Chapter 2
Linear Algebra

Fix a fractional $\mathscr{O}_F$-ideal $\mathfrak{c} \supset \mathscr{O}_F$. In this chapter we introduce the linear algebraic notions of $\mathfrak{c}$-polarized RM modules and $\mathfrak{c}$-polarized CM modules, and show that certain spaces of special endomorphisms of these objects carry natural quadratic forms. The modules themselves will reappear in Chap. 3 as the first homology of abelian surfaces over $\mathbb{C}$ with real and complex multiplication, and the quadratic spaces of special endomorphisms will underlie the construction of Hilbert modular Eisenstein series in Sect. 4.5.

2.1 The Reflex Algebra

A *CM type* of E is an unordered pair $\Sigma = \{\pi_1, \pi_2\}$ of $\mathbb{Q}$-algebra homomorphisms $\pi_1, \pi_2 : E \to \mathbb{C}$ whose restrictions to F are related by

$$\pi_1|_F = \pi_2|_F \circ \sigma.$$

By Galois theory, $B \mapsto \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(B, \mathbb{Q}^{\mathrm{alg}})$ establishes an equivalence between the category of étale $\mathbb{Q}$-algebras and the category of finite sets with a continuous action of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})$. If we fix an embedding $\mathbb{Q}^{\mathrm{alg}} \to \mathbb{C}$, the set of all CM types of E becomes a $G_{\mathbb{Q}}$-set, and so determines an étale $\mathbb{Q}$-algebra which we call $E^\sharp$. Thus there is a canonical bijection $\Sigma \mapsto \phi_\Sigma$

$$\{\text{CM types of } E\} \cong \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(E^\sharp, \mathbb{C}). \tag{2.1}$$

The algebra $E^\sharp$ and the bijection (2.1) can be made more explicit as follows. Consider the commutative $\mathbb{Q}$-algebra

$$M = E \otimes_{\mathrm{id}, F, \sigma} E.$$

B. Howard and T. Yang, *Intersections of Hirzebruch–Zagier Divisors and CM Cycles*, Lecture Notes in Mathematics 2041, DOI 10.1007/978-3-642-23979-3_2,

On the left we view E as an F-algebra via the inclusion $x \mapsto x$ of F into E, and on the right we view E as an F-algebra via the conjugate embedding $x \mapsto x^\sigma$. Thus for any $a, b \in E$ and $x \in F$ we have the relation $(xa) \otimes b = a \otimes (x^\sigma b)$. Define $\mathbb{Q}$-algebra automorphisms $\rho, \tau \in \mathrm{Aut}(M)$ by

$$\rho(a \otimes b) = \overline{b} \otimes a \qquad \tau(a \otimes b) = b \otimes a.$$

Viewing E as a subalgebra of M via the embedding $a \mapsto a \otimes 1$, we define $\mathbb{Q}$-algebras $E^\sharp$ and $F^\sharp$ by

$$\begin{tikzcd}
 & M \arrow[dl, dash] \arrow[dr, dash] & \\
E^\sharp = M^{\langle \tau \rangle} \arrow[d, dash] & & E = M^{\langle \tau\rho \rangle} \arrow[d, dash] \\
F^\sharp = M^{\langle \tau, \rho^2 \rangle} \arrow[dr, dash] & & F = M^{\langle \tau\rho, \rho^2 \rangle} \arrow[dl, dash] \\
 & \mathbb{Q}. &
\end{tikzcd}$$

The $\mathbb{Q}$-algebra $E^\sharp$ is the *reflex algebra* of E. The *reflex homomorphism* $\phi_\Sigma : E^\sharp \to \mathbb{C}$ associated to the CM type $\Sigma = \{\pi_1, \pi_2\}$ is defined as the restriction to $E^\sharp$ of the $\mathbb{Q}$-algebra homomorphism $M \to \mathbb{C}$ defined by

$$a \otimes b \mapsto \pi_1(a) \cdot \pi_2(b).$$

The *reflex field* of Σ is $E_\Sigma = \phi_\Sigma(E^\sharp)$, and $\mathcal{O}_\Sigma$ denotes the ring of integers of E_Σ. For a prime $\mathfrak{q}$ of $\mathcal{O}_\Sigma$ let $\mathbb{F}_\mathfrak{q}$ be the residue field of $\mathfrak{q}$.

Let $x \mapsto x^\dagger$ denote the restriction to $E^\sharp$ of the automorphism $a \otimes b \mapsto \overline{a} \otimes \overline{b}$ of M, so that $F^\sharp$ is the subalgebra of $E^\sharp$ fixed by $x \mapsto x^\dagger$.

Lemma 2.1.1 *1. In case (**cyclic**) $E^\sharp$ is isomorphic to E, and $x \mapsto x^\dagger$ is complex conjugation.*

*2. In case (**biquad**) $E^\sharp$ is isomorphic to $E_1 \times E_2$, and $x \mapsto x^\dagger$ is the product of the complex conjugations.*

*3. In case (**nongal**) $E^\sharp$ is a quartic CM field which is not Galois over $\mathbb{Q}$ and is not isomorphic to E. The automorphism $x \mapsto x^\dagger$ is complex conjugation.*

*In particular in case (**biquad**) $F^\sharp \cong \mathbb{Q} \times \mathbb{Q}$, and in cases (**cyclic**) and (**nongal**) $F^\sharp$ is a real quadratic field.*

Proof. This as an easy exercise in Galois theory, and is left to the reader. □

A *Hermitian form* on an $E^\sharp$-module V is a pairing $\langle\ ,\ \rangle : V \times V \to E^\sharp$ that is $E^\sharp$-linear in the first variable and satisfies $\langle v, w\rangle = \langle w, v\rangle^\dagger$.

2.2 Polarized RM Modules

Definition 2.2.1 *An* RM module *is a pair* (T, κ_T) *in which* T *is a* $\mathbb{Z}$*-module, and* $\kappa_T : \mathscr{O}_F \to \mathrm{End}_{\mathbb{Z}}(T)$ *is a ring homomorphism making* T *into a projective* $\mathscr{O}_F$*-module of rank* 2*.*

The *polarization module* $P(T, \kappa_T)$ is the $\mathscr{O}_F$-module of alternating $\mathbb{Z}$-bilinear forms $\lambda_T : T \times T \to \mathbb{Z}$ satisfying

$$\lambda_T(\kappa_T(x)t_1, t_2) = \lambda_T(t_1, \kappa_T(x)t_2)$$

for every $x \in \mathscr{O}_F$. A $\mathfrak{c}$-*polarization* of (T, κ_T) is a $\lambda_T \in P(T, \kappa_T)$ satisfying

$$\mathfrak{c}T = \{t_1 \in T \otimes_{\mathbb{Z}} \mathbb{Q} : \lambda_T(t_1, t_2) \in \mathbb{Z} \text{ for all } t_2 \in T\}.$$

The $\mathscr{O}_F$-module $P(T, \kappa_T)$ is projective of rank one. Given a $\mathfrak{c}$-polarized RM module $\mathbf{T} = (T, \kappa_T, \lambda_T)$, let $j \mapsto j^*$ be the involution of $\mathrm{End}_{\mathbb{Z}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ determined by

$$\lambda_T(jt_1, t_2) = \lambda_T(t_1, j^*t_2).$$

A *special endomorphism* of $\mathbf{T}$ is a $j \in \mathrm{End}_{\mathbb{Z}}(T)$ satisfying

$$\kappa_T(x) \circ j = j \circ \kappa_T(x^\sigma)$$

for all $x \in \mathscr{O}_F$, and satisfying $j^* = j$. The $\mathbb{Z}$-module of all special endomorphisms of $\mathbf{T}$ is denoted $L(\mathbf{T})$, and we set

$$V(\mathbf{T}) = L(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For a prime ℓ, abbreviate $L_\ell(\mathbf{T}) = L(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $V_\ell(\mathbf{T}) = V(\mathbf{T}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Let $J \mapsto J^\iota$ be the main involution on $M_2(F)$, characterized by $JJ^\iota = \det(J)$, and define a $\mathbb{Q}$-vector space

$$\begin{aligned} W_{M_2(\mathbb{Q})} &= \{J \in M_2(F) : J^\sigma = J^\iota\} \\ &= \left\{ \begin{pmatrix} a & \delta b \\ \delta c & a^\sigma \end{pmatrix} \in M_2(F) : a \in F \text{ and } b, c \in \mathbb{Q} \right\}. \end{aligned}$$

Here $\delta \in F$ is any nonzero element satisfying $\delta^\sigma = -\delta$. The determinant det is a quadratic form on $W_{M_2(\mathbb{Q})}$.

Proposition 2.2.2 *1. Up to isomorphism there is a unique $\mathfrak{c}$-polarized RM module,* $\mathbf{T}$.
2. The function $Q_{\mathbf{T}}(j) = j \circ j$ defines a quadratic form on $L(\mathbf{T})$.
3. There is an isomorphism of $\mathbb{Q}$-quadratic spaces

$$\big(V(\mathbf{T}), Q_{\mathbf{T}}\big) \cong \big(W_{M_2(\mathbb{Q})}, \det\big).$$

4. The $\mathbb{Q}$-quadratic space $\big(V(\mathbf{T}), Q_{\mathbf{T}}\big)$ has rank 4*, signature* $(2,2)$*, determinant* d_F*, and Hasse invariant (normalized as in [35])*

$$\mathrm{hasse}(V(\mathbf{T}), Q_{\mathbf{T}}) = \left(\frac{-d_F, -1}{\mathbb{Q}}\right) \in \mathrm{Br}_2(\mathbb{Q}).$$

Here $\mathrm{Br}_2(\mathbb{Q})$ *is the* 2*-torsion subgroup of the Brauer group of* $\mathbb{Q}$*.*

Proof. Let $\mathbf{T}$ be a $\mathfrak{c}$-polarized RM module. The polarization λ_T has the form $\lambda_T = \mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda_T$ for a unique $\mathscr{O}_F$-symplectic form $\Lambda_T : T \times T \to \mathfrak{D}_F^{-1}$. As T is projective of rank two as an $\mathscr{O}_F$-module we may fix an $\mathscr{O}_F$-linear isomorphism $T \cong \mathscr{O}_F \oplus \mathfrak{a}$ for some fractional $\mathscr{O}_F$-ideal $\mathfrak{a}$ whose image in $\mathrm{Pic}(\mathscr{O}_F)$ is traditionally called the *Steinitz class* of T. Writing elements of $\mathscr{O}_F \oplus \mathfrak{a} \subset F \oplus F$ as column vectors, the fractional ideal $\mathfrak{a}$ and the isomorphism may be chosen in such a way that

$$\Lambda_T(a, b) = {}^t a \cdot \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) \cdot b.$$

The condition that λ_T is a $\mathfrak{c}$-polarization is then equivalent to $\mathfrak{a} \cdot \mathfrak{c} = \mathfrak{D}_F^{-1}$. This proves the uniqueness of $\mathbf{T}$.

Using the above isomorphism $T \cong \mathscr{O}_F \oplus \mathfrak{a}$ to view elements of T as column vectors, any $j \in V(\mathbf{T})$ can be written uniquely in the form $t \mapsto J \cdot t^\sigma$ for some $J \in M_2(F)$. The condition $j = j^*$ translates to the condition $J^\sigma = J^\iota$, and the rule $j \mapsto J$ establishes a bijection $V(\mathbf{T}) \cong W_{M_2(\mathbb{Q})}$ identifying $Q_{\mathbf{T}}$ with det. All of the remaining claims are now elementary calculations. □

Let Λ_T be the F-symplectic form on $T_{\mathbb{Q}} = T \otimes_{\mathbb{Z}} \mathbb{Q}$ determined by $\lambda_T = \mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda_T$, and define algebraic groups over $\mathbb{Q}$

$$G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{Sp}(T_{\mathbb{Q}}, \Lambda_T)$$
$$H = \mathrm{SO}(V(\mathbf{T}), Q_{\mathbf{T}}).$$

The group G acts on $V(\mathbf{T})$ through orthogonal transformations by the rule $g \bullet j = g \circ j \circ g^{-1}$, and this defines a homomorphism $G \to H$. In this way one sees that the construction of $V(\mathbf{T})$ from $\mathbf{T}$ gives a concrete way of realizing the exceptional isomorphism of real Lie algebras $\mathfrak{sp}(2) \times \mathfrak{sp}(2) \to \mathfrak{so}(2,2)$. For any choice of $j \in V(\mathbf{T})$ with $Q_{\mathbf{T}}(j) > 0$ the inclusion $H_j \to H$ of the isotropy subgroup of j in H gives a concrete way of realizing the inclusion of real Lie algebras $\mathfrak{so}(1,2) \to \mathfrak{so}(2,2)$. The above exceptional isomorphism will allow us to identify a Hilbert modular surface with an orthogonal Shimura variety. The inclusions $\mathfrak{so}(1,2) \to \mathfrak{so}(2,2)$ for varying j will then have a moduli-theoretic incarnation in the form of a family of special cycles of codimension one, the Hirzebruch–Zagier divisors, on this Shimura variety.

2.3 Polarized CM Modules

Definition 2.3.1 *A* CM module *is a pair* (T, κ_T) *in which* T *is a* $\mathbb{Z}$*-module and* $\kappa_T : \mathscr{O}_E \to \mathrm{End}_{\mathbb{Z}}(T)$ *is a ring homomorphism making* T *into a projective* $\mathscr{O}_E$*-module of rank* 1.

A $\mathfrak{c}$*-polarization* of (T, κ_T) is a $\mathfrak{c}$-polarization λ_T of the underlying RM module. Let $\mathbf{T} = (T, \kappa_T, \lambda_T)$ be a $\mathfrak{c}$-polarized CM module. Elementary linear algebra shows that the $\mathfrak{c}$-polarization λ_T satisfies

$$\lambda_T(\kappa_T(x)t_1, t_2) = \lambda_T(t_1, \kappa_T(\overline{x})t_2)$$

for all $x \in \mathscr{O}_E$. If $\Sigma = \{\pi_1, \pi_2\}$ is a CM type of E then the homomorphism of $\mathbb{Q}$-vector spaces $E \to \mathbb{C} \times \mathbb{C}$ defined by $x \mapsto (\pi_1(x), \pi_2(x))$ extends to an isomorphism of real vector spaces $E_{\mathbb{R}} \cong \mathbb{C} \times \mathbb{C}$. We therefore acquire an action $\kappa_{T,\Sigma}$ of $\mathbb{C} \times \mathbb{C}$ on $T_{\mathbb{R}}$, and in particular the diagonal embedding $\mathbb{C} \to \mathbb{C} \times \mathbb{C} \cong E_{\mathbb{R}}$ makes $T_{\mathbb{R}}$ into a $\mathbb{C}$-vector space. There is a unique choice of CM type Σ for which the Hermitian form on $T_{\mathbb{R}}$

$$H_T(x, y) = \lambda_T(i \cdot x, y) + i\lambda_T(x, y) \tag{2.2}$$

(the scalar multiplication $i \cdot x$ of $\mathbb{C}$ on $T_{\mathbb{R}}$ depends on Σ, as just explained) is positive definite.

Definition 2.3.2 *Given a* $\mathfrak{c}$*-polarized CM module* $\mathbf{T}$ *the* CM type *of* $\mathbf{T}$ *is the unique CM type* $\Sigma = \Sigma(\mathbf{T})$ *for which the Hermitian form (2.2) has positive definite real part.*

Remark 2.3.3 *Let* $\mathbf{T}$ *be a* $\mathfrak{c}$*-polarized CM module. If we fix an isomorphism of* E*-modules* $E \cong T_{\mathbb{Q}}$*, then there is a unique* $\omega_{\mathbf{T}} \in E^{\times}$ *such that* $\overline{\omega}_{\mathbf{T}} = -\omega_{\mathbf{T}}$ *and*

$$\lambda_T(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\omega_{\mathbf{T}} x \overline{y}).$$

If one makes a different choice of isomorphism $E \cong T_{\mathbb{Q}}$ *then* $\omega_{\mathbf{T}}$ *is multiplied by an element of* $\mathrm{Nm}_{E/F}(E^{\times})$*. The CM type of* $\mathbf{T}$ *is characterized as the unique CM type for which the induced* $\mathbb{C}$*-module structure on* $E_{\mathbb{R}}$ *makes* $i \cdot \omega_{\mathbf{T}} \in F_{\mathbb{R}}$ *totally positive.*

Now fix a $\mathfrak{c}$-polarized CM module $\mathbf{T}$ and recall the $\mathbb{Q}$-quadratic space $\big(V(\mathbf{T}), Q_{\mathbf{T}}\big)$ of Sect. 2.2 associated to the underlying RM module. We will use the action of $\mathscr{O}_E$

on $\mathbf{T}$ to make $V(\mathbf{T})$ into a Hermitian $E^\sharp$-module. First define an action of the $\mathbb{Q}$-algebra M of Sect. 2.1 on

$$\tilde{V}(\mathbf{T}) = \{j \in \mathrm{End}_{\mathbb{Z}}(T) \otimes_{\mathbb{Z}} \mathbb{Q} : \kappa_T(x) \circ j = j \circ \kappa_T(x^\sigma) \text{ for all } x \in \mathscr{O}_F\}$$

by

$$(a \otimes b) \bullet j = \kappa_T(a) \circ j \circ \kappa_T(\overline{b}).$$

The subspace $V(\mathbf{T}) \subset \tilde{V}(\mathbf{T})$ of $*$-fixed endomorphisms is stable under the action of the subalgebra $E^\sharp \subset M$, although it is generally false that the $\mathbb{Z}$-lattice $L(\mathbf{T}) \subset V(\mathbf{T})$ is stable under the action of $\mathscr{O}_{E^\sharp}$. If $\mathfrak{l}$ is a place of $F^\sharp$ abbreviate $V_{\mathfrak{l}}(\mathbf{T}) = V(\mathbf{T}) \otimes_{F^\sharp} F^\sharp_{\mathfrak{l}}$.

Lemma 2.3.4 *The $\mathbb{Q}$-bilinear form on $V(\mathbf{T})$ defined by*

$$[j_1, j_2]_{\mathbf{T}} = Q_{\mathbf{T}}(j_1 + j_2) - Q_{\mathbf{T}}(j_1) - Q_{\mathbf{T}}(j_2)$$

satisfies $[x \bullet j_1, j_2]_{\mathbf{T}} = [j_1, x^\dagger \bullet j_2]_{\mathbf{T}}$ *for every* $x \in E^\sharp$.

Proof. We may assume that $x = a \otimes b + b \otimes a$ for some $a, b \in E$, as elements of this form generate $E^\sharp$ as a $\mathbb{Q}$-module. In the interest of simplifying the notation we suppress κ_T, and simply view E as embedded in $\mathrm{End}_{\mathbb{Z}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The essential point is that $F = \{f \in \mathrm{End}_{\mathscr{O}_F}(T) \otimes_{\mathbb{Z}} \mathbb{Q} : f^* = f\}$. In particular, as $j_1 \circ \overline{b} \circ j_2 + j_2 \circ b \circ j_1$ is both $*$-fixed and F-linear, it belongs to F, and so commutes with a. Thus

$$a \circ j_1 \circ \overline{b} \circ j_2 - j_1 \circ \overline{b} \circ j_2 \circ a = j_2 \circ b \circ j_1 \circ a - a \circ j_2 \circ b \circ j_1$$

and similar reasoning shows that

$$j_2 \circ a \circ j_1 \circ \overline{b} - \overline{b} \circ j_2 \circ a \circ j_1 = \overline{b} \circ j_1 \circ \overline{a} \circ j_2 - j_1 \circ \overline{a} \circ j_2 \circ \overline{b}.$$

Using these relations, direct calculation shows

$$[x \bullet j_1, j_2]_{\mathbf{T}} - [j_1, x^\dagger \bullet j_2]_{\mathbf{T}} = 0.$$

□

It follows from Lemma 2.3.4 that there is a unique $E^\sharp$-Hermitian form $\langle j_1, j_2 \rangle_{\mathbf{T}}$ on $V(\mathbf{T})$ satisfying

$$[j_1, j_2]_{\mathbf{T}} = \mathrm{Tr}_{E^\sharp/\mathbb{Q}} \langle j_1, j_2 \rangle_{\mathbf{T}},$$

and that $Q^\sharp_{\mathbf{T}}(j) = \langle j, j \rangle_{\mathbf{T}}$ is the unique $F^\sharp$-quadratic form on $V(\mathbf{T})$ satisfying

$$Q_{\mathbf{T}} = \mathrm{Tr}_{F^\sharp/\mathbb{Q}} \circ Q^\sharp_{\mathbf{T}}.$$

For any CM type Σ of E the restriction of ϕ_Σ to $F^\sharp$ is an archimedean place of $F^\sharp$ denoted ∞^-_Σ. Let ∞^+_Σ be the other archimedean place of $F^\sharp$.

Proposition 2.3.5 *Suppose* $\mathbf{T}$ *has CM type* Σ. *The* $F^\sharp$*-quadratic space* $(V(\mathbf{T}), Q_{\mathbf{T}}^\sharp)$ *has signature* $(2,0)$ *at* ∞_Σ^+, *and has signature* $(0,2)$ *at* ∞_Σ^-.

Proof. Abbreviate $\infty^\pm = \infty_\Sigma^\pm$. Let $\Sigma = \{\pi_1, \pi_2\}$ be the CM type of $\mathbf{T}$, and identify $E_\mathbb{R} \cong \mathbb{C}\times\mathbb{C}$ using the isomorphism $z \mapsto (\pi_1(z), \pi_2(z))$. This makes $T_\mathbb{R}$ into a $\mathbb{C}\times\mathbb{C}$-module, and the idempotents $e_1, e_2 \in F_\mathbb{R}$ induce a decomposition $T_\mathbb{R} \cong T_1 \oplus T_2$ in which each T_k is a one-dimensional $\mathbb{C}$-vector space on which E acts through $\pi_k : E \to \mathbb{C}$. Each T_k comes with an $\mathbb{R}$-symplectic form λ_k (the restriction of λ_T to T_k) for which $x \mapsto \lambda_k(ix, x)$ is positive definite. For any $f \in \mathrm{Hom}_\mathbb{R}(T_1, T_2)$ define $f^\vee \in \mathrm{Hom}_\mathbb{R}(T_2, T_1)$ by the relation $\lambda_1(t_1, f^\vee(t_2)) = \lambda_2(f(t_1), t_2)$ for all $t_k \in T_k$. Using the relation $e_1^\sigma = e_2$, we see that $j \mapsto (j|_{T_1}, j|_{T_2})$ defines an injection

$$V(\mathbf{T})_\mathbb{R} \to \mathrm{Hom}_\mathbb{R}(T_1, T_2) \times \mathrm{Hom}_\mathbb{R}(T_2, T_1),$$

whose image is the space of pairs $(f, f^\vee)$. The quadratic form on $Q_\mathbf{T}$ is identified with $f^\vee \circ f$. In particular restriction to T_1 defines an isomorphism

$$V(\mathbf{T})_\mathbb{R} \cong \mathrm{Hom}_\mathbb{R}(T_1, T_2) = \mathrm{Hom}_\mathbb{C}(T_1, T_2) \oplus \mathrm{Hom}_{\overline{\mathbb{C}}}(T_1, T_2),$$

where the two spaces in the direct sum are the spaces of $\mathbb{C}$-linear and $\mathbb{C}$-conjugate-linear maps. Tracing through these isomorphisms, one sees that the action of $E^\sharp$ is through the reflex homomorphism $\phi_{\{\overline{\pi}_1, \pi_2\}} : E^\sharp \to \mathbb{C}$ on the first summand and through the reflex homomorphism $\phi_{\{\pi_1, \pi_2\}} : E^\sharp \to \mathbb{C}$ on the second summand. The first of these reflex homomorphisms restricts to the place ∞^+ of $F^\sharp$, while the second restricts to the place ∞^-. In other words

$$V(\mathbf{T}) \otimes_{F^\sharp, \infty^+} \mathbb{R} \cong \mathrm{Hom}_\mathbb{C}(T_1, T_2) \tag{2.3}$$

$$V(\mathbf{T}) \otimes_{F^\sharp, \infty^-} \mathbb{R} \cong \mathrm{Hom}_{\overline{\mathbb{C}}}(T_1, T_2). \tag{2.4}$$

Fix isomorphisms of $\mathbb{C}$-vector spaces $T_1 \cong \mathbb{C} \cong T_2$ in such a way that the $\mathbb{R}$-symplectic forms λ_1 and λ_2 are each identified with the form $\lambda_k(x, y) = -\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(ix\overline{y})$ (this is possible because $\lambda_k(ix, x)$ is positive definite). Every

$$f \in \mathrm{Hom}_\mathbb{C}(T_1, T_2) \cong \mathrm{Hom}_\mathbb{C}(\mathbb{C}, \mathbb{C})$$

then has the form $f(t_1) = z \cdot t_1$ for some $z \in \mathbb{C}$, and $f^\vee(t_1) = \overline{z} \cdot t_2$. Thus $f^\vee \circ f = z\overline{z}$ proving that (2.3) is a positive definite $\mathbb{R}$-quadratic space of rank 2. Similarly every

$$f \in \mathrm{Hom}_{\overline{\mathbb{C}}}(T_1, T_2) \cong \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}, \mathbb{C})$$

then has the form $f(t_1) = z \cdot \overline{t}_1$ for some $z \in \mathbb{C}$, and $f^\vee(t_1) = -\overline{z} \cdot \overline{t}_2$. Thus $f^\vee \circ f = -z\overline{z}$ proving that (2.4) is negative definite of rank 2. This completes the proof. $\square$

Propositions 2.2.2 and 2.3.5 imply that $V(\mathbf{T})$ is free of rank one over $E^\sharp$, and that the $E^\sharp$-Hermitian form $\langle\cdot,\cdot\rangle_{\mathbf{T}}$ on $V(\mathbf{T})$ is nondegenerate. It follows that there is an $E^\sharp$-linear isomorphism of $F^\sharp$-quadratic spaces

$$\big(V(\mathbf{T}), Q^\sharp_{\mathbf{T}}\big) \cong \big(E^\sharp, \beta(\mathbf{T})xx^\dagger\big), \tag{2.5}$$

for some $\beta(\mathbf{T}) \in (F^\sharp)^\times$.

The importance of the $F^\sharp$-quadratic space structure on the space $V(\mathbf{T})$ may be understood by considering the algebraic group over $\mathbb{Q}$

$$H^\sharp = \mathrm{Res}_{F^\sharp/\mathbb{Q}}\mathrm{SO}(V(\mathbf{T}), Q^\sharp_{\mathbf{T}}).$$

This group is naturally a subgroup of

$$H = \mathrm{SO}(V(\mathbf{T}), Q_{\mathbf{T}}),$$

and the inclusion $H^\sharp \to H$ gives a concrete way of realizing the inclusion of real Lie algebras $\mathfrak{so}(2) \times \mathfrak{so}(2) \to \mathfrak{so}(2,2)$. In the discussion of moduli problems in Chap. 3, this inclusion will have a moduli-theoretic incarnation in the form of a codimension two cycle on a Hilbert modular surface: the cycle of points with complex multiplication by $\mathscr{O}_E$.

2.4 Algebraic Groups and Class Groups

In this subsection we construct generalized class groups

$$C_0(E) \subset C_+(E) \subset C(E)$$

that act on the set of all $\mathfrak{c}$-polarized CM modules, and algebraic groups S_E and T_E that act on the space of special endomorphisms of a $\mathfrak{c}$-polarized CM module. Let S_E be the algebraic group over $\mathbb{Q}$ whose functor of points is

$$S_E(A) = \{x \in (E^\sharp \otimes_{\mathbb{Q}} A)^\times : xx^\dagger = 1\}$$

for any $\mathbb{Q}$-algebra A. Let T_E be the algebraic group over $\mathbb{Q}$ with functor of points

$$T_E(A) = \{x \in (E \otimes_{\mathbb{Q}} A)^\times : x\overline{x} \in A^\times\}.$$

Let $\mathbb{G}_m$ be the multiplicative over $\mathbb{Q}$, and view $\mathbb{G}_m$ as a subgroup of T_E using the inclusion $A^\times \to (E \otimes_{\mathbb{Q}} A)^\times$. There is a natural group homomorphism $E^\times \to (E^\sharp)^\times$ defined by $x \mapsto x \otimes x$. This homomorphism may be modified, as in the following lemma, to yield a homomorphism of algebraic groups $T_E \to S_E$.

Lemma 2.4.1 *Define a homomorphism* $\nu_E : T_E \to S_E$ *by*

$$\nu_E(x) = \frac{x \otimes x}{x\overline{x}}.$$

If k *is a field of characteristic* 0*, or* $k = \mathbb{A}$*, or* $k = \mathbb{A}_f$*, then the sequence*

$$1 \to \mathbb{G}_m(k) \to T_E(k) \xrightarrow{\nu_E} S_E(k) \to 1$$

is exact.

Proof. See the proof of [22, Proposition 2.13] □

For every prime $\ell < \infty$ define a compact open subgroup $U_E = \prod_\ell U_{E,\ell}$ of $T_E(\mathbb{A}_f)$ by

$$U_E = T_E(\mathbb{A}_f) \cap \widehat{\mathscr{O}}_E^\times.$$

The map $\nu_E : T_E \to S_E$ of Lemma 2.4.1 induces an isomorphism

$$T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_f)/U_E \cong S_E(\mathbb{Q})\backslash S_E(\mathbb{A}_f)/\nu_E(U_E).$$

Let $I(E)$ be the set of all pairs $\mathbf{Z} = (\mathfrak{z}, \zeta)$ in which $\mathfrak{z}$ is a fractional ideal of $\mathscr{O}_E$ and $\zeta \in F^\times$ satisfies $\mathfrak{z}\overline{\mathfrak{z}} = \zeta\mathscr{O}_E$. Then $I(E)$ is a group under componentwise multiplication, and $P(E) = \{(z\mathscr{O}_E, z\overline{z}) : z \in E^\times\}$ is a subgroup. Define a generalized class group

$$C(E) = I(E)/P(E)$$

and let $C_+(E) \subset C(E)$ be the subgroup consisting of those $(\mathfrak{z}, \zeta)$ for which ζ is totally positive. The function $(\mathfrak{z}, \zeta) \mapsto \mathfrak{z}$ defines a homomorphism $C(E) \to \mathrm{Pic}(\mathscr{O}_E)$ with finite kernel, and so $C(E)$ is finite. Given a $t \in T_E(\mathbb{A}_f)$ let ζ be the unique positive rational number that satisfies $\zeta\widehat{\mathbb{Z}} = (t\overline{t})\widehat{\mathbb{Z}}$, and let $\mathfrak{z}$ be the fractional $\mathscr{O}_E$-ideal defined by $\mathfrak{z}\widehat{\mathscr{O}}_E = t\widehat{\mathscr{O}}_E$. Then $t \mapsto (\mathfrak{z}, \zeta)$ determines an injective homomorphism

$$T_E(\mathbb{Q})\backslash T_E(\mathbb{A}_f)/U_E \to C_+(E) \tag{2.6}$$

whose image is denoted $C_0(E) \subset C_+(E)$.

Let $\mathbf{T} = (T, \kappa_T, \lambda_T)$ be a $\mathfrak{c}$-polarized CM module. Given a pair $\mathbf{Z} = (\mathfrak{z}, \zeta) \in I(E)$ define a new $\mathfrak{c}$-polarized CM module

$$(T, \kappa_T, \lambda_T) \otimes \mathbf{Z} = (S, \kappa_S, \lambda_S)$$

as follows. The underlying $\mathbb{Z}$-module is $S = T \otimes_{\mathscr{O}_E} \mathfrak{z}$, the action $\kappa_S : \mathscr{O}_E \to \mathrm{End}(S)$ is $\kappa_S(x)(t \otimes z) = t \otimes (xz)$, and λ_S is defined by

$$\lambda_S\big(t_1 \otimes z_1, t_2 \otimes z_2\big) = \lambda_T(\kappa_T(\zeta^{-1}z_1\overline{z}_2)t_1, t_2).$$

The right hand side makes sense as $\zeta^{-1}z_1\bar{z}_2 \in \mathcal{O}_E$. The construction $\mathbf{T} \mapsto \mathbf{T} \otimes \mathbf{Z}$ defines an action of $C(E)$ on the set of isomorphism classes of $\mathfrak{c}$-polarized CM modules. Using the notation of Remark 2.3.3, a simple calculation shows that $\omega_{\mathbf{T}\otimes\mathbf{Z}} = \zeta^{-1} \cdot \omega_{\mathbf{T}}$ from which it follows that

$$\Sigma(\mathbf{T} \otimes \mathbf{Z}) = \Sigma(\mathbf{T}) \iff \mathbf{Z} \in C_+(E). \tag{2.7}$$

Proposition 2.4.2 *1. The set X of isomorphism classes of $\mathfrak{c}$-polarized CM modules is a simply transitive $C(E)$-set.*

2. The set X_Σ of isomorphism classes of $\mathfrak{c}$-polarized CM modules with a fixed CM type Σ is either empty or is a simply transitive $C_+(E)$-set. If there is a finite prime of F ramified in E then X_Σ is nonempty.

Proof. First we show that the set of $\mathfrak{c}$-polarized CM modules is nonempty. Let $\mathfrak{A}$ be any fractional $\mathcal{O}_E$-ideal, and fix an $\omega \in E^\times$ such that $\overline{\omega} = -\omega$. Define a $\mathbb{Z}$-bilinear alternating form

$$\lambda(x, y) = \mathrm{Tr}_{E/\mathbb{Q}}(\omega x \overline{y})$$

on $\mathfrak{A}$. If $\kappa : \mathcal{O}_E \to \mathrm{End}_{\mathbb{Z}}(\mathfrak{A})$ is the natural action, the triple $(\mathfrak{A}, \kappa, \lambda)$ is a $\mathfrak{b}$-polarized CM module, where $\mathfrak{b}^{-1} = \omega\mathfrak{A}\overline{\mathfrak{A}}\mathfrak{D}_E$. Here $\mathfrak{D}_E$ is the different of $E/\mathbb{Q}$. The Hilbert class field of F is linearly disjoint from E (as E is ramified at the archimedean places), and so class field theory implies that the norm map from the ideal class group of E to the ideal class group of F is surjective. Therefore we may factor $\mathfrak{c}\mathfrak{b}^{-1} = y\mathfrak{Y}\overline{\mathfrak{Y}}$ for some $y \in F^\times$ and some fractional $\mathcal{O}_E$-ideal $\mathfrak{Y}$. If ω is replaced by $y\omega$ and $\mathfrak{A}$ is replaced by $\mathfrak{Y}\mathfrak{A}$, then $\mathbf{T} = (\mathfrak{A}, \kappa, \lambda)$ is a $\mathfrak{c}$-polarized CM module. In the notation of Remark 2.3.3, $\omega = \omega_{\mathbf{T}}$.

The proof that the action of $C(E)$ on X is simply transitive is a routine exercise, which we leave to the reader. This, together with (2.7), implies that X_Σ is either empty or a simply transitively $C_+(E)$-set.

Now use Σ to view $E_\mathbb{R}$ as a $\mathbb{C}$-vector space, as in Sect. 2.3. We may repeat the argument of the first paragraph, but choose the initial the traceless $\omega \in E^\times$ so that $i\omega \in F_\mathbb{R}$ is totally positive. If there is at least one finite prime of F that is ramified in E then the narrow Hilbert class field of F is linearly disjoint from E, and class field theory implies that the norm map from the ideal class group of E to the narrow ideal class group of F is surjective. This allows us to choose y to be totally positive, and Remark 2.3.3 then shows that the $\mathbf{T}$ constructed above has CM type Σ. □

The remainder of this subsection is devoted to the proof of the following proposition, which will be a crucial ingredient in the proof of Theorem 5.3.4. For a $\mathfrak{c}$-polarized CM module $\mathbf{T}$ set

$$\widehat{L}(\mathbf{T}) = L(\mathbf{T}) \otimes_\mathbb{Z} \widehat{\mathbb{Z}} \qquad \widehat{V}(\mathbf{T}) = V(\mathbf{T}) \otimes_\mathbb{Q} \widehat{\mathbb{Q}}.$$

Proposition 2.4.3 *Assume either (**cyclic**) or (**nongal**). There is a*

$$\mathbf{Z} = (\mathfrak{Z}, \zeta) \in C(E)$$

such that $\mathrm{Nm}_{F/\mathbb{Q}}(\zeta) < 0$*, and such that for any* $\mathfrak{c}$*-polarized CM module* $\mathbf{T}$ *there is an isomorphism of* $\widehat{F}^{\sharp}$*-quadratic spaces*

$$\big(\widehat{V}(\mathbf{T}), Q^{\sharp}_{\mathbf{T}}\big) \cong \big(\widehat{V}(\mathbf{T}\otimes\mathbf{Z}), Q^{\sharp}_{\mathbf{T}\otimes\mathbf{Z}}\big)$$

identifying $\widehat{L}(\mathbf{T})$ *with* $\widehat{L}(\mathbf{T}\otimes\mathbf{Z})$*. For any such* $\mathbf{Z}$ *the reflex homomorphisms*

$$\phi_{\Sigma(\mathbf{T}\otimes\mathbf{Z})}, \phi_{\Sigma(\mathbf{T})} : E^{\sharp} \to \mathbb{C}$$

have distinct restrictions to $F^{\sharp}$ *(equivalently, the CM types* $\Sigma(\mathbf{T}\otimes\mathbf{Z})$ *and* $\Sigma(\mathbf{T})$ *are neither equal nor complex conjugates).*

Before the proof, we need some technical preparation. Letting ∞ denote the archimedean place of $\mathbb{Q}$, define finite groups of exponent 2

$$\mathrm{Gen}_{\infty}(E/F) = F^{\times}_{\infty}/\mathrm{Nm}_{E/F}(E^{\times}_{\infty})$$
$$\mathrm{Gen}_{f}(E/F) = \widehat{\mathscr{O}}^{\times}_{F}/\mathrm{Nm}_{E/F}(\widehat{\mathscr{O}}^{\times}_{E}),$$

and the *genus group*

$$\mathrm{Gen}(E/F) = \mathrm{Gen}_{\infty}(E/F) \times \mathrm{Gen}_{f}(E/F).$$

The projections to the two factors are denoted $\mathbf{z} \mapsto \mathbf{z}_{\infty}$ and $\mathbf{z} \mapsto \mathbf{z}_{f}$. Given $\mathbf{Z} = (\mathfrak{z}, \zeta) \in I(E)$ we may choose an idele $z \in \mathbb{A}^{\times}_{E}$ such that $z\widehat{\mathscr{O}}_{E} = \mathfrak{z}\widehat{\mathscr{O}}_{E}$. Then $\mathrm{gen}(\mathbf{Z}) = \zeta^{-1}z\overline{z}$ defines the *genus invariant*

$$\mathrm{gen} : C(E) \to \mathrm{Gen}(E/F).$$

The subgroup $C_{+}(E) \subset C(E)$ is precisely the kernel of $\mathbf{Z} \mapsto \mathrm{gen}(\mathbf{Z})_{\infty}$. If $\chi : \mathbb{A}^{\times}_{F} \to \{\pm 1\}$ denotes the idele class character corresponding to the extension E/F, a brief exercise in class field theory shows that the sequence

$$C(E) \xrightarrow{\mathrm{gen}} \mathrm{Gen}(E/F) \xrightarrow{\chi} \{\pm 1\} \to 1 \tag{2.8}$$

is exact, where the arrow labeled χ is the composition

$$\mathrm{Gen}(E/F) \to \mathbb{A}^{\times}_{F}/\mathrm{Nm}_{E/F}(\mathbb{A}^{\times}_{E}) \xrightarrow{\chi} \{\pm 1\}.$$

Lemma 2.4.4 *Assuming either (**cyclic**) or (**nongal**), there is a* $\mathbf{Z} \in C(E)$ *and a* $u \in \widehat{\mathbb{Z}}^{\times}$ *such that* $\mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_{\infty}) < 0$ *and*

$$u^{2} \cdot \mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_{f}) \in \mathrm{Nm}_{E/\mathbb{Q}}(\widehat{\mathscr{O}}^{\times}_{E}),$$

where $\mathbf{z} = \mathrm{gen}(\mathbf{Z})$.

Proof. If we choose a totally negative $\Delta \in F^\times$ such that $E = F(\sqrt{\Delta})$ then our hypothesis that $E/\mathbb{Q}$ is not a biquadratic extension implies $\mathrm{Nm}_{F/\mathbb{Q}}(\Delta) \notin (\mathbb{Q}^\times)^2$. Let p be any prime such that $\mathrm{ord}_p(\mathrm{Nm}_{F/\mathbb{Q}}(\Delta))$ is odd. Then p is either split or ramified in F, and in either case there is a place v_0 of F above p for which $\mathrm{ord}_{v_0}(\Delta)$ is odd. The place v_0 is necessarily ramified in E, and if w_0 denotes the place of E above v_0 then we may choose a $\mathbf{z}_{v_0} \in \mathscr{O}_{F,v_0}^\times$ that is not a norm from $\mathscr{O}_{E,w_0}^\times$.

If p is split in F then let $v_1 \neq v_0$ be the other place above p. Then $\mathrm{ord}_{v_1}(\Delta)$ is even, and class field theory (or a Hilbert symbol calculation) gives the first equality in

$$\mathbb{Z}_p^\times = \mathrm{Nm}_{E_{v_0}/\mathbb{Q}_p}(\mathscr{O}_{E,v_0}^\times) \cdot \mathrm{Nm}_{E_{v_1}/\mathbb{Q}_p}(\mathscr{O}_{E,v_1}^\times) = \mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathscr{O}_{E,p}^\times).$$

Thus

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathscr{O}_{E,p}^\times).$$

If v is a finite place of F with $v \neq v_0$ then set $\mathbf{z}_v = 1 \in \mathscr{O}_{F,v}^\times$. Now define

$$\mathbf{z}_f = \prod_v \mathbf{z}_v \in \mathrm{Gen}_f(E/F)$$

and set

$$\mathbf{z}_\infty = (1, -1) \in \{\pm 1\} \times \{\pm 1\} \cong \mathrm{Gen}_\infty(E/F).$$

and $\mathbf{z} = (\mathbf{z}_\infty, \mathbf{z}_f) \in \mathrm{Gen}(E/F)$. By construction $\chi(\mathbf{z}) = 1$, and so by the exactness of 2.8 there is a $\mathbf{Z} \in C(E)$ such that $\mathrm{gen}(\mathbf{Z}) = \mathbf{z}$. This choice of $\mathbf{Z}$ has the desired properties.

Now assume that p is totally ramified in E. If $E_{w_0}/\mathbb{Q}_p$ is a biquadratic field extension then $\mathrm{Nm}_{F/\mathbb{Q}}(\Delta) \in (\mathbb{Q}_p^\times)^2$, contradicting the choice of p. Thus either $E_{w_0}/\mathbb{Q}_p$ is not Galois, or $E_{w_0}/\mathbb{Q}_p$ is Galois with cyclic Galois group. Assume first that $E_{w_0}/\mathbb{Q}_p$ is Galois with cyclic Galois group. The Artin symbol $[\mathbf{z}_{v_0}; E_{w_0}/F_{v_0}]$ is the nontrivial element of $\mathrm{Gal}(E_{w_0}/F_{v_0})$. By local class field theory the inclusion $\mathrm{Gal}(E_{w_0}/F_{v_0}) \to \mathrm{Gal}(E_{w_0}/\mathbb{Q}_p)$ satisfies

$$[\mathbf{z}_{v_0}; E_{w_0}/F_{v_0}] \mapsto [\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}); E_{w_0}/\mathbb{Q}_p]$$

and we deduce that the element

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathbb{Z}_p^\times/\mathrm{Nm}_{E_{w_0}/\mathbb{Q}_p}(\mathscr{O}_{E,w_0}^\times) \cong \mathrm{Gal}(E_{w_0}/\mathbb{Q}_p)$$

has order 2, and hence is a square. Thus for some $u_p \in \mathbb{Z}_p^\times$ we have

$$u_p^2 \cdot \mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathscr{O}_{E,p}^\times).$$

We now set $\mathbf{z}_v = 1$ for every finite place $v \neq v_0$ and construct $\mathbf{z}$ and $\mathbf{Z}$ exactly as in the previous paragraph. It remains to treat the case in which $E_{w_0}/\mathbb{Q}_p$ is not Galois. In this case if we set $L = F_{v_0}(\sqrt{\Delta^\sigma})$ then $L \ncong E_{w_0}$, and so class field theory implies

$$F_{v_0}^\times = \mathrm{Nm}_{E_{w_0}/F_{v_0}}(E_{w_0}^\times) \cdot \mathrm{Nm}_{L/F_{v_0}}(L^\times).$$

If we now factor

$$\mathbf{z}_{v_0} = \mathrm{Nm}_{E_{w_0}/F_{v_0}}(a) \cdot \mathrm{Nm}_{L/F_{v_0}}(b)$$

with $a \in E_{w_0}^\times$ and $b \in L^\times$ then

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) = \mathrm{Nm}_{E_{w_0}/\mathbb{Q}_p}(a) \cdot \mathrm{Nm}_{L/\mathbb{Q}_p}(b).$$

By construction of L the norm maps $E_{w_0}^\times \to \mathbb{Q}_p^\times$ and $L^\times \to \mathbb{Q}_p^\times$ have the same image, and so

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(E_{w_0}^\times).$$

But $\mathbf{z}_{v_0} \in \mathbb{Z}_p^\times$, and hence

$$\mathrm{Nm}_{F_{v_0}/\mathbb{Q}_p}(\mathbf{z}_{v_0}) \in \mathrm{Nm}_{E_p/\mathbb{Q}_p}(\mathscr{O}_{E,p}^\times).$$

The construction of $\mathbf{z}$ and $\mathbf{Z}$ now proceeds as in the previous paragraph. $\square$

Lemma 2.4.5 *Fix a $\mathbf{Z} \in C(E)$ and a $\mathfrak{c}$-polarized CM module $\mathbf{T}$. If we set $\mathbf{S} = \mathbf{T} \otimes \mathbf{Z}$, then there is an isomorphism of $\widehat{F}^\sharp$-quadratic spaces*

$$\big(\widehat{V}(\mathbf{T}), Q_{\mathbf{T}}^\sharp\big) \cong \big(\widehat{V}(\mathbf{S}), \mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_f) \cdot Q_{\mathbf{S}}^\sharp\big)$$

identifying $\widehat{L}(\mathbf{T})$ with $\widehat{L}(\mathbf{S})$. Here $\mathbf{z}_f \in \widehat{\mathscr{O}}_F^\times$ is any representative of the finite part of $\mathbf{z} = \mathrm{gen}(\mathbf{Z})$.

Proof. This is a simple calculation. Fix a representative $(\mathfrak{Z}, \zeta) \in I(E)$ of $\mathbf{Z}$ and let $z \in \mathbb{A}_E^\times$ satisfy $z \cdot \widehat{\mathscr{O}}_E = \mathfrak{Z}\widehat{\mathscr{O}}_E$. There is an $\widehat{\mathscr{O}}_E$-linear isomorphism $\psi : \widehat{T} \to \widehat{S}$ defined by $\psi(t) = t \otimes z_f$. Given a $j \in \widehat{L}(\mathbf{T})$ one checks directly that

$$\psi_* j = \psi \circ \kappa_T(\mathbf{z}_f^{-1}) \circ j \circ \psi^{-1}$$

defines an element of $\widehat{L}(\mathbf{S})$, and that $j \mapsto \psi_* j$ is the desired isomorphism. $\square$

Proof (of Proposition 2.4.3). Let $\mathbf{Z}$ be as in Lemma 2.4.4, and set $\mathbf{S} = \mathbf{T} \otimes \mathbf{Z}$. By Lemma 2.4.5 there is an $r \in \widehat{\mathscr{O}}_E^\times$, and an isomorphism

$$(\widehat{V}(\mathbf{T}), Q_{\mathbf{T}}^\sharp) \cong (\widehat{V}(\mathbf{S}), \mathrm{Nm}_{E/\mathbb{Q}}(r) \cdot Q_{\mathbf{S}}^\sharp)$$

identifying the $\widehat{\mathbb{Z}}$-lattices $\widehat{L}(\mathbf{T})$ and $\widehat{L}(\mathbf{S})$. If we set $s = r \otimes r \in (\widehat{E}^\sharp)^\times$ then $\mathrm{Nm}_{E^\sharp/F^\sharp}(s) = \mathrm{Nm}_{E/\mathbb{Q}}(r)$, and

$$s \bullet \widehat{L}(\mathbf{S}) = \kappa_S(r) \circ \widehat{L}(\mathbf{S}) \circ \kappa_S(\overline{r}) = \widehat{L}(\mathbf{S}).$$

Using the relation $Q^{\sharp}_{\mathbf{S}}(s \bullet x) = \mathrm{Nm}_{E^{\sharp}/F^{\sharp}}(s) \cdot Q^{\sharp}_{\mathbf{S}}(x)$ we see that $x \mapsto s \bullet x$ defines an isomorphism

$$(\widehat{V}(\mathbf{S}), \mathrm{Nm}_{E/\mathbb{Q}}(r) \cdot Q^{\sharp}_{\mathbf{S}}) \cong (\widehat{V}(\mathbf{S}), Q^{\sharp}_{\mathbf{S}}),$$

which preserves $\widehat{L}(\mathbf{S})$.

If we represent $\mathbf{Z} \in C(E)$ by a pair $(\mathfrak{Z}, \zeta) \in I(E)$ then $\mathrm{Nm}_{F/\mathbb{Q}}(\mathbf{z}_{\infty}) < 0$ implies that $\zeta \in F^{\times}$ is neither totally positive nor totally negative. From Remark 2.3.3 and the discussion preceding (2.7), it follows that the CM types of $\mathbf{S}$ and $\mathbf{T}$ are neither equal nor complex conjugates. $\square$

Springer

http://www.springer.com/978-3-642-23978-6

Intersections of Hirzebruch–Zagier Divisors and CM Cycles
Howard, B.; Yang, T.
2012, VIII, 140 p., Softcover
ISBN: 978-3-642-23978-6