

# Preface

The goal of this monograph is to prove new instances of relations between special cycles on integral models of Shimura varieties, and the Fourier coefficients of derivatives of Eisenstein series. The prototype of such a relation appears in the famous work of Gross and Zagier [17], where the special cycles are linear combinations of Heegner points on the integral model of the modular curve  $X_0(N)$ , and the Eisenstein series in question appears in the integral representation of a Rankin–Selberg  $L$ -function.

A program to generalize such relations to other Shimura varieties, and to Eisenstein series on higher rank groups, has been initiated by Kudla [26, 28], and in many special cases such relations have been proved. To give few examples, the book [34] of Kudla, Rapoport, and the second author relates the intersections of cycles of CM points on Shimura curves to a Siegel Eisenstein series of genus two, and uses these relations to prove new formulas of Gross–Zagier type. Here one may think of the Shimura curve as being associated with the  $\mathrm{GSpin}$  cover of a reductive group of type  $\mathrm{SO}(1, 2)$ , and of the CM points as arising from a family of embeddings  $\mathrm{SO}(0, 2) \rightarrow \mathrm{SO}(1, 2)$ . In a similar spirit, Kudla and Rapoport study in [29] the triple intersection of divisors on an  $\mathrm{SO}(2, 2)$  Shimura variety arising from embeddings  $\mathrm{SO}(1, 2) \rightarrow \mathrm{SO}(2, 2)$ , and relate these intersections to the Fourier coefficients of a Siegel Eisenstein series of genus three. In [30] those same authors study the fourfold intersection of divisors on a Shimura variety of type  $\mathrm{SO}(3, 2)$  arising from embeddings  $\mathrm{SO}(2, 2) \rightarrow \mathrm{SO}(3, 2)$ , and relate these intersections to a Siegel Eisenstein series of genus four. When  $n > 3$ , the interpretation of orthogonal Shimura varieties of signature  $(n, 2)$  as moduli spaces of abelian varieties breaks down, slowing further progress in this direction. A conjectural picture for all  $n$  is described in [28]. Fortunately, the Shimura varieties of type  $\mathrm{GU}(p, q)$  have a moduli interpretation for all signatures  $(p, q)$ , providing fertile ground for future research. Recent work of Kudla and Rapoport [31, 32] treats the  $n$ -fold intersections of divisors arising from embeddings  $\mathrm{GU}(n-2, 1) \rightarrow \mathrm{GU}(n-1, 1)$ , and their relation to Fourier coefficients of Eisenstein series on  $\mathrm{U}(n, n)$ .

In the series of papers [5, 57, 58], Bruinier and the second author study a different problem, in which a family of divisors on a Hilbert modular surface is intersected

with a fixed cycle of codimension two. If one views the Hilbert modular surface as the Shimura variety associated with the  $\mathrm{GSpin}$  cover of a reductive group of type  $\mathrm{SO}(2, 2)$ , then the family of divisors arise from embeddings  $\mathrm{SO}(1, 2) \rightarrow \mathrm{SO}(2, 2)$ , and are commonly known as *Hirzebruch–Zagier divisors*. The fixed cycle of codimension two arises from an embedding  $\mathrm{SO}(2, 0) \times \mathrm{SO}(0, 2) \rightarrow \mathrm{SO}(2, 2)$ . Under the moduli interpretation of the Hilbert modular surface, the Hirzebruch–Zagier divisors correspond to embedded Shimura curves, and the codimension two cycle corresponds to a collection of complex multiplication points. The papers just cited relate these intersection multiplicities of these cycles to the Fourier coefficients of the *pullback* of a Hilbert modular Eisenstein series via the diagonal embedding  $\mathfrak{H} \rightarrow \mathfrak{H} \times \mathfrak{H}$  of the complex upper half plane.

The main result of this monograph is an arithmetic interpretation of the original Fourier coefficients of the Hilbert modular Eisenstein series, rather than the coefficients of its diagonal restriction. Many of the main results of [57, 58] then follow as easy corollaries, and with fewer unwanted hypotheses. In particular, we obtain results in the case where the field of complex multiplication is a biquadratic extension of  $\mathbb{Q}$ . In this case, excluded in the work of Bruinier and the second author cited above, the cycle of complex multiplication points and the Hirzebruch–Zagier divisors may have nonempty intersection in the complex fiber of the Hilbert modular surface.

The relations we prove among intersection multiplicities and Eisenstein series strongly suggest (and are a step toward proving) a Gross–Zagier style theorem for Hilbert modular surfaces, as explained later in the introduction.

Finally, the methods used here to study cycles on Hilbert modular surfaces apply equally well to cycles on a product of modular curves. In this setting, the Hirzebruch–Zagier divisors are replaced by the classical Hecke correspondences, and our methods yield a refined version of the results of Gross and Zagier on the prime factorizations of singular moduli [16]. Because the proofs simplify drastically in this degenerate case, it is treated in a separate paper [22].

Intersections of Hirzebruch–Zagier Divisors and CM  
Cycles

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