

Contents

1	Introduction	1
	References	2
2	Basic Differential Geometry	5
2.1	Introduction	5
2.2	Differentiable Manifolds	6
2.2.1	Notion of Manifold.	6
2.2.2	Vectors on a Manifold	8
2.2.3	Linear Forms	10
2.2.4	Tensors	12
2.2.5	Fields on a Manifold.	13
2.3	Pseudo-Riemannian Manifolds	13
2.3.1	Metric Tensor	13
2.3.2	Signature and Orthonormal Bases.	14
2.3.3	Metric Duality	15
2.3.4	Levi-Civita Tensor	17
2.4	Covariant Derivative.	17
2.4.1	Affine Connection on a Manifold.	17
2.4.2	Levi-Civita Connection.	20
2.4.3	Curvature	22
2.4.4	Weyl Tensor	24
2.5	Lie Derivative	25
2.5.1	Lie Derivative of a Vector Field.	25
2.5.2	Generalization to Any Tensor Field	27
	References	28
3	Geometry of Hypersurfaces	29
3.1	Introduction	29
3.2	Framework and Notations	29
3.3	Hypersurface Embedded in Spacetime.	30

3.3.1	Definition	30
3.3.2	Normal Vector	32
3.3.3	Intrinsic Curvature	33
3.3.4	Extrinsic Curvature	34
3.3.5	Examples: Surfaces Embedded in the Euclidean Space \mathbb{R}^3	36
3.3.6	An Example in Minkowski Spacetime: The Hyperbolic Space \mathbb{H}^3	40
3.4	Spacelike Hypersurfaces	43
3.4.1	The Orthogonal Projector	44
3.4.2	Relation Between K and ∇n	46
3.4.3	Links Between the ∇ and D Connections	47
3.5	Gauss–Codazzi Relations	49
3.5.1	Gauss Relation	50
3.5.2	Codazzi Relation	52
	References	54
4	Geometry of Foliations	55
4.1	Introduction	55
4.2	Globally Hyperbolic Spacetimes and Foliations	55
4.2.1	Globally Hyperbolic Spacetimes	55
4.2.2	Definition of a Foliation	56
4.3	Foliation Kinematics	57
4.3.1	Lapse Function	57
4.3.2	Normal Evolution Vector	57
4.3.3	Eulerian Observers	60
4.3.4	Gradients of n and m	63
4.3.5	Evolution of the 3-Metric	64
4.3.6	Evolution of the Orthogonal Projector	66
4.4	Last Part of the 3+1 Decomposition of the Riemann Tensor	67
4.4.1	Last Non Trivial Projection of the Spacetime Riemann Tensor	67
4.4.2	3+1 Expression of the Spacetime Scalar Curvature	69
	References	71
5	3+1 Decomposition of Einstein Equation	73
5.1	Einstein Equation in 3+1 form	73
5.1.1	The Einstein Equation	73
5.1.2	3+1 Decomposition of the Stress-Energy Tensor	74
5.1.3	Projection of the Einstein Equation	76
5.2	Coordinates Adapted to the Foliation	78
5.2.1	Definition	78
5.2.2	Shift Vector	79
5.2.3	3+1 Writing of the Metric Components	82

5.2.4	Choice of Coordinates via the Lapse and the Shift . . .	85
5.3	3+1 Einstein Equation as a PDE System	86
5.3.1	Lie Derivatives Along m as Partial Derivatives	86
5.3.2	3+1 Einstein System	87
5.4	The Cauchy Problem	88
5.4.1	General Relativity as a Three-Dimensional Dynamical System	88
5.4.2	Analysis Within Gaussian Normal Coordinates	89
5.4.3	Constraint Equations	92
5.4.4	Existence and Uniqueness of Solutions to the Cauchy Problem	92
5.5	ADM Hamiltonian Formulation	94
5.5.1	3+1 form of the Hilbert Action	94
5.5.2	Hamiltonian Approach	95
	References	98
6	3+1 Equations for Matter and Electromagnetic Field	101
6.1	Introduction	101
6.2	Energy and Momentum Conservation	102
6.2.1	3+1 Decomposition of the 4-Dimensional Equation	102
6.2.2	Energy Conservation	102
6.2.3	Newtonian Limit	104
6.2.4	Momentum Conservation	105
6.3	Perfect Fluid	106
6.3.1	Kinematics	106
6.3.2	Baryon Number Conservation	109
6.3.3	Dynamical Quantities	111
6.3.4	Energy Conservation Law	112
6.3.5	Relativistic Euler Equation	113
6.3.6	Flux-Conservative Form	114
6.3.7	Further Developments	117
6.4	Electromagnetism	117
6.4.1	Electromagnetic Field	117
6.4.2	3+1 Maxwell Equations	119
6.4.3	Electromagnetic Energy, Momentum and Stress	122
6.5	3+1 Ideal Magnetohydrodynamics	123
6.5.1	Basic Settings	123
6.5.2	Maxwell Equations	125
6.5.3	Electromagnetic Energy, Momentum and Stress	127
6.5.4	MHD-Euler Equation	127
6.5.5	MHD in Flux-Conservative Form	129
	References	130

7	Conformal Decomposition	133
7.1	Introduction	133
7.2	Conformal Decomposition of the 3-Metric	135
7.2.1	Unit-Determinant Conformal “Metric”	135
7.2.2	Background Metric	135
7.2.3	Conformal Metric	136
7.2.4	Conformal Connection	138
7.3	Expression of the Ricci Tensor	141
7.3.1	General Formula Relating the Two Ricci Tensors	141
7.3.2	Expression in Terms of the Conformal Factor	142
7.3.3	Formula for the Scalar Curvature	142
7.4	Conformal Decomposition of the Extrinsic Curvature	143
7.4.1	Traceless Decomposition	143
7.4.2	Conformal Decomposition of the Traceless Part	144
7.5	Conformal Form of the 3+1 Einstein System	147
7.5.1	Dynamical Part of Einstein Equation	147
7.5.2	Hamiltonian Constraint	150
7.5.3	Momentum Constraint	151
7.5.4	Summary: Conformal 3+1 Einstein System	151
7.6	Isenberg–Wilson–Mathews Approximation to General Relativity	152
	References	156
8	Asymptotic Flatness and Global Quantities	159
8.1	Introduction	159
8.2	Asymptotic Flatness	159
8.2.1	Definition	160
8.2.2	Asymptotic Coordinate Freedom	161
8.3	ADM Mass	162
8.3.1	Definition from the Hamiltonian Formulation of GR	162
8.3.2	Expression in Terms of the Conformal Decomposition	167
8.3.3	Newtonian Limit	169
8.3.4	Positive Energy Theorem	170
8.3.5	Constancy of the ADM Mass	171
8.4	ADM Momentum	171
8.4.1	Definition	171
8.4.2	ADM 4-Momentum	172
8.5	Angular Momentum	172
8.5.1	The Supertranslation Ambiguity	172
8.5.2	The “Cure”	173
8.5.3	ADM Mass in the Quasi-Isotropic Gauge	174
8.6	Komar Mass and Angular Momentum	176

8.6.1	Komar Mass	176
8.6.2	3+1 Expression of the Komar Mass and Link with the ADM Mass	179
8.6.3	Komar Angular Momentum	182
	References	185
9	The Initial Data Problem	187
9.1	Introduction	187
9.1.1	The Initial Data Problem	187
9.1.2	Conformal Decomposition of the Constraints	188
9.2	Conformal Transverse-Traceless Method	189
9.2.1	Longitudinal / Transverse Decomposition of $\hat{A}^{\hat{i}\hat{j}}$	189
9.2.2	Conformal Transverse-Traceless Form of the Constraints	191
9.2.3	Decoupling on Hypersurfaces of Constant Mean Curvature	192
9.2.4	Existence and Uniqueness of Solutions to Lichnerowicz Equation	193
9.2.5	Conformally Flat and Momentarily Static Initial Data	194
9.2.6	Bowen–York Initial Data	200
9.3	Conformal Thin Sandwich Method	204
9.3.1	The Original Conformal Thin Sandwich Method	204
9.3.2	Extended Conformal Thin Sandwich Method	205
9.3.3	XCTS at Work: Static Black Hole Example	207
9.3.4	Uniqueness Issue	210
9.3.5	Comparing CTT, CTS and XCTS	210
9.4	Initial Data for Binary Systems	211
9.4.1	Helical Symmetry	211
9.4.2	Helical Symmetry and IWM Approximation	213
9.4.3	Initial Data for Orbiting Binary Black Holes	214
9.4.4	Initial Data for Orbiting Binary Neutron Stars	216
9.4.5	Initial Data for Black Hole: Neutron Star Binaries	217
	References	217
10	Choice of Foliation and Spatial Coordinates	223
10.1	Introduction	223
10.2	Choice of Foliation	224
10.2.1	Geodesic Slicing	224
10.2.2	Maximal Slicing	225
10.2.3	Harmonic Slicing	231
10.2.4	1+log Slicing	233
10.3	Evolution of Spatial Coordinates	235
10.3.1	Normal Coordinates	236

10.3.2	Minimal Distortion	236
10.3.3	Approximate Minimal Distortion	241
10.3.4	Gamma Freezing	242
10.3.5	Gamma Drivers	244
10.3.6	Other Dynamical Shift Gauges	246
10.4	Full Spatial Coordinate-Fixing Choices	247
10.4.1	Spatial Harmonic Coordinates	247
10.4.2	Dirac Gauge	248
	References	249
11	Evolution schemes	255
11.1	Introduction	255
11.2	Constrained Schemes	255
11.3	Free Evolution Schemes	256
11.3.1	Definition and Framework	256
11.3.2	Propagation of the Constraints	257
11.3.3	Constraint-Violating Modes	261
11.3.4	Symmetric Hyperbolic Formulations	262
11.4	BSSN Scheme	262
11.4.1	Introduction	262
11.4.2	Expression of the Ricci Tensor of the Conformal Metric	262
11.4.3	Reducing the Ricci Tensor to a Laplace Operator	265
11.4.4	The Full Scheme	267
11.4.5	Applications	269
	References	269
	Appendix A: Conformal Killing Operator and Conformal Vector Laplacian	273
	Appendix B: Sage Codes	281
	Index	287

3+1 Formalism in General Relativity

Bases of Numerical Relativity

Gourgoulhon, É.

2012, XVII, 294 p. 29 illus., Softcover

ISBN: 978-3-642-24524-4