Chapter 1
Basic (Elementary) Inequalities and Their Application

There are many trivial facts which are the basis for proving inequalities. Some of them are as follows:

1. If $x \geq y$ and $y \geq z$ then $x \geq z$, for any $x, y, z \in \mathbb{R}$.
2. If $x \geq y$ and $a \geq b$ then $x + a \geq y + b$, for any $x, y, a, b \in \mathbb{R}$.
3. If $x \geq y$ then $x + z \geq y + z$, for any $x, y, z \in \mathbb{R}$.
4. If $x \geq y$ and $a \geq b$ then $xa \geq yb$, for any $x, y \in \mathbb{R}^+$ or $a, b \in \mathbb{R}^+$.
5. If $x \in \mathbb{R}$ then $x^2 \geq 0$, with equality if and only if $x = 0$. More generally, for $A_i \in \mathbb{R}^+$ and $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ holds $A_1x_1^2 + A_2x_2^2 + \cdots + A_nx_n^2 \geq 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n = 0$.

These properties are obvious and simple, but are a powerful tool in proving inequalities, particularly *Property 5*, which can be used in many cases.

We'll give a few examples that will illustrate the strength of *Property 5*.

Firstly we'll prove few "elementary" inequalities that are necessary for a complete and thorough upgrade of each student who is interested in this area.

To prove these inequalities it is sufficient to know elementary inequalities that can be used in a certain part of the proof of a given inequality, but in the early stages, just basic operations are used.

The following examples, although very simple, are the basis for what follows later. Therefore I recommend the reader pay particular attention to these examples, which are necessary for further upgrading.

Exercise 1.1 Prove that for any real number $x > 0$, the following inequality holds

$$x + \frac{1}{x} \geq 2.$$

Solution From the obvious inequality $(x - 1)^2 \geq 0$ we have

$$x^2 - 2x + 1 \geq 0 \quad \Leftrightarrow \quad x^2 + 1 \geq 2x,$$

and since $x > 0$ if we divide by x we get the desired inequality. Equality occurs if and only if $x - 1 = 0$, i.e. $x = 1$.

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Exercise 1.2 Let $a, b \in \mathbb{R}^+$. Prove the inequality

$$\frac{a}{b} + \frac{b}{a} \ge 2.$$

Solution From the obvious inequality $(a-b)^2 \ge 0$ we have

$$a^2 - 2ab + b^2 \ge 0 \quad \Leftrightarrow \quad a^2 + b^2 \ge 2ab \quad \Leftrightarrow \quad \frac{a^2+b^2}{ab} \ge 2 \quad \Leftrightarrow \quad \frac{a}{b} + \frac{b}{a} \ge 2.$$

Equality occurs if and only if $a - b = 0$, i.e. $a = b$.

Exercise 1.3 (Nesbitt's inequality) Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}.$$

Solution According to Exercise 1.2 it is clear that

$$\frac{a+b}{b+c} + \frac{b+c}{a+b} + \frac{a+c}{c+b} + \frac{c+b}{a+c} + \frac{b+a}{a+c} + \frac{a+c}{b+a} \ge 2+2+2 = 6. \tag{1.1}$$

Let us rewrite inequality (1.1) as follows

$$\left(\frac{a+b}{b+c} + \frac{a+c}{c+b}\right) + \left(\frac{c+b}{a+c} + \frac{b+a}{a+c}\right) + \left(\frac{b+c}{a+b} + \frac{a+c}{b+a}\right) \ge 6,$$

i.e.

$$\frac{2a}{b+c} + 1 + \frac{2b}{c+a} + 1 + \frac{2c}{a+b} + 1 \ge 6$$

or

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2},$$

a s required.

Equality occurs if and only if $\frac{a+b}{b+c} = \frac{b+c}{a+b}$, $\frac{a+c}{c+b} = \frac{c+b}{a+c}$, $\frac{b+a}{a+c} = \frac{a+c}{b+a}$, from where easily we deduce $a = b = c$.

The following inequality is very simple but it has a very important role, as we will see later.

Exercise 1.4 Let $a, b, c \in \mathbb{R}$. Prove the inequality

$$a^2 + b^2 + c^2 \ge ab + bc + ca.$$

Solution Since $(a-b)^2 + (b-c)^2 + (c-a)^2 \ge 0$ we deduce

$$2(a^2 + b^2 + c^2) \ge 2(ab + bc + ca) \quad \Leftrightarrow \quad a^2 + b^2 + c^2 \ge ab + bc + ca.$$

Equality occurs if and only if $a = b = c$.

As a consequence of the previous inequality we get following problem.

Exercise 1.5 Let $a, b, c \in \mathbb{R}$. Prove the inequalities

$$3(ab+bc+ca) \leq (a+b+c)^2 \leq 3(a^2+b^2+c^2).$$

Solution We have

$$\begin{aligned} 3(ab+bc+ca) &= ab+bc+ca+2(ab+bc+ca) \\ &\leq a^2+b^2+c^2+2(ab+bc+ca) = (a+b+c)^2 \\ &= a^2+b^2+c^2+2(ab+bc+ca) \\ &\leq a^2+b^2+c^2+2(a^2+b^2+c^2) = 3(a^2+b^2+c^2). \end{aligned}$$

Equality occurs if and only if $a=b=c$.

Exercise 1.6 Let $x, y, z > 0$ be real numbers such that $x+y+z=1$. Prove that

$$\sqrt{6x+1}+\sqrt{6y+1}+\sqrt{6z+1} \leq 3\sqrt{3}.$$

Solution Let $\sqrt{6x+1}=a$, $\sqrt{6y+1}=b$, $\sqrt{6z+1}=c$.
Then

$$a^2+b^2+c^2 = 6(x+y+z)+3 = 9.$$

Therefore

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2) = 27, \quad \text{i.e.} \quad a+b+c \leq 3\sqrt{3}.$$

Exercise 1.7 Let $a, b, c \in \mathbb{R}$. Prove the inequality

$$a^4+b^4+c^4 \geq abc(a+b+c).$$

Solution By Exercise 1.4 we have that: If $x, y, z \in \mathbb{R}$ then

$$x^2+y^2+z^2 \geq xy+yz+zx.$$

Therefore

$$\begin{aligned} a^4+b^4+c^4 &\geq a^2b^2+b^2c^2+c^2a^2 = (ab)^2+(bc)^2+(ca)^2 \\ &\geq (ab)(bc)+(bc)(ca)+(ca)(ab) = abc(a+b+c). \end{aligned}$$

Exercise 1.8 Let $a, b, c \in \mathbb{R}$ such that $a+b+c \geq abc$. Prove the inequality

$$a^2+b^2+c^2 \geq \sqrt{3}abc.$$

Solution We have

$$(a^2+b^2+c^2)^2 = a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2$$
$$= a^4+b^4+c^4+a^2(b^2+c^2)+b^2(c^2+a^2)+c^2(a^2+b^2). \tag{1.2}$$

By Exercise 1.7, it follows that

$$a^4+b^4+c^4 \geq abc(a+b+c). \tag{1.3}$$

Also

$$b^2+c^2 \geq 2bc, \qquad c^2+a^2 \geq 2ca, \qquad a^2+b^2 \geq 2ab. \tag{1.4}$$

Now by (1.2), (1.3) and (1.4) we deduce

$$(a^2+b^2+c^2)^2 \geq abc(a+b+c)+2a^2bc+2b^2ac+2c^2ab$$
$$= abc(a+b+c)+2abc(a+b+c) = 3abc(a+b+c). \tag{1.5}$$

Since $a+b+c \geq abc$ in (1.5) we have

$$(a^2+b^2+c^2)^2 \geq 3abc(a+b+c) \geq 3(abc)^2,$$

i.e.

$$a^2+b^2+c^2 \geq \sqrt{3}abc.$$

Equality occurs if and only if $a=b=c=\sqrt{3}$.

Exercise 1.9 Let $a, b, c > 1$ be real numbers. Prove the inequality

$$abc+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > a+b+c+\frac{1}{abc}.$$

Solution Since $a, b, c > 1$ we have $a > \frac{1}{b}, b > \frac{1}{c}, c > \frac{1}{a}$, i.e.

$$\left(a-\frac{1}{b}\right)\left(b-\frac{1}{c}\right)\left(c-\frac{1}{a}\right) > 0.$$

After multiplying we get the required inequality.

Exercise 1.10 Let a, b, c, d be real numbers such that $a^4+b^4+c^4+d^4 = 16$. Prove the inequality

$$a^5+b^5+c^5+d^5 \leq 32.$$

Solution We have $a^4 \leq a^4+b^4+c^4+d^4 = 16$, i.e. $a \leq 2$ from which it follows that $a^4(a-2) \leq 0$, i.e. $a^5 \leq 2a^4$.

Similarly we obtain $b^5 \leq 2b^4$, $c^5 \leq 2c^4$ and $d^5 \leq 2d^4$.

Hence

$$a^5+b^5+c^5+d^5 \le 2(a^4+b^4+c^4+d^4)=32.$$

Equality occurs iff $a=2, b=c=d=0$ (up to permutation).

Exercise 1.11 Prove that for any real number x the following inequality holds

$$x^{12}-x^9+x^4-x+1>0.$$

Solution We consider two cases: $x<1$ and $x\ge 1$.

(1) Let $x<1$. We have

$$x^{12}-x^9+x^4-x+1=x^{12}+(x^4-x^9)+(1-x).$$

Since $x<1$ we have $1-x>0$ and $x^4>x^9$, i.e. $x^4-x^9>0$, so in this case

$$x^{12}-x^9+x^4-x+1>0,$$

i.e. the desired inequality holds.

(2) For $x\ge 1$ we have

$$\begin{aligned}x^{12}-x^9+x^4-x+1&=x^8(x^4-x)+(x^4-x)+1\\&=(x^4-x)(x^8+1)+1=x(x^3-1)(x^8+1)+1.\end{aligned}$$

Since $x\ge 1$ we have $x^3\ge 1$, i.e. $x^3-1\ge 0$.

Therefore

$$x^{12}-x^9+x^4-x+1>0,$$

and the problem is solved.

Exercise 1.12 Prove that for any real number x the following inequality holds

$$2x^4+1\ge 2x^3+x^2.$$

Solution We have

$$\begin{aligned}2x^4+1-2x^3-x^2&=1-x^2-2x^3(1-x)=(1-x)(1+x)-2x^3(1-x)\\&=(1-x)(x+1-2x^3)=(1-x)(x(1-x^2)+1-x^3)\\&=(1-x)\left(x(1-x)(1+x)+(1-x)(1+x+x^2)\right)\\&=(1-x)\left((1-x)(x(1+x)+1+x+x^2)\right)\\&=(1-x)^2((x+1)^2+x^2)\ge 0.\end{aligned}$$

Equality occurs if and only if $x=1$.

Exercise 1.13 Let $x, y \in \mathbb{R}$. Prove the inequality

$$x^4 + y^4 + 4xy + 2 \geq 0.$$

Solution We have

$$\begin{aligned} x^4 + y^4 + 4xy + 2 &= (x^4 - 2x^2y^2 + y^4) + (2x^2y^2 + 4xy + 2) \\ &= (x^2 - y^2)^2 + 2(xy + 1)^2 \geq 0, \end{aligned}$$

as desired.

Equality occurs if and only if $x = 1$, $y = -1$ or $x = -1$, $y = 1$.

Exercise 1.14 Prove that for any real numbers x, y, z the following inequality holds

$$x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1).$$

Solution We have

$$\begin{aligned} &x^4 + y^4 + z^2 + 1 - 2x(xy^2 - x + z + 1) \\ &\quad = (x^4 - 2x^2y^2 + x^4) + (z^2 - 2xz + x^2) + (x^2 - 2x + 1) \\ &\quad = (x^2 - y^2)^2 + (x - z)^2 + (x - 1)^2 \geq 0, \end{aligned}$$

from which we get the desired inequality.

Equality occurs if and only if $x = y = z = 1$ or $x = z = 1$, $y = -1$.

Exercise 1.15 Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove the inequality

$$xy + yz + 2zx \leq \frac{1}{2}.$$

Solution We will prove that

$$2xy + 2yz + 4zx \leq (x + y + z)^2,$$

from which, since $x + y + z = 1$ we'll obtain the required inequality.

The last inequality is equivalent to

$$x^2 + y^2 + z^2 - 2zx \geq 0, \quad \text{i.e.} \quad (x - z)^2 + y^2 \geq 0,$$

which is true.

Equality occurs if and only if $x = z$ and $y = 0$, i.e. $x = z = \frac{1}{2}$, $y = 0$.

Exercise 1.16 Let $a, b \in \mathbb{R}^+$. Prove the inequality

$$a^2 + b^2 + 1 > a\sqrt{b^2 + 1} + b\sqrt{a^2 + 1}.$$

Solution From the obvious inequality

$$(a-\sqrt{b^2+1})^2+(b-\sqrt{a^2+1})^2 \geq 0, \tag{1.6}$$

we get the desired result.

Equality occurs if and only if

$$a=\sqrt{b^2+1} \quad \text{and} \quad b=\sqrt{a^2+1}, \quad \text{i.e.} \quad a^2=b^2+1 \quad \text{and} \quad b^2=a^2+1,$$

which is impossible, so in (1.6) we have strictly inequality.

Exercise 1.17 Let $x, y, z \in \mathbb{R}^+$ such that $x+y+z=3$. Prove the inequality

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx.$$

Solution We have

$$3(x+y+z)=(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx).$$

Hence it follows that

$$xy+yz+zx=\frac{1}{2}(3x-x^2+3y-y^2+3z-z^2).$$

Then

$$\begin{aligned}
&\sqrt{x}+\sqrt{y}+\sqrt{z}-(xy+yz+zx)\\
&\quad=\sqrt{x}+\sqrt{y}+\sqrt{z}+\frac{1}{2}(x^2-3x+y^2-3y+z^2-3z)\\
&\quad=\frac{1}{2}((x^2-3x+2\sqrt{x})+(y^2-3y+2\sqrt{y})+(z^2-3z+2\sqrt{z}))\\
&\quad=\frac{1}{2}(\sqrt{x}(\sqrt{x}-1)^2(\sqrt{x}+2)+\sqrt{y}(\sqrt{y}-1)^2(\sqrt{y}+2)\\
&\qquad+\sqrt{z}(\sqrt{z}-1)^2(\sqrt{z}+2)) \geq 0,
\end{aligned}$$

i.e.

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx.$$

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