Bartels (2001) and Torres et al. (2010).

2.5 Calculation of H_2 and H_∞ Norms of Fractional Order Systems

For FO LTI systems, the H_2 and H_∞ norms are similar to their corresponding integer order analogues with slight modifications as detailed in Malti et al. (2003) and Sabatier et al. (2005).

2.5.1 H_2 Norm of FO Systems

The H_2 norm of a transfer function $F(s)$, i.e., $\|F\|_{H_2} := \|F\|_2$ which is analogous to the impulse response energy $\|f\|_2$ or L_2 norm of a real signal $f(t)$ can be stated as:

$$\begin{aligned} \|f(t)\|_2 &= \left(\int_0^\infty f^T(t) f(t) dt \right)^{1/2} \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^\infty F^H(j\omega) F(j\omega) d\omega \right)^{1/2} = \|F(j\omega)\|_2 \end{aligned} \quad (2.50)$$

If it be assumed that $F(s)$ be a Laplace-domain BIBO-stable explicit commensurate fractional order (n) transfer function and q, ρ are the integer and non-integer parts of $1/n$ respectively, i.e. $q := \lfloor \frac{1}{n} \rfloor$, $\rho := \frac{1}{n} - \lfloor \frac{1}{n} \rfloor$ and let,

$$G(\omega^n) := F(j\omega), \frac{A(x)}{B(x)} := G(x) \overline{G(x)} \quad (2.51)$$

Then the H_2 norm of $F(s)$ is given by the following conditions:

1. If $\deg(B) \leq \deg(A) + 1/n$, then $\|F\|_2 = \infty$.
2. If $\deg(B) > \deg(A) + 1/n$ and $\rho \neq 0$, then

$$\|F\|_2 = \frac{\sum_{k=1}^r \sum_{l=1}^{v_k} (-1)^{l-1} a_{k,l} s_k^{\rho-1} \binom{\rho-1}{l-1}}{n \sin(\rho\pi)} \quad (2.52)$$

where $a_{k,l}$, $(-s_k)$ and v_k are co-efficients, poles and their multiplicities of the partial fraction expansion of

$$x^q \frac{A(x)}{B(x)} = \sum_{k=1}^r \sum_{l=1}^{v_k} \frac{a_{k,l}}{(x + s_k)^l}$$

3. If $\deg(B) > \deg(A) + \frac{1}{n}$ and $\rho = 0$:

$$\|F\|_2^2 = \sum_{k=2}^r \frac{c_k (\ln(s_k) - \ln(s_1))}{n\pi (s_k - s_1)} + \sum_{k=1}^r \sum_{l=2}^{v_k} \frac{b_{k,l} s_k^{1-l}}{n\pi (l-1)} \quad (2.53)$$

where $(-s_k)$ and v_k are poles and their multiplicities of $x^{q-1}Q(x)$. s_1 is an arbitrary chosen pole. c_k and $b_{k,l}$ represent coefficients of the following expansion of $x^{q-1}Q(x)$:

$$x^{q-1} \frac{A(x)}{B(x)} = \sum_{k=2}^r \frac{c_k}{(x + s_1)(x + s_k)} + \sum_{k=1}^r \sum_{l=2}^{v_k} \frac{b_{k,l}}{(x + s_k)^l} \quad (2.54)$$

All poles of order one are in the left sum; all poles of order greater than one are in the right double sum.

2.5.2 H_∞ Norm of FO Systems

The H_∞ -norm of a continuous LTI system model $G(s)$ is defined through the L_2 norm as:

$$\|G(s)\|_\infty = \sup_{U(s) \in H_2} \frac{\|Y(s)\|_2}{\|U(s)\|_2} \quad (2.55)$$

Here, $Y(s)$, $U(s) \in H_2$ represent the Laplace transform of the output signal and input signal respectively. Also,

$$\|G(s)\|_{\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega)) \quad (2.56)$$

where $\sigma_{\max}$ denotes the greatest singular value of $G(j\omega)$ defined by

$$\sigma(G(j\omega)) = \sqrt{\text{spec}(G^T(-j\omega)G(j\omega))} \quad (2.57)$$

The H_{∞} -norm of a fractional system may not always be bounded. For the commensurate fractional state space system S_f

$$(S_f) : \begin{cases} D^v x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (2.58)$$

where $v \in \mathbb{R}_+$ denotes the fractional order of the system,

$$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{p \times l}, C \in \mathbb{R}^{m \times n} \text{ and } D \in \mathbb{R}^{m \times p}$$

The H_{∞} -norm is bounded by a real positive number γ if and only if the eigenvalues of the matrix A_{γ} lie in the stable domain defined by $\{s \in \mathbb{C} : |\arg(s)| > v\pi/2\}$. Here,

$$A_{\gamma} = \begin{pmatrix} A + B(\gamma^2 I - D^T D)^{-1} D^T C & e^{-vj\pi} B(\gamma^2 I - D^T D)^{-1} B^T \\ C^T (I + D(\gamma^2 I - D^T D)^{-1} D^T) C & e^{-vj\pi} (A^T + C^T D(\gamma^2 I - D^T D)^{-1} B^T) \end{pmatrix}$$

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