Chapter 2
Stationary and Periodic Solutions of Differential Equations

2.1 Stationary and Periodic Stochastic Processes. Convergence of Stochastic Processes

A stochastic process $\xi(t) = \xi(t, \omega)$ $(-\infty < t < \infty)$ with values in $\mathbb{R}^l$ is said to be stationary (in the strict sense) if for every finite sequence of numbers $t_1, \dots, t_n$ the joint distribution of the random variables $\xi(t_1 + h), \dots, \xi(t_n + h)$ is independent of h. If we replace the arbitrary number h by a multiple of a fixed number θ, $h = k\theta$ $(k = \pm 1, \pm 2, \dots)$, we get the definition of a periodic stochastic process with period θ, or a θ-periodic stochastic process.[1] Stationary and periodic stochastic processes constitute a mathematical idealization of physical noise acting on linear and nonlinear devices functioning in a medium with unvarying or periodically varying properties.

Let $\xi(t)$ be a stationary stochastic process with finite variance. By the definition of stationarity,

$$\begin{gathered} \mathbf{E}\xi(t) = m = \text{const}, \qquad \operatorname{var}\xi(t) = \mathbf{D} = \text{const}, \\ K(s,t) = \operatorname{cov}(\xi(s), \xi(t)) = K(t-s). \end{gathered} \tag{2.1}$$

As already mentioned in Chap. 1, a process satisfying conditions (2.1) is said to be stationary in the wide sense. An important characteristic of stationary processes is their spectral density (see Sect. 1.1).

If $\xi(t)$ is a θ-periodic stochastic process, then $\mathbf{E}\xi(t) = m(t)$ and $\operatorname{var}\xi(t) = D(t)$ are evidently periodic with the same period, i.e.

$$m(t+\theta) = m(t), \qquad \mathbf{D}(t+\theta) = \mathbf{D}(t). \tag{2.2}$$

[1] There is an enormous literature on the properties of stationary stochastic processes. Among others, we might mention the paper [283] and the books [56], [241], [99]. The properties of periodic processes to be discussed below may be found, e.g., in a paper [57] and in [254].

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The matrix-valued function $K(s,t)$ satisfies then the condition

$$K(s+\theta, t+\theta) = K(s,t) \tag{2.3}$$

for all s, t. A process whose moments satisfy (2.2) and (2.3) is said to be periodic in the wide sense.

It is obvious that a stationary process is periodic with arbitrary period. Conversely, a periodic process can be made stationary by a simple transformation (shift of the argument). Indeed, if τ is a random variable uniformly distributed on the interval $[0,\theta]$ and independent of the θ-periodic stochastic process $\xi(t)$, then the process $\eta(t) = \xi(t+\tau)$ is stationary. To prove this it suffices to observe that for every $t_1, \dots, t_n, A_1, \dots, A_n$ the function $\mathbf{P}\{\xi(t_1+h) \in A_1, \dots, \xi(t_n+h) \in A_n\}$ is θ-periodic with respect to h, and therefore, for every h,

$$\begin{aligned}
&\mathbf{P}\{\eta(t_1+h) \in A_1, \dots, \eta(t_n+h) \in A_n\} \\
&\quad = \frac{1}{\theta}\int_0^\theta \mathbf{P}\{\xi(t_1+s+h) \in A_1, \dots, \xi(t_n+s+h) \in A_n\}\, ds \\
&\quad = \frac{1}{\theta}\int_0^\theta \mathbf{P}\{\xi(t_1+s) \in A_1, \dots, \xi(t_n+s) \in A_n\}\, ds \\
&\quad = \mathbf{P}\{\eta(t_1) \in A_1, \dots, \eta(t_n) \in A_n\}.
\end{aligned}$$

It is easily verified that by averaging the moments of the process $\xi(t)$ over the period we obtain the corresponding moments of the process $\eta(t)$. For example,

$$\mathbf{E}\eta(t) = \frac{1}{\theta}\int_0^\theta \mathbf{E}\xi(s)\, ds,$$

$$\operatorname{cov}(\eta(s), \eta(t)) = \frac{1}{\theta}\int_0^\theta \operatorname{cov}(\xi(s+h), \xi(t+h))\, dh.$$

It is evident that a deterministic periodic function can be regarded as a periodic stochastic process. After a suitable shift of the argument we get a stationary process.

Let $f(t,x)$ be a Borel-measurable function, θ-periodic in t, and $\xi(t)$ a θ-periodic stochastic process. It is then readily seen that the process $f(t,\xi(t))$ is also θ-periodic. For example, if τ is a random variable uniformly distributed on the interval $[0, 2\pi]$, then the process $\xi \sin(t+\tau)$ is stationary for every random variable ξ independent of τ, while the process $\xi \cos t \sin(t+\tau)$ is 2π-periodic. The sample functions of the processes in these examples are periodic. It is easy to construct also examples of periodic processes which almost surely have no periodic sample functions (paths).

In this chapter we shall frequently have to deal with sequences of random variables and with stochastic processes converging in various senses. Therefore let us first recall various definitions of convergence and some results connected with them.[2]

[2]See [232], [251], [92].

A sequence of measures $\{\mu_n\}$ defined in $(\mathbb{R}^l, \mathfrak{B})$ is said to be weakly convergent to a measure μ if

$$\lim_{n\to\infty}\int_{\mathbb{R}^l} f(x)\mu_n(dx) = \int_{\mathbb{R}^l} f(x)\mu(dx)$$

for every continuous and bounded function $f(x)$ on $\mathbb{R}^l$.

A sequence of random variables ξ_n is *weakly convergent* to ξ if the sequence of measures $\mathbf{P}_n(A) = \mathbf{P}\{\xi_n \in A\}$ converges weakly to the measure $\mathbf{P}(A) = \mathbf{P}\{\xi \in A\}$.

A sequence of random variables ξ_n is said to be *weakly compact* if it contains a weakly convergent subsequence. A sufficient condition for a sequence ξ_n to be weakly compact is that the random variables ξ_n be uniformly bounded in probability, i.e.,

$$\sup_n \mathbf{P}\{|\xi_n| > R\} \to 0 \quad \text{as } R \to \infty.$$

A sequence ξ_n is said to *converge in probability* to ξ if $\mathbf{P}\{|\xi_n - \xi| > \delta\} \to 0$ as $n \to \infty$ for each $\delta > 0$.

Given a sequence ξ_n which converges weakly to ξ_0, one can construct a sequence $\tilde{\xi}_n$ $(n = 0, 1, 2, \ldots)$ in another probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}}, \widetilde{\mathbf{P}})$ such that $\tilde{\xi}_n \to \tilde{\xi}_0$ in probability and the variables ξ_n and $\tilde{\xi}_n$ have the same distribution function for every $n \geq 0$. Skorokhod [251] has generalized these results to stochastic processes as follows.

Theorem 2.1 *Let $\xi_n(t,\omega)$ $(n = 1, 2, \ldots)$ be a sequence of stochastic processes in $\mathbb{R}^l$ such that for every $t_1, \ldots, t_k$ the joint distribution of $\xi_n(t_1), \ldots, \xi_n(t_k)$ is weakly convergent to some limit, and the sequence $\xi_n(t)$ is uniformly stochastically continuous, i.e.,*

$$\sup_{n,\ |s_1 - s_2| < h} \mathbf{P}\{|\xi_n(s_1) - \xi_n(s_2)| > \varepsilon\} \underset{h\to 0}{\longrightarrow} 0. \tag{2.4}$$

Then one can construct a sequence of stochastic processes $\tilde{\xi}_n(s)$ $(n = 0, 1, 2, \ldots)$ in another probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}}, \widetilde{\mathbf{P}})$ such that the process $\tilde{\xi}_0(s)$ is stochastically continuous, $\tilde{\xi}_n(s) \to \tilde{\xi}_0(s)$ in probability for all s and the finite-dimensional distributions of the processes $\xi_n(s)$ and $\tilde{\xi}_n(s)$ coincide for $n > 0$.

Theorem 2.2 *A sufficient condition for a sequence of stochastic processes $\xi_n(t)$ to contain a subsequence of processes with weakly convergent finite-dimensional distributions is that the sequence satisfies condition* (2.4) *and is uniformly bounded in probability*:

$$\sup_{t,n} \mathbf{P}\{|\xi_n(t)| > R\} \to 0 \quad \textit{as } R \to \infty. \tag{2.5}$$

Let the processes $\xi_n(t)$, $\xi(t)$ be continuous on the interval $[a, b]$. Let $\mathcal{C}[a, b]$ denote the space of all continuous functions on $[a, b]$; all the sample functions of the processes $\xi_n(t)$, $\xi(t)$ are almost surely in this class.

A sequence $\xi_n(t)$ is said to be *weakly convergent* to $\xi(t)$ as $n \to \infty$ if for every functional f continuous on $\mathcal{C}[a, b]$

$$\mathbf{E} f(\xi_n(t)) \underset{n\to\infty}{\longrightarrow} \mathbf{E} f(\xi(t)).$$

Prokhorov [232] has proved the following theorem.

Theorem 2.3 *If the finite-dimensional distributions of the processes $\xi_n(t)$ are weakly convergent to some limit and there exist $\alpha > 1$, $k > 0$ and $a > 0$ such that for all t_1, t_2 and n*

$$\mathbf{E}|\xi_n(t_2) - \xi_n(t_1)|^a < k|t_2 - t_1|^\alpha,$$

then the sequence of processes $\xi_n(t)$ is weakly convergent to a process $\xi(t)$ whose finite-dimensional distributions coincide with the above-mentioned limit distributions.

2.2 Existence Conditions for Stationary and Periodic Solutions[3]

An important part of the qualitative theory of differential equations is the study of periodic solutions of systems with periodic right-hand sides.

In a more general setting, this corresponds to the study of existence conditions and properties of periodic and stationary solutions of differential equations whose right-hand side is a periodic or stationary process in t for fixed values of the space variable.

In this section we shall present a general, but not sufficiently effective for applications, solution of this problem. In the next section we shall use this result to derive effective sufficient conditions for the existence of stationary and periodic solutions in terms of auxiliary functions.

Theorem 2.4 *Let $G(x, z)$ $(x \in \mathbb{R}^l,\ z \in \mathbb{R}^k)$ be a function and $\xi(t)$ a stationary stochastically continuous process in $\mathbb{R}^k$, satisfying conditions* (1.23), (1.24). *Then there exists a stationary solution of the equation*

$$\frac{dx}{dt} = G(x, \xi(t)) \tag{2.6}$$

which is stationarily related to $\xi(t)$ if and only if this equation has at least one solution $y(t, \omega)$ satisfying the condition

[3] Existence conditions for stationary and periodic solutions of differential equations with random right-hand side have been investigated under different assumptions and by other methods by Vorovich [269] and Dorogovtsev [57].

$$\frac{1}{T}\int_0^T \mathbf{P}\{|y(t,\omega)| > R\}\,dt \to 0 \quad as\ R \to \infty \tag{2.7}$$

uniformly in $T > T_0$ *(or* $T < -T_0$*).*

Proof Necessity is obvious, since every stationary solution $y(t,\omega)$ satisfies condition (2.7). To prove sufficiency, we first make the following observation. Solving (2.6) with initial condition $x(0) = x_0(\omega)$ by successive approximations, one may readily verify that the random variable $x(t,\omega)$ is measurable with respect to the minimal σ-algebra containing all possible events $\{\xi(s) \in A_1\}$ ($s \in [0,t]$) and $\{x_0(\omega) \in A_2\}$. Here and below, $A_i \in \mathfrak{B}$, where $\mathfrak{B}$ is the σ-algebra of Borel sets in Euclidean space. Therefore, in order to prove the existence of a stationary process $(X(t), \xi(t))$ satisfying (2.6) it will suffice to show that there exists a random variable $\eta(\omega)$ such that for all $t > 0$, $A_0, A_1, \ldots, A_m, s_1, \ldots, s_m$,

$$\begin{aligned}&\mathbf{P}\{\eta(\omega) \in A_0,\ \xi(s_1) \in A_1, \ldots, \xi(s_m) \in A_m\}\\&\quad = \mathbf{P}\{X(t) \in A_0,\ \xi(s_1 + t) \in A_1, \ldots, \xi(s_m + t) \in A_m\},\end{aligned} \tag{2.8}$$

where $X(t)$ is the solution of (2.6) with initial condition $x(0) = \eta(\omega)$.

Assume for definiteness that condition (2.7) holds with $T > 0$. Let $\tau_k(\omega)$ be a random variable, uniformly distributed on $[0,k]$ and independent of $\xi(t)$ and $y(0,\omega)$. We set $x_0^{(k)}(\omega) = y(\tau_k(\omega), \omega)$ and

$$x_k(t,\omega) = y(t + \tau_k(\omega), \omega), \qquad \xi_k(t,\omega) = \xi(t + \tau_k(\omega), \omega).$$

It is obvious that

$$\begin{aligned}&\mathbf{P}\{x_k(t) \in A_0, \xi_k(s_1) \in A_1, \ldots, \xi_k(s_m) \in A_m\}\\&\quad = \frac{1}{k}\int_0^k \mathbf{P}\{y(t+s) \in A_0, \xi(s_1 + s) \in A_1, \ldots, \xi(s_m + s) \in A_m\}\,ds.\end{aligned} \tag{2.9}$$

It follows from (2.9) that for every k the distribution of the process $\xi_k(t)$ is the same as that of the process $\xi(t)$. It also follows from (2.7) that uniformly in $k > 0$,

$$\mathbf{P}\{|x_0^{(k)}(\omega)| > R\} = \frac{1}{k}\int_0^k \mathbf{P}\{|y(t)| > R\}\,dt \underset{R\to\infty}{\longrightarrow} 0. \tag{2.10}$$

By the stochastic continuity of the process $\xi(t)$ and by (2.10), the family $(x_0^{(k)}(\omega), \xi^{(k)}(t,\omega))$ satisfies conditions (2.4) and (2.5). Applying Theorems 2.1 and 2.2, we see that in some probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}}, \widetilde{\mathbf{P}})$ there is a sequence $(\tilde{x}_0^{(k)}(\widetilde{\omega}), \tilde{\xi}^{(k)}(t,\widetilde{\omega}))$ with the same distribution as $(x_0^{(k)}(\omega), \xi^{(k)}(t,\omega))$, such that some subsequence $(\tilde{x}_0^{(n_k)}(\widetilde{\omega}), \tilde{\xi}^{(n_k)}(t,\widetilde{\omega}))$ converges in probability to $(\tilde{x}(\widetilde{\omega}), \tilde{\xi}(t,\widetilde{\omega}))$. Obviously, the finite-dimensional distributions of the processes $\tilde{\xi}(t,\widetilde{\omega})$ and $\xi(t,\omega)$ are the same.

We can now construct on the original probability space random variables $x(\omega)$ and $x^{(n_k)}(\omega)$ whose joint distribution with $\xi(t,\omega)$ is the same as the joint distribution of

$$\tilde{x}(\widetilde{\omega}), \quad \tilde{x}_0^{(n_k)}(\widetilde{\omega}), \quad \tilde{\xi}(t,\widetilde{\omega}).$$

We shall prove that (2.8) holds for $\eta(\omega) = x(\omega)$. Let $X_{n_k}(t)$ $(k = 1, 2, \ldots)$ denote the solution of (2.6) with initial condition $X_{n_k}(0) = x^{(n_k)}(\omega)$. Now conditions (1.23), (1.24) and the Gronwall–Bellman lemma imply the inequality

$$|X_{n_k}(t) - X(t)| < |x^{(n_k)}(\omega) - x(\omega)| \exp\left\{\int_0^t B(u,\omega)\,du\right\},$$

and so $X_{n_k}(t) \to X(t)$ in probability for every t. Let f be a continuous bounded function. Then, by what we have proved it follows from (2.9) that for each t and $s_1, \ldots, s_m$,

$$\begin{aligned}
&\mathbf{E} f(\xi(s_1+t), \ldots, \xi(s_m+t), X(t)) \\
&\quad = \lim_{k\to\infty} \mathbf{E} f(\xi(s_1+t), \ldots, \xi(s_m+t), X_{n_k}(t)) \\
&\quad = \lim_{k\to\infty} \mathbf{E} f(\xi_{n_k}(s_1+t), \ldots, \xi_{n_k}(s_m+t), x_{n_k}(t)) \\
&\quad = \lim_{k\to\infty} \frac{1}{n_k}\int_0^{n_k} \mathbf{E} f(\xi(s_1+t+u), \ldots, \xi(s_m+t+u), y(t+u))\,du \\
&\quad = \lim_{k\to\infty} \frac{1}{n_k}\int_0^{n_k} \mathbf{E} f(\xi(s_1+s), \ldots, \xi(s_m+s), y(s))\,ds \\
&\quad = \mathbf{E} f(\xi(s_1), \ldots, \xi(s_m), x(\omega)).
\end{aligned} \tag{2.11}$$

This implies (2.8), and hence the assertion of the theorem. □

The analogous result for a periodic process $\xi(t)$ is given by the following theorem.

Theorem 2.5 *Let $G(x,z)$ $(x \in \mathbb{R}^\ell, z \in \mathbb{R}^k)$ be a given function and $\xi(t)$ a θ-periodic stochastically continuous process in $\mathbb{R}^k$ satisfying conditions* (1.23), (1.24). *Then there exists a periodic solution of* (2.6) *which is periodically related to $\xi(t)$ if and only if the equation has at least one solution $y(t,\omega)$ satisfying the condition*

$$\frac{1}{|k|+1}\sum_{n=0}^{k} \mathbf{P}\{|y(n\theta)| > R\} \to 0 \quad \text{as } R \to \infty \tag{2.12}$$

uniformly in $k = 1, 2, \ldots$ (or $k = -1, -2, \ldots$).

Proof The proof is entirely analogous to that of Theorem 2.4. The only difference is that instead of the processes $x_k(t) = y(t + \tau_k)$ one must consider a sequence

$Y_k(t) = y(t + \chi_k)$, where χ_k is a random variable independent of $\xi(t)$ and $y(0, \omega)$ such that $\mathbf{P}\{\chi_k = n\theta\} = 1/(k+1)$ $(n = 0, 1, \ldots, k)$. □

As we shall see in the next section, the advantage of condition (2.7) over (2.12) is that it is easier to verify whether (2.7) holds even if no solutions of (2.6) are known. Thus, the following lemma may be sometimes useful.

Lemma 2.1 *Condition* (2.12) *of Theorem* 2.5 *can be replaced by condition* (2.7).

Proof The necessity of condition (2.7) is obvious. Let us prove the sufficiency. Let $y(t) = y(t, \omega)$ be a solution of (2.6) satisfying condition (2.7). Then for each τ, $z(t) = y(t + \tau)$ is a solution of the equation

$$\frac{dz}{dt} = G(z, \xi(t + \tau)). \tag{2.13}$$

Now let τ be a random variable uniformly distributed on $[0, \theta]$ and independent of the process $\xi(t)$. Then, as shown in Sect. 2.1, $\xi(t + \tau)$ is a stationary process. Moreover, the solution $z(t)$ of (2.13) satisfies condition (2.7), since

$$\begin{aligned}\frac{1}{T}\int_0^T \mathbf{P}\{|z(t)| > R\}\,dt &= \frac{1}{\theta}\int_0^\theta ds\,\frac{1}{T}\int_0^T \mathbf{P}\{|y(t+s)| > R\}\,dt\\ &\le \frac{T+\theta}{T}\,\frac{1}{\theta}\int_0^\theta ds\,\frac{1}{T+s}\int_0^{T+s} \mathbf{P}\{|y(u)| > R\}\,du.\end{aligned}$$

Applying Theorem 2.4, we see that (2.13) has a solution $Z_1(t, \omega)$ which is a stationary process. It follows from Theorem 1.5 that

$$\sup_{0\le t\le\theta} |Z_1(t)| \le |Z_1(0)| + \int_0^\theta |G(Z_1(0), \xi(s+\tau))|\,ds \exp\left\{\int_0^\theta B(s+\tau, \omega)\,ds\right\}.$$

By conditions (1.23), (1.24) and the stationarity of the process Z_1, the probability of the event

$$\left\{\sup_{s\le t\le s+\theta} |Z_1(t)| > R\right\}$$

is independent of s and

$$\mathbf{P}\left\{\sup_{s\le t\le s+\theta} |Z_1(t)| > R\right\} \to 0 \quad \text{as } R \to \infty. \tag{2.14}$$

It is clear now that the function $y_1(t, \omega) = Z_1(t - \tau(\omega), \omega)$ is a solution of (2.6). By (2.14), this solution satisfies condition (2.12). Hence, by Theorem 2.5, it follows that (2.6) has a periodic solution. This completes the proof of the lemma. □

Remark 2.1 The global Lipschitz condition (1.23) is sometimes too restrictive. It can be seen from the proofs of Theorem 2.4 and Lemma 2.1 that this condition is used only to verify (2.14) and the relation

$$X_{n_k}(t) \to X(t) \quad \text{in probability as } k \to \infty. \tag{2.15}$$

These relations hold if the solutions of (2.6) are uniformly unboundedly continuable in the sense of Remark 1.4 and conditions (1.24), (1.28) are satisfied.

In fact, by conditions (1.24), (1.28) and the Gronwall–Bellman lemma, for every fixed t_0 and all sample functions $X_{n_k}(t,\omega)$, $X(t,\omega)$ satisfying the conditions

$$\sup_{0\le t\le t_0} |X_{n_k}(t)| \le R, \qquad \sup_{0\le t\le t_0} |X(t)| \le R, \tag{2.16}$$

we get the inequality

$$|X_{n_k}(t_0) - X(t_0)| \le |x^{(n_k)}(\omega) - x(\omega)| \exp\left\{ \int_0^{t_0} B_R(t,\omega)\,dt \right\}. \tag{2.17}$$

Let $\varepsilon > 0$ be arbitrary. Since the solutions of (2.6) are uniformly unboundedly continuable, the probability of the events (2.16) can be made greater than $1 - \varepsilon/2$ by choosing R sufficiently large. Hence and by considering (2.17) for sufficiently large k we get the inequalities

$$\begin{aligned} &\mathbf{P}\{|X_{n_k}(t_0) - X(t_0)| > \varepsilon\} \\ &\quad \le \frac{\varepsilon}{2} + \mathbf{P}\left\{ |x^{(n_k)}(\omega) - x(\omega)| \exp\left(\int_0^{t_0} B_R(t,\omega)\,dt \right) > \frac{\varepsilon}{2} \right\} \le \varepsilon. \end{aligned}$$

This proves (2.15). The proof of (2.14) is analogous.

This remark, together with Theorem 1.7, implies the following corollaries.

Corollary 2.1 *Let the function $F(x,t)$, $\sigma(x,t)$ and the stochastic process $\xi(t)$ be θ-periodic and satisfy the assumptions of Theorem* 1.7. *Assume also that the equation $dx/dt = F(x,t) + \sigma(x,t)\xi(t)$ has a solution satisfying condition* (2.7). *Then this equation also has a θ-periodic solution. Similarly, if F and σ are independent of t and $\xi(t)$ is a stationary process, then the above conditions imply the existence of a stationary solution.*

Corollary 2.2 *Conditions* (2.7) *and* (2.12) *are valid if the system* (2.6) *is dissipative. Therefore, if the system* (2.6) *is dissipative, $\xi(t,\omega)$ is a stationary* (*periodic*) *stochastically continuous process and conditions* (1.23), (1.24) *are satisfied, then the system* (2.6) *has a stationary* (*periodic*) *solution.*

Example 2.1 Let $G(x,t)$ be a deterministic function which is θ-periodic in t and such that conditions (1.23), (1.24) are satisfied and the equation ($x, G \in \mathbb{R}^l$)

$$\frac{dx}{dt} = G(x,t) \tag{2.18}$$

has at least one bounded solution. It follows from Theorem 2.5 that for some (generally random) initial condition the solution of (2.18) is a periodic stochastic process. For $l \leq 2$ this follows also from a well-known theorem of Massera (see [228, p. 186]). Of course, this result does not guarantee the existence of a deterministic periodic solution of (2.18), since a periodic stochastic process need not have periodic sample functions.

2.3 Special Existence Conditions for Stationary and Periodic Solutions

For systems of the special form

$$\frac{dx}{dt} = F(x,t) + \sigma(x,t)\xi(t) \tag{2.19}$$

one can derive effective conditions which are sufficient for the existence of periodic and stationary solutions.

Theorem 2.6 *Suppose that the vector $F(x,t)$ and the matrix $\sigma(x,t)$ are θ-periodic in t and that they satisfy a local Lipschitz condition*; *let further $F(0,t) \in \mathbf{L}$ and*

$$\sup_{x,t} \|\sigma(x,t)\| < \infty. \tag{2.20}$$

Assume moreover that the truncated system

$$\frac{dx}{dt} = F(x,t)$$

has a Lyapunov function $V(x,t) \in \mathbf{C}_0$ satisfying the following conditions:

1. *$V(x,t)$ is nonnegative*, *and*

$$\inf_{t>0} V(x,t) \to \infty \quad \textit{as } |x| \to \infty.$$

2. *d^0V/dt is bounded above*, *and* $\sup_{t>0} d^0V/dt \to -\infty$ *as* $|x| \to \infty$.

Then (2.19) has a θ-periodic solution for each θ-periodic stochastically continuous process $\xi(t)$ with finite expectation. If F and σ independent of t and $\xi(t)$ is a stationary process, *then the same conditions imply the existence of a stationary solution.*

Proof Let $x(t) = x(t,\omega)$ be a solution of (2.19) satisfying the condition $x(t_0) = x_0$. Using Condition 1 of the theorem, inequality (2.20) and Lemma 1.3, we see that almost surely, for $t > t_0$ and some constant $k > 0$,

$$-V(x_0,t_0) \leq V(x(t),t) - V(x_0,t_0)$$

$$\leq \int_{t_0}^{t} \frac{d^0 V(x(s), s)}{ds}\, ds + k \int_{t_0}^{t} |\xi(s)|\, ds. \tag{2.21}$$

We set $k_1 = \sup_E d^0V/dt$, $-c_r = \sup_{|x|>r} d^0V/dt$. It follows from the assumptions of the theorem that

$$k_1 < \infty, \qquad c_r \to \infty \quad \text{as } r \to \infty. \tag{2.22}$$

Replacing for $|x(s)| > r$ the function d^0V/ds in (2.21) by the bound $-c_r$ and for $|x(s)| \leq r$ by the bound k_1 and then taking expectations, we get

$$-V(x_0, t_0) \leq -c_r \int_{t_0}^{t} \mathbf{P}\{|x(s)| > r\}\, ds + k_1(t - t_0) + k \int_{t_0}^{t} \mathbf{E}|\xi(s)|\, ds.$$

Hence it follows by (2.22) that for some constant k_2

$$\frac{\int_{t_0}^{t} \mathbf{P}\{|x(s)| > r\}\, ds}{t - t_0} < \frac{k_2}{c_r} \to 0 \quad \text{as } r \to \infty. \tag{2.23}$$

Condition (2.23) is equivalent to (2.7). Applying Lemma 2.1 and Corollary 2.1, we get the first assertion of the theorem. The second assertion can be proved in the same way. □

Remark 2.2 The assertion of the theorem is valid when the assumption that $\inf_{t>0} V(x, t) \to \infty$ as $|x| \to \infty$ is replaced by the assumption that the solutions of (2.19) are uniformly unboundedly continuable for $t > 0$. It is also sufficient to require that the solutions be unboundedly continuable for $t < 0$ and that the following condition holds: The function d^0V/dt is bounded below and $d^0V/dt \to \infty$ as $|x| \to \infty$. (This case reduces to the preceding one if we set $s = -t$.)

Example 2.2 If the system (2.19) is one-dimensional ($x \in \mathbb{R}^1$), then, considering the Lyapunov function $V = |x|$, we have $d^0V/dt = F(x, t) \operatorname{sign} x$. Hence Theorem 2.6 and Remark 2.2 yield the following result.

If F and σ are periodic functions of t such that

$$F \in \mathbf{C}_0, \qquad \sigma \in \mathbf{C}_0, \qquad \sup |\sigma| < \infty$$

and either $F(x, t) \operatorname{sign} x \to -\infty$ or $F(x, t) \operatorname{sign} x \to \infty$ as $|x| \to \infty$, then (2.19) has a periodic solution in $\mathbb{R}^1$ for every periodic process $\xi(t)$ with bounded expectation. An analogous conclusion holds for stationary solutions as well.

For example, if $f(t)$ is a θ-periodic continuous function and $\xi(t)$ a θ-periodic process, then the equation $dx/dt = xf(t) + \xi(t)$ always has a periodic solution, provided $\mathbf{E}|\xi(t)| < \infty$ and $f(t)$ does not vanish. On the other hand it is obvious that if $F(x, t) > k > -\infty$ (or $F(x, t) < k < \infty$), then (2.19) need not have periodic solutions, since for a suitable choice of σ and ξ the right-hand side of (2.19) will have fixed sign. A more general result is given by the following

Lemma 2.2 *Let $F(x) \in \mathbf{C}$ be a function for which none of the conditions*

$$F(x)\operatorname{sign} x \underset{|x|\to\infty}{\longrightarrow} \pm\infty \tag{2.24}$$

is valid. Then there exists a stationary stochastic process $\xi(t)$ with finite expectation such that the equation

$$\frac{dx}{dt} = F(x) + \xi(t) \tag{2.25}$$

has no stationary solution.

Proof As already mentioned, the assertion is obvious if the function $F(x)$ is bounded above or below. If it is neither and conditions (2.24) do not hold, then there exist an infinite sequence of points α_k ($k = 1, 2, \dots$) and a number c such that $\alpha_k \to \infty$ or $\alpha_k \to -\infty$ as $k \to \infty$, $F(\alpha_k) = c$ and each α_k is a stable equilibrium point of the equation

$$\frac{dx}{dt} = F(x) - c. \tag{2.26}$$

To be more specific, suppose that $\alpha_k \to \infty$. Then the following three cases are possible:

(a) $F(x) \geq c$ for $x < x'$,
(b) $F(x) \leq c$ for $x < x'$,
(c) there exists a sequence $x'_k \to -\infty$ such that $F(x'_k) = c$ and the x'_k are stable equilibrium points of (2.26).

Case (a). We may assume without loss of generality that $x' = \alpha_1$. We claim that in this case (2.25) has no stationary solutions if $\xi(t) = -c + |\eta(t)|$, where $\eta(t)$ is a stationary stochastic process such that for every constant A

$$\mathbf{P}\left\{\sup_{0\leq u\leq t} \int_u^{u+1} |\eta(s)|\,ds > A\right\} \to 1 \quad \text{as } t \to \infty. \tag{2.27}$$

(Condition (2.27) holds for instance if $\eta(t)$ is a Gaussian stationary Markov process governed by the generator $d^2/dx - x\,d/dx$.)

Suppose that there exists a stationary process $x(t)$ satisfying (2.25). Since $F(x) \geq c$ for $x < \alpha_1$, the function $x(t)$ is monotone increasing for $x(t) < \alpha_1$, and therefore

$$\mathbf{P}\{x(0,\omega) < \alpha_1\} = 0. \tag{2.28}$$

We shall prove that $\mathbf{P}\{\alpha_1 \leq x(0,\omega) < \alpha_2\} = 0$. To this end, we first observe that, by construction, the points α_k have the following property: once the sample function $x(t_0) = \alpha_k$ at some time t_0, it "cannot" go to the left of α_k for $t > t_0$. Hence in this case it follows from (2.25) that either $X(t+1) > \alpha_2$ or

$$X(t+1) - X(t) \geq \int_t^{t+1} |\eta(s)|\,ds + \min_{x\in[\alpha_1,\alpha_2]} (F(x) - c).$$

Hence follows the relation

$$\{x(0,\omega) \geq \alpha_1\} \cap \left\{ \sup_{0\leq u\leq t-1} \int_u^{u+1} |\eta(s)|\,ds + \min_{x\in[\alpha_1,\alpha_2]} (F(x)-c) > \alpha_2 - \alpha_1 \right\}$$
$$\subset \{x(t,\omega) \geq \alpha_2\}. \tag{2.29}$$

By (2.27), (2.28) and (2.29),

$$\mathbf{P}\{x(0,\omega) \geq \alpha_2\} = \lim_{t\to\infty} \mathbf{P}\{x(t,\omega) \geq \alpha_2\} = 1.$$

Similarly, we show that $\mathbf{P}\{x(0,\omega) \geq a_k\} = 1$ for every k. This contradiction shows that a stationary solution does not exist. The proof for cases (b) and (c) is similar. (In case (b) one sets $\xi(t) = -c - |\eta(t)|$.) □

Example 2.3 [4] Suppose that for $|x| > x_0$ and some positive integers n and k the coefficients of the equation

$$x'' + f(x)x' + g(x) = \sigma(x,x')\xi(t,\omega) \tag{2.30}$$

satisfy the conditions

$$0 < g(x)/x^{2n+1} < c, \qquad 0 < f(x)/x^{2k} < c,$$

and also the conditions

$$|\sigma| < c; \qquad g(x)F(x) \to \infty \quad \text{as } |x| \to \infty \quad \left(F(x) = \int_0^x f(t)\,dt\right),$$
$$F(x)\operatorname{sign} x > \delta > 0 \quad \text{for } |x| > x_0.$$

Let $\xi(t)$ be a periodic (stationary) stochastic process with finite expectation. Then (2.30) has a periodic (stationary) solution. The proof utilizes Theorem 2.6 applied to the system of equations derived from (2.30), where we set $x' = y$, and take the Lyapunov function

$$V(x,y) = \left[\frac{y^2}{2} + (F(x)-p(x))y + G(x) + \int_0^x f(t)(F(t)-p(t))\,dt + 1\right]^\alpha - c_1,$$

with $G(x) = \int_0^x g(t)\,dt$, $p(x) = \gamma \arctan x$, and the positive constants γ, c_1, α so chosen that

$$\min V(x,y) = 0, \qquad V \in \mathbf{C}_0,$$
$$d^0V/dt \to -\infty \quad \text{as } x^2 + y^2 \to \infty.$$

Note that the conditions of this example hold for a Van der Pol equation in which $f(x) = x^2 - 1$, $g(x) = x$.

[4] This example is due to Nevelson.

2.4 Conditions for Convergence to a Periodic Solution

Hitherto we have dealt only with conditions concerning the existence of a periodic (stationary) solution of a differential equation whose right-hand side is a periodic (stationary) process for fixed x. However only those periodic solutions are of practical interest which are stable, in the sense that if the initial conditions lie in a certain class, then the solutions ultimately converge to periodic solutions. In most cases it is sufficient to consider stability for initial conditions which are independent of the right-hand side of the system.

In some cases a periodic solution of a differential equation turns out to be stable in the large, i.e., every solution ultimately converges to a periodic solution. It is clear that if a periodic solution is stable in the large it is unique. These definitions are rather vague, for it is not clear in what sense one should understand the concepts "ultimately" and "convergence to a periodic process". The first of these concepts can be made rigorous as follows.

Definition 2.1 A periodic (stationary) solution $x^0(t,\omega)$ of (2.6) is stable in a certain sense for initial conditions belonging to a class $\mathbf{K}$ if for all random variables $x_0(t_0,\omega) \in \mathbf{K}$, a.s. the solution $x(t, x_0(t_0,\omega), t_0, \omega)$ of (2.6) with initial condition $x(t_0) = x_0(t_0,\omega)$ converges to $x^0(t,\omega)$ in that same given sense as $t_0 \to -\infty$.

In accordance with the various types of convergence (see Sect. 2.1), we can consider almost sure stability, stability in probability, weak stability, and so on. In this section we shall establish some sufficient conditions for almost sure stability.

The following theorem indicates the connection between the asymptotically stable compact invariant set of a deterministic equation and the periodic (stationary) solutions of the perturbed system obtained when a small stochastic process is superimposed on the deterministic system. To simplify the exposition, we shall confine ourselves to the case in which the invariant set is an equilibrium point, the system of equations is autonomous and the random perturbation stationary.

Theorem 2.7 *Let y_0 be an asymptotically stable singular point of the system*

$$\frac{dx}{dt} = F(x), \tag{2.31}$$

where $F(x) \in \mathbf{C}$. Let $g(x,z)$ $(x \in \mathbb{R}^l, z \in \mathbb{R}^k)$ be a bounded Borel-measurable function such that $\|\partial g(x,z)/\partial x\|$ is bounded in a neighborhood of the point y_0, and $\xi(t,\omega)$ a stochastically continuous stationary stochastic process in $\mathbb{R}^k$. Then for all sufficiently small $|\varepsilon|$ the equation

$$\frac{dx}{dt} = F(x) + \varepsilon g(x, \xi(t,\omega)) \tag{2.32}$$

has a stationary solution which almost surely satisfies the condition

$$\sup_{-\infty<t<\infty} |x(t,\omega) - y_0| < \delta(\varepsilon) \quad (\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0).$$

If moreover the point y_0 is asymptotically stable for the system (2.31) *in the linear approximation, then a sufficiently small neighborhood of the point y_0 contains a unique stationary solution of* (2.32) *which is almost surely stable for every initial condition $x_0(t_0, \omega)$ such that for some $\delta_1(\varepsilon)$*

$$\mathbf{P}\{|x_0(t_0, \omega) - y_0| < \delta_1(\varepsilon)\} = 1. \tag{2.33}$$

Proof Suppose y_0 is asymptotically stable for the system (2.31) and consider a fixed neighborhood of y_0. If $|\varepsilon|$ and $|x(t_0) - y_0|$ are sufficiently small, then no solution of the system (2.32) can leave this neighborhood for $t > t_0$. This follows directly from the stability of the solution $x(t) \equiv y_0$ of (2.31) with respect to continually acting perturbations (see [191]). This together with Theorem 2.4 implies the first part of the theorem.

Since the linear system

$$\frac{dz}{dt} = \frac{\partial F}{\partial x}(y_0)z$$

is asymptotically stable and the matrix $((\partial g/\partial x))$ is bounded in a neighborhood of y_0, there exists a constant $\delta_1(\varepsilon)$ such that for $|x_i - y_0| < \delta_1(\varepsilon)$, all $t > t_0$ and certain positive constants c and λ,

$$|x^{(2)}(t) - x^{(1)}(t)| < ce^{-\lambda(t-t_0)}, \tag{2.34}$$

where $x^{(i)}(t)$ is a solution of (2.32) with initial condition $x^{(i)}(t_0) = x_i$, $i = 1, 2$.

Let $X(t, \omega)$ be some stationary solution of (2.32) in the $\delta_1(\varepsilon)$-neighborhood of the point y_0, and $Y^{(t_0)}(t, \omega)$ a solution of (2.32) satisfying the initial condition $Y^{(t_0)}(t_0, \omega) = x_0(t_0, \omega)$, where $x_0(t_0, \omega)$ satisfies condition (2.33). Setting $x^{(1)} = X(t_0, \omega)$, $x^{(2)} = x_0(t_0, \omega)$ in (2.34), we see that

$$\mathbf{P}\left\{\lim_{t_0 \to -\infty} Y^{(t_0)}(t, \omega) = X(t, \omega)\right\} = 1$$

as $t_0 \to -\infty$ which implies the required assertions. □

Note that if we set $x^{(1)} = X(t_0, \omega)$ in (2.34) and let $t_0 \to -\infty$, the evolution of the process $X(t, \omega)$ for $t \in (-\infty, s)$ is determined by that of the process $\xi(t, \omega)$ on the same interval. If moreover $g(x, z)$ is invertible as a function of z, the converse is also true. Thus the process $X(t, \omega)$ has the same regularity and mixing properties (see [241]) as the process $\xi(t, \omega)$.

Theorem 2.8 *Let G be a function which is θ-periodic in t* (*independent of t*) *and satisfies the assumptions of Theorem* 1.11, *and $\xi(t, \omega)$ a θ-periodic* (*stationary*) *stochastic process. Then the equation*

$$\frac{dx}{dt} = G(x, t, \xi(t, \omega))$$

has a unique periodic (*stationary*) *solution which is almost surely stable for any initial conditions such that* $\mathbf{P}\{|x_0(t_0, \omega)| < c\} = 1$ *for some c.*

The reader should have no difficulty in proving this theorem, employing the arguments used in the proofs of Theorems 2.7 and 1.11.

We conclude this chapter with the following comments.

Remark 2.3 1. Theorem 2.4, which is the fundamental theorem of this chapter, admits various generalizations. For example, it is not hard to prove a corresponding result for equations with delayed argument (see [142]) and for Itô (stochastic) equations (see Chap. 3).

In [106] similar methods were used to prove an analogous theorem for Itô equations with delay.

2. The problem of the existence and stability of stationary (periodic) solutions is also of interest for partial differential equations. For example, let us consider a simple model problem in the strip $0 < x < 1$, $-\infty < t < \infty$:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + c(x)u + f(x, \xi(t,\omega)) = Lu + f(x, \xi(t,\omega)),\\ &u(0,t) = u(1,t) = 0. \end{aligned}\right\} \tag{2.35}$$

It is readily shown that if $\xi(t)$ is a continuous stochastic process and $\mathbf{E} f(x, \xi(t,\omega))$ is bounded uniformly in $x \in [0,1]$, then problem (2.35) has a stationary solution in the following sense: There exists a function $u(x,t,\omega)$ satisfying the equation and the boundary conditions of (2.35) for almost all ω, which for each fixed x is a stationary stochastic process stationarily related to $\xi(t,\omega)$.

Let $p(x,t,y)$ denote the Green function of the problem

$$\frac{\partial u}{\partial t} = Lu, \qquad u(0,t) = u(1,t) = 0.$$

Then the above-mentioned stationary solution can be determined from the formula

$$u(x,t,\omega) = \int_{-\infty}^{t} ds \int_0^1 p(x, t-s, y) f(y, \xi(s,\omega))\, dy.$$

It is easy to show that this stationary solution is stable in the sense that every solution of problem (2.35) satisfying the initial condition $u(x,t_0) = \varphi(x,t_0)$ converges almost surely to $u(x,t,\omega)$ as $t_0 \to -\infty$, for every bounded function $\varphi(x,t_0)$.

This model can be readily generalized; for example, instead of homogeneous boundary conditions one can consider conditions of the form

$$u(0,t) = \xi_1(t,\omega), \qquad u_1(1,t) = \xi_2(t,\omega),$$

where $\xi_1(t,\omega)$, $\xi_2(t,\omega)$ are stationary and stationarily related stochastic processes.

It is also easy to prove the existence of a stationary solution in case of an unbounded domain, provided the coefficient $c(x)$ in the operator L satisfies the condition $c(x) \le c_0 < 0$. There is an analogous result for periodic solutions.

Apparently far more interesting but not so well investigated is the existence problem for stationary solutions of nonlinear partial differential equations. A few papers have been devoted to the solutions of the equations of hydrodynamics with stochastic coefficients (see the survey article [114] which includes a detailed bibliography).

3. We have established above certain results concerning the almost sure stability of stationary and periodic solutions. Although it seems that weak stability is rather more frequently encountered, no general conditions for weak stability are presently known. In particular, the following well-known problem seems to be yet unsolved. Let $F(x,t)$ $(x \in \mathbb{R}^1)$ be a periodic function such that $F(x,t)\,\mathrm{sign}\,x \to -\infty$ as $|x| \to \infty$. Consider the equation $dx/dt = F(x,t) + \xi(t,\omega)$. What restrictions do we have to impose on the periodic process $\xi(t)$ in order to ensure that every solution of this equation defined by an initial condition independent of $\xi(t)$ converges to some periodic solution? It seems probable that this property is shared by quite a broad class of processes $\xi(t)$. For example, it is known that even in the relatively "unfavorable" case of a deterministic process $\xi(t)$ the property always holds (see [228, Theorem 9.2]).

4. The question of stability of stationary and periodic solutions is intimately connected with the investigation of the properties of a stationary (periodic) solution of (2.19). Suppose that (2.19) has a stationary solution $x(t)$. To simplify matters, assume that F and σ are independent of t and $\xi(t)$ is a stationary process which is ergodic, regular, satisfies a mixing condition, etc. Under what restrictions on F, σ will the process $x(t)$ possess the analogous properties?

In the proof of Theorem 2.7 above we have answered this question only in the simplest case.

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