Chapter 3
The Equation of Cinquini-Cibrario

3.1 Very Brief Historical Remarks

Although the study of mixed elliptic–hyperbolic equations goes back at least to Riemann's computation of the Laplacian in toroidal coordinates (c.f. [46] or p. 461, (B) of [7]), the first systematic study of well-posedness for boundary value problems appears to be the memoir by Tricomi [49]. Reasoning from purely mathematical assumptions, Tricomi studied the equation

$$yu_{xx} + u_{yy} = 0, \tag{3.1}$$

where $u = u(x, y)$. This equation is typically defined on a domain $\Omega \subset\subset \mathbf{R}^2$ bounded by a smooth Jordan curve γ in the upper half plane and the characteristic lines

$$x + \frac{2}{3}(-y)^{3/2} = 1, \;\; x > 0, \tag{3.2}$$

$$x - \frac{2}{3}(-y)^{3/2} = -1, \;\; x < 0, \tag{3.3}$$

in the lower half-plane (Fig. 3.1). Tricomi proved that a unique solution to (3.1) exists on Ω provided the values of $u(x, y)$ are smoothly prescribed on the elliptic arc γ and the characteristic line (3.3).

Tricomi further claimed that any sufficiently smooth equation having the form

$$\alpha(x, y)\, u_{xx} + 2\beta(x, y)\, u_{xy} + \gamma(x, y)\, u_{yy} + \text{lower order} = 0,$$

for which the discriminant $\beta^2 - \alpha\gamma$ changes sign along a smooth curve, is locally equivalent to (3.1) plus lower order terms. This erroneous statement (which is repeated, for example, on pp. 81 and 82 of the classic text [11]) was corrected in the early 1930s by Cinquini-Cibrario, who studied the equation [18]

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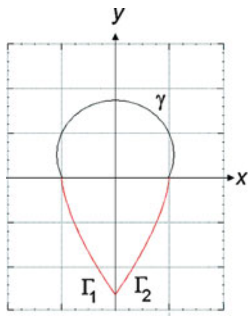

Fig. 3.1 A typical Tricomi domain. The elliptic region is bounded by the curve γ and lies above the x-axis; the hyperbolic region is bounded by the characteristic lines Γ_1, Γ_2 and lies below the x-axis. The hyperbolic boundary is the graph of the equation $y = -\{(3/2)\,[(1-x)]\}^{2/3}$

$$xu_{xx} + u_{yy} = 0. \tag{3.4}$$

Her investigations led to the discovery of a small number of canonical forms, slightly generalizing (3.1) and (3.4), to which any linear second-order elliptic–hyperbolic equation is reducible near a point of $\mathbf{R}^2$ (see [17], and Chap. 1 of [12]).

It was shortly after Cinquini-Cibrario abandoned her study of elliptic–hyperbolic equations in favor of other areas of analysis that von Kármán and Frankl' [22] independently drew attention to the physical significance of equations closely related to (3.1), having the forms

$$yu_{xx} - u_{yy} = 0 \tag{3.5}$$

and

$$\mathscr{K}(y)u_{xx} + u_{yy} = 0, \tag{3.6}$$

where

$$\mathscr{K}(0) = 0 \tag{3.7}$$

and

$$y\mathscr{K}(y) > 0 \text{ for } y \neq 0. \tag{3.8}$$

Such equations arise as linear approximations to the quasilinear *continuity equation* for steady ideal potential flow in the plane (Sect. 2.7.2, application (*i*)). The approximation is valid near the sonic curve, at which the character of the continuity equation changes from elliptic to hyperbolic type. Equations having the form (3.6) – possibly including lower-order terms – and satisfying conditions (3.7) and (3.8) are said to be of Tricomi type (c.f. (1.1)). Despite the importance of Cinquini-Cibrario's contribution, equations having the form

$$\mathscr{K}(x)u_{xx} + u_{yy} + \text{ lower-order terms } = 0, \tag{3.9}$$

satisfying (3.7) and

$$x\mathscr{K}(x) > 0 \text{ for } x \neq 0, \tag{3.10}$$

are said to be of Keldysh type (c.f. (1.2)), in honor of the mathematician who studied the degeneration of ellipticity in such equations in the early 1950s [30]. We will use this established terminology, except that (3.4) will be called *Cinquini-Cibrario's equation,* rather than the *Keldysh equation* as in, e.g., [16].

3.2 Transformation to Canonical Form

In considering canonical forms for elliptic–hyperbolic equations, we ignore lower-order terms as they do not significantly affect the analysis. Lower-order terms are, however, often crucial to the existence of solutions to equations having the form (3.9), as will be shown.

Suppose our equation originally has the form

$$\alpha\,(x,z)\,\varphi_{xx} + \gamma\,(x,z)\,\varphi_{zz} = 0, \tag{3.11}$$

where γ is nonvanishing on the domain. (All the models considered in the sequel will have highest-order terms of this form, either in cartesian or polar coordinates. The coefficient of φ_{xz} can be set to zero by a standard coordinate change; see Sect. 1.2 of [12].) Perform the coordinate transformation $(x,z) \to (\xi\,(x,z), \eta\,(x,z))$, where

$$\xi = \alpha\,(x,z).$$

In these coordinates,

$$\begin{aligned}\alpha\varphi_{xx} + \gamma\varphi_{zz} = {} & \left(\xi\xi_x^2 + \gamma\xi_z^2\right)\varphi_{\xi\xi} \\ & +2\left(\xi\xi_x\eta_x + \gamma\xi_z\eta_z\right)\varphi_{\xi\eta} + \left(\xi\eta_x^2 + \gamma\eta_z^2\right)\varphi_{\eta\eta}.\end{aligned} \tag{3.12}$$

If the transformation $(x,z) \to (\xi,\eta)$ is to be nonsingular, its Jacobian must be nonvanishing, *i.e.,*

$$\xi_x\eta_z - \xi_z\eta_x \neq 0. \tag{3.13}$$

In addition, we want the coefficients of the cross term $\varphi_{\xi\eta}$ to be zero in the new coordinates; so, taking into account (3.12), we impose the condition that

$$\xi\xi_x\eta_x + \gamma\xi_z\eta_z = 0. \tag{3.14}$$

It is easy for the two first derivatives of η to satisfy (3.13) and (3.14) simultaneously for given ξ and γ.

Either

i) ξ and ξ_z never vanish simultaneously on the sonic curve, or else
ii) There exist one or more points (x,z) on the sonic curve at which

$$\xi\,(x,z) = \xi_z\,(x,z) = 0. \tag{3.15}$$

Because $\xi = \alpha(x, z)$ and γ does not vanish, case *ii*) is equivalent to the case in which the characteristics of (3.11) are tangent to the sonic curve; c.f. (2.3).

In case *i)*, in order for ξ to vanish we would need, given condition (3.14) and the assumption that γ is positive, the additional condition $\eta_z = 0$. But if ξ and η_z both vanish, then the coefficient of $\varphi_{\eta\eta}$ in (3.12) also vanishes. That is,

$$\xi \eta_x^2 + \gamma \eta_z^2 = 0 \tag{3.16}$$

whenever $\xi = 0$.

Applying (3.14) to (3.12), we obtain from (3.11) the equation

$$\varphi_{\xi\xi} + \frac{\xi \eta_x^2 + \gamma \eta_z^2}{\xi \xi_x^2 + \gamma \xi_z^2} \varphi_{\eta\eta} = 0. \tag{3.17}$$

The denominator in the coefficient of $\varphi_{\eta\eta}$ cannot be zero: ξ and ξ_z cannot vanish simultaneously, and if ξ_x vanishes, then ξ_z must be nonzero in order to preserve condition (3.13). The numerator in the coefficient of $\varphi_{\eta\eta}$ vanishes whenever $\xi = 0$, by the arguments leading to (3.16). Thus (3.17) is an equation having the form

$$\varphi_{\xi\xi} + \mathscr{K}(\xi, \eta)\, \varphi_{\eta\eta} = 0, \tag{3.18}$$

where $\mathscr{K}(\xi, \eta) = 0$ when $\xi = 0$, an equation of *Tricomi type*.

In case *ii)*, condition (3.13) prevents η_z from vanishing when ξ_z vanishes. Thus in case *ii)* we obtain from (3.11), (3.12), and (3.14) an equation of the form

$$\frac{\xi \xi_x^2 + \gamma \xi_z^2}{\xi \eta_x^2 + \gamma \eta_z^2} \varphi_{\xi\xi} + \varphi_{\eta\eta} = 0, \tag{3.19}$$

where the numerator in the coefficient of $\varphi_{\xi\xi}$, but not the denominator, is zero at one or more points on the sonic curve when ξ is zero there. That is, (3.19) is an equation of the form

$$\mathscr{K}(\xi, \eta)\, \varphi_{\xi\xi} + \varphi_{\eta\eta} = 0, \tag{3.20}$$

where $\mathscr{K}(\xi, \eta) = 0$ when $\xi = 0$, an equation of *Keldysh type*. That is what we would expect from our interpretation of case *ii*) as requiring that the characteristics of (3.11) be tangent to the sonic curve.

This line of reasoning establishes the two canonical forms for equations originally having the form (3.11). See Sect. 1.2 of [12, 17], Sect. 1 of [40], and (75)–(78) of [52] for similar arguments.

3.3 A Closed Dirichlet Problem Which Is Classically Ill-Posed

In 1956, Morawetz [38] proved the uniqueness of sufficiently smooth solutions to *open* Dirichlet problems, having data prescribed on only part of the boundary, for certain mixed elliptic–hyperbolic equations of Tricomi type. That result implied that the closed Dirichlet problem is over-determined for sufficiently smooth solutions of such equations.

Morawetz's result was later extended to a large class of boundary value problems for Tricomi-type equations, by Manwell ([36] and Sect. 16 of [37]) and by Morawetz herself [39]. In this section we extend the result to a large class of equations of Keldysh type.

For given $\mathscr{K}(x)$, define constants a, b, d, and m, where $m < a \leq 0 < d$ and $b > 0$. Consider the domain $\mathscr{D}$ formed by the line segments

$$\mathscr{L}_1 = \{(x,y)\,|a \leq x \leq d, y = -b\};$$
$$\mathscr{L}_2 = \{(x,y)\,|x = d, -b \leq y \leq b\};$$
$$\mathscr{L}_3 = \{(x,y)\,|a \leq x \leq d, y = b\};$$

the characteristic line Γ_1 joining the points $(m,0)$ and $(a,-b)$; and the characteristic line Γ_2 joining the points $(m,0)$ and (a,b); see Fig. 3.2.

We consider equations having the form

$$Lu \equiv \mathscr{K}(x)u_{xx} + u_{yy} + \frac{\mathscr{K}'(x)}{2}u_x = 0, \tag{3.21}$$

where $\mathscr{K}$ satisfies conditions (3.7) and (3.10). We assume for convenience that $\mathscr{K}$ is C^1, and monotonic on the hyperbolic region, but the result clearly extends to weaker hypotheses on $\mathscr{K}$. For example, the monotonicity hypothesis on $\mathscr{K}$ is imposed only in order to simplify the graphs of the characteristic lines, which in turn simplifies the proof of Theorem 3.1 in the hyperbolic region. See also the discussion of the type-change function (3.32), below.

As an example, choose $\mathscr{K}(x) = x^{2k_0-1}$ for $k_0 \in \mathbf{Z}^+$. The operator L under this choice of type-change function is an analogue, for equations of Keldysh type,

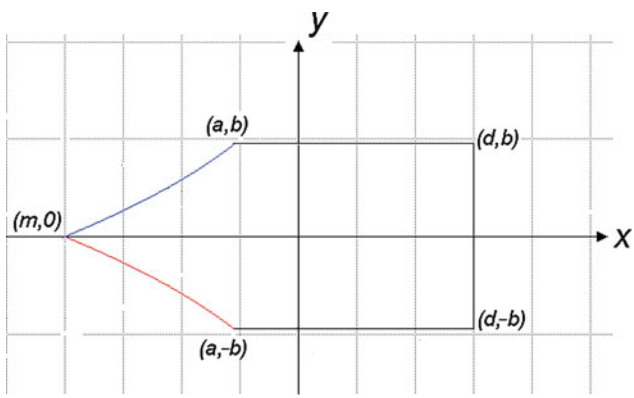

Fig. 3.2 The domain $\mathscr{D}$

of the well known *Gellerstedt operator* [24] for equations of Tricomi type. Other roughly equivalent examples are the polar-coordinate forms of an equation arising from a uniform asymptotic approximation of high-frequency waves near a caustic (Chap. 5) and of the equation for harmonic fields on the extended projective disc (Chap. 6). Both equations can be put into the form (3.21) in the cartesian $r\theta$-plane, with the x-axis replaced by the line $r = 1$ as a sonic line.

The y-axis divides the domain $\mathscr{D}$ of (3.21) into the subdomains

$$\mathscr{D}^+ = \{(x, y) \in \mathscr{D} | x \geq 0\},$$

and $\mathscr{D}^- = \mathscr{D} \backslash \mathscr{D}^+$. Equation (3.21) is (non-uniformly) elliptic for $(x, y) \in \mathscr{D}^+$.

Theorem 3.1. *Let $u(x, y)$ be a sufficiently smooth solution of (3.21) on $\mathscr{D} \cup \partial\mathscr{D}$, with $\mathscr{K}$ satisfying conditions (3.7) and (3.10). Assume that $\mathscr{K}$ is C^1, and monotonic on $\mathscr{D}^-$. If u vanishes identically on the non-characteristic boundary, then $u \equiv 0$ on all of $\mathscr{D}$.*

Remark. Regarding the required smoothness of the solution u, in the proof we assume as in [40] that u is a twice-continuously differentiable solution of (3.21) up to the boundary, and is identically zero on the noncharacteristic boundary; see also [39]. However, the theorem remains true under the weaker assumption that u is continuous on $\mathscr{D} \cup \partial\mathscr{D}$ and has first partial derivatives which are sufficiently smooth so that the integral

$$I = \int_0^{(x,y)} \left[\mathscr{K}(x)u_x^2 - u_y^2\right] dy - 2u_x u_y dx \tag{3.22}$$

is continuous on $\mathscr{D} \cup \partial\mathscr{D}$; c.f. [38].

Proof. We follow the approach of [38–40]. Differentiating the auxiliary function I under the integral sign, we observe that

$$\begin{aligned}
\frac{\partial}{\partial y}\left(-2u_x u_y\right) - \frac{\partial}{\partial x}\left[\mathscr{K}(x)u_x^2 - u_y^2\right] &= \left(-2u_{xy}u_y - 2u_x u_{yy}\right) \\
&\quad - \left[\mathscr{K}'(x)u_x^2 + 2\mathscr{K}(x)u_x u_{xx} - 2u_y u_{yx}\right] \\
&= -2u_x u_{yy} - \left[\mathscr{K}'(x)u_x^2 + 2\mathscr{K}(x)u_x u_{xx}\right] \\
&= -2u_x\left(u_{yy} + \frac{\mathscr{K}'(x)}{2}u_x + \mathscr{K}(x)u_{xx}\right) = 0,
\end{aligned}$$

using (3.21) and the equivalence of mixed partial derivatives. We conclude that there exists a function $\xi(x, y)$ such that

$$\xi_x = -2u_x u_y \tag{3.23}$$

and

$$\xi_y = \mathscr{K}(x)u_x^2 - u_y^2. \tag{3.24}$$

We will first show that u vanishes identically in $\mathscr{D}^+$. To accomplish this, we must show that u vanishes identically on the sonic line $x = 0$. Once we have shown that, we will have zero boundary conditions on $\mathscr{D}^+$. We will complete the proof for the elliptic region by invoking Proposition 2.1.

Because $u \equiv 0$ on $\mathscr{L}_1$, we conclude that u_x vanishes identically on that horizontal line. Thus we have, by (3.23),

$$\xi_x = 0 \text{ on } \mathscr{L}_1. \tag{3.25}$$

Also, $u \equiv 0$ on $\mathscr{L}_3$, so $u_x = 0$ on that horizontal line as well, implying that

$$\xi_x = 0 \text{ on } \mathscr{L}_3. \tag{3.26}$$

Equations (3.25) and (3.26) imply that on $\mathscr{L}_1$ and $\mathscr{L}_3$, ξ is a function of y only. But y is constant on those two horizontal lines, implying that

$$\xi = c_1 \text{ on } \mathscr{L}_1$$

and

$$\xi = c_2 \text{ on } \mathscr{L}_3,$$

where c_1 and c_2 are constants. On the line $\mathscr{L}_2$, $u \equiv 0$, implying that $u_y = 0$ on that vertical line. Also, $\mathscr{K}(x) > 0$ on $\mathscr{L}_2$. These facts imply, using (3.24), that $\xi_y \geq 0$ on $\mathscr{L}_2$, which in turn implies that

$$c_2 \geq c_1. \tag{3.27}$$

On the line $x = 0$, $\mathscr{K} = 0$, implying by (3.24) that $\xi_y \leq 0$ on that vertical line. This is turn implies that

$$c_2 \leq c_1. \tag{3.28}$$

Inequalities (3.27) and (3.28) are in contradiction unless $c_1 = c_2$. Taking into account that ξ cannot increase with increasing y on the line $x = 0$, it also cannot decrease with increasing y, as it would then have to increase in order to return to its initial value at the endpoint. This implies that $\xi_y = 0$ on the y-axis. Using (3.24) again, we find that on the y-axis,

$$-u_y^2 = 0, \tag{3.29}$$

so the function $u(0, y)$ is constant there. Because

$$u(0, -b) = u(0, b) = 0,$$

that constant is zero. Thus on the rectangle $\partial\mathscr{D}^+$ we have a closed Dirichlet problem having homogeneous boundary conditions.

Proposition 2.1 of Sect. 2.4.1 implies that the C^2 function u attains both its maximum and minimum values on the boundary. Because it is identically zero there, u must be zero in all of $\mathscr{D}^+$.

We obtain the identical vanishing of u on the hyperbolic region by integration along characteristic lines as in [1]. We have

$$d\xi = \xi_x dx + \xi_y dy = \left(-2u_x u_y\right) dx + \left[\mathscr{K}(x)u_x^2 - u_y^2\right] dy.$$

On characteristic lines,

$$dx = \pm\sqrt{-\mathscr{K}(x)}dy$$

and

$$\begin{aligned} d\xi &= \left[\mp 2u_x u_y \sqrt{-\mathscr{K}(x)} + \mathscr{K}(x)u_x^2 - u_y^2\right] dy \\ &= -\left[\sqrt{-\mathscr{K}(x)}u_x \pm u_y\right]^2 dy \le 0. \end{aligned} \tag{3.30}$$

Thus ξ is nonincreasing in y on any arbitrarily chosen characteristic.

Initially, take $a = 0$.

Because $u \equiv 0$ on the sonic line $x = 0$, we conclude that $u_y = 0$ on that vertical line. So $\xi_x = 0$ on the sonic line by (3.23). Because $K(0) = 0$, we conclude that $\xi_y = 0$ on the sonic line by (3.24) and (3.29). Beginning at the point $(0, -b)$, proceed along Γ_1 to $(m, 0)$ and then along Γ_2 to $(0, b)$. Expression (3.30) implies that ξ will not increase in y along this path from $(0, -b)$ and $(0, b)$. Because ξ is equal to the same constant at those two points, ξ must be constant in y along $\Gamma_1 \cup \Gamma_2$. (If ξ decreased in y at any point along such a path, it would have to increase in y at a later point in order to return to its constant value at $(0, b)$. And it cannot increase in y along a characteristic.) Ascending along the y-axis from the point $(0, -b)$, for any initial point above $(0, -b)$ and any terminal point below $(0, b)$ on the y-axis we can always find a pair of characteristic lines intersecting at some point on the x-axis to the right of $(m, 0)$. We conclude that $\xi_y = 0$ on $\mathscr{D}^-$. But then (3.24) implies that

$$\mathscr{K}(x)u_x^2 = u_y^2 \text{ on } \mathscr{D}^-. \tag{3.31}$$

Because $\mathscr{K} < 0$ on $\mathscr{D}^-$, we are forced to conclude from (3.31) that $u_x = u_y = 0$ on $\mathscr{D}^-$. This is turn implies that u is constant on $\mathscr{D}^-$. Because $u \equiv 0$ on the sonic line, that constant must be zero by the smoothness of u.

Now take $a < 0$. Because $\xi_x = 0$ on $\mathscr{L}_1$ and $\mathscr{L}_3$, ξ remains constant between $(0, -b)$ and $(a, -b)$ and between $(0, b)$ and (a, b). Moreover, ξ_y remains non-positive along Γ_1 and Γ_2. As we move the initial and terminal points to the right along $\mathscr{L}_1$ and $\mathscr{L}_3$ in $\mathscr{D}^-$, we can always find a pair of characteristic lines which intersect at a point on the x-axis to the right of $(m, 0)$. Arguing as in the case $a = 0$, we again conclude that $u \equiv 0$ on $\mathscr{D}^-$. This completes the proof of Theorem 3.1. □

In the special case in which $\mathscr{K}$ is an analytic function, we do not require a maximum principle, so we do not need to show that $u = 0$ on the line $x = 0$. Rather, we observe that $u_y = 0$ on $\mathscr{L}_2$ as $u \equiv 0$ on that vertical line. Our analysis of the constants c_1 and c_2 implies that $\xi_y = 0$ on $\mathscr{L}_2$ as well. Because in addition, $\mathscr{K} > 0$ on $\mathscr{L}_2$, (3.24) implies that $u_x = 0$ on $\mathscr{L}_2$. We use this last identity as Cauchy data for the Cauchy–Kowalevsky Theorem, to argue that u remains equal to zero as one moves in the negative x-direction away from $\mathscr{L}_2$ along the rectangle $\mathscr{D}^+$. This argument was applied in [40].

An example of a natural type-change function which is *not* analytic is the function

$$\mathscr{K}(x) = \mathrm{sgn}[x], \tag{3.32}$$

which yields an analogue, for equations of Keldysh type, of the Lavrent'ev–Bitsadze equation (2.5). Although such $\mathscr{K}(x)$ is also not C^1, our proof will work for this choice of $\mathscr{K}$ provided (3.21) is suitably interpreted.

Corollary 3.1. *The closed Dirichlet problem for (3.21) on $\mathscr{D}$ is over-determined for u and $\mathscr{D}$ defined as in Theorem 3.1.*

Proof. Suppose that u_1 and u_2 are two such solutions of the open Dirichlet problem for (3.21) on $\mathscr{D}$, with data prescribed only on $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_3$. Then $U \equiv u_2 - u_1$ satisfies the hypotheses of Theorem 3.1. We conclude that $u_1 = u_2$ in $\mathscr{D}$. That is, any sufficiently smooth solution to (3.21) is uniquely determined by data given on the non-characteristic boundary. So the problem is over-determined for sufficiently smooth solutions if data are given on the entire boundary. This completes the proof. □

Theorem 3.2. *The conclusion of Theorem 3.1 remains true if the Dirichlet conditions on the non-characteristic boundary of $\mathscr{D}$ are replaced by the following mixed Dirichlet–Neumann conditions: $u_y \equiv 0$ on $\mathscr{L}_1$ and $\mathscr{L}_3$; $u \equiv 0$ on $\mathscr{L}_2$.*

Proof. The existence of ξ satisfying (3.23) and (3.24) is established by the same arguments as in the proof of Theorem 3.1. The condition that $u_y = 0$ on $\mathscr{L}_1$ and $\mathscr{L}_3$ implies that $\xi_x = 0$ on those horizontal lines. So the proof of Theorem 3.1 implies that ξ is equal to a constant c_0 on $\mathscr{L}_1$ and $\mathscr{L}_3$. Because $u = 0$ on $\mathscr{L}_2$, (3.23) and (3.24) imply that ξ is equal to c_0 on $\mathscr{L}_2$ as well. The arguments leading to (3.29) imply that u is constant on the line $x = 0$ (but not necessarily equal to zero, as we no longer assume the vanishing of u on the lines $\mathscr{L}_1$ and $\mathscr{L}_3$). So (3.23) and (3.24) imply that ξ is constant on the line $x = 0$. Because of the conditions on $\mathscr{L}_1$ and $\mathscr{L}_3$, that constant is equal to c_0. Thus we conclude that ξ is equal to c_0 on the rectangle $\partial\mathscr{D}^+$.

A direct calculation, using (3.21), (3.23), (3.24), and the identity of mixed partial derivatives, shows that ξ satisfies

$$\mathscr{K}(x)\xi_{xx} + \xi_{yy} + \frac{\mathscr{K}'(x)}{2}\xi_x = 0.$$

Now Proposition 2.1 implies that ξ is a constant (not necessarily zero) in $\mathscr{D}^+$. In particular, (3.23) implies that

$$\xi_x = -2u_x u_y = 0,$$

so $u_x = 0$ and/or $u_y = 0$. If $u_x = 0$, then $u \equiv 0$ in $\mathscr{D}^+$ because $u = 0$ on $\mathscr{L}_2$. If $u_y = 0$, then (3.24) and the constancy of ξ imply that

$$\xi_y = \mathscr{K} u_x^2 = 0.$$

Because $\mathscr{K} > 0$ on $\mathscr{D}^+ \backslash \{x = 0\}$, we conclude that $u_x = 0$ on $\mathscr{D}^+ \backslash \{x = 0\}$. Because $u = 0$ on $\mathscr{L}_2$, we again conclude that $u \equiv 0$ on $\mathscr{D}^+ \backslash \{x = 0\}$. The smoothness of u implies that u is also zero on the line $x = 0$.

The proof that $u \equiv 0$ in $\mathscr{D}^-$ is the same as in the proof of Theorem 3.1. □

Corollary 3.2. *Let f_1, f_2, and f_3 be given functions defined on the arcs $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_3$, respectively. The mixed Dirchlet–Neumann problem in which $u_y = f_1$ on $\mathscr{L}_1$, $u_y = f_3$ on $\mathscr{L}_3$, $u = f_2$ on $\mathscr{L}_2$, and any boundary conditions at all are imposed on the characteristic lines, is over-determined for sufficiently smooth solutions of (3.21) in $\mathscr{D}$.*

Regarding the material in this section, see also Theorems 6.1 and 6.2, below, in which an equation of the form (3.21) is defined on domains having geometry which differs from that of the domain of Fig. 3.2. Note that conditions (3.7) and (3.10) are not satisfied in those cases, but the arguments of this section can still be applied.

3.4 A Closed Dirichlet Problem for Distribution Solutions

Although the closed Dirichlet problem is ill-posed for classical solutions, it may be well-posed for solutions having weaker properties.

3.4.1 Almost-Correct *Boundary Conditions*

As we mentioned in Sect. 2.3, the solutions to closed boundary value problems that we will obtain for equations of Keldysh type are a little smoother than generic distribution solutions. The latter fail to lie in a classical function space, whereas our distribution solutions will lie in L^2. They are called *weak solutions* by Berezanskii – c.f. (2.13) of [10], Sect. II.2; but they do not correspond to weak solutions as defined in other standard texts, e.g., [31] or [41].

Define, for a given C^1 function $\mathscr{K}(x, y)$, the space $L^2(\Omega; |\mathscr{K}|)$ and its dual as in Sect. 2.4.2. In this chapter we take the type-change function to depend only on

x, as in Cinquini-Cibrario's original papers. But many of the results will generalize to more complicated choices of $\mathscr{K}$. For example, the results of Sects. 3.3 and 3.4 extend easily to the cold plasma model as presented in Chap. 4, below, in which $\mathscr{K} = x - y^2$; see Sect. 3 of [40] and Sects. 2 and 3 of [43].

Recall that a Hilbert space $\mathscr{H}$ is said to be *rigged* if there is a subspace $\mathscr{V} \subset \mathscr{H}$ for which

$$\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}^*,$$

where $\mathscr{V}^*$ is the space dual to $\mathscr{V}$ in the "test function" topology. If L is formally self-adjoint on distributions, there exists a unique, continuous, self-adjoint extension

$$L : H_0^1(\Omega; \mathscr{K}) \to H^{-1}(\Omega; \mathscr{K}),$$

leading to the rigged triple of Hilbert spaces

$$H_0^1(\Omega; \mathscr{K}) \subset L^2(\Omega) \subset H^{-1}(\Omega; \mathscr{K}).$$

The dual space $H^{-1}(\Omega; K)$ is defined via the negative norm

$$||w||_{H^{-1}(\Omega;K)} = \sup_{0 \neq \varphi \in C_0^\infty(\Omega)} \frac{|\langle w, \varphi \rangle|}{||\varphi||_{H_0^1(\Omega,K)}},$$

Here $\langle\, ,\, \rangle$ is the Lax *duality bracket* (or *duality pairing*), motivated by the Schwarz inequality

$$|\langle w, \varphi \rangle| \leq ||w||_{H^{-1}(\Omega;\mathscr{K})} ||\varphi||_{H_0^1(\Omega;\mathscr{K})} \tag{3.33}$$

for $w \in H^{-1}(\Omega; \mathscr{K})$ and $\varphi \in H_0^1(\Omega; \mathscr{K})$; see [9] for a detailed discussion.

The key technical step in the majority of existence proofs in these notes is an *energy inequality* having the form

$$||u||_U \leq C ||Lu||_V, \tag{3.34}$$

where U and V are appropriately chosen spaces of functions or distributions. (In the more general case in which L is not formally self-adjoint, the term Lu on the right-hand side of (3.34) is replaced by L^*u; see Appendix A.2.) The choices in our case reduce to:

i) $U = H_0^1(\Omega; \mathscr{K})$, $V = L^2(\Omega)$,
or
ii) $U = L^2(\Omega; |\mathscr{K}|)$, $V = H^{-1}(\Omega; \mathscr{K})$.

Roughly speaking, choice *i*) leads to considerably less regularity than choice *ii*); but (3.34) is considerably harder to establish for choice *ii*) than for choice *i*).

In particular, it is sufficient for the existence of an L^2 solution to boundary value problems for the operator equation

$$Lu = f, \tag{3.35}$$

where f is a given distribution depending on (x, y), that inequality (3.34) hold on Ω with U and V defined as in case *i*). This argument seems to have originated in the work of Berezanskii [8] on so-called "almost-correct" boundary conditions.

In order to prove uniqueness of the solution and the satisfaction of boundary values in a stronger sense than duality, one would like to establish (3.34) for U and V defined according to choice *ii*). This will be accomplished in Sect. 4.3, for a different choice of type-change function. The failure of the methods of that section to apply in an obvious way to the type-change functions of the present chapter will be analyzed in Sect. 4.5.

3.4.2 The Existence of Distribution Solutions

Consider equations having the form (3.35), where f is a given function or distribution defined on points $(x, y) \in \Omega$. By a *distribution solution* of (3.35) with the boundary condition

$$u(x, y) = 0 \, \forall (x, y) \in \partial\Omega \tag{3.36}$$

we mean a function $u \in L^2(\Omega)$ such that $\forall \xi \in H_0^1(\Omega; \mathscr{K})$ for which $L^*\xi \in L^2(\Omega)$, we have

$$\big(u, L^*\xi\big) = \langle f, \xi \rangle. \tag{3.37}$$

Notice that distribution solutions to a homogeneous Dirichlet problem need not vanish on the boundary. (An example of one that does not is based on the fundamental solution to the Tricomi equation: Example 2.4 of [32].)

We will restrict our discussion to the formally self-adjoint operator

$$L_{(\mathscr{K}=x;k=1)}u = xu_{xx} + u_x + u_{yy}, \tag{3.38}$$

which is a special case of the operator introduced in (2.12).

Lemma 3.1. *Denote by Ω any bounded, connected subdomain of $\mathbf{R}^2$ having piecewise smooth boundary with counter-clockwise orientation. Let u be any C_0^2 function on Ω. Then*

$$||u||_{H_0^1(\Omega;x)} \leq C||L_{(x;1)}u||_{L^2(\Omega)}. \tag{3.39}$$

Remark. In general the operator on the right is the adjoint of L, which in the case of $L_{(\mathscr{K};1)}$ is L itself. In the proof we will suppress the subscripts of L.

Proof. The proof is similar to the proof of Theorem 2 of [43], and is based on ideas in Sect. 2 of [32].

For a positive constant $\delta \ll 1$, define the function

$$Mu = au + bu_x + cu_y, \tag{3.40}$$

where $a = -1$, $c = 2(2\delta - 1)y$, and

$$b(x) = \begin{cases} \exp(2\delta x/Q_1) & \text{if } x \in \Omega^+ \\ \exp(6\delta x/Q_2) & \text{if } x \in \Omega^- \end{cases}.$$

Here $\Omega^+ = \{x \in \Omega \mid x \geq 0\}$ and $\Omega^- = \Omega \backslash \Omega^+$. Choose $Q_1 = \exp(2\delta\mu_1)$, where $\mu_1 = \max_{x \in \overline{\Omega^+}} x$. Define the negative number μ_2 by $\mu_2 = \min_{x \in \overline{\Omega^-}} x$ and let $Q_2 = \exp(\mu_2)$. For example, if $\Omega = \mathscr{D}$, where $\mathscr{D}$ is the domain of Sect. 3.3, then $\mu_1 = d$ and $\mu_2 = m$. (The constants a and b defined in this section have nothing to do with the constants a and b defined in the preceding section.)

Notice that on Ω^+,

$$2\delta x \leq 2\delta\mu_1 \leq 2\delta\mu_1 e^{2\delta\mu_1} = Q_1 \log Q_1,$$

or

$$\frac{2\delta x}{Q_1} \leq \log Q_1.$$

Exponentiating both sides, we conclude that $b \leq Q_1$ on Ω^+.

Choose $\delta = \delta(\Omega)$ to be sufficiently small so that $6\delta < Q_2$. Then on Ω^-,

$$6\delta x \geq 6\delta\mu_2 = 6\delta \log Q_2 > Q_2 \log Q_2,$$

so $b > Q_2$ on Ω^-.

The coefficient $b(x)$ exceeds zero and is continuous but not differentiable on the y-axis. When we integrate over Ω, it is necessary to introduce a cut along the y-axis, which is analogous to the procedure employed in [32, 33]. The boundary integrals involving a, b, and c on either side of this line will cancel. Integrating by parts using Proposition 2.3 with $\mathscr{K}(x) = x$ and $k = 1$, we obtain

$$(Mu, Lu) = \int\int_{\Omega^+ \cup \Omega^-} \alpha u_x^2 + \gamma u_y^2 \, dxdy, \tag{3.41}$$

where

$$\alpha_{\Omega^+} = \delta\left[2 - \frac{b}{Q_1}\right]x + \frac{b}{2} \geq \delta x;$$

$$\alpha_{\Omega^-} = \delta\left[2 - 3\frac{b}{Q_2}\right]x + \frac{b}{2} \geq \delta|x|;$$

if δ is sufficiently small, then there is a positive constant ε such that

$$\gamma_{\Omega^+} = 2 + \delta\left(\frac{b}{Q_1} - 2\right) \geq \varepsilon$$

and

$$\gamma_{\Omega^-} = 2 + \delta\left(\frac{3b}{Q_2} - 2\right) \geq \varepsilon.$$

The path integral in Proposition 2.3 does not appear in (3.41) because u has compact support in Ω.

Let

$$\delta' = \min\{\delta, \varepsilon\}.$$

Then

$$\begin{aligned}\delta' \int\int_\Omega \left(|x|u_x^2 + u_y^2\right) dxdy &\leq (Mu, Lu)\\ &\leq ||Mu||_{L^2}||Lu||_{L^2}\\ &\leq C(\Omega)\left[\int\int_\Omega \left(|x|u_x^2 + u_y^2\right) dxdy\right]^{1/2} ||Lu||_{L^2(\Omega)},\end{aligned}$$

where we have used Proposition 2.2 in obtaining the bound on the L^2-norm of Mu. (In the proof of that proposition it is sufficient for u to be C^1 and to vanish on $\partial\Omega$.) Dividing through by the weighted double integral on the right completes the proof of Lemma 3.1. □

This leads to the following existence result:

Theorem 3.3. *The Dirichlet problem $L_{(\mathscr{K};k)}u = f$ with boundary condition (3.36) possesses a distribution solution $u \in L^2(\Omega)$ for every $f \in H^{-1}(\Omega;\mathscr{K})$ whenever $\mathscr{K} = x$ and $\kappa = 1$.*

Proof. The proof for our case is essentially identical to the proof in the well known case of Tricomi-type operators (c.f. [32], Theorem 2.2). Again we suppress the subscripts of L; but for generality we distinguish L from its adjoint L^*, although under the hypotheses of the theorem the two operators are equal. Define for $\xi \in C_0^\infty$ a linear functional

$$J_f\left(L^*\xi\right) \equiv \langle f, \xi\rangle. \tag{3.42}$$

This functional is bounded on a subspace of L^2 by the inequality

$$|\langle f, \xi\rangle| \leq ||f||_{H^{-1}(\Omega;x)}||\xi||_{H_0^1(\Omega;\mathscr{K})}, \tag{3.43}$$

provided we apply Lemma 3.1 to the variable ξ in the second term on the right. Precisely, J_f is a bounded linear functional on the subspace $\mathscr{M}$ of $L^2(\Omega)$ consisting of elements having the form $L^*\xi$ with $\xi \in C_0^\infty(\Omega)$. The Hahn–Banach Theorem – Theorem 2.1 of Sect. 2.4.4 – allows us to extend J_f to an operator $\mathscr{J}_f$ defined on the closure $\overline{\mathscr{M}}$ of $\mathscr{M}$ in $L^2(\Omega)$, where in the notation of the theorem, $\mathscr{B} = \overline{\mathscr{M}}$, $\ell = J_f$, and $\mathscr{L} = \mathscr{J}_f$. Now we can take ξ to lie in $H_0^1(\Omega;\mathscr{K})$, on the basis of arguments such as those in Sect. 2.4.2, and conclude that $\mathscr{J}_f$ is bounded on $\overline{\mathscr{M}}$

for any $\xi \in H_0^1(\Omega; \mathscr{K}) \ni L\xi \in L^2(\Omega)$. Extending by zero on the orthogonal complement of $\overline{\mathscr{M}}$, we obtain a bounded linear functional on all of $L^2(\Omega)$.

Apply the Riesz Representation Theorem – Theorem 2.2 of Sect. 2.4.4, taking $\mathscr{H} = L^2(\Omega)$, $T = \mathscr{J}_f$, and $X = L^*\xi$. We conclude that there is a vector $u \in L^2$ such that

$$\left(u, L^*\xi\right) = \mathscr{J}_f\left(L^*\xi\right).$$

But the extension of (3.42) to the functional $\mathscr{J}$ implies that

$$\left(u, L^*\xi\right) = \langle f, \xi\rangle$$

$\forall\, \xi \in H_0^1(\Omega; \mathscr{K}) \ni L\xi \in L^2(\Omega)$. This is our definition (3.37) of distribution solution. □

These results have been extended to elliptic–hyperbolic operators having a wide range of lower-order terms, which may not be associated with a formally self-adjoint differential operator, but which use the type-change function of Chap. 4; see [43], Sect. 2. Note, however, that the extension of type-change functions of the form $\mathscr{K}(x)$ to type-change functions of the form $\mathscr{K}(x, y)$, whatever their other complications, may simplify the absorption of a nonzero value for the coefficient β in (3.41).

3.5 A Strong Solution to an Open Dirichlet–Neumann Problem

In order to find solutions having stronger regularity properties, we restrict our attention to a favorable class of domains.

3.5.1 D-Star-Shaped Domains

Following Lupo and Payne (Sect. 2 of [34]), we consider a one-parameter family $\psi_\lambda(x, y)$ of inhomogeneous dilations given by

$$\psi_\lambda(x, y) = \left(\lambda^{-\alpha}x, \lambda^{-\beta}y\right)$$

for $\alpha, \beta, \lambda \in \mathbf{R}^+$, and the associated family of operators

$$\Psi_\lambda u = u \circ \psi_\lambda \equiv u_\lambda.$$

Denote by D the vector field

$$Du = \left[\frac{d}{d\lambda}u_\lambda\right]_{|\lambda=1} = -\alpha x\partial_x - \beta y\partial_y. \tag{3.44}$$

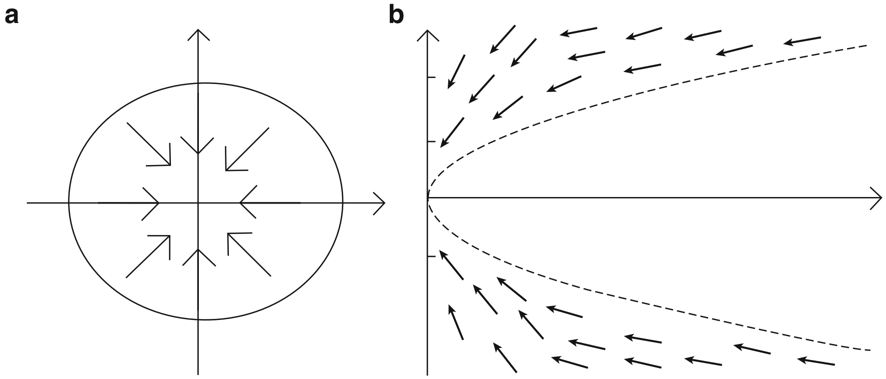

Fig. 3.3 (**a**) The *left-hand domain* is *star-shaped* in the conventional sense; (**b**) the complement of the region enclosed by the parabola in the *right-hand domain* is *star-shaped* with respect to a vector field of the form (3.44) for which $\alpha \neq \beta$

An open set $\Omega \subseteq \mathbf{R}^2$ is said to be *star-shaped* with respect to the flow of D if $\forall\,(x_0, y_0) \in \overline{\Omega}$ and each $t \in [0, \infty]$ we have $F_t\,(x_0, y_0) \subset \overline{\Omega}$, where

$$F_t\,(x_0, y_0) = (x(t), y(t)) = \left(x_0 e^{-\alpha t}, y_0 e^{-\beta t}\right).$$

If a domain is star-shaped with respect to a vector field D, then it is possible to "float" from any point of the domain to the origin along the flow lines of the vector field. If these flow lines are straight lines through the origin ($\alpha = \beta$), then we recover the conventional notion of a star-shaped domain. (See Fig. 3.3.) By an appropriate translation, the origin can be replaced by any point (x_s, y_s) in the plane as a source (or limit point) of the flow. In that case we obtain a translated function $\tilde{F}_t$ for which

$$\lim_{t\to\infty} \tilde{F}_t\,(x_0, y_0) = (x_s, y_s)\;\;\forall\,(x_0, y_0) \in \overline{\Omega}.$$

Moreover, whenever a domain is star-shaped with respect to the flow of a vector field satisfying (3.44), the domain boundary will be *starlike* in the sense that

$$(\alpha x, \beta y) \cdot \hat{\mathbf{n}}\,(x, y) \geq 0,$$

where $\hat{\mathbf{n}}$ is the outward-pointing normal vector on $\partial\Omega$; see Lemma 2.2 of [34]. In equivalent notation, given a vector field $V = -(b, c)$ and a boundary arc Γ which is starlike with respect to V, the inequality

$$bn_1 + cn_2 \geq 0 \tag{3.45}$$

is satisfied on Γ.

3.5.2 Admissible Domains

The following theorem is an “admissibility result” in the sense that we assume that the domain Ω supports a host of inequalities with respect to various parameters of the problem and show that solutions exist on such a domain. In that case it is necessary to show that reasonable examples of admissible domains exist, and this will be done subsequently. Note that the term “admissible,” when applied to a domain, does not mean exactly the same thing as it does when applied to boundary conditions.

Theorem 3.4 ([44]). *Let Ω be a bounded, connected domain of $\mathbf{R}^2$ having C^2 boundary $\partial\Omega$, oriented in a counterclockwise direction. Let $\partial\Omega_1^+$ be a (possibly empty and not necessarily proper) subset of $\partial\Omega^+$. Let inequality (3.45) be satisfied on $\partial\Omega^+\backslash\partial\Omega_1^+$. On $\partial\Omega_1^+$ let*

$$bn_1 + cn_2 \le 0 \tag{3.46}$$

and on $\partial\Omega\backslash\partial\Omega^+$, let

$$-bn_1 + cn_2 \ge 0. \tag{3.47}$$

Let $b(x,y)$ and $c(x,y)$ satisfy

$$b^2 + c^2\mathcal{K} \ne 0 \tag{3.48}$$

on Ω, with neither b nor c vanishing on Ω^+, and let

$$\mathcal{K}\,(bn_1 - cn_2)^2 + (c\mathcal{K}n_1 + bn_2)^2 \le 0 \text{ on } \partial\Omega\backslash\partial\Omega^+. \tag{3.49}$$

Let L be given by (2.13), (2.14) and let EL be symmetric positive, where

$$E = \begin{pmatrix} b & -c\mathcal{K} \\ c & b \end{pmatrix}.$$

Let the Dirichlet condition

$$-u_1n_2 + u_2n_1 = 0 \tag{3.50}$$

be satisfied on $\partial\Omega^+\backslash\partial\Omega_1^+$ and let the Neumann condition

$$\mathcal{K}u_1n_1 + u_2n_2 = 0 \tag{3.51}$$

be satisfied on $\partial\Omega_1^+$. Then (2.13), (2.14) possess a strong solution on Ω, in the sense of Sect. 2.5, for every $\mathbf{f} \in L^2(\Omega)$.

Remark. Let the operator L be given by

$$(L\mathbf{u})_1 = xu_{1x} + u_{2y} + \kappa_1 u_1 + \kappa_2 u_2, \tag{3.52}$$

$$(L\mathbf{u})_2 = u_{1y} - u_{2x}, \tag{3.53}$$

where κ_1 and κ_2 are constants. Sufficient conditions for the system to be symmetric positive are

$$2b\kappa_1 - b_x\mathscr{K} - b + c_y\mathscr{K} > 0 \text{ in } \Omega \tag{3.54}$$

and

$$\begin{aligned}&\left(2b\kappa_1 - b_x\mathscr{K} - b + c_y\mathscr{K}\right)\left(2c\kappa_2 + b_x - c_y\right)\\ &\quad - \left(b\kappa_2 + c\kappa_1 - c_x\mathscr{K} - c - b_y\right)^2 > 0 \text{ in } \Omega.\end{aligned} \tag{3.55}$$

Proof. For all points $(\tilde{x}, \tilde{y}) \in \partial\Omega$, decompose the matrix

$$\beta\left(\tilde{x}, \tilde{y}\right) = \begin{pmatrix} \mathscr{K}\left(bn_1 - cn_2\right) & c\mathscr{K}n_1 + bn_2 \\ c\mathscr{K}n_1 + bn_2 & -\left(bn_1 - cn_2\right) \end{pmatrix}$$

into a matrix sum having the form $\beta = \beta_+ + \beta_-$.

On $\partial\Omega^+\backslash\partial\Omega_1^+$, decompose β into the submatrices

$$\beta_+ = \begin{pmatrix} \mathscr{K}bn_1 & bn_2 \\ \mathscr{K}cn_1 & cn_2 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} -\mathscr{K}cn_2 & \mathscr{K}cn_1 \\ bn_2 & -bn_1 \end{pmatrix}.$$

Then $\beta_-\mathbf{u} = 0$ under boundary condition (3.50). We have

$$\mu^* = (bn_1 + cn_2)\begin{pmatrix} \mathscr{K} & 0 \\ 0 & 1 \end{pmatrix},$$

so condition (3.45) implies that the Dirichlet condition (3.50) is semi-admissible on $\partial\Omega\backslash\partial\Omega_1^+$.

On $\partial\Omega_1^+$, choose

$$\beta_+ = \begin{pmatrix} -\mathscr{K}cn_2 & \mathscr{K}cn_1 \\ bn_2 & -bn_1 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} \mathscr{K}bn_1 & bn_2 \\ \mathscr{K}cn_1 & cn_2 \end{pmatrix}.$$

Then $\beta_-\mathbf{u} = 0$ under the Neumann boundary condition (3.51), and

$$\mu^* = -(bn_1 + cn_2)\begin{pmatrix} \mathscr{K} & 0 \\ 0 & 1 \end{pmatrix}$$

is positive semi-definite under condition (3.46).

On $\partial\Omega\backslash\partial\Omega^+$, choose $\beta_+ = \beta$ and take β_- to be the zero matrix. Then $\mu = \mu^* = \beta$ and

$$\mu_{11} = \mathscr{K}\,(bn_1 - cn_2).$$

Because μ_{11} is non-negative by (3.47), μ^* is positive semi-definite by inequality (3.49); so no conditions need be imposed outside the elliptic portion of the boundary. We conclude that the boundary conditions are semi-admissible.

Now we prove admissibility:

On $\partial\Omega^+\backslash\partial\Omega_1^+$ the null space of β_- is composed of vectors satisfying the Dirichlet condition (3.50), which is imposed on that boundary arc. The null space of β_+ is composed of vectors satisfying the adjoint condition (3.51). On $\partial\Omega_1^+$, this relation is reversed. In order to show that the direct sum of these null spaces spans the two-dimensional space $\mathscr{V}_{|\partial\Omega^+}$, it is sufficient to show that the set

$$\left\{\begin{pmatrix} 1 \\ n_2/n_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\mathscr{K}n_1/n_2 \end{pmatrix}\right\}$$

is linearly independent there. Setting

$$c_1\begin{pmatrix} 1 \\ n_2/n_1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -\mathscr{K}n_1/n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we find that $c_1 = -c_2$ and

$$-c_2\left(\frac{n_2^2 + \mathscr{K}n_1^2}{n_1 n_2}\right) = 0. \tag{3.56}$$

Recall that the elliptic boundary is that part of the domain boundary on which the type-change function K is positive. Equation (3.56) can only be satisfied on the elliptic boundary if $c_2 = 0$, implying that $c_1 = 0$. Thus the direct sum of the null spaces of $\beta_\pm$ on $\partial\Omega^+$ is linearly independent and must span $\mathscr{V}$ over that portion of the boundary.

On $\partial\Omega\backslash\partial\Omega^+$, the null space of β_- contains every 2-vector and the null space of β_+ contains only the zero vector; so on that boundary arc, their direct sum spans $\mathscr{V}$.

On $\partial\Omega^+\backslash\partial\Omega_1^+$, the range $\mathscr{R}_+$ of β_+ is the subset of the range $\mathscr{R}$ of β for which

$$v_2 n_1 - v_1 n_2 = 0 \tag{3.57}$$

for $(v_1, v_2) \in \mathscr{V}$; the range $\mathscr{R}_-$ of β_- is the subset of $\mathscr{R}$ for which

$$\mathscr{K}v_1 n_1 + v_2 n_2 = 0 \tag{3.58}$$

for $(v_1, v_2) \in \mathscr{V}$. Analogous assertions hold on $\partial\Omega_1^+$, in which the ranges of $\mathscr{R}_+$ and $\mathscr{R}_-$ are interchanged. Because if n_1 and n_2 are not simultaneously zero the system (3.57), (3.58) has only the trivial solution $v_2 = v_1 = 0$ on $\partial\Omega^+$, we conclude that $\mathscr{R}_+ \cap \mathscr{R}_- = \{0\}$ on $\partial\Omega^+$.

On $\partial\Omega\backslash\partial\Omega^+$, $\mathscr{R}_- = \{0\}$, so $\mathscr{R}_+ \cap \mathscr{R}_- = \{0\}$ trivially.

The invertibility of E under condition (3.48) completes the proof of Theorem 3.4. $\square$

By taking $\partial\Omega_1^+$ to be either the empty set or all of $\partial\Omega^+$, Theorem 3.4 implies the existence of strong solutions for either the open Dirichlet problem or the open Neumann problem for the system (2.13), (2.14).

The argument leading to (3.56) suggests that the Tricomi problem (Sect. 3.1) is strongly ill-posed under the hypotheses of the theorem. This is because in the Tricomi problem, data are given on both the elliptic boundary and a characteristic curve; but on characteristic curves, $\mathscr{K}$ satisfies

$$\mathscr{K} = -\frac{n_2^2}{n_1^2}. \tag{3.59}$$

Substituting this equation into (3.56), we find that the equation is satisfied on characteristic curves without requiring the constants c_1 and c_2 to be zero.

However, the theorem is less restrictive if the operator in (2.13) is given by

$$\begin{aligned}(L\mathbf{u})_1 &= xu_{1x} - u_{2y} + \kappa_1 u_1 + \kappa_2 u_2,\\ (L\mathbf{u})_2 &= -u_{1y} + u_{2x},\end{aligned} \tag{3.60}$$

where, again, κ_1 and κ_2 are constants. This variant is analogous to the variant of the Tricomi equation,

$$yu_{xx} - u_{yy} + \text{ zeroth-order terms} = 0,$$

studied in various contexts by Friedrichs [23], Katsanis [28], Sorokina [47, 48], and Didenko [20]. In that case, choose

$$E = \begin{pmatrix} b & c\mathscr{K} \\ c & b \end{pmatrix}.$$

Obvious modifications of conditions (3.54) and (3.55) guarantee that the equation

$$EL\mathbf{u} = E\mathbf{f}$$

will be symmetric positive. Condition (3.48) must be replaced by the invertibility condition

$$b^2 - c^2\mathscr{K} \neq 0,$$

which is restrictive on the subdomain Ω^+ rather than on Ω^- as in (3.48). Most importantly, the discussion leading to Table 1 of [29] now applies, with only minor changes, and one can obtain a long list of possible starlike boundaries which result in strong solutions to suitably formulated problems of Dirichlet or Neumann type. In particular, one can formulate a Tricomi problem which is strongly well-posed.

The hypotheses of Theorem 3.4 have a rather formal appearance, but many of the conditions have natural interpretations. For example, inequalities (3.45), (3.46), and (3.47) are satisfied whenever boundary arcs are starlike with respect to an appropriate vector field. Moreover, (3.49) is always satisfied on the characteristic boundary:

Proposition 3.1. *Let Γ be a characteristic curve for (2.13), with the higher-order terms of the operator L satisfying (2.14). Then the left-hand side of inequality (3.49) is identically zero on Γ.*

Proof. We have, using (3.59),

$$\begin{aligned}(c\mathcal{K}n_1 + bn_2)^2 &= c^2\mathcal{K}^2 n_1^2 + 2\mathcal{K}cbn_1n_2 + b^2n_2^2\\ &= -c^2\mathcal{K}^2\frac{n_2^2}{\mathcal{K}} + 2\mathcal{K}cbn_1n_2 - b^2\mathcal{K}n_1^2\\ &= -\mathcal{K}\left(c^2n_2^2 - 2cbn_1n_2 + b^2n_1^2\right)\\ &= -\mathcal{K}\left(cn_2 - bn_1\right)^2.\end{aligned}$$

Substituting the extreme right-hand side of this equation into the second term of (3.49) completes the proof. □

3.5.3 An Example of an Admissible Domain

A simple example which illustrates the hypotheses of Theorem 3.4 can be constructed for the case $\mathcal{K}(x) = x$. In that case the system (2.13), (2.14) can be reduced, by taking $u_1 = u_x$, $u_2 = u_y$, and $\mathbf{f} = 0$, to the original Cinquini-Cibrario equation (3.4). We will consider the formally self-adjoint case, corresponding to the equation

$$Lu = \mathcal{K}(x)u_{xx} + u_{yy} + \mathcal{K}'(x)u_x,$$

with $\mathcal{K}(x) = x$ ((3.52) with $\kappa_1 = 1$ and $\kappa_2 = 0$). Choose $b = x + M$, where M is a positive constant which is assumed to be large in comparison with all other parameters of the problem – in particular, $b > 0\ \forall x \in \overline{\Omega}$; $c = \epsilon(y + 5)$, where ϵ is a small positive constant; $(n_1, n_2) = (dy/ds, -dx/ds)$, where s is arc length on the boundary. We obtain a symmetric-positive system satisfying inequality (3.48) for M sufficiently large.

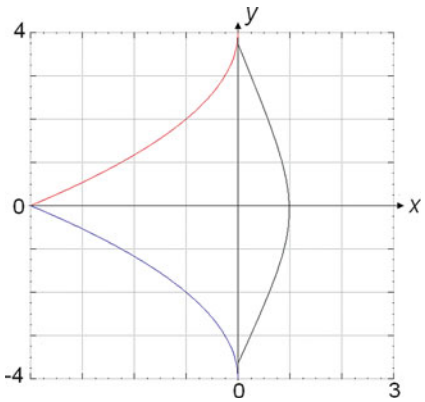

Fig. 3.4 The domain Ω. The cusps at the points $(-4, 0)$ and $(0, \pm 4)$ can be smoothed out as described in the text. The *upper curve* is the characteristic $y = -2\sqrt{-x} + 4$. The *lower curve* is the characteristic $y = 2\sqrt{-x} - 4$

Let the hyperbolic region Ω^- be bounded by intersecting characteristic curves originating on the sonic line. Condition (3.49) is satisfied on $\partial\Omega^-$ by Proposition 3.1. Condition (3.47) is satisfied for M sufficiently large. As a concrete example, let $\partial\Omega^- = \Gamma^- \cup \Gamma^+$, where

$$\Gamma^{\pm} = \left\{(x, y) \in \Omega^- | y = \pm 2\left(\sqrt{-x} - 2\right)\right\}.$$

These curves intersect at the point $(-4, 0)$. Their intersection is not C^2, but it can be easily "smoothed out" (by the addition of a small noncharacteristic curve connecting the points $(-4 + \delta_0, \pm\delta_1)$ for $0 < \delta_0, \delta_1 \ll 1$) without violating either of the governing inequalities. Let the elliptic boundary $\partial\Omega^+$ be a smooth convex curve, symmetric about the x-axis, with endpoints at $(0, \pm 4)$ on the sonic line. Let the disconnected subset $\partial\Omega_1^+$ of $\partial\Omega^+$ take the form of two small "smoothing curves," on which the slope of the tangent line to $\partial\Omega^+$ changes sign in order to prevent a cusp at the two endpoints. Inequality (3.45) is satisfied on $\partial\Omega^+ \backslash \Omega_1^+$ and, again assuming that M is sufficiently large, inequality (3.46) is satisfied on the two smoothing curves comprising $\partial\Omega_1^+$.

The domain Ω of this construction, shown in Fig. 3.4, is nearly identical to one originally considered by Cinquini-Cibrario – which is illustrated in Fig. 2 on p. 277 of [19]. However, in order for the analysis to apply, the cusps near the points $(-4, 0)$ and $(0, \pm 4)$ should be smoothed out as described above.

Theorem 3.4 implies that strong solutions to an open, homogeneous, mixed Dirichlet–Neumann problem for the first-order inhomogeneous form of Cinquini-Cibrario's equation exist on this natural class of domains for any square-integrable forcing function.

3.6 Relation to Magnetically Dominated Plasmas

Maxwell's equations for a magnetic field **B** (with $\mu_0 = 1$ and ignoring the electric field **E**) assume the form

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{3.61}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{3.62}$$

The first equation gives the relation between the field and the current $\mathbf{j}$, while the second equation asserts the absence of magnetic monopoles. Equation (3.61) implies that

$$\mathbf{j} \times \mathbf{B} = (\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{3.63}$$

In, e.g., atmospheric plasma or in magnetized thermonuclear fusion in toroidal geometries, the left-hand-side of (3.63) is equal to ∇p, where p is atmospheric pressure in the former case, and the kinetic pressure of a magnetized confined plasma in the latter case. One then obtains the full *magnetostatic equations*, in which the equation

$$\mathbf{j} \times \mathbf{B} = \nabla p \tag{3.64}$$

is appended to (3.61) and (3.62).

In the special case of zero pressure, we obtain the *Beltrami equations*, which consist of (3.62) and the additional equation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0. \tag{3.65}$$

These describe so-called *force-free fields*. Examples include equilibrium magnetic fields in the solar corona.

In the solar physics literature, one usually encounters (3.65) in the form of an eigenvalue equation for the curl operator,

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \tag{3.66}$$

where α is a scalar function. Taking into account (3.61), we observe that α has a physical interpretation as the proportionality factor between the magnetic field $\mathbf{B}$ and the current density $\mathbf{j}$ of the medium. In models for the equilibrium magnetic field of the solar corona, α is often taken to be constant or to depend on $\mathbf{r}$. But in general, α varies with position within the magnetic field. Note that any solution of (3.66) automatically satisfies (3.65) by the skew-symmetry of the cross product.

Taking the divergence of both sides in (3.66), one obtains

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\alpha \mathbf{B}) = \mathbf{B} \cdot \nabla \alpha, \tag{3.67}$$

using (3.62). But the identity of mixed partial derivatives insures that the left-hand side of (3.67) vanishes for continuously differentiable $\mathbf{B}$, leading to the equation

$$\mathbf{B} \cdot (\nabla \alpha) = 0. \tag{3.68}$$

Equation (3.68) implies that α is constant along the *magnetic lines of force*: curves for which the tangent line at any point lies in the direction of the magnetic field at that point.

If α is known and $\mathbf{B}$ is unknown, then (3.66) is an elliptic partial differential equation for which Dirichlet or Neumann boundary value problems are expected. If $\mathbf{B}$ is known and α is unknown, then (3.68) is typically a hyperbolic equation for which the Cauchy problem is expected. Physicists refer to this (somewhat misleadingly) as a *mixed elliptic–hyperbolic structure*; see, e.g., Sect. 2.1 of [3] or (2.1), (2.2) of [5].

In applications to models of the solar corona, one would like to solve (3.66) with condition (3.62) subject to boundary conditions derived from spectro-polarimetric measurements at lower altitudes. In addition to the considerable practical difficulties associated with such measurements (see, e.g., [53]), there is concern that the resulting boundary conditions might not form a well-posed boundary value problem.

Classically, models of the solar corona based on (3.66) and condition (3.62) have been accompanied by Neumann-like boundary conditions having the form [25]

$$B_{n|\partial\Omega} = g,$$

$$\alpha_{|\partial\Omega^+} = h,$$

where g and h are sufficiently smooth functions; B_n is the component of $\mathbf{B}$ normal to the boundary $\partial\Omega$; and $\partial\Omega^+$ is the part of $\partial\Omega$ on which B_n has a fixed sign – say, the part of the boundary on which B_n exceeds zero. The latter unusual condition corresponds to the prescription of α over one polarity of the magnetic field.

Recently [3], a Dirichlet-type problem was posed for a model of the solar corona based on (3.66) with condition (3.62). In that case one prescribes each of the three cartesian components of the magnetic field $\mathbf{B}$ on the domain boundary $\partial\Omega$:

$$B_{i|\Omega} = f_i, \; i = 1, 2, 3,$$

where f_i are three regular functions.

In either case, physicists like to prescribe the asymptotic condition

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{B}| = 0, \tag{3.69}$$

in order that the solution remain physically meaningful as the outer boundary of the domain is extended out to infinity.

Equations (3.66), (3.62) for axially symmetric fields also arise in plasma models of ball lightning in the Earth's atmosphere [27, 50] and of certain extra-galactic jets [51]. But those cases differ in details of the domain geometry, and also with respect to conditions on the fluid pressure, which we have been taking to be zero but which is not zero in the terrestrial atmosphere. In all cases, Neumann-like and/or Dirichlet-like boundary conditions are physically meaningful, but are manifestly well-posed only in the case of α equal to a constant (in which case (3.62) and (3.66) are a first-order variant of the Helmholtz equation). Selections from the vast mathematical literature on physical applications of Beltrami fields include [2, 4, 13, 14, 21, 26].

3.6.1 The Axisymmetric Case

Because force-free magnetic fields in models of the solar corona are axially symmetric, the magnetic field $\mathbf{B}$ can always be written in the form

$$\mathbf{B}(r,\theta) = B_r\hat{\mathbf{e}}_r + B_\theta\hat{\mathbf{e}}_\theta + B_\varphi\hat{\mathbf{e}}_\varphi, \tag{3.70}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\varphi$ are an orthonormal basis for spherical coordinates; in this context, the subscripts r, θ and φ denote components of $\mathbf{B}$ in spherical coordinates (*not* partial differentiation in the direction of those variables); θ denotes the polar spherical coordinate and φ the equatorial spherical coordinate; and

$$B_r = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[P(r,\theta)\right], \tag{3.71}$$

$$B_\theta = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left[P(r,\theta)\right], \tag{3.72}$$

$$B_\varphi = \frac{1}{r\sin\theta}Q(r,\theta), \tag{3.73}$$

for scalar functions $P(r,\theta)$ and $Q(r,\theta)$.

Notice that

$$\begin{aligned}\nabla\cdot\mathbf{B} &\equiv \left(\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\mathbf{e}}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(B_r\hat{\mathbf{e}}_r + B_\theta\hat{\mathbf{e}}_\theta + B_\varphi\hat{\mathbf{e}}_\varphi\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial P}{\partial\theta}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial P}{\partial r}\right) = 0,\end{aligned}$$

so (3.62) is satisfied automatically. Moreover,

$$\begin{aligned}\nabla\times\mathbf{B} = {}& \frac{1}{r^2\sin\theta}\left(\frac{\partial Q}{\partial\theta}\right)\hat{\mathbf{e}}_r - \frac{1}{r\sin\theta}\left(\frac{\partial Q}{\partial r}\right)\hat{\mathbf{e}}_\theta\\ & -\left\{\frac{1}{r\sin\theta}\left(\frac{\partial^2 P}{\partial r^2}\right) + \frac{1}{r^3}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\left(\frac{\partial P}{\partial\theta}\right)\right]\right\}\hat{\mathbf{e}}_\varphi.\end{aligned} \tag{3.74}$$

Thus in order for $\mathbf{B}$ to satisfy (3.66), we require that:

$$\frac{\partial Q}{\partial\theta} = \alpha\frac{\partial P}{\partial\theta} \tag{3.75}$$

in order for the coefficients of $\hat{\mathbf{e}}_r$ in (3.74) to equal those in (3.71);

$$\frac{\partial Q}{\partial r} = \alpha \frac{\partial P}{\partial r}, \tag{3.76}$$

in order for the coefficients of $\hat{\mathbf{e}}_\theta$ in (3.74) to equal those in (3.72). Conditions (3.75) and (3.76) can be satisfied by choosing Q to be a function of P for which

$$\frac{dQ}{dP} = \alpha. \tag{3.77}$$

In order for the components of $\hat{\mathbf{e}}_\varphi$ in (3.74) to equal those in (3.73) we require, taking into account (3.77), that

$$r^2 \frac{\partial^2 P}{\partial r^2} + \frac{\partial^2 P}{\partial \theta^2} - \cot\theta \frac{\partial P}{\partial \theta} + r^2 \alpha Q = 0. \tag{3.78}$$

Thus in the axisymmetric case, taking α to be independent of $\mathbf{B}$, the system (3.62), (3.66) reduces to the scalar equation (3.78).

3.6.2 *An Open Dirichlet Problem for Cinquini-Cibrario's Equation Which Is Classically Well-Posed*

Equation (3.78) in the special case $\alpha = 0$ can be derived from Cinquini-Cibrario's equation (3.4) in the half-plane $x \geq 0$ by the coordinate transformation

$$x = \frac{r^2 \sin^2 \theta}{4}, \quad y = r \cos\theta, \tag{3.79}$$

or equivalently,

$$r = \sqrt{4x + y^2}, \quad \theta = \arctan \frac{2\sqrt{x}}{y}. \tag{3.80}$$

Realizing that, up to a constant, $Q = \alpha P$ by (3.77), we will eventually find that the limitation $\alpha = 0$ is relatively unimportant.

It is possible to solve (3.78) by separation of variables and then paste the solution, which will be defined only within the upper half-plane, onto a particular solution of (3.4) in the lower half-plane.

Proceeding as in Sect. 3 of [19] for the remainder of this section, we seek a solution in the form

$$P(r, \theta) = R(r)T(\theta). \tag{3.81}$$

Substituting (3.81) into (3.78), we obtain

$$r^2 R'' T + R T'' - (\cot\theta)\, R T'' = 0,$$

implying that

$$\frac{r^2 R''}{R} = \frac{T' \cot\theta - T''}{T} = k,$$

where k is an arbitrary constant, which we will write in the form

$$k = \lambda\,(\lambda - 1).$$

We obtain the two ordinary differential equations

$$r^2 R'' - \lambda\,(\lambda - 1)\, R = 0, \tag{3.82}$$

$$T'' - (\cot\theta)\, T' + \lambda\,(\lambda - 1)\, T = 0. \tag{3.83}$$

Equation (3.82) admits particular solutions of the form $R = r^{\lambda}$ (and also of the form $R = r^{1-\lambda}$, although solutions of the latter form will not be used). Substituting $\tau = \cos\theta$ into (3.83), that equation assumes the form

$$\left(1 - \tau^2\right) \frac{d^2 T}{d\tau^2} + \lambda\,(\lambda - 1)\, T = 0. \tag{3.84}$$

If we write

$$T(\tau) = \left(1 - \tau^2\right) S(\tau),$$

then $S\,(\tau)$ satisfies the equation

$$\left(1 - \tau^2\right) \frac{d^2 S}{d\tau^2} - 4\tau \frac{dS}{d\tau} + (\lambda - 2)\,(\lambda + 1)\, S = 0. \tag{3.85}$$

Choosing λ to be an integer, we find that solutions to (3.85) are well known in the literature on special functions. For example, (3.85) reduces to (13.03) in Chap. 5 of Olver's standard reference [42], taking the parameters μ and ν of that work to equal 1 and $\lambda - 1$, respectively. (Or, in the display equation immediately preceding (15.09) in the same reference, take $\mu = -1$ and $\nu = \lambda - 1$.) The solution can be written in terms of Legendre functions, which we denote by $C_{\lambda-2}^{3/2}(\tau)$. In terms of these functions, we can write a particular solution to (3.84) in the form

$$T(\tau) = \left(1 - \tau^2\right) C_{\lambda-2}^{3/2}(\tau) = \left(\sin^2\theta\right) C_{\lambda-2}^{3/2}\,(\cos\theta).$$

Setting $n = \lambda - 2$, we obtain a particular solution to (3.78) in the case $\alpha = 0$:

$$P_n = r^{n+2} \left(\sin^2\theta\right) C_n^{3/2}\,(\cos\theta), \tag{3.86}$$

where $C_n^{3/2}$ has the explicit representation

$$C_n^{3/2}(\cos\theta) = \frac{2}{\sqrt{n}} \sum_{s=0}^{m} \frac{(-1)^s \Gamma\left(\frac{3}{2}+n-s\right)}{s!\,(n-2s)!} 2^{n-2s} (\cos\theta)^{n-2s},$$

for $m = n/2$ or $m = (n-1)/2$, depending on whether n is even or odd.

The analysis is only slightly different in the case $Q = \alpha P$, $\alpha \neq 0$, as the zeroth-order term only enters into the simpler of the two equations, (3.82). We obtain, in place of (3.82), the equation

$$r^2 R'' + \left[\alpha^2 r^2 - \lambda(\lambda-1)\right] R = 0, \tag{3.87}$$

which can also be solved in terms of special functions, for example,

$$R(r) = (\alpha r)^{1/2} \mathscr{J}_{n+1/2}(\alpha r) \tag{3.88}$$

and

$$T(\tau) = -n(n+1) \int_1^\tau \mathscr{P}_n(x)dx. \tag{3.89}$$

In (3.88) and (3.89): $n = -\lambda$, $\mathscr{J}$ is now a standard spherical Bessel function, and $\mathscr{P}$ is a Legendre polynomial – c.f. (18a) and (18b) of [50]; (19b) and (19c′) of [51]; and Chap. 5, Sect. 13.1, of [42].

In the other half-plane, $x \leq 0$, a particular solution to (3.4) is given by

$$u_n(x,y) = \frac{4}{\pi}(n+1)(n+2)\,x \int_0^1 t^{1/2}(1-t)^{1/2}[Y(t)]^n\,dt, \tag{3.90}$$

where

$$Y(t) = y - 2(-x)^{1/2} + 4(-x)^{1/2}t.$$

In terms of x and y, the solution in the half-plane $x \geq 0$ can be written in the form

$$u_n(x,y) = \frac{2x}{\sqrt{\pi}} \sum_{s=0}^{m} \frac{(-1)^s \Gamma\left(\frac{3}{2}+n-s\right)}{s!\,(n-2-s)!} 2^{n-2-s}\left(4x+y^2\right)^s y^{n-2s}. \tag{3.91}$$

It has been shown by classical arguments ([19], Theorem 9) that the existence of solutions (3.90) and (3.91) in the two half-planes can be used to construct a solution to the following open Dirichlet problem:

Let the domain be given by the region shown in Fig. 3.4, with the boundary arc in the half-plane $x \geq 0$ given explicitly by the curve

$$4x + y^2 = 1. \tag{3.92}$$

(The same arguments will work for boundary arcs of the slightly more general form $4x + (y - y_0)^2 = k$, where k is a positive constant and y_0 is a constant.) It can be shown that a unique solution to (3.4) exists which is equal on the curve C given by (3.92) to a function of the form $x\varphi(y)$, where $\varphi(y)$ is finite and continuous on C, including the endpoints, and vanishes on the y-axis. The solution is analytic in the interior of the domain.

In summary, Cinquini-Cibrario's equation in a half-plane, with a zeroth-order perturbation, is related by a coordinate transformation to an equation for axisymmetric magnetically dominated plasmas. Moreover, solutions in the two half-planes can be pasted together to provide a solution to an open Dirichlet problem for the fully elliptic–hyperbolic equation (3.4).

Boundary value problems for (3.4) restricted to the half plane $x \geq 0$ were studied by Keldysh [30] and have applications to high-speed flow, particularly in certain nonlinear generalizations; see, e.g., [15].

3.7 The Fundamental Solution

The topic of fundamental solutions has potential applications to boundary value problems of all kinds. In particular, it is the main ingredient for solving the Dirichlet problem by the Green's function method. This brief discussion follows the work of S-X. Chen [16].

Recall that a distribution $\mathscr{E}$ is a *fundamental solution* of the operator equation (3.35), for a given operator L and function f, if

$$L\mathscr{E} = \delta. \tag{3.93}$$

Here δ is the *Dirac distribution,* which can be defined as the singular measure (with respect to Lebesgue integration) for which

$$\int_{\mathbf{R}^n} f(x)\delta\,(dx) = f(0),\ \ x \in \mathbf{R}^n,$$

for all compactly supported continuous functions f; see, e.g., Sect. I.8, Examples 2 and 3, of [54]. The Dirac distribution arises naturally as an element of the negative Sobolev space H^{-s} for $s > n/2$; see Sect. 3.1 of [9] or Example 7.43 of [45] for details of this interpretation.

If $\mathscr{E}$ is a fundamental solution for the operator L, then $\mathscr{E} * f$ is a solution for (3.35), where in this context $*$ denotes the *convolution operator*: for $x, y \in \mathbf{R}^n$,

$$\mathscr{E} * f \equiv \int_{\mathbf{R}^n} \mathscr{E}(y) f\,(x - y)\,dy = \int_{\mathbf{R}^n} \mathscr{E}\,(x - y)\,f(y)dy.$$

Taking $u = \mathscr{E} * f$, we have $\forall\, x \in \mathbf{R}^n$,

$$Lu = L\,(\mathscr{E} * f) = L\mathscr{E} * f = \delta * f = f,$$

where we have used (3.93) and the elementary identities $\delta * g = g$ and

$$\partial_i\,(g * h) = (\partial_i g) * h = g * (\partial_i h).$$

The first step in constructing a fundamental solution is often a search for an expression having the right invariance properties. In this case, Cinquini-Cibrario's equation and the Dirac δ-function are both invariant under the coordinate rescaling $s : (x, y) \to (w, z)$, for

$$w = t^2 x,\ \ z = ty,\ \ t > 0. \tag{3.94}$$

Thus we expect to be able to find *similarity solutions* having the form

$$\varphi = y^\nu F\left(A\frac{x}{y^2} + B\right), \tag{3.95}$$

where A and B are constants to be chosen. Let

$$\mu = A\frac{x}{y^2} + B.$$

Then

$$\begin{aligned} \varphi_x &= Ay^{\nu-2}F'(\mu) \\ \varphi_{xx} &= A^2 y^{\nu-4}F''(\mu), \\ \varphi_y &= \nu y^{\nu-1}F(\mu) - 2Axy^{\nu-3}F'(\mu), \end{aligned}$$

and

$$\begin{aligned} \varphi_{yy} = {} & \nu\,(\nu-1)\,y^{\nu-2}F(\mu) \\ & -2Ax\,(2\nu-3)\,y^{\nu-4}F'(\mu) + 4A^2x^2y^{\nu-6}F''(\mu). \end{aligned}$$

We obtain

$$\begin{aligned} \mathscr{K}\,(x, y)\,\varphi_{xx} + \varphi_{yy} = {} & y^{\nu-2}\left\{A^2\left[\frac{\mathscr{K}\,(x, y)}{y^2} + 4\frac{x^2}{y^4}\right]F''(\mu)\right. \\ & \left. -2A\frac{x}{y^2}\,(2\nu-3)\,F'(\mu) + \nu\,(\nu-1)\,F(\mu)\right\}. \end{aligned} \tag{3.96}$$

Taking $\mathscr{K}(x, y) = x$, then

$$\frac{\mathscr{K}}{y^2} = \frac{x}{y^2} = \frac{\mu - B}{A}.$$

Taking $A = 4,\ B = 1$ in (3.96) allows us to write the equation

$$x\varphi_{xx} + \varphi_{yy} = 0$$

as a particular ordinary differential equation, obtaining

$$y^{\nu-2}\left\{4\mu\left(\mu-1\right)F''(\mu) - 2\left(2\nu-3\right)\left(\mu-1\right)F'(\mu) + \nu\left(\nu-1\right)F(\mu)\right\} = 0. \tag{3.97}$$

Recall that the *hypergeometric equation* is an expression of the form

$$z\left(1-z\right)w''(z) + \left[c - \left(a+b+1\right)z\right]w'(z) - ab\cdot w(z) = 0, \tag{3.98}$$

where $a,\ b,\ c$ are given parameters. Applying the method of Frobenius, (3.98) is easily reduced to a series

$$\sum_{n=0}^{\infty}\left\{\left(n+1\right)\left(n+c\right)\alpha_{n+1} - \left[n^2 + \left(a+b\right)n + ab\right]\alpha_n\right\}z^n = 0,$$

having indicial equation

$$\alpha_{n+1} = \frac{\left(n+a\right)\left(n+b\right)}{\left(n+1\right)\left(n+c\right)}\alpha_n. \tag{3.99}$$

Writing

$$w(z) = \sum_{n=0}^{\infty}\alpha_n z^n,$$

we obtain a *regular solution,* having the form

$$w(z) = \alpha_0 F\left(a; b; c; z\right) \equiv \alpha_0\left[1 + \frac{ab}{1!c}z + \frac{a\left(a+1\right)b\left(b+1\right)}{2!c\left(c+1\right)}z^2 + \cdots\right]. \tag{3.100}$$

A complete solution is provided by all linear combinations of the two independent solutions $F\left(a; b; c; z\right)$ and $z^{1-c}F\left(a+1-c; b+1-c; 2-c; z\right)$. The resulting series converges, for $z \in \mathbf{R}$, on $-1 < z < 1$. It converges for $z = \pm 1$ provided $c > a + b$. See, e.g., Sect. 2.1 of [35].

If $y \neq 0$, then multiplying (3.97) by $-1/4$ and taking $\nu = -1$ puts the equation in the form (3.98) with $a = 1/2$, $b = 1$, and $c = 5/2$; c.f. Sect. 4.3.2. This equation, derived in [16], has the two linearly independent solutions

$$F(a;b;c;\mu) = F\left(\frac{1}{2};1;\frac{5}{2};\mu\right)$$

and

$$\mu^{1-c}F(a-c+1;b-c+1;2-c;\mu) = \mu^{-3/2}F\left(-1;-\frac{1}{2};-\frac{1}{2};\mu\right).$$

Evaluating (3.100) for this choice of parameters, we find that only the terms in (3.99) corresponding to $n = 0$ and $n = 1$ are nonzero. That is,

$$F\left(-1;-\frac{1}{2};-\frac{1}{2};\mu\right) = 1 + \frac{(-1)(-1/2)}{(-1/2)}\mu = 1-\mu,$$

and

$$\varphi = y^{\nu}F(\mu) = y^{-1}\mu^{-3/2}(1-\mu) =$$

$$y^{-1}\left(4\frac{x}{y^2}+1\right)^{-3/2}\left(-4\frac{x}{y^2}\right) = -\frac{4x}{(4x+y^2)^{3/2}}. \tag{3.101}$$

To prove that (3.101) is a fundamental solution, one would ordinarily show that

$$\langle L\mathscr{E}, \psi\rangle = \langle \delta, \psi\rangle = \psi(0,0) \tag{3.102}$$

$\forall\psi(x,y) \in C_0^{\infty}\left(\mathbf{R}^2\right)$, where in this context angle brackets denote the duality product of Sect. 2.4.5. However, only a (slightly) weaker assertion is true in this case, because the function φ given by (3.101) has a singularity of order 3/2 on the characteristic line

$$y^2 + 4x = 0. \tag{3.103}$$

Thus by a *fundamental solution* of (3.4) on any domain including the curve (3.103) we will mean only a distribution $\mathscr{E}$ satisfying the expression

$$\lim_{\epsilon\to 0}\left(\langle\mathscr{E}, L^*\psi\rangle_{|y^2+4x>\epsilon^2} + \langle\mathscr{E}, L^*\psi\rangle_{|y^2+4x<-\epsilon^2}\right) = \psi(0,0) \tag{3.104}$$

for all smooth, compactly supported test functions ψ, where L^* is the operator adjoint to L. In the language of Sect. 2.4.5, the *finite part* of the divergent integral describing the distribution $\mathscr{E}$ (or, in other words, the principal value of the distribution, abbreviated *p.v.*) is a fundamental solution in the ordinary sense of (3.102).

Theorem 3.5 (S-X. Chen [16]). *The distribution*

$$\mathscr{E} = \mathscr{C} \times p.v.\frac{4x\, sgn\left(y^2+4x\right)}{|y^2+4x|^{3/2}}$$

is the fundamental solution of (3.4).

Here $\mathscr{C}$ is a constant coefficient that is computed in the course of the proof.

Proof. We only give the idea of the proof, and refer the reader to Sect. 4 of [16] for the remaining details. (Note the difference in notation between (3.4) and (1) of [16].) For

$$Lu = xu_{xx} + u_{yy},$$

the adjoint operator is given by

$$L^*u = (xu)_{xx} + u_{yy} = xu_{xx} + u_{yy} + 2u_x.$$

In that case

$$\begin{aligned} uLv - vL^*u &= u\left(xv_{xx} + v_{yy}\right) - v\left(xu_{xx} + u_{yy} + 2u_x\right) \\ &= (uxv_x)_x - [v\,(xu)_x]_x + \left(uv_y\right)_y - \left(vu_y\right)_y. \end{aligned} \tag{3.105}$$

Integrating the above identity over an appropriate domain of $\mathbf{R}^2$, Stokes' Theorem can be applied on the right-hand side in order to express the left-hand side as a boundary integral.

Now we will construct the boundary of the domain in a way that will be convenient for evaluating (3.104), using (3.105).

Introduce the *characteristic coordinates* [6]

$$\ell = y + 2\sqrt{-x}$$

and

$$m = y - 2\sqrt{-x}$$

on the subdomain $x < 0$. Then the curves $\ell =$ constant and $m =$ constant are the characteristic lines for (3.4). Define the curves $A : m = \epsilon$; $A' : m = -\epsilon$; $B : \ell = -\epsilon$; $B' : \ell = \epsilon$; and

$$\gamma : x = \frac{1}{4}\epsilon^2 \sin^2\theta,\ y = \epsilon\cos\theta,\ 0 \leq \theta \leq \pi.$$

(Here θ is the polar angle in the sense that a ray lying along $\theta = 0$ lies along the positive y-axis and a ray lying along $\theta = \pi$ lies along the negative y-axis.) Finally, choose R so large that the interior of the circle $C_R : x^2 + y^2 = R^2$ encloses the support of the test function ψ. Denote by D^+ the domain enclosed by the curves C_R, γ, A, and B. Denote by D^- the domain enclosed by the curves C_R, A', and B' (Fig. 3.5).

Taking $u = \psi$ and $v = \mathscr{E}$ in (3.105), we evaluate this equation on each of the boundary curves in D^+ and D^-, taking into account that ψ has support in the interior of C_R.

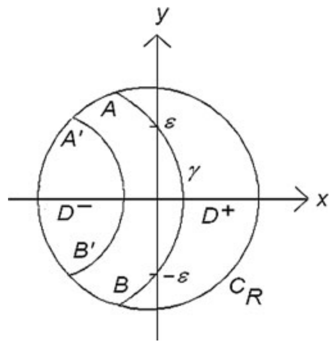

Fig. 3.5 The domain of Theorem 3.5

In particular,

$$\int_{D+} \mathscr{E} L^* \psi \, dxdy = \int_{B \cup \gamma \cup A} \left[x\psi \mathscr{E}_x - \mathscr{E} (x\psi)_y \right] dy + \left(\mathscr{E} \psi_y - \psi \mathscr{E}_y \right) dx. \tag{3.106}$$

We will completely describe the evaluation of the line integral along the curve γ, and then give a few hints about how the calculation would be done on the remaining curves, referring the reader to [16] for the remainder of the details.

On γ,

$$\mathscr{E} = \frac{4x}{(y^2+4x)^{3/2}} = \frac{\epsilon^2 \sin^2\theta}{\left(\epsilon^2\cos^2\theta + \epsilon^2\sin^2\theta\right)^{3/2}} = \frac{\sin^2\theta}{\epsilon}.$$

$$\mathscr{E}_y = -\frac{3}{2}\frac{4x}{(y^2+4x)^{5/2}} \cdot 2y = -\frac{3\sin^2\theta\cos\theta}{\epsilon^2}.$$

$$\mathscr{E}_x = \frac{4}{(y^2+4x)^{3/2}} - \frac{24x}{(y^2+4x)^{5/2}} = \frac{2\left(2-3\sin^2\theta\right)}{\epsilon^3}.$$

Moreover,

$$dx = \frac{\epsilon^2}{2}\sin\theta\cos\theta \, d\theta$$

and

$$dy = -\epsilon \sin\theta \, d\theta.$$

Integrating over γ, we write

$$I_\gamma(\epsilon) \equiv I_1 + I_2,$$

where

$$
\begin{aligned}
I_1 &= \int_\gamma -\mathscr{E}\,(x\psi)_y\,dy + \mathscr{E}\psi_y\,dx \\
&= \int_0^\pi \frac{\sin^2\theta}{\epsilon}\left(\psi_x \frac{\epsilon^2\sin^2\theta}{4} + \psi\right)\epsilon\sin\theta\,d\theta + \int_0^\pi \frac{\sin^2\theta}{\epsilon}\psi_y\frac{\epsilon^2}{2}\sin\theta\cos\theta\,d\theta \\
&= \int_0^\pi \sin^3\theta\left[\psi_x\frac{\epsilon^2\sin^2\theta}{4} + \psi + \frac{\epsilon}{2}\psi_y\cos\theta\right]d\theta \\
&= \int_0^\pi \psi\sin^3\theta\,d\theta + O\,(\varepsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_\gamma \psi x\mathscr{E}_x\,dy - \psi\mathscr{E}_y\,dx \\
&= -\int_0^\pi \psi\left(\frac{1}{4}\varepsilon^2\sin^2\theta\right)\frac{2\left(2-3\sin^2\theta\right)}{\varepsilon^3}\varepsilon\sin\theta\,d\theta \\
&\quad + \int_0^\pi \psi\left(\frac{3\sin^2\theta\cos\theta}{\varepsilon^2}\right)\frac{\varepsilon^2}{2}\sin\theta\cos\theta\,d\theta \\
&= -\int_0^\pi \psi\sin^3\theta\left[1-\frac{3}{2}\left(\sin^2\theta+\cos^2\theta\right)\right]d\theta = \frac{1}{2}\int_0^\pi \psi\sin^3\theta\,d\theta.
\end{aligned}
$$

In the limit as ε tends to zero, γ tends to $(0,0)$ and we obtain

$$
I_\gamma(0) = \frac{3}{2}\psi\,(0,0)\int_0^\pi \sin^3\theta\,d\theta = 2\psi\,(0,0). \tag{3.107}
$$

Integrating over the curves A, A', B, and B' in characteristic coordinates yields similar estimates. Use the fact that

$$
y = \frac{\ell+m}{2}
$$

and

$$
x = -\left(\frac{\ell-m}{4}\right)^2.
$$

Then in characteristic coordinates,

$$\begin{aligned}\mathscr{E} &= \frac{4x}{(y^2+4x)^{3/2}}\\ &= \frac{-4\left[(\ell-m)/4\right]^2}{\left\{\left[(\ell+m)/2\right]^2-4\left[(\ell-m)/4\right]^2\right\}^{3/2}} = -\frac{(\ell-m)^2}{4\,(\ell m)^{3/2}};\\ \mathscr{E}_\ell &= \frac{1}{2}\frac{(\ell-m)}{(\ell m)^{3/2}}\left[-1+\frac{3}{4}\frac{(\ell-m)\,m}{\ell m}\right];\\ \mathscr{E}_m &= \frac{1}{2}\frac{(\ell-m)}{(\ell m)^{3/2}}\left[1+\frac{3}{4}\frac{(\ell-m)\,\ell}{\ell m}\right].\end{aligned}$$

Thus

$$\mathscr{E}_y = \mathscr{E}_\ell \ell_y + \mathscr{E}_m m_y = \frac{2\,(m+\ell)}{8}\frac{(\ell-m)^2}{(\ell m)^{5/2}}.$$

and

$$\mathscr{E}_x = \mathscr{E}_\ell \ell_x + \mathscr{E}_m m_x = \frac{4}{(\ell m)^{3/2}} + \frac{3}{2}\frac{(\ell-m)^2}{(\ell m)^{5/2}}.$$

Substituting these formulas into the right-hand side of (3.106), we obtain a result analogous to (3.107). A similar result can be obtained for the integral over D^-. The sum of the coefficients of $\psi\,(0,0)$ over the union of the individual boundary arcs is $\mathscr{C}^{-1}$. See Sect. 4 of [16] for details. □

3.7.1 The Effect of a First-Order Term

As is typical of equations of Keldysh type, the regularity of the solution depends on the magnitude of a lower-order term. To see this, we consider in place of (3.4) the equation

$$L_\kappa u = 0, \tag{3.108}$$

where

$$L_\kappa u = x u_{xx} + \kappa u_x + u_{yy} \tag{3.109}$$

for a constant $\kappa \in [0,2]$.

Arguing exactly as in (3.96)–(3.101) but including the lower-order term, we obtain in place of (3.101) the expression

$$\varphi = \frac{(-4x)^{1-\kappa}}{(4x+y^2)^{\frac{3}{2}-\kappa}}.$$

Consequently, it is possible to prove the following:

Theorem 3.6 (S-X. Chen [16]). *Let $\kappa > -1/2$ and*

$$\mathscr{E}_\kappa = \begin{cases} 0 & \text{for } 4x + y^2 \geq 0, \\ \frac{(-4x)^{1-\kappa}}{(-4x-y^2)^{\frac{3}{2}-\kappa}} & \text{for } 4x + y^2 < 0. \end{cases}$$

Then the distribution $T_\kappa = \mathscr{C}_\kappa F.P.(E_\kappa)$ is the fundamental solution of (3.108). Here $\mathscr{C}_\kappa$ is a constant depending on κ.

Notice that, not for the first time, the coefficients $\kappa = 1/2$ and $\kappa = 1$ both have special significance, but in this case as "turning points" in the properties of the solution:

At points for which $4x + y^2 < 0$,

$$|4x + y^2| = -\left(4x + y^2\right) \leq -4x.$$

Thus if $\kappa \geq 1$ we have, on the support of E_κ, the identity

$$\frac{(-4x)^{1-\kappa}}{(-4x - y^2)^{\frac{3}{2}-\kappa}} = \frac{(-4x)^{1-\kappa}}{(-4x - y^2)^{1-\kappa}} \cdot \frac{1}{(-4x - y^2)^{\frac{1}{2}}},$$

from which we obtain

$$|E_\kappa| = (-4x)^{1-\kappa}\left(-4x - y^2\right)^{\kappa-1} \cdot \frac{1}{(-4x - y^2)^{\frac{1}{2}}} \leq \frac{1}{(-4x - y^2)^{\frac{1}{2}}}.$$

This greatly simplifies the integrals in the proof. But if $\kappa \leq 1/2$, then the distribution E_κ is singular at points for which $4x + y^2 = 0$, as the associated integrals fail to converge.

References

1. Agmon, S., Nirenberg, L., Protter, M.H.: A maximum principle for a class of hyperbolic equations and applications to equations of mixed elliptic–hyperbolic type. Commun. Pure Appl. Math. **6**, 455–470 (1953)
2. Aly, J.J.: On the uniqueness of the determination of the coronal potential magnetic field from line-of-sight boundary conditions. Solar Phys. **111**, 287–296 (1987)
3. Amari, T., Aly, J.J., Luciani, J.F., Boulmezaoud, T.Z., Mikic, Z.: Reconstructing the solar coronal magnetic field as a force-free magnetic field. Solar Phys. **174**, 129–149 (1997)
4. Amari, T., Boulbe, C., Boulmezaoud, T.Z.: Computing Beltrami fields. SIAM J. Sci. Comput. **31**(5), 3217–3254 (2009)
5. Amari, T., Luciani, J.F., Mikick, Z.: Magnetohydrodynamic models of solar coronal magnetic fields. Plasma Phys. Control. Fusion **41**, A779–A786 (1999)

6. Barros-Neto, J., Gelfand, I.M.: Fundamental solutions for the Tricomi operator. Duke Math. J. **98**, 465–483 (1999)
7. Bateman, H.: Partial Differential Equations. Dover, New York (1944)
8. Berezanskii, Yu.M.: Energy inequalities for some classes of equations of mixed type [in Russian]. Dokl. Akad. Nauk SSSR **132**, 9–12 (1960) [Soviet Math. Doklady **1**, 447–451 (1960)]
9. Berezanskii, Yu.M.: Spaces with negative norms. Russian Math. Surveys **18**(1), 63–95 (1963)
10. Berezanskii, Ju.M.: Expansions in Eigenfunctions of Selfadjoint Operators. American Mathematical Society, Providence (1968)
11. Bers, L.: Mathematical Aspects of Subsonic and Transonic Gas Dynamics. Wiley, New York (1958)
12. Bitsadze, A.V.: Equations of the Mixed Type, Zador, P. (trans). Pergammon, New York (1964)
13. Boulmezaoud, T.A., Amari, T.: On the existence of non-linear force-free fields in three-dimensional domains. Z. Angew. Math. Phys. **51**, 942–967 (2000)
14. Boulmezaoud, T-Z., Maday, Y., Amari, T.: On the linear force-free fields in bounded and unbounded three-dimensional domains. Math. Modelling Numer. Anal. **33**, 359–393 (1999)
15. Čanić, S., Keyfitz, B.: An elliptic problem arising from the unsteady transonic small disturbance equation J. Diff. Equations **125**, 548–574 (1996)
16. Chen, S-X.: The fundamental solution of the Keldysh type operator, Science in China, Ser. A: Mathematics **52**, 1829–1843 (2009)
17. Cibrario, M.: Sulla riduzione a forma canonica delle equazioni lineari alle derivate parziali di secondo ordine di tipo misto. Rendiconti del R. Insituto Lombardo, Ser. II **65**, 889–906 (1932)
18. Cibrario, M.: Alcuni teoremi di esistenza e di unicita per l'equazione $xz_{xx}, +z_{yy} = 0$. Atti R. Acc. Sci. Torino **68** 35–44 (1932–1933)
19. Cibrario, M.: Intorno ad una equazione lineare alle derivate parziali del secondo ordine di tipe misto iperbolico-ellittica. Ann. Sc. Norm. Sup. Pisa, Cl. Sci., Ser. 2 **3**(3, 4), 255–285 (1934)
20. Didenko, V.P.: On the generalized solvability of the Tricomi problem. Ukrain. Math. J. **25**, 10–18 (1973)
21. Flyer, N., Fornberg,B., Thomas, S., Low, B.C.: Magnetic field confinement in the solar corona. I. Force-free magnetic fields. Astrophys. J. **606**, 1210–1222 (2004)
22. Frankl', F.L.: Problems of Chaplygin for mixed sub- and supersonic flows [in Russian]. Izv. Akad. Nauk SSSR, ser. mat. **9**(2), 121–143 (1945)
23. Friedrichs, K.O.: Symmetric positive linear differential equations. Commun. Pure Appl. Math. **11**, 333–418 (1958)
24. Gellerstedt, S.: Quelques problèmes mixtes lour l'équation $y^m z_{xx} + z_{yy} = 0$. Arkiv für Matematik, Astronomi och Fusik **26A**(3), 1–32 (1937)
25. Grad, H., Rubin, H.: Hydromagnetic equilibria and force free fields. In: Proceedings of the 2nd International Conference on Peaceful Uses of Atomic Energy, vol. 31, p. 190. United Nations, Geneva (1958)
26. Hudson, S.R., Hole, M.J., Dewar, R.L.: Eigenvalue problems for Beltrami fields arising in a three-dimensional toroidal magnetohydrodynamic equilibrium problem. Phys. Plasmas **14**, 052505 (2007)
27. Janhunen, P.: Magnetically dominated plasma models of ball lightning. Annales Geophysicae. Atmos. Hydrospheres Space Sci. **9**, 377–380 (1991)
28. Katsanis, T.: Numerical techniques for the solution of symmetric positive linear differential equations. Ph.D. thesis, Case Institute of Technology (1967)
29. Katsanis, T.: Numerical solution of Tricomi equation using theory of symmetric positive differential equations. SIAM J. Numer. Anal. **6**, 236–253 (1969)
30. Keldysh, M.V.: On certain classes of elliptic equations with singularity on the boundary of the domain [in Russian]. Dokl. Akad. Nauk SSSR **77**, 181–183 (1951)
31. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and Quasilinear Elliptic Equations. Academic Press, New York (1968)
32. Lupo, D., Morawetz, C.S., Payne, K.R.: On closed boundary value problems for equations of mixed elliptic-hyperbolic type. Commun. Pure Appl. Math. **60**, 1319–1348 (2007)

33. Lupo, D., Morawetz, C.S., Payne, K.R.: Erratum: On closed boundary value problems for equations of mixed elliptic-hyperbolic type [Commun. Pure Appl. Math. **60**, 1319–1348 (2007)]. Commun. Pure Appl. Math. **61**, 594 (2008)
34. Lupo, D., Payne, K.R.: Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types. Commun. Pure Appl. Math. **56**, 403–424 (2003)
35. Magnus, W., Oberhettinger, F.: Formulas and Theorems for the Special Functions of Mathematical Physics, J. Werner, trans. Chelsea, New York (1949)
36. Manwell, A.R.: On locally supersonic plane flows with a weak shock wave, J. Math. Mech. **16**, 589–638 (1966)
37. Manwell, A.R.: The Hodograph Equations. Hafner Publishing, New York (1971)
38. Morawetz, C.S.: Note on a maximum principle and a uniqueness theorem for an elliptic-hyperbolic equation. Proc. R. Soc. London, Ser. A **236**, 141–144 (1956)
39. Morawetz, C.S.: Non-existence of transonic flow past a profile. Commun. Pure Appl. Math. **17**, 357–367 (1964)
40. Morawetz, C.S., Stevens, D.C., Weitzner, H.: A numerical experiment on a second-order partial differential equation of mixed type. Commun. Pure Appl. Math. **44**, 1091–1106 (1991)
41. Morrey, C.B.: Multiple Integrals in the Calculus of Variations. Springer, Berlin (1966)
42. Olver, F.W.J.: Asymptotics and Special Functions. A K Peters, Natick (1997)
43. Otway, T.H.: Energy inequalities for a model of wave propagation in cold plasma. Publ. Mat. **52**, 195–234 (2008)
44. Otway, T.H.: Unique solutions to boundary value problems in the cold plasma model. SIAM J. Math. Anal. **42**, 3045–3053 (2010)
45. Renardy, M., Rogers, R.C.: An Introduction to Partial Differential Equations. Springer, Berlin (2004)
46. Riemann, B.: Partielle Differentialgleichungen und deren Anwendung auf Physikalische Fragen. K. Hattendorff, ed. (1861). Nabu Press, 2010
47. Sorokina, N.G.: Strong solvability of the Tricomi problem [in Russian]. Ukrain. Mat. Zh. **18**, 65–77 (1966)
48. Sorokina, N.G.: Strong solvability of the generalized Tricomi problem [in Russian]. Ukrain. Mat. Zh. **24**, 558–561 (1972) [Ukrainian Math. J. **24**, 451–453 (1973)]
49. Tricomi, F.: Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto. Rendiconti Atti dell' Accademia Nazionale dei Lincei Ser. 5 **14**, 134–247 (1923)
50. Tsui, K.H.: A self-similar magnetohydrodynamic model for ball lightnings. Phys. Plasmas **13**, 072102 (2006)
51. Tsui, K.H., Serbeto, A.: Time-dependent magnetohydrodynamic self-similar extragalactic jets. Astrophys. J. **658**, 794–803 (2007)
52. Weitzner, H.: "Wave propagation in a plasma based on the cold plasma model." Courant Inst. Math. Sci. Magneto-Fluid Dynamics Div. Report MF–103, August, 1984
53. Wheatland, M.S.: Reconstruction of nonlinear force-free fields and solar flare prediction. In: Duldig, M. (ed.) Advances in Geosciences, vol. 8: Solar Terrestrial, pp. 123–137. World Scientific, Singapore (2006)
54. Yosida, K.: Functional Analysis. Springer, Berlin (1980)

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