

Chapter 2

Numerical Algorithms for ESM: State of the Art

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Numerical simulation of geophysical flows has been historically one of the earliest instances in which the power of electronic computers was employed successfully to the quantitative prediction of natural phenomena. Initially, the main focus was indeed on numerical weather forecasting, for which the ideas of Richardson had already provided an appropriate conceptual framework, that was subsequently strengthened by the analyses and the numerical simulations of Charney and von Neumann. References to numerical methods applied in early numerical weather prediction models can be found e.g. in Mesinger and Arakawa (1976), Williamson (1979), Haltiner and Williamson (1980) and Williamson (2007), while references for the oceanic modelling literature can be found in Haidvogel and Beckmann (1999) and Kantha and Clayson (2000). The first attempts at numerical simulation of climate took place in the 1960s. In the pioneering works of Smagorinsky (1963), Kasahara and Washington (1967), Manabe and Bryan (1969), Manabe et al. (1975) and Bryan et al. (1975), the outline for many later modelling attempts was laid out. At the heart of all climate and ESM lies the numerical solution of systems of highly coupled and nonlinear fluid flow equations. The equations of motion for atmospheric flow can be written, in a rather general form, as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla K - (2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \times \mathbf{v} - \frac{1}{\rho} \nabla \cdot p + \nabla \Phi + \mathbf{F}_{ext}^v + \mathbf{F}_{par}^v \quad (2.2)$$

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$$\frac{\partial(\rho\epsilon)}{\partial t} + \nabla \cdot [(\rho\epsilon + p + \mathbf{R}) \cdot \mathbf{v}] = 0 \quad (2.3)$$

$$p = RT\rho \quad (2.4)$$

These model equations are obtained by considering air as an ideal gas, where ρ denotes the air density, p the pressure, R the ideal gas constant. Furthermore, \mathbf{v} denotes the air velocity, ϵ is the total energy, K is kinetic energy, $\mathbf{\Omega}$ denotes the rotation velocity of the Earth, $\boldsymbol{\zeta} = \nabla \times \mathbf{v}$ is the relative vorticity, Φ denotes the normal gravity potential, \mathbf{F}_{ext}^v denotes the resultant of the external forces, \mathbf{F}_{par}^v represents the effect of parameterized subgrid processes and \mathbf{R} denotes the radiation heat flux. On the other hand, the equations of motion for oceanic flows can be written as

$$\nabla \cdot \mathbf{v} = 0 \quad (2.5)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla K - (2\mathbf{\Omega} + \boldsymbol{\zeta}) \times \mathbf{v} - \frac{1}{\rho} \nabla \cdot p + \nabla \Phi + \mathbf{F}_{par}^v \quad (2.6)$$

$$\frac{\partial(\theta)}{\partial t} + \nabla \cdot (\theta \mathbf{v}) = \mathbf{F}_{par}^\theta \quad (2.7)$$

$$\frac{\partial S}{\partial t} + \nabla \cdot (S \mathbf{v}) = \mathbf{F}_{par}^S \quad (2.8)$$

$$\rho = \rho(S, T, p) \quad (2.9)$$

where θ is potential temperature, S is salinity and the equation of state for water is known only by fitting to experimental data. It can be observed that water is modelled as an incompressible fluid, which has several implications on the numerical solution procedure for the equations of motion of the ocean. The task of solving such complex equation systems is shared with other important areas of applied mathematics and engineering, but the specific problems encountered in atmospheric and oceanic flow modelling have led to the development of original approaches for its solution. These problems are mostly connected to the peculiar nature of the forces acting on oceanic and atmospheric flows, such as gravity and rotation. The boundaries of the domains where these flows take place are described by complex orography, and, for the ocean, highly variable topography that extends all through the water column. Finally, it is necessary to parameterize a large number of relevant physical processes that take place on spatial scales largely unresolved on the typical computational grids that can be employed in practice.

In the following, we will review the basic features of the numerical methods that were eventually included in the most commonly used ESM. We will present

methods for space and time discretization separately in [Sects. 2.1–2.3](#), respectively, while [Sect. 2.4](#) will be devoted to the approximation of parameterized processes.

2.1 Space Discretization: Finite Difference and Finite Volume Methods

The set of equations introduced in the previous section are examples of systems of nonlinear partial differential equations (PDE), that can be written in general form as

$$\frac{\partial \psi}{\partial t} = \mathcal{L}(\psi) \quad (2.10)$$

where $\psi = \psi(\mathbf{x}, t)$ is a function of space and time variables, respectively, and \mathcal{L} is a (generally nonlinear) differential operator, whose solution needs to be computed on some spatial domain Ω and time interval $[0, T]$.

The finite difference and finite volume methods are among the most widely used approximation techniques for problems like (2.10). In both types of methods, a spatially discretized domain Ω_h is introduced, where h denotes the typical size of the discrete spatial elements. For each discrete element $i = 1, \dots, m$, a numerical approximation ξ_i to the solution is introduced, along with discrete approximations \mathcal{L}_h of the continuous differential operators, so as to obtain a spatially discretized problem

$$\frac{\partial \xi_i}{\partial t} = \mathcal{L}_h(\xi)_i \quad i = 1, \dots, m. \quad (2.11)$$

Problem (2.11) constitutes a set of nonlinear ordinary differential equations, whose unknowns are approximations of the continuous solution over each discrete element. This equation set can be solved by application of the time discretization methods described in [Sect. 2.3](#).

In the finite difference method, the differential operators in the continuous PDE are replaced by discrete approximations that only involve approximate solution values at a finite set of appropriately chosen points (*mesh* or *grid*). More specifically, finite difference approximations are obtained by introducing first a discrete approximation $\Omega_h = \{\mathbf{x}_i, i = 1, \dots, m\}$ of the spatial domain, usually consisting in a regular array of discrete locations \mathbf{x}_i (mesh nodes). The parameter h denotes in this case a measure of the distance between neighbouring elements of Ω_h . Typically, the mesh nodes are regularly spaced along the coordinate directions and h is related to the average spacing along these directions. For each $\mathbf{x}_i \in \Omega_h$, the discrete approximation (2.11) is defined by taking ξ_i as the discrete approximations to the values $\psi(\mathbf{x}_i, t)$ taken by the continuous solutions at the discrete nodes. \mathcal{L}_h denotes a discrete operator that is obtained by replacing the derivatives present in \mathcal{L} with finite difference quotients. In a one dimensional context, for example, if the interval $[0, L]$ is approximated by

introducing equally spaced nodes $x_i = ih, i = 0, \dots, m$, with $h = L/m$, first order derivatives can be approximated by centered differences

$$\psi'(x_i) \approx \frac{\xi_{i+1} - \xi_{i-1}}{2h}.$$

In order to be useful, the resulting discretization must represent a *consistent* approximation of (2.10), in the sense that $\lim_{h \rightarrow 0} |\mathcal{L}_h(\psi)_i - \mathcal{L}(\psi)_i| = 0$.

This consistency property ensures that the discrete problem (2.11) is a good approximation of the continuous problem (2.10), but it is not sufficient to guarantee that accumulation of numerical errors will not take place. In order to make sure that a proper approximation of the solution to the continuous problem has been achieved, *convergence* is indeed required, i.e. the fact that $\lim_{h \rightarrow 0} |\xi_i(t) - \psi_i(t)| = 0$.

These concepts also extend to finite volume methods. For a complete and mathematically rigorous review of the finite difference method and of the related concepts one may refer to Leveque (2007) and Strikwerda (1989).

In the finite volume discretization approach, the discrete approximation $\Omega_h = \{\mathbf{E}_i, i = 1, \dots, m\}$ of the spatial domain is usually given by a set of non overlapping *cells* (or *control volumes*) \mathbf{E}_i . The parameter h denotes in this case a measure of the typical size of the elements of Ω_h , while the discrete values ξ_i are defined as discrete approximations of the averaged values $\int_{\mathbf{E}_i} \psi(\mathbf{x}, t) d\mathbf{x} / |\mathbf{E}_i|$. In order to derive a discretized problem, the continuous system (2.10) is rewritten in divergence form and averaged over each control volume, so as to obtain, by application of Gauss theorem, a set of equations in which the evolution of the averages ξ_i of the prognostic variables is determined by their *fluxes* through the edges of each control volume. These fluxes are not prognostic variables of the set of discretized equations, but must be recovered by appropriate interpolation procedures from the cell averaged values. For a complete and mathematically rigorous review of the finite volume method and of the related concepts one may refer to Fletcher (1997) and Hirsch (1990). Finite volume approaches yield naturally mass and energy conserving methods, in contrast to standard finite difference discretizations, for which these properties are less obvious to achieve.

While these discretization techniques are applicable to a large number of science and engineering problems, we would now like to stress a couple of main features specific to numerical methods introduced in the climate modelling literature.

Firstly, while models for fluid dynamics at laboratory scale are quite well established, such as e.g. the Reynolds averaged Navier–Stokes equations, in atmospheric and oceanic modelling a great attention has been devoted to the formulation of the continuous equations themselves, as well as to the choice of the most appropriate vertical coordinate system. The vertical coordinate choice is especially important because of the stratification induced by gravity and of the difference in spatial scales between the horizontal and vertical directions. In contrast to classical fluid dynamics, mass based coordinates have been used extensively in AGCMs. A review of different formulations for the equations of atmospheric motion can be found in Kasahara (1974). For ocean models, three different approaches are available, but

two are most widely used: geopotential (z -, level-) coordinates and isopycnic coordinates (Griffies et al. 2000). The third, terrain-following (so-called σ) coordinate system involves a non-orthogonal transformation (Haidvogel and Beckmann 1999) and has only recently been applied in global configurations. Recent years have also seen a *hybridization* of vertical coordinates, in which various concepts are combined to improve the representation of processes in critical regions: for example, rotated mixing tensors for level models to represent the predominantly isopycnic mixing in the ocean interior and vertically resolved surface mixed layer modules for isopycnic models.

Until quite recently, Cartesian latitude–longitude meshes were used almost exclusively for the discretization of the spatial domain. Latitude–longitude grids with constant spacing, however, suffer from the so called *pole problem*, due to the convergence of the meridians at the poles, which results in computational cells of very small width that restrict the time step choice because of stability constraints (see Sect. 2.3) and that introduce small scale features in the solutions that cannot be properly resolved on the same grid away from the poles. This implied that, even at relatively coarse resolution, some form of spatial smoothing or filtering was necessary in order to obtain correct results. For ocean models, these problems can be avoided by rotating the coordinate systems and placing the poles out of the computational domain of interest (i.e. on land). Thus, in this context the horizontal grid structure evolved towards Mercator projections (yielding isotropic grids that exclude the pole) and orthogonal curvilinear coordinate systems, with telescoping capabilities and rotated poles, e.g., tripolar grids, which avoid the computational singularity at the North pole. Furthermore, for ocean models *masking* of land areas is necessary, causing an overhead in memory/storage and computing time. For atmospheric models, on the other hand, the pole problem constituted one of the main motivations for studying alternative approaches, which began already in the late 1960s (see e.g. Sadourny et al. 1968; Williamson 1968). These developments have found their way in complete CGMs only much later (see the discussion in Sect. 2.3), also because of the subsequent development of spectral transform methods, which can easily use more uniform grids and do not require special pole treatment (see the discussion in Sect. 2.2).

Concerning the numerical methods chosen to approximate the equations of motion, simple centered finite differences were usually chosen in the earliest modelling attempts. However, the numerical stability problems and the difficulties in reproducing correctly gravity waves and the geostrophic adjustment process motivated two important developments, around which research on numerical methods for geophysical flows was centered for a couple of decades. On one hand, alternative discretizations based on finite difference approximations using different sets of nodes for each prognostic variable (*staggering*) were investigated in great detail, see e.g. Mesinger and Arakawa (1976) and the more modern presentations in Randall (1994) and Dukowicz (1995). This allowed to identify those discrete variable arrangements that are more effective in reproducing basic dynamical processes at minimal computational cost. It is remarkable that similar analyses have been carried out only much later for more advanced discretizations derived e.g. by application of the finite

element method. In finite difference ocean models, horizontal grids are usually of B or C type, according to the classification proposed by Arakawa. Both grids have advantages and disadvantages, because the individual terms in the equations can be either evaluated directly or only after averaging of values at neighboring points, which reduces resolution and accuracy. In addition to affecting wave propagation, the chosen grid has consequences for the implementation of lateral boundary conditions (no slip or free slip) and the specification of coastal geometry (passages and islands). A great attention was also devoted to the derivation of numerical methods which guaranteed preservation of global invariants of motion at a discrete level. Starting with the pioneering work of Arakawa (1966) on two dimensional, incompressible flows, a number of numerical methods were proposed that ensured either energy or enstrophy preservation or both (see e.g. Sadourny 1975; Arakawa and Lamb 1981; Mesinger 1981; Simmons and Burridge 1981; Janjic 1984). These methods allowed to minimize the need for numerical diffusion to control nonlinear instabilities and to obtain physically correct results even at relatively coarse resolutions.

Another important development, which will be discussed in greater detail in Sect. 2.3, consists in the so called semi-Lagrangian method. This method, which links the spatial and temporal discretization of advection equations, is especially efficient and accurate for advection dominated flows. After earlier proposals, it was strongly advocated for numerical weather prediction (NWP) applications in Robert (1982), see also the general review in Staniforth and Côté (1991). Its potential advantages for climate modelling were also investigated (see e.g. Williamson and Rasch 1989) and introduced in some climate models. The original semi-Lagrangian method is neither mass nor energy conservative, but after the development of locally conservative semi-Lagrangian methods in flux form (Leonard et al. 1996; Lin and Road 1996) and their subsequent integration in more complete models (see e.g. Lin and Road, 1997; Lin 2004; Roeckner et al. 2003), that they started to become an important component of ESM as they are of some NWP models.

2.2 Space Discretization: Spectral Transform Methods

The limitations of finite difference approaches were soon apparent to numerical modelers. Because of the nonlinearity of the equations of motion, errors due to the numerical approximation associated to different wavenumbers interact quickly and lead to potentially large amplification of the original phase and amplitude errors, if the spatial resolution is not sufficiently high. Given the limitations of the hardware available in the 1960s, this motivated the investigation of more effective numerical methods, that could produce reasonably accurate solutions for large scale dynamics even employing a relatively small number of discrete degrees of freedom. Thus, almost as soon as Fast Fourier Transform algorithms became widely available, the first concepts of spectral discretization were proposed for the equations of atmospheric motion. Since the application of these models requires periodic computational domains without lateral boundaries or islands, their diffusion was

exclusive to the atmospheric component of ESM. Indeed, most leading climate and NWP centers were running operational spectral models by the early 1990s, about twenty years after their introduction. The outline of the spectral transform approach for atmospheric flow equations was introduced at the beginning of the 1970s, see, among others, Orszag (1970), Bourke (1972, 1974), Hoskins and Simmons (1975) and a more complete review in Williamson (2007).

In the spectral transform approach, model unknowns are represented by truncated expansions in appropriate systems of orthogonal functions, so that the unknowns of the discretized problem are the expansion coefficients rather than the gridpoint values of the approximated solutions. *Completeness* of these systems guarantees that all functions within an appropriate functional space (usually, the set of square integrable functions) can be well approximated in the mean square sense, if sufficiently many basis functions are employed. The equations are rewritten so as to isolate as much as possible linear terms, for which the computation is carried out in spectral space in terms of appropriate combinations of orthogonal function coefficients. Since derivatives can be computed exactly (up to the accuracy of the spectral truncation) this allows to avoid aliasing errors. For the quadratic nonlinear terms that arise in fluid motion equations, the two factors are reconstructed from the spectral coefficients and their product is computed in physical space, yielding again an exact result up to the truncation accuracy, provided that a sufficient number of gridpoints is used for the numerical computation of the spectral coefficient.

Due to the spherical geometry of the Earth, a natural choice for the orthogonal basis functions are spherical harmonic functions (see e.g. Abramowitz and Stegun 1965). These functions are eigenfunctions of the Laplace operator in spherical coordinates and can be denoted as $P_n^m(\mu)e^{im\lambda}$, where λ denotes longitude, $\mu = \sin \theta$ and θ denotes latitude, m is the zonal wave number, n is the meridional index and P_n^m are the associated Legendre functions of the first kind. These are in turn defined as

$$P_n^m(\mu) = \sqrt{(2n+1) \frac{(n-m)!}{(n+m)!} \frac{1}{2^n n!}} (1-\mu^2)^{\frac{m}{2}} \frac{d^{(n+m)}}{d\mu^{(n+m)}} (\mu^2-1), \quad (m \geq 0) \quad (2.12)$$

where $P_n^{-m}(\mu) = P_n^m(\mu)$.

With these definitions, the approximation by truncated expansions in spherical harmonic functions of a generic function of the horizontal variables and time $X = X(\lambda, \mu, t)$ can be defined as:

$$X(\lambda, \mu, t) = \sum_{m=-M}^M \sum_{n=m}^{N(M)} X_n^m(t) P_n^m(\mu) e^{im\lambda} \quad (2.13)$$

where X_n^m are the (generally complex-valued) spectral coefficients of the field X . As remarked above, a great advantage of the spectral transform approach is due to the fact that the Fourier representation allows to compute horizontal derivatives analytically, so that

$$\left(\frac{dX}{d\lambda}\right)_m = imX_m \quad \text{and} \quad \left(\frac{dX}{d\mu}\right)_m = \sum_{n=m}^{N(m)} X_n^m \frac{dP_n^m}{d\mu} \quad (2.14)$$

where the derivative of the Legendre functions are recovered by appropriate recurrence relations (see e.g. Abramowitz and Stegun 1965).

The spatial resolution of the spectral transform discretizations is determined by the procedure chosen to truncate the spectral representation of the unknowns. Most AGCMs employ the *triangular* truncation technique, for which the maximum meridional index $N(m)$ is taken to be equal to the maximum zonal wavenumber M . This truncation is commonly identified as TM, where values for M commonly used up to now in climate modelling have varied between 21 and 159.

As previously remarked, the grid on which the calculations are performed has to be chosen to give an exact (given the spectral truncation of the fields, and within round-off error) contribution to spectral tendencies from quadratic non-linear terms. The integrals with respect to λ involve the product of three trigonometric functions and can be evaluated exactly using a regularly-spaced grid of at least $3M + 1$ points. It is to be remarked that, once this requirement is satisfied, there is no necessity for the number of grid nodes to be equal along each parallel. As a result, *reduced* Gaussian grids can be effectively employed (see e.g. Hortal and Simmons 1991; Williamson and Rosinski 2000), which do not exhibit pole convergence in the longitudinal direction and allow to increase substantially the efficiency of spectral transform models, without degrading their accuracy. For the latitudinal integrals, quadratic nonlinear terms lead to integrands that are polynomials in μ . They may thus be computed exactly using Gaussian quadrature with points located at the (approximately equally-spaced) latitudes that satisfy $P_{N_G}^0(\mu) = 0$, for sufficiently large integer N_G . These latitudes are referred to as the Gaussian latitudes.

2.3 Time Discretizations

The time continuous solution $\xi_i(t)$, $t \in [0, T]$ of (2.11) is approximated by introducing a time step $\Delta t = T/n$ and a set of discrete time levels $t_k = k\Delta t$, $k = 0, \dots, n$ for which a numerical method for systems of ordinary differential equations provides an approximated solution ξ_i^k , $k = 0, \dots, n$. Time dependent problems such as those present in ESM require careful choice of the time discretization approach, both for accuracy and efficiency reasons. The number of degrees of freedom necessary for the discrete problem (2.11) to be a reasonably realistic approximation of geophysical scale fluid motions is relatively high, implying an equally high computational cost, which grows nonlinearly with spatial resolution, due to the multidimensional nature of the problem. Furthermore, as well known from the theory of numerical methods for ordinary differential equations (see e.g. LeVeque 2007), the choice of the timestep cannot generally be made based only on accuracy or efficiency considerations, but must also comply with numerical *stability* criteria. Such restrictions represent necessary (and, in linear cases, sufficient) conditions for the accumulation of round-off and

truncation errors not to result in a catastrophic error growth (*numerical instability*) that leads in a relatively small number of timesteps to completely incorrect results. Some stability criteria, such as the well known Courant–Friedrichs–Lewy condition (Courant et al. 1952; Strikwerda 1989), also have a physical interpretation related to the hyperbolic nature of the equations of fluid motion considered, which have among their possible solutions fast wave motions (sound waves for the atmosphere, surface gravity waves for ocean models including a free surface upper boundary condition).

Among *explicit* time discretization methods, which do not require the solution of a system of equations at each timestep to update the solution from one discrete time level to the next, the *leapfrog* discretization method has been for a long time one of the most widely used, since it allows to achieve second order accuracy in time in a very straightforward way. It is a *multistep* scheme derived by approximating the time derivative by a centered difference approximation, which leads to the following time discretization of equation (2.11):

$$\frac{\xi_i^{k+1} - \xi_i^{k-1}}{2\Delta t} = \mathcal{L}_h(\xi^k) \quad i = 1, \dots, m \quad k = 1, \dots, n. \quad (2.15)$$

A complete analysis of this method can be found in Strikwerda (1989). Here, it will be sufficient to remark that, due to the presence of a computational mode, a filtering procedure has to be employed to avoid decoupling of the odd and even labelled time steps. The most popular procedure for this is the so-called Asselin filter (Asselin 1972). Several alternatives to the leapfrog method have been considered and evaluated, among these the third order Adams–Bashforth scheme, which was analyzed in Durran (1991).

Several time discretization algorithms have been proposed and employed in ESM for the purpose of enhancing the efficiency of simple explicit schemes. One approach, known as the *split explicit* or *mode splitting* technique in the atmospheric and oceanic literature, respectively, consists in a separate treatment of the terms responsible for the fastest wave motions. For these terms, an explicit scheme with a smaller time step is employed in a substepping procedure (see e.g. Klemp and Wilhelmson 1978; Haidvogel et al. 2004), while all the terms which do not imply too severe stability restrictions are discretized with a single, larger time step. Other effective techniques that achieve the same goal by different means are the so called *semi-implicit* time discretizations, in which the terms responsible for faster propagating wavelike solutions are discretized implicitly in time. Originally proposed for NWP applications in Robert (1981), the semi-implicit approach has proven to be very effective, especially when combined to the spectral discretization technique. Due to its good stability properties, one of the most widely used implicit methods is the Crank–Nicolson scheme, (see LeVeque 2007), that can be defined as

$$\frac{\xi_i^{k+1} - \xi_i^k}{\Delta t} = \alpha \mathcal{L}_h(\xi^{k+1}) + (1 - \alpha) \mathcal{L}_h(\xi^k) \quad i = 1, \dots, m \quad k = 1, \dots, n, \quad (2.16)$$

where α is an averaging parameter. Second order accuracy is achieved for $\alpha = 1/2$ and, in the linear case, unconditional stability is guaranteed for $\alpha \in [1/2, 1]$.

As already pointed out in [Sect. 2.1](#), the semi-Lagrangian method is a discretization approach that links the spatial and temporal discretizations for advection equations. A more comprehensive review of the specific features of this method can be found e.g. in Staniforth and Côté (1991). Indeed, the typical set of fluid motion equations can be rewritten as

$$\frac{d\psi}{dt} = \mathcal{L}(\psi) \quad (2.17)$$

where the Lagrangian derivative is defined as

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla c \quad (2.18)$$

for a generic scalar quantity c and $\mathbf{V}(\mathbf{x}, t)$ denotes a space and time dependent velocity field, which may as well be among the components of the solution vector \mathbf{u} . Under some regularity assumptions on the velocity field, the streamline or characteristic line functions can be defined as the solutions $\mathbf{X}(t; s, \mathbf{x})$ of the ordinary differential equations $d\mathbf{X}(t; s, \mathbf{x})/dt = \mathbf{V}(\mathbf{X}(t; s, \mathbf{x}), t)$ with initial datum at time s given by $\mathbf{X}(t; s, \mathbf{x}) = \mathbf{x}$. Based on the fact that purely advected quantities are constant along the streamlines, the semi-Lagrangian method leads to the space-time discretization

$$\frac{\xi_i^{k+1} - \xi_{i,*}^k}{\Delta t} = \mathcal{L}_h(\xi^k)_{i,*} \quad i = 1, \dots, m \quad k = 1, \dots, n. \quad (2.19)$$

where $\xi_{i,*}^k$ denotes the evaluation of the discrete solution at time level k at the spatial location \mathbf{x}_* , defined as the foot of the numerically approximated streamline $\mathbf{X}(t^k; t^{k+1}, \mathbf{x}_i)$. Since \mathbf{x}_* does not generally belong to the computational mesh, a multidimensional interpolation has to be performed to reconstruct its value. The resulting method is stable under very mild stability restrictions, which imply in practice that it can be used with a timestep that is much larger than standard time discretizations without decreasing the overall accuracy. The combination of semi-implicit and semi-Lagrangian discretizations has also proven to be very effective and has been the basis for the dynamical core of the Met Office Unified Model (Davies et al. 2005).

2.4 Numerical Algorithms for Parameterized Physical Processes

Parameterized physical processes are a key component of any ESM and definitely the most complex one, since effective parameterizations are hard to devise and require complicated physical reasoning. In some cases, such as for moist processes in the atmosphere or turbulent mixing in the ocean, spatial scales smaller by orders of magnitude than the resolved ones are involved in the physical processes whose

bulk effect must be described as accurately as possible. Other processes, such as radiation in the atmosphere, are much less scale dependent, but very simplified models must be used for purely computational reasons. A complete review of the relevant literature would be an almost impossible task. Some attempts at partial literature surveys can be found in Morcrette (1991), Garrat (1993), Large et al. (1994), Arakawa 2004, Beljaars et al. (2004), Marshall (2005), Collins et al. (2006), and Wild (2008).

The capability of an ESM to describe correctly atmospheric or oceanic dynamics depends in an essential way on the description of the parameterized process. Climate simulations are highly sensitive to changes in the description of parameterized processes (see e.g. Zhang and McFarlane 1995), and the uncertainties related to cloud parameterization schemes are widely acknowledged to contribute significantly to the difficulty of achieving reliable climate change forecasts. For atmospheric models, one way to reduce such uncertainties is by carrying out systematic verification of shorter time range forecasts (see e.g. Phillips et al. 2004).

The computational cost of this ESM component is a large share of the total, although a precise estimate is highly resolution dependent and the precise amount differs a lot between atmosphere and ocean models and within each of these groups. On the other hand, the numerical approximation of the terms representing parameterized physical processes in the equations of atmospheric and oceanic motions has certainly been the least carefully analysed ESM component. The effect of most of these processes is usually taken into account by an operator splitting approach, in which tendencies due to each single process are computed separately and added up to compute the complete forcing. This necessarily leads to decoupling and splitting errors whose impact on the effective accuracy of the models is hard to measure. Many parameterized processes are described via bulk vertical diffusion coefficients, whose spatial discretization is usually performed by finite difference or finite volume approaches that, when employed with the typical irregular vertical spacing of most atmospheric or oceanic models, are barely first order accurate and can induce large amounts of spurious numerical diffusion.

In ocean models, the main issue is the unresolved part of the continuous spectrum of oceanic scales and variability that needs to be parameterized properly. Often, the net effect of subgrid scale processes is assumed to be diffusive. In that case, a second order Laplacian diffusion term is used, with adaptive (spatially and temporally varying) coefficients, see e.g. the classical approach in Smagorinsky (1963). Proportionality to grid spacing, current velocities (or their gradients) are common for lateral (isopycnic) mixing schemes; proportionality to current shear or stratification are used for the vertical (diapycnic) direction (see e.g. Gent and McWilliams 1990). In boundary layers at the surface and bottom of the ocean, more sophisticated *turbulent closure* schemes are necessary (see e.g. Warner et al. 2005). Vertical homogenization of part of the water column due to *convection* as the result of static instability is either achieved by instantaneous adjustment at the end of each time step, or through substantially increased vertical mixing coefficients that are used in combination with implicit time stepping schemes for vertical diffusion. More on the orientation of mixing tensors can be found in Griffies (2004). Non-diffusive effects

of subgrid scale processes include *topographic stress* (which results as a rectification due to fluctuating flow fields over variable topography) and non-local vertical fluxes in boundary layer turbulence.

For atmospheric models, research on numerical problems issuing from parameterized processes has mostly focused in two areas. On one hand, robust, nonlinearly stable schemes for the discretization of the vertical turbulent diffusion equations have been proposed (see e.g. Kalnay and Kanamitsu 1988; Girard and Delage 1990; Teixeira 1999; Wood et al. 2007). These attempts have been aimed at devising numerical methods that provide a reasonable trade-off between improved accuracy of the time discretization of nonlinear turbulent diffusions and the computational efficiency required to carry out long range numerical simulations.

On the other hand, in the last decade increasing attention has been devoted to the analysis of the splitting approaches that are almost universally employed to couple the dynamical cores of the ESM to the terms describing parameterized processes. This issue has been studied e.g. in Caya et al. (1998), Williamson (2002), Dubal et al. (2005, 2006), and Staniforth and Wood (2008), highlighting the importance of a closer integration of the numerical methods employed in the dynamical cores and in physical parameterization packages.

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