Chapter 2
Semiclassical Techniques

We start this chapter with describing the main features of classical and quantum systems. We focus in contrast to the previous chapter here on expressing them in terms of formulas. Afterwards we introduce basic semiclassical techniques yielding approximations for the time evolution operator, the Green function, the density of states and the conductance. In the next two sections we explain how to calculate semiclassically the contributions from correlated orbit pairs using the configuration- [1, 2] and the phase-space approach [3–6] in the case of the conductance and the spectral form factor.

2.1 Basic Features of Classical and Quantum Systems

Classical mechanics completely describes the state of a particle at time t by its position and its momentum. These are solutions of Newton's equations of motion at time t for given initial conditions.[1] The motion of a particle in classical mechanics is thus deterministic. This however does not imply that the motion of a particle is easy to predict, for example in the case of a strong dependence of the motion on the initial conditions. This strong dependence is one main characteristic of classically chaotic systems. In order to quantify this dependence one studies how the motion changes under the influence of a perturbation of the initial conditions. We therefore consider one orbit in a system with initial conditions $(\mathbf{r}, \mathbf{p})$ with position $\mathbf{r}$ and with momentum $\mathbf{p}$ and one with slightly perturbed initial conditions $(\mathbf{r} + \epsilon\delta\mathbf{r}, \mathbf{p} + \epsilon\delta\mathbf{p})$ with $\epsilon \ll 1$. We calculate now the difference between the two phase-space coordinates at time t and expand it in a Taylor series in ϵ

$$(\mathbf{r} + \epsilon\delta\mathbf{r}, \mathbf{p} + \epsilon\delta\mathbf{p})(t) - (\mathbf{r}, \mathbf{p})(t) = \epsilon D\phi^t(\mathbf{r}, \mathbf{p})(\delta\mathbf{r}, \delta\mathbf{p}) + \mathcal{O}(\epsilon^2) \tag{2.1}$$

[1] In this work we only consider Hamiltonian systems, that are not explicitly time dependent. In particular this excludes dissipation.

D. Waltner, *Semiclassical Approach to Mesoscopic Systems*,
Springer Tracts in Modern Physics 245, DOI: 10.1007/978-3-642-24528-2_2,

with the differential of the Hamiltonian flow $D\phi^t(\mathbf{r}, \mathbf{p})$ defined as

$$D\phi^t(\mathbf{r}, \mathbf{p}) = \begin{pmatrix} \frac{\partial \mathbf{r}(t)}{\partial \mathbf{r}} & \frac{\partial \mathbf{r}(t)}{\partial \mathbf{p}} \\ \frac{\partial \mathbf{p}(t)}{\partial \mathbf{r}} & \frac{\partial \mathbf{p}(t)}{\partial \mathbf{p}} \end{pmatrix} \tag{2.2}$$

with $(\mathbf{r}, \mathbf{p})(t) \equiv (\mathbf{r}(t), \mathbf{p}(t))$ and $(\mathbf{r} + \epsilon\delta\mathbf{r}, \mathbf{p} + \epsilon\delta\mathbf{p})(t)$ defined analogously. The differential in Eq. (2.2) containing the derivatives of the momenta and positions at time t with respect to the initial momenta and positions thus determines the stability of an orbit. Of special interest in chaotic systems are orbits deviating from each other or approaching each other exponentially with t. In order to quantify this exponential separation one usually considers the $2d$ positive eigenvalues of the symmetric matrix $D\phi^t(\mathbf{r}, \mathbf{p})^T D\phi^t(\mathbf{r}, \mathbf{p})$ denoted by $\mu_i^2(\mathbf{r}, \mathbf{p}, t)$ in a d-dimensional system. The exponent T denotes here the transposed of a matrix. A measure for the exponential separation is then obtained by defining the Lyapunov exponent

$$\lambda_i(\mathbf{r}, \mathbf{p}) \equiv \lim_{t\to\infty} \left(\frac{1}{t} \ln \mu_i(\mathbf{r}, \mathbf{p}, t) \right). \tag{2.3}$$

The above quantity has the following properties: If $\lambda_i > 0$, the initial perturbation $(\delta\mathbf{r}, \delta\mathbf{p})$ increases exponentially like $e^{\lambda_i t}$, for $\lambda_i < 0$ it decreases exponentially and for $\lambda_i = 0$ there is a non-exponential change of the perturbation with time. The Lyapunov exponents always come in pairs. For each $\lambda_i > 0$ there exists an exponent $-\lambda_i < 0$. Two exponents vanish in every system, the one in the direction of the flow and the one perpendicular to the surface of constant energy. Definition (2.3) leads us to a first property of chaotic systems: Hyperbolicity, meaning that $(2d-2)$ Lyapunov exponents $\lambda_i(\mathbf{r}, \mathbf{p})$ are different from zero for almost all $(\mathbf{r}, \mathbf{p})$.

We define here the monodromy matrix $\mathbb{M}$ that will appear later in this work: It is a $(2d-2) \times (2d-2)$-matrix in the case of a d-dimensional system and is obtained from the differential of the flow $D\phi^t(\mathbf{r}, \mathbf{p})$ by restricting it to the directions where the Lyapunov exponents do not vanish thus reducing the dimension by two.

A further property of chaotic system is ergodicity: This means that almost all orbits with duration $t \to \infty$ are equidistributed in the part of the energy surface at energy E filled by the system. This implies for almost all initial conditions in closed systems, that time averages are equal to phase-space averages, in terms of formulas,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t dt' f\left(\mathbf{r}(t'), \mathbf{p}(t')\right) = \frac{\int d\mathbf{r} \int d\mathbf{p} f(\mathbf{r}, \mathbf{p}) \delta(E - H(\mathbf{r}, \mathbf{p}))}{\Omega(E)} \tag{2.4}$$

for a smooth function $f(\mathbf{r}, \mathbf{p})$ of position and momentum and the Hamilton function $H(\mathbf{r}, \mathbf{p})$ of the considered system. The phase-space integrals here are performed with respect to the phase-space parts filled by the system. We defined in this context the volume of the energy shell of the system

$$\Omega(E) \equiv \int d\mathbf{r} \int d\mathbf{p}\, \delta\left(E - H\left(\mathbf{r}, \mathbf{p}\right)\right). \tag{2.5}$$

These two conditions are the main characteristics of chaotic systems that we will use in this book. There are however also stronger conditions as mixing, that implies ergodicity: This means that correlations between the starting point of a trajectory and the phase-space point one obtains for the trajectory at time t can be neglected for $t \to \infty$ implying in terms of formulas

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{\Omega(E)} \int d\mathbf{r} \int d\mathbf{p}\, h\left(\mathbf{r}(t), \mathbf{p}(t)\right) f\left(\mathbf{r}, \mathbf{p}\right) \delta\left(E - H\left(\mathbf{r}, \mathbf{p}\right)\right) \\ = \frac{1}{\Omega(E)^2} \int d\mathbf{r} \int d\mathbf{p}\, h\left(\mathbf{r}, \mathbf{p}\right) \delta\left(E - H\left(\mathbf{r}, \mathbf{p}\right)\right) \\ \times \int d\mathbf{r}' \int d\mathbf{p}' f\left(\mathbf{r}', \mathbf{p}'\right) \delta\left(E - H\left(\mathbf{r}', \mathbf{p}'\right)\right)\end{aligned} \tag{2.6}$$

with another smooth function $h(\mathbf{r}, \mathbf{p})$ of position and momentum.

Up to now we only considered one class of classical systems, chaotic ones. The counterpart, integrable systems, possess in d-dimensions apart from the Hamilton function $d - 1$ conserved quantities $F_i\left(\mathbf{r}, \mathbf{p}\right)$. This means that all F_i, F_j $(i, j \in \{1, \ldots, d\})$ are in involution, i.e. that their Poisson brackets

$$\left\{F_i, F_j\right\}\left(\mathbf{r}, \mathbf{p}\right) \equiv \frac{\partial F_i\left(\mathbf{r}, \mathbf{p}\right)}{\partial \mathbf{r}} \frac{\partial F_j\left(\mathbf{r}, \mathbf{p}\right)}{\partial \mathbf{p}} - \frac{\partial F_i\left(\mathbf{r}, \mathbf{p}\right)}{\partial \mathbf{p}} \frac{\partial F_j\left(\mathbf{r}, \mathbf{p}\right)}{\partial \mathbf{r}} \tag{2.7}$$

vanish. In this context we defined $F_d\left(\mathbf{r}, \mathbf{p}\right) \equiv H\left(\mathbf{r}, \mathbf{p}\right)$. These systems show linear stability, i.e. neighbouring trajectories deviate from each other linearly with time. An ergodicity relation as in (2.4) can be also obtained for these systems, however the integration range is no longer given by the energy shell but further restricted by additional conserved quantities. The motion in phase space can be shown by studying the dynamics in terms of action-angle variables to take place on tori.

In the intermediate case of mixed dynamics the two kinds of phase-space dynamics coexist in a single system: The phase space contains regions of chaotic, i.e. ergodic, motion and regular motion, i.e. motion on tori.

In contrast to macroscopic systems, the theory to describe the dynamics of microscopic systems is quantum mechanics. In this case the state of the system at time t is described by a wavefunction obtained as solution of the Schrödinger equation for a certain initial state at time zero. The state of the underlying system cannot, in contrast to classical mechanics, be described by a point in phase space due to the Heisenberg uncertainty principle. It is not even possible to derive joint probability distributions for position and momentum, only functions that are similar to these distributions, for example Wigner or Husimi functions. Another difference between classical and quantum mechanics is that the Schrödinger equation is, in contrast to the Newton equation, always a linear differential equation leading only to a linear dependence of the state at time t on a perturbation of the initial state.

2.2 Introduction into Semiclassical Techniques

After having introduced the methods to describe classical and quantum systems we now want to explain in more detail the method used in this book for describing mesoscopic systems, that contains features of both theories: semiclassics. How does this method really work? We explain the procedure in detail for the quantum propagator. The propagator $K\left(\mathbf{r}, \mathbf{r}', t\right)$ evolves the initial quantum state $\psi_0\left(\mathbf{r}'\right)$ describing a system at time $t = 0$ into the state at time t, $\psi\left(\mathbf{r}, t\right)$, i.e.

$$\psi\left(\mathbf{r}, t\right) = \int d\mathbf{r}' K\left(\mathbf{r}, \mathbf{r}', t\right) \psi_0\left(\mathbf{r}'\right), \tag{2.8}$$

with the integration over the configuration space of the considered system. In order to find a semiclassical approximation for $K\left(\mathbf{r}, \mathbf{r}', t\right)$ we start by making a WKB-ansatz

$$K\left(\mathbf{r}, \mathbf{r}', t\right) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbb{R}^d} d\mathbf{p}' \sum_{\nu \in \mathbb{N}_0} \hbar^\nu a_\nu\left(\mathbf{r}, \mathbf{p}', t\right) \exp\left[(i/\hbar)\left(R\left(\mathbf{r}, \mathbf{p}', t\right) - \mathbf{r}'\mathbf{p}'\right)\right], \tag{2.9}$$

with the (for the moment) arbitrary scalar functions $R\left(\mathbf{r}, \mathbf{p}', t\right)$ and $a_\nu\left(\mathbf{r}, \mathbf{p}', t\right)$. The initial condition $K\left(\mathbf{r}, \mathbf{r}', 0\right) = \delta^{(d)}\left(\mathbf{r} - \mathbf{r}'\right)$ that guarantees that Eq. (2.8) is fulfilled for $t = 0$ leads to $R\left(\mathbf{r}, \mathbf{p}', 0\right) = \mathbf{r}\mathbf{p}'$, $a_0\left(\mathbf{r}, \mathbf{p}', 0\right) = 1$ and $a_\nu\left(\mathbf{r}, \mathbf{p}', 0\right) = 0$ for $\nu > 0$. The ansatz (2.9) is then inserted into the Schrödinger equation for the propagator with the Hamilton operator $\hat{H}$ governing the dynamics of the quantum system

$$\left(\hat{H} - i\hbar\frac{\partial}{\partial t}\right) K\left(\mathbf{r}, \mathbf{r}', t\right) = 0 \tag{2.10}$$

with $\hat{H}$ acting on the $\mathbf{r}$-coordinate. In the next step the result is sorted by powers of $\hbar$. This yields that $R\left(\mathbf{r}, \mathbf{p}', t\right)$ solves to leading order in $\hbar$ the classical Hamilton-Jacobi equation [7]

$$H\left(\mathbf{r}, \frac{\partial R}{\partial \mathbf{r}}\right) + \frac{\partial R}{\partial t} = 0 \tag{2.11}$$

with the classical Hamilton function $H\left(\mathbf{r}, \partial R/\partial \mathbf{r}\right)$ corresponding to the Hamilton-operator $\hat{H}$, that is obtained by replacing the position operator $\hat{\mathbf{r}}$ by the position coordinate $\mathbf{r}$ and the momentum operator $\hat{\mathbf{p}}$ by the momentum $\partial R/\partial \mathbf{r}$. Following Hamilton-Jacobi theory there exists a classical trajectory of duration t connecting $\left(\partial R/\partial \mathbf{p}', \mathbf{p}'\right)$ and $\left(\mathbf{r}, \partial R/\partial \mathbf{r}\right)$. Furthermore this implies that the Legendre transform of $R\left(\mathbf{r}, \mathbf{p}', t\right)$ from $\mathbf{p}'$ to $\mathbf{r}$ given by

$$S\left(\mathbf{r}, \mathbf{r}', t\right) = R\left(\mathbf{r}, \mathbf{p}', t\right) - \mathbf{r}'\mathbf{p}' \tag{2.12}$$

is Hamilton's principal function [8]

$$S\left(\mathbf{r},\mathbf{r}',t\right)=\int_0^t dt' L\left(\mathbf{r}(t'),\dot{\mathbf{r}}(t'),t'\right) \tag{2.13}$$

with the Lagrange function $L\left(\mathbf{r}(t'),\dot{\mathbf{r}}(t'),t'\right)$ and the velocity $\dot{\mathbf{r}}(t')$ of the particle at time t'.

By considering now the equation next-to-leading order in $\hbar$ the prefactor $a_0\left(\mathbf{r},\mathbf{p}',t\right)$ can be determined to be given by [7]

$$a_0\left(\mathbf{r},\mathbf{p}',t\right)=\sqrt{\det\left(\frac{\partial R\left(\mathbf{r},\mathbf{p}',t\right)}{\partial\mathbf{r}\partial\mathbf{p}'}\right)} \tag{2.14}$$

with $R\left(\mathbf{r},\mathbf{p}',t\right)$ again the solution of the Hamilton Jacobi equation (2.11).

From now on we will neglect the higher-order terms $a_\nu\left(\mathbf{r},\mathbf{p}',t\right)$ with $\nu>0$ and insert the expressions for $a_0\left(\mathbf{r},\mathbf{p}',t\right)$ and $R\left(\mathbf{r},\mathbf{p}',t\right)$ into the ansatz (2.9).

Next we perform the $\mathbf{p}'$-integral. Therefore we use the method of stationary phase that extracts from integrals containing rapidly oscillating phases the leading contributions in $\hbar$ yielding [9]

$$\begin{aligned}\int_{\mathbb{R}^d} d\mathbf{r}\, a\left(\mathbf{r},\mathbf{r}'\right)\mathrm{e}^{(i/\hbar)\phi(\mathbf{r},\mathbf{r}')}&=(2\pi\hbar)^{d/2}\sum_j a\left(\mathbf{r}_{0,j},\mathbf{r}'\right)\mathrm{e}^{(i/\hbar)\phi(\mathbf{r}_{0,j},\mathbf{r}')}\\&\times\frac{\exp\left(i\frac{\pi}{4}\mathrm{sign}\left(\frac{\partial^2\phi}{\partial r_k\partial r_l}\left(\mathbf{r}_{0,j},\mathbf{r}'\right)\right)\right)}{\left|\det\frac{\partial^2\phi}{\partial r_k\partial r_l}\left(\mathbf{r}_{0,j},\mathbf{r}'\right)\right|}+\mathcal{O}\left(\hbar^{d/2+1}\right)\end{aligned} \tag{2.15}$$

with the scalar functions $a\left(\mathbf{r},\mathbf{r}'\right)$ and $\phi\left(\mathbf{r},\mathbf{r}'\right)$. The sum on the right hand side of the last equation runs over all stationary points $\mathbf{r}_{0,j}$ of the integral on the left hand side defined as $\partial\phi\left(\mathbf{r},\mathbf{r}'\right)/\partial\mathbf{r}=0$. Here $\mathrm{sign}(A)$ denotes the difference of the number of positive and negative eigenvalues of the matrix A. The intuitive interpretation of this method is: For $\hbar\to 0$ the integrand on the left hand side of Eq. (2.15) is rapidly oscillating in dependence of $\mathbf{r}$ everywhere except for regions close to points where $\partial\phi\left(\mathbf{r},\mathbf{r}'\right)/\partial\mathbf{r}=0$. Thus only those points contribute in leading order in $\hbar$. To apply relation (2.15) these points need to be isolated and the determinant of second derivatives needs to be non-zero at the stationary points

$$\det\frac{\partial^2\phi}{\partial r_k\partial r_l}\left(\mathbf{r}_{0,j},\mathbf{r}'\right)\neq 0. \tag{2.16}$$

For a more detailed description of the assumptions, see for example [8, 9].

Using now the method of stationary phase we find the semiclassical expression for the quantum propagator, the so-called van Vleck-Gutzwiller propagator [10, 11]

$$K_{\mathrm{sc}}\left(\mathbf{r},\mathbf{r}',t\right)=\left(\frac{1}{\sqrt{2\pi i\hbar}}\right)^d\sum_{\gamma(\mathbf{r}'\to\mathbf{r},t)}A_\gamma\mathrm{e}^{(i/\hbar)S_\gamma\left(\mathbf{r},\mathbf{r}',t\right)-i(\pi/2)\mu_\gamma}, \tag{2.17}$$

given by a sum over classical orbits from $\mathbf{r}'$ to $\mathbf{r}$ of duration t. Each summand contains the prefactor $A_\gamma = \sqrt{\left|\det\left(-\frac{\partial^2 S_\gamma(\mathbf{r},\mathbf{r}',t)}{\partial\mathbf{r}\partial\mathbf{r}'}\right)\right|} = \sqrt{\left|\det\left(\frac{\partial\mathbf{p}_0^\gamma}{\partial\mathbf{r}}\right)\right|}$ obtained first by Morette and van Hove [12] with the initial momentum $\mathbf{p}_0^\gamma$ and a phase with the aforementioned principal function. The phase factor $\mathrm{e}^{-i\pi\mu_\gamma/2}$ results from the $\hbar$-independent phase in Eq. (2.15) and the sign of $a_0\left(\mathbf{r},\mathbf{p}',t\right)$ in Eq. (2.14). This so-called Maslov phase can be expressed by the number of conjugated points along the orbit, so-called caustics, i.e. by the number of times the prefactor A_γ gets singular along the orbit plus twice the number of hard wall boundary reflections [11]. The number of vanishing eigenvalues of the inverse matrix $\frac{\partial\mathbf{r}}{\partial\mathbf{p}_0^\gamma}$ determines the order of a caustic. If the number is equal to the dimension of configuration space, we have a focal point: Changing $\mathbf{p}_0^\gamma$ leaves all components of $\mathbf{r}$ unchanged.

The expression (2.17) contains as classical element the sum over classical orbits and as quantum mechanical one phases with the principal functions of the underlying trajectories in the exponentials, that can cause interference effects.

Starting from the semiclassical formula for the propagator, semiclassical expressions can be also obtained for other quantities that we will need here like Green functions, or the density of states of a system. In that cases, relation (2.17) is again the building block: One uses in the derivation of semiclassical expressions for these quantities the exact relation between the propagator and the Green function (a Fourier transformation) and the exact relation between the Green function and the density of states (a trace integral) plus the semiclassical form of the propagator to obtain an approximate formula for these quantities. This yields for the (retarded) Green function $G\left(\mathbf{r},\mathbf{r}',E\right)$ at energy E given in terms of the propagator by

$$G\left(\mathbf{r},\mathbf{r}',E\right) = \frac{1}{i\hbar}\lim_{\epsilon\to 0}\int_0^\infty dt\, K\left(\mathbf{r},\mathbf{r}',t\right)\mathrm{e}^{(i/\hbar)(E+i\epsilon)t} \tag{2.18}$$

the semiclassical expression [11] by performing the t-integral by stationary phase

$$G_{\mathrm{sc}}\left(\mathbf{r},\mathbf{r}',E\right) = \frac{1}{i\hbar}\left(\frac{1}{\sqrt{2\pi i\hbar}}\right)^{d-1}\sum_{\gamma(\mathbf{r}'\to\mathbf{r},E)} B_\gamma \mathrm{e}^{(i/\hbar)S_\gamma(\mathbf{r},\mathbf{r}',E)-i(\pi/2)\nu_\gamma} \tag{2.19}$$

with the orbits γ connecting $\mathbf{r}'$ and $\mathbf{r}$ at energy E, the prefactors B_γ calculated in [11] are given by $B_\gamma = \left(1/\left(|\dot{\mathbf{r}}|\,|\dot{\mathbf{r}}'|\right)\left|\det\left(\partial^2 S_\gamma/\partial\mathbf{r}_\perp\partial\mathbf{r}'_\perp\right)\right|\right)^{1/2}$ and the classical actions of γ, $S_\gamma\left(\mathbf{r},\mathbf{r}',E\right) = \int_{\mathbf{r}'}^{\mathbf{r}} d\mathbf{y}\,\mathbf{p}\left(\mathbf{y},E\right)$, in the exponentials. Here $\dot{\mathbf{r}}$ and $\dot{\mathbf{r}}'$ are the final and initial velocities and $\mathbf{r}_\perp$ and $\mathbf{r}'_\perp$ denote the $(d-1)$ final and initial spatial coordinates perpendicular to the trajectory, respectively. The Maslov phases ν_γ are determined by the ones of the propagator and the phases resulting from performing the t-integral in (2.18) within stationary phase [8].

The spectral density is defined in terms of the spectrum $\{E_n\}$ as

$$d(E) = \sum_{n=1}^\infty \delta\left(E - E_n\right). \tag{2.20}$$

It is proportional to the imaginary part of trace of $G\left(\mathbf{r}, \mathbf{r}', E\right)$

$$d\left(E\right) = -\frac{1}{\pi}\Im \operatorname{Tr}\left[G\left(E\right)\right] = -\frac{1}{\pi}\Im \int d\mathbf{r}\, G\left(\mathbf{r}, \mathbf{r}, E\right) \tag{2.21}$$

with $\Im$ denoting the imaginary part, the real part will be denoted by $\Re$. The famous Gutzwiller trace formula [11] for chaotic systems is obtained by performing the $\mathbf{r}$-integral within stationary phase yielding

$$\Im \mathrm{Tr} G_{\mathrm{sc}}\left(E\right) = -\frac{\pi \Omega(E)}{(2\pi\hbar)^d} + \underbrace{\Im \frac{-i}{\hbar} \sum_{\gamma} T_{\gamma}^{\mathrm{prim}} C_{\gamma} \mathrm{e}^{(i/\hbar) S_{\gamma}(E) - i(\pi/2)\tau_{\gamma}}}_{\mathrm{Tr} G_{\mathrm{sc}}^{+\mathrm{osc}}(E)} \tag{2.22}$$

with the oscillatory part of the trace of the Green function $\mathrm{Tr} G_{\mathrm{sc}}^{+\mathrm{osc}}(E)$ defined for later reference. The spectral density can thus be expressed semiclassically as

$$d_{\mathrm{sc}}\left(E\right) = \frac{\Omega(E)}{(2\pi\hbar)^d} + \frac{1}{\pi\hbar}\Re \sum_{\gamma} T_{\gamma}^{\mathrm{prim}} C_{\gamma} \mathrm{e}^{(i/\hbar) S_{\gamma}(E) - i(\pi/2)\tau_{\gamma}}. \tag{2.23}$$

The first summand, the so-called Weyl term, follows after performing the $\mathbf{r}$-integral by stationary phase from very short trajectories contained in (2.19). It can be interpreted as the number of Planck cells of the volume $(2\pi\hbar)^d$ in d dimensions fitting into the volume of the energy shell $\Omega(E)$. The second summand is rapidly oscillating as function of the energy and characterises the oscillations of the density of states around the mean part. It contains a sum over periodic orbits γ of primitive duration $T_{\gamma}^{\mathrm{prim}}$ with the prefactor $C_{\gamma} = 1/\left|\det\left(\mathbb{M}_{\gamma} - \mathbb{1}_{2d-2}\right)\right|^{1/2}$ containing the monodromy matrix $\mathbb{M}_{\gamma}$ defined before Eq. (2.4) and the actions $S_{\gamma}(E) = \oint d\mathbf{y}\, \mathbf{p}(\mathbf{y}, E)$. The Maslov phases τ_{γ} contain the phases μ_{γ} from the Green function and the additional phases from the additional stationary phase.

Note that the stationary phase approximation in Eq. (2.22) can only be performed for isolated stationary points, i.e. isolated orbits. For integrable systems they form families covering the tori, here a different formula developed by Berry, Tabor and Balian, Bloch applies, see [13–17].

Whereas the semiclassical expression for the spectral density is of interest itself, the semiclassical Green function will be mainly used here to derive a semiclassical form of the transmission coefficients; i.e. the elements of the scattering matrix $S(E)$ at energy E. Considering a two lead geometry this matrix contains the reflection and transmission subblocks $r(E), t(E)$ for the incoming wave in the lead 1 and $r'(E), t'(E)$ for the incoming wave in the lead 2, respectively

$$S(E) = \begin{pmatrix} r(E) & t'(E) \\ t(E) & r'(E) \end{pmatrix}. \tag{2.24}$$

The dimension of the matrices is determined by the condition that the longitudinal wavefunction possesses positive energy, i.e. by the largest number N_i fulfilling

$E - \hbar^2/(2m)(N_i\pi/W_i)^2 > 0$ with the total energy E and the energy of the transverse wavefunction $\hbar^2/(2m)(N_i\pi/W_i)^2$ containing the width of the ith lead, W_i. The dimension of $S(E)$ is then given by $N \equiv N_1 + N_2$. Using the Landauer-Büttiker formalism [18–20] an expression for the conductance G can be obtained in terms of the subblock t

$$G = 2\frac{e^2}{h}\mathrm{Tr}\left(tt^\dagger\right) \tag{2.25}$$

with the factor 2 accounting for spin degeneracy in the absence of spin-orbit interaction. These scattering-matrix elements $t_{\alpha,\beta}(E)$, that we will in the following consider as a function of the absolute value of the wave vector $k = \sqrt{2mE}/\hbar$ with the particle mass m, are related to the Green function by the Fisher Lee relations [21]

$$t_{\alpha,\beta}(k) = -i\hbar\sqrt{v_\alpha v_\beta}\int_0^{W_1} dy \int_0^{W_2} dy' \Phi_\alpha^*(y)\, G\left(y, y', E\right) \Phi_\beta\left(y'\right), \tag{2.26}$$

i.e. the scattering-matrix elements are given by the projections of the Green function onto the transverse eigenfunctions in the leads of width W_1 and W_2. The star denotes here complex conjugation. These eigenfunctions are, for hard wall boundary conditions, given by $\Phi_\alpha(y) = \sqrt{2/W_1}\sin(\alpha\pi y/W_1)$ and $\Phi_\beta\left(y'\right) = \sqrt{2/W_2}\sin\left(\beta\pi y'/W_2\right)$. The longitudinal velocities in the left and right lead are denoted v_β and v_α, respectively. For the matrix elements of the reflection subblock a relation similar to Eq. (2.26) can be obtained [21]

$$r_{\alpha,\beta}(k) = \delta_{\alpha,\beta} - i\hbar\sqrt{v_\alpha v_\beta}\int_0^{W_1} dy \int_0^{W_1} dy' \Phi_\alpha^*(y)\, G\left(y, y', E\right) \Phi_\beta\left(y'\right) \tag{2.27}$$

with y, y' now both lying in the same lead.

To obtain a semiclassical form of Eq. (2.26) [22, 23], the semiclassical expression of the Green function (2.19) is used and the integrals with respect to y and y' are performed within stationary phase yielding the condition

$$\frac{\partial S_\gamma}{\partial y'} = -p_{y'} = -\frac{\overline{\beta}\pi\hbar}{W_2} \tag{2.28}$$

with $\overline{\beta} \equiv \pm\beta$. An analogous condition is obtained for the momenta p_y. Thus only those paths which enter into the cavity at (x', y') with a fixed angle $\sin\theta' = \pm\beta\pi/(kW_2)$ and exit the cavity at (x, y) with angle $\sin\theta = \pm\alpha\pi/(kW_1)$ contribute to $t_{\alpha,\beta}(k)$. There is an intuitive explanation for this condition: The trajectories are those whose transverse wave vectors on entrance and exit match the wave vectors of the modes in the leads. One then obtains for the semiclassical expression for the transmission amplitudes $t_{\alpha,\beta}(k)$

$$t^{\mathrm{sc}}_{\alpha,\beta}(k) = -\frac{\sqrt{\pi i\hbar}}{\sqrt{2W_1W_2}}\sum_{\gamma(\overline{\alpha},\overline{\beta})} \mathrm{sign}(\overline{\alpha})\mathrm{sign}(\overline{\beta}) D_\gamma \exp\left(\frac{i}{\hbar}\widetilde{S}_\gamma\left(\overline{\alpha}, \overline{\beta}, E\right) - i\frac{\pi}{2}\eta_\gamma\right) \tag{2.29}$$

with the reduced actions

$$\widetilde{S}_\gamma\left(\overline{\alpha}, \overline{\beta}, E\right) \equiv S_\gamma(\mathbf{r}, \mathbf{r}', E) + \hbar k y' \sin\theta' - \hbar k y \sin\theta. \tag{2.30}$$

Furthermore one obtains $D_\gamma = \sqrt{|(\partial y/\partial\theta')|\,/(\hbar k|\cos\theta'|)}$; the index η_γ contains the Maslov index appearing in the Green function and additional phases due to the two stationary phase integrals. For the reflection (2.27) an analogous expression in terms of trajectories returning to the same lead holds. The $\delta_{\alpha,\beta}$ in Eq. (2.27) is cancelled by direct trajectories not entering the cavity.

The three Eqs. (2.17, 2.23, 2.29) are the basic semiclassical formulas we will use in this book.

In the next two sections we want to present the main advances in showing how the RMT results can be derived with semiclassical methods: First we analyse the transmission through a chaotic system based on the Landauer-Büttiker formalism and afterwards we consider the spectral form factor.

2.3 Quantum Corrections to the Transmission

In this section we want to study the expression obtained by inserting the semiclassical approximation (2.29) into the Landauer-Büttiker formula (2.25). Here we obtain with $G \equiv 2\frac{e^2}{h}T$

$$T^{\mathrm{sc}} = \sum_{\alpha=1}^{N_1}\sum_{\beta=1}^{N_2}\left|t^{\mathrm{sc}}_{\alpha,\beta}(k)\right|^2 = \frac{\pi\hbar}{2W_1W_2}\sum_{\alpha=1}^{N_1}\sum_{\beta=1}^{N_2}\sum_{\gamma,\gamma'}F^{\gamma,\gamma'}_{\alpha,\beta} \tag{2.31}$$

with

$$F^{\gamma,\gamma'}_{\alpha,\beta} \equiv D_\gamma D_{\gamma'} \exp\left[\frac{i}{\hbar}\left(S_\gamma - S_{\gamma'} - i\frac{\pi}{2}\mu_{\gamma,\gamma'}\right)\right] \tag{2.32}$$

with $\mu_{\gamma,\gamma'}$ containing the indices η_γ and $\eta_{\gamma'}$, the sign-factors in Eq. (2.29) and the phases due to the two last summands in Eq. (2.30). For $\hbar \to 0$ the function in (2.32) is rapidly oscillating in dependence of the energy due to the factor $\mathrm{e}^{(i/\hbar)\left(S_\gamma - S_{\gamma'}\right)}$. In the following we wish to identify those contributions to Eq. (2.31) which survive an average over a classically small but quantum mechanically large k-window Δk. Important contributions to (2.31) will result from very similar trajectories γ and γ'. Afterwards we wish to evaluate their contributions to $\left|t_{\alpha,\beta}(k)\right|^2$ using basic principles of chaotic dynamics: For our calculation we will need hyperbolicity and ergodicity.

This section is divided into three subsections: After showing how to obtain the diagonal contribution in the first subsection we describe the calculation of the simplest non-diagonal contribution in the second part and study its behaviour as a function of a magnetic field. In the third subsection we introduce an approach to calculate further quantum corrections.

2.3.1 Diagonal Contribution

The simplest approximation is $\gamma = \gamma'$, the so-called diagonal approximation, yielding

$$\left|t_{\alpha,\beta}(k)\right|^2_{\text{diag}} = \frac{\pi\hbar}{2W_1W_2}\sum_{\gamma}\left|D_\gamma\right|^2. \tag{2.33}$$

The remaining sum over classical trajectories in Eq. (2.33) can be calculated using a classical sum rule [24] that can be derived using in the ergodicity relation (2.4) a proper function $f(\mathbf{r}, \mathbf{p})$, see for a derivation of a similar rule [25] and for a detailed derivation of this sum rule [26]. It yields

$$\sum_{\gamma}\left|D_\gamma\right|^2 \approx \frac{4W_1W_2}{\Omega(E)}\int_0^\infty dT\rho(T), \tag{2.34}$$

where $\Omega(E)$ again denotes the phase-space volume of the system at energy E, and $\rho(T)$ is the classical probability to find a particle still inside an open system after a time T if it was inside at $T = 0$. For long times this function decays exponentially $\rho(T) \sim e^{-T/\tau_D}$ for $T \to \infty$ for a chaotic system, with the dwell time $\tau_D = \frac{\Omega(E)}{2\pi\hbar(N_1+N_2)}$. This exponential decay can be easily understood based on the equidistribution of trajectories: The number ΔN of particles leaving the system during ΔT is given by the overall number of particles N times the ratio of the phase-space volume from which the particles leave during ΔT and the whole phase-space volume of the system. The differential equation for N, obtained in the case of infinitesimal ΔT, has obviously an exponential solution.

By inserting Eq. (2.34) into Eq. (2.33), we obtain

$$\left|t_{\alpha,\beta}(k)\right|^2_{\text{diag}} = \frac{1}{N_1+N_2}. \tag{2.35}$$

Its derivation required ergodicity valid only for long trajectories. We will assume that the classical dwell time is large enough, i.e. the opening is small enough, in order to have a statistically relevant number of long trajectories left after time T. The result in Eq. (2.35) allows for a very simple interpretation: It is just the probability of reaching one of the $N_1 + N_2$ channels if each of the channels can be reached equally likely.

Next we want to evaluate off-diagonal (interference) contributions to the quantum transmission. We will present here two approaches: The first one, the configuration-space approach, that was also chronologically the first one to calculate off-diagonal contributions is very illustrative as it measures the difference between the orbit and its partner in terms of the crossing angle of the crossing orbit and not in terms of the phase-space difference as the second one, the phase-space approach [3–6]. The latter one is however more general as the configuration-space approach relies on assumptions valid only for two-dimensional systems. Originally developed for surfaces of constant negative curvature [1, 2], it could later be generalised to general hyperbolic chaotic systems with two degrees of freedom [27].

2.3.2 First Quantum Correction: Configuration-Space Approach

In the following we will describe the calculation of the contributions from pairs of different trajectories, however with similar actions.

We want to remark that this calculation was motivated by the existence of RMT results for $\left|t_{\alpha,\beta}(k)\right|^2$ [28]

$$\left|t_{\alpha,\beta}(k)\right|^2_{\mathrm{RMT}} = \frac{1}{N_1 + N_2} \tag{2.36}$$

for the circular unitary ensemble, i.e. for systems without time-reversal symmetry, and

$$\left|t_{\alpha,\beta}(k)\right|^2_{\mathrm{RMT}} = \frac{1}{N_1 + N_2 + 1} = \frac{1}{N_1 + N_2} \sum_{k=0}^{\infty} \left(\frac{-1}{N_1 + N_2}\right)^k \tag{2.37}$$

for the circular orthogonal ensemble, i.e. for systems with time-reversal symmetry. The result (2.36) shows that in the absence of time-reversal symmetry semiclassically there are no additional contributions expected when semiclassical and RMT results coincide apart from the one obtained in diagonal approximation above. In the presence of time-reversal symmetry there exist apart from the diagonal contribution—the $k = 0$-term in (2.37)—further contributions that need to be explained semiclassically, if semiclassics reproduces RMT results.

For a long time it was not clear how orbits could look like that are on the one hand *correlated* but on the other hand *different*. There are many orbit pairs in a chaotic system having accidentally nearly equal actions. Their number is usually exponentially increasing as a function of their duration T_γ, as can be seen by applying the sum rule (2.34) directly in Eq. (2.29) and taking into account that usually $D_\gamma \propto \mathrm{e}^{-\lambda T_\gamma/2}$ for large T_γ. These contributions are however assumed to appear randomly and to be cancelled by the energy average. In order to describe universal features of a chaotic system after energy averaging, one has to find orbits that are correlated in a systematic way. These orbits were first identified and analysed in 2000 in the context of spectral statistics [1]. There, *periodic* orbits were studied to compute correlations between energy eigenvalues of quantum systems with classically chaotic counterpart, we will come to this point in the next section. Based on *open*, lead-connecting trajectories, in Ref. [24] this approach was generalised to the conductance we study here. Still, the underlying mechanism to form pairs of classically correlated orbits is the same in the two cases. In Fig. 2.1 we show a representative example of such a correlated (periodic) orbit pair in the chaotic hyperbola billiard. The two partner orbits are topologically the same up to the region marked by the circle where one orbit exhibits a self-crossing (left panel) while the partner orbit an "avoided crossing" (right panel). Usually, such trajectory pairs are drawn schematically as shown in Fig. 2.2.

We here consider very long orbits with self-crossings characterised by crossing angles $\epsilon \ll \pi$. In Refs. [1, 2] it was shown that there exists for each orbit a partner orbit starting and ending (exponentially) close to the first one. It follows the first

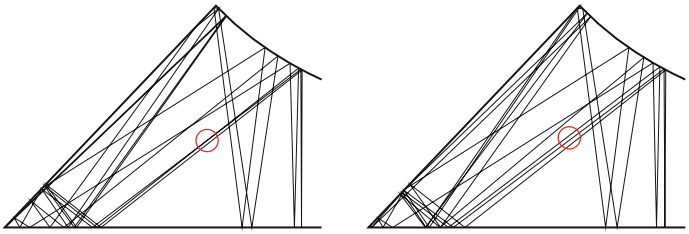

Fig. 2.1 Pair of two periodic orbits in the hyperbola billiard differing from each other essentially in the region marked by the *circle*, where the *left* orbit exhibits a self-crossing while the *right* partner orbit shows an "avoided crossing" (taken from [29])

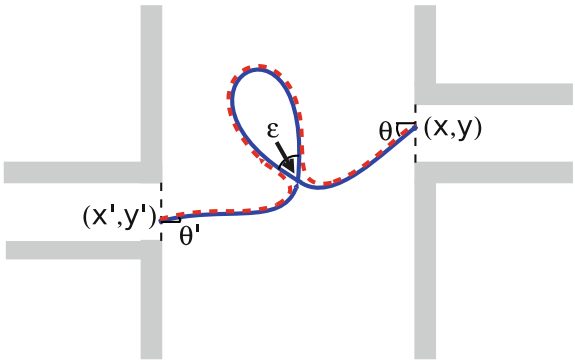

Fig. 2.2 Schematic drawing of a pair of orbits yielding the first non-diagonal contribution to the transmission considered in Ref. [24]. One of the orbits crosses itself under an angle ϵ, the other one possesses an "avoided crossing". Except for the crossing region both orbits are almost identical

orbit until the crossing, avoids this, however, traverses the loop in reversed direction and avoids the crossing again.

In order to quantify the contribution of these trajectory pairs to Eq. (2.31) we need two inputs: an expression for the action difference and for the density quantifying how often an orbit of time T exhibits a self-intersection, both quantities expressed as a function of the parameter ϵ. The formula for the action difference ΔS can be derived by linearising the dynamics of the orbit without crossing around the reference orbit with crossing giving, in the limit $\epsilon \ll \pi$, [1, 2]

$$\Delta S \approx \frac{p^2 \epsilon^2}{2m\lambda} \tag{2.38}$$

with p the absolute value of the momentum. At this point we can justify our assumption of small crossing angles ϵ: In the limit $\hbar \to 0$, we expect important contributions to Eq. (2.31) only from orbit pairs with small action differences, larger action differences will yield rapidly oscillating contributions as a function of the energy. This implies that the crossing angles have to be small, as we see from Eq. (2.38).

Before deriving the number of self-crossings, $P(\epsilon, T)\, d\epsilon$, in the range between ϵ and $\epsilon + d\epsilon$ for an orbit of time T, we give rough arguments how this expression depends on ϵ and T for trajectories in billiards. There, each orbit is composed of a chain of N chords connecting the reflection points. Following an orbit, the first two

chords cannot intersect, the third chord can cross with up to one, the fourth chord with up to two segments, and so on. Hence, the overall number of self-crossings will be proportional to $\sum_{n=3}^{N}(n-2) \propto N^2$, to leading order in N, i.e. proportional to T^2.

The crossing-angle dependence of $P(\epsilon, T)$ can be estimated for small ϵ as follows: Given a trajectory chord of length L, a second chord, tilted by an angle ϵ with respect to the first one, will cross it inside the billiard (with area of order L^2) only if the distance between the reflection points of the two chords at the boundary is smaller than $L \sin\epsilon$. The triangle formed in the latter case includes a fraction $\sin\epsilon$ of the entire billiard size. From this rough estimation we expect $P(\epsilon, T) \propto T^2 \sin\epsilon$.

More rigorously, the quantity $P(\epsilon, T)\, d\epsilon$, can be expressed for an arbitrary orbit γ as [1, 2]

$$P(\epsilon, T)\, d\epsilon = \left\langle \int_{T_{\min}(\epsilon)}^{T-T_{\min}(\epsilon)} dt_l \int_{T_{\min}(\epsilon)/2}^{T-t_l-T_{\min}(\epsilon)/2} dt_s\, |J|\, \delta\left(\mathbf{r}(t_s) - \mathbf{r}(t_s+t_l)\right) \times \delta\left(\epsilon - \alpha(t_s, t_s+t_l)\right) \right\rangle d\epsilon \tag{2.39}$$

with the average $\langle \ldots \rangle$ taken over different initial conditions $(\mathbf{x}_0, \mathbf{p}_0)$. The time of the closed loop of the trajectory is denoted by t_l and the time before the loop by t_s. $\alpha(t_s, t_s+t_l)$ denotes the absolute value of the angle between the velocities $\mathbf{v}(t_s)$ and $\mathbf{v}(t_s+t_l)$. $|J|$ is the Jacobian for the transformation from the argument of the first delta function to t_l and t_s ensuring that $P(\epsilon, T)\, d\epsilon$ yields a 1 for each crossing of γ. With the absolute value of the velocity, v, it can be expressed as

$$|J| = |\mathbf{v}(t_s) \times \mathbf{v}(t_s+t_l)| = v^2 \sin\alpha(t_s, t_s+t_l). \tag{2.40}$$

As the derivation [1] of the formula for $P(\epsilon, T)$ for a chaotic system, starting from the formal expression (2.39), is instructive to see how information can be extracted from the basic principles of chaotic dynamics beyond the diagonal approximation, we will present it here in detail. Hyperbolicity will yield a justification for the minimal time $T_{\min}(\epsilon)$ already introduced in Eq. (2.39); we will come back to that point later and first study the effect of ergodicity. To proceed we interchange the phase-space integral of the average with the time integrals, substitute $(\mathbf{r}(t_s), \mathbf{p}(t_s)) \longmapsto (\mathbf{r}_0, \mathbf{p}_0)$ in Eq. (2.39) and obtain

$$P(\epsilon, T) = 2m \int_{T_{\min}(\epsilon)}^{T-T_{\min}(\epsilon)} dt_l v^2 \sin\epsilon\, p_E(\epsilon, t_l)\, (T - t_l - T_{\min}(\epsilon)), \tag{2.41}$$

with the averaged classical return probability density

$$p_E(\epsilon, t_l) = \frac{1}{2m} \langle \delta(\mathbf{r}_0 - \mathbf{r}(t_l))\, \delta(\epsilon - |\measuredangle(\mathbf{v}_0, \mathbf{v}(t_l))|) \rangle. \tag{2.42}$$

This yields the probability density that a particle possessing the energy E returns after the time t_l to its starting point with the angle $|\measuredangle(\mathbf{v}_0, \mathbf{v}(t_l))| = \epsilon$. For long times this can be replaced by $1/\Omega(E)$, assuming ergodicity. Then we obtain

$$P(\epsilon, T) \approx 2m \int_{T_{\min}(\epsilon)}^{T-T_{\min}(\epsilon)} dt_l v^2 \sin\epsilon \frac{1}{\Omega(E)} (T - t_l - T_{\min}(\epsilon)) = \frac{mv^2}{\Omega(E)} \sin\epsilon (T - 2T_{\min}(\epsilon))^2 . \tag{2.43}$$

Now we return to our assumption of hyperbolicity and explain the cutoff time $T_{\min}(\epsilon)$, introduced in the equations above. To this end, we consider two classical paths leaving their crossing with a small angle ϵ. The initial deviation of their velocities is $\delta v_i \approx \epsilon v$. In order to form a closed loop, the deviation of the velocities δv_f, when both paths have traversed half of the closed loop, has to be given by $\delta v_f = cv$ with c of the order unity. Then we get for the minimal time $T_{\min}(\epsilon)$ to form a closed loop, due to the maximally exponential divergence of neighboring orbits for $T_{\min}(\epsilon) \to \infty$,

$$c \sim \epsilon e^{(\lambda T_{\min}(\epsilon))/2}, \tag{2.44}$$

implying

$$T_{\min}(\epsilon) \sim \frac{2}{\lambda} \ln\left(\frac{c}{\epsilon}\right). \tag{2.45}$$

An argument similar to the one used here for the closed loop can also be applied to the other two parts of the trajectory leaving the crossing with an angle ϵ towards the opening of the conductor. Suppose t_s in Eq. (2.39) has a length between 0 and $T_{\min}(\epsilon)/2$, then both parts have to be so close together that they must leave both through the same lead. As we are interested in the transmission, we also have to exclude in that case a length of t_s between 0 and $T_{\min}(\epsilon)/2$. A similar argument holds for the case where the last part of the orbit has a length between 0 and $T_{\min}(\epsilon)/2$, in this case the orbit has to come very close to the opening already before the crossing and leave before it could have crossed. Accounting for all these restrictions, gives the integration limits in Eq. (2.39).

We want to remark here that the restrictions of the time integrals given above only hold for the transmission T. If we would calculate the reflection $R \equiv \sum_{\alpha,\beta=1}^{N_1} |r_{\alpha,\beta}(k)|^2$ with the reflection elements of the scattering matrix $r_{\alpha,\beta}(k)$, the effect of short legs—to the corresponding contribution is referred as coherent backscattering—has to be taken into account, see for example [23, 30]. This contribution was actually the first off-diagonal contribution that could be calculated semiclassically and that is changed by a magnetic field as the orbit and its partner enclose non-zero flux, because both traverse the loop in different directions. In the articles [23, 30] this contribution was then used to explain the Lorentzian lineshap of the reflection in dependence of the magnetic field for weak magnetic fields that do

not change the shape of the the trajectories significantly. Via the current conservation relation

$$T + R = N, \tag{2.46}$$

that follows from the unitarity of the scattering matrix—for a proof of the latter relation see for example [26, 31]—the lineshape of the transmission T was then predicted semiclassically.

Now we are prepared to calculate the contribution of the considered trajectory pairs to the transmission. Therefore we keep in Eq. (2.31) one sum over trajectories that we will perform using the same classical sum rule as in diagonal approximation, the other we can replace by a sum over all the partner trajectories of one trajectory, which can be calculated using $P(\epsilon, T)$. There is, however, one subtlety concerning the survival probability $\rho(T)$ in the sum rule: We argued already, that if the crossing happens near the opening, both parts of the orbit act in a correlated way; $\rho(T)$ is changed in the case of the trajectory pairs considered here for a similar reason: Because we know that the two parts of the orbit leaving the crossing on each side are very close to each other, the orbit can either leave the cavity during the first stretch, i.e. during the first time it traverses the crossing region, or cannot leave at all. This implies that we have to change[2] the survival probability from $\rho(T)$ to $\rho(T - T_{\min}(\epsilon))$. Then we arrive at the loop (L) contribution

$$\begin{aligned}
\left|t_{\alpha,\beta}(k)\right|^2_{\mathrm{L}} &= \frac{\pi\hbar}{2W_1W_2}\sum_{\gamma}\sum_{P}\left|D_\gamma\right|^2 2\Re\exp\left(i\frac{p^2\epsilon_P^2}{2m\lambda\hbar}\right)\\
&= \frac{4\pi\hbar}{\Omega(E)}\int_0^{\pi} d\epsilon\int_{2T_{\min}(\epsilon)}^{\infty} dT\,\mathrm{e}^{-(T-T_{\min}(\epsilon))/\tau_D}\,P(\epsilon, T)\cos\left(\frac{p^2\epsilon^2}{2m\lambda\hbar}\right)\\
&= \frac{8\pi\hbar m v^2\tau_D^3}{\Omega(E)^2}\int_0^{\pi} d\epsilon\,\mathrm{e}^{-T_{\min}(\epsilon)/\tau_D}\sin\epsilon\cos\left(\frac{p^2\epsilon^2}{2m\lambda\hbar}\right)
\end{aligned} \tag{2.47}$$

with the sum over the partner trajectories P in the first line. As the important contributions require very small action differences, i.e. very similar trajectories, and as the prefactor D_γ is not as sensitive as the actions to small changes of the trajectories, we can neglect differences between γ and γ' in the prefactor.[3] In the second line we

[2] This effect together with the requirement of a finite length of the orbit parts leaving towards the opening was originally not taken into account in Ref. [24]. In that calculation, the contributions from these two effects cancel each other, they will be only important when considering more complicated diagrams as in the next subsection.

[3] We assumed here that the two trajectories possess the same Maslov index: Although this was not yet shown for pairs of open orbits considered here there are several reasons why the Maslov indices can be assumed not to influence properties of chaotic systems showing behaviour predicted by RMT: For special chaotic systems like surfaces of constant negative curvature the Maslov indices do not depend on the specific orbit and thus drop out in Eq. (2.47). Furthermore the following interpretation is helpful: The Maslov index is given by the number of times the stable and unstable manifolds rotate by half a turn plus twice the number of reflections on walls with Dirichlet boundary

applied the classical sum rule with the modification explained before Eq. (2.47) and used $P(\epsilon, T)$ to evaluate the sum over P. After performing the simple time integral in the third line, we can do the ϵ-integration as for example in Ref. [34] by taking into account that the important contributions come from very small ϵ, yielding

$$\begin{aligned}\left|t_{\alpha,\beta}(k)\right|_{\mathrm{L}}^2 &= \frac{8\pi\hbar m v^2 \tau_D^3}{\Omega(E)^2}\int_0^\pi d\epsilon (\epsilon/c)^{\frac{2}{\lambda\tau_D}} \sin\epsilon \cos\left(\frac{p^2\epsilon^2}{2m\lambda\hbar}\right)\\ &= \frac{8\pi\hbar m v^2 \tau_D^3}{\Omega(E)^2}\int_0^\infty dz \frac{m\lambda\hbar}{p^2}\left(\frac{1}{c}\right)^{\left(\frac{2}{\lambda\tau_D}\right)}\left(\frac{2m\lambda\hbar z}{p^2}\right)^{\frac{1}{\lambda\tau_D}}\cos z\\ &= -\frac{8\pi\hbar m v^2 \tau_D^2}{\Omega(E)^2}\int_0^\infty dz \frac{m\hbar}{p^2}\left(\frac{1}{c}\right)^{\left(\frac{2}{\lambda\tau_D}\right)}\left(\frac{2m\lambda\hbar z}{p^2}\right)^{\frac{1}{\lambda\tau_D}}\frac{\sin z}{z}. \end{aligned} \tag{2.48}$$

In the first line we already rewrote $e^{-T_{\min}(\epsilon)/\tau_D}$ as $(\epsilon/c)^{\frac{2}{\lambda\tau_D}}$, and in the second line we approximated $\sin\epsilon \approx \epsilon$ and substituted $z = p^2\epsilon^2/(2m\lambda\hbar)$. Then we perform a partial integration with respect to z neglecting rapidly oscillating terms that are cancelled by the k-average, introduced after Eq. (2.32). Eventually, we perform the z-integral by pushing the upper limit to infinity, i.e. $\hbar \to 0$ and taking into account our assumption of large dwell times, i.e. $\lambda\tau_D \to \infty$. Additionally we assume $\left(2m\lambda\hbar/p^2\right)^{\frac{1}{\lambda\tau_D}} \approx 1$; we will return to the last point in Chap. 4

Finally, we arrive at the leading non-diagonal contribution to the quantum transmission [24],

$$\left|t_{\alpha,\beta}(k)\right|_{\mathrm{L}}^2 = -\frac{1}{(N_1+N_2)^2}, \tag{2.49}$$

that agrees with the $k = 1$-term in the RMT expression given in (2.37).

2.3.2.1 Magnetic Field Dependence of the Non-diagonal Contribution

Up to now we assumed time-reversal symmetry. If this symmetry is destroyed, for example by applying a strong magnetic field, the latter contribution will vanish, because the closed loop has to be traversed in different directions by the trajectory and its partner. Here we study the transition region between zero and weak magnetic field. In particular, we consider a homogeneous magnetic field B_z perpendicular to the sample that is assumed weak enough not to change the classical trajectories, but

Footnote 3 (continued)
conditions [32]. This yields for periodic orbits that the Maslov indices of the orbit and its partner are equal: First both are classically very similar and second the manifolds have to return to their original positions for the orbits to become periodic. These facts exclude that stable and unstable manifolds of orbit and partner differ by at least half a turn. For open orbits similar arguments were applied in [33]. In the following we will thus always assume that there are no additional contributions due to Maslov phases.

only the actions in the exponents. Since the closed loop is traversed in different directions by the two trajectories, we obtain an additional phase difference $(4\pi A B_z/\phi_0)$ between the two trajectories with the enclosed area A of the loop and the flux quantum $\phi_0 = (hc/e)$. We further need the distribution of enclosed areas for a trajectory with a closed loop of time T in chaotic systems, given by

$$P(A, T) = \frac{1}{\sqrt{2\pi T\beta'}} \exp\left(-\frac{A^2}{2T\beta'}\right) \tag{2.50}$$

with a system specific parameter β'. A derivation of this formula can be found for example in Ref. [35]. Equation (2.50) can be interpreted in the following way [36]: Long orbits accumulate their enclosed areas in a random way leading due to the central limit theorem to a Gaussian area distribution. Including the phase difference due to the magnetic field and the area distribution in a modified $P(\epsilon, T)$ yields

$$\begin{aligned} P_B(\epsilon, T) &\approx \frac{2mv^2}{\Omega(E)} \sin\epsilon \int_{T_{\min}(\epsilon)}^{T-T_{\min}(\epsilon)} dt_l \, (T - t_l - T_{\min}(\epsilon)) \\ &\quad \times \int_{-\infty}^{\infty} dA \, P(A, t_l - T_{\min}(\epsilon)) \cos\frac{4\pi A B_z}{\phi_0} \\ &= \frac{2mv^2}{\Omega(E)} \sin\epsilon \int_{T_{\min}(\epsilon)}^{T-T_{\min}(\epsilon)} dt_l \, (T - t_l - T_{\min}(\epsilon)) \, \mathrm{e}^{-(t_l - T_{\min}(\epsilon))/t_B} \end{aligned} \tag{2.51}$$

with $t_B = \frac{\phi_0^2}{8\pi^2\beta' B_z^2}$. In the first step we used that paths leaving the crossing to form a closed loop, enclose a negligible flux, as long as they are correlated; for a more detailed analysis see Appendix D of Ref. [37]. Performing the T- and ϵ-integrals similar to the case without magnetic field, yields [24]

$$\left|t_{\alpha,\beta}(k, B_z)\right|_{\mathrm{L}}^2 = -\frac{1}{(N_1 + N_2)^2} \frac{1}{1 + \tau_D/t_B}. \tag{2.52}$$

We obtain in dependence of B_z an inverted Lorentzian with minimum at zero magnetic field, implying that the transmission through our sample increases with increasing magnetic field. This weak-localisation phenomenon is visible as the reduction of the average quantum transmission in Fig. 2.3.

After this introduction into the semiclassical methods for the evaluation of non-diagonal contributions in configuration space, we will now turn to the generalisation to phase space which also allows for an elegant way to compute higher-order corrections in $1/N$ than the leading weak-localisation contribution presented above.

2.3.3 Quantum Transmission: Phase-Space Approach

The above configuration-space treatment, based on self-crossings, is restriced to systems with two degrees of freedom. For higher-dimensional, dynamical chaotic systems one cannot assume to find a one-to-one correspondence between partner

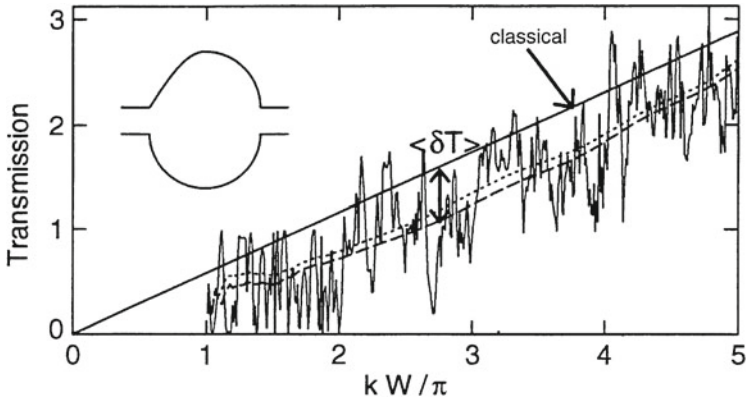

Fig. 2.3 Transmission of the depicted geometry as a function of kW/π for $W_1 = W_2 \equiv W$. The *straight full line* is the resulting transmission in diagonal approximation, the *fluctuating full line* the numerically determined quantum transmission. The *dashed* and the *dotted curve* are smoothed curves for the quantum transmission without and with a weak magnetic field, respectively (taken from [23]). Note the large fluctuations around these curves. They are universal in chaotic systems and referred to as universal conductance fluctuations. We will come back to their semiclassical description in Chap. 4

orbits and crossings of an orbit [3, 4]. In order to overcome these difficulties, a phase-space approach was developed for calculating the spectral form factor in spectral statistics in [3–5] involving periodic orbits. The next challenge was the generalisation of this theory to trajectory pairs differing from each other at several places, solved again first for the spectral form factor [6] and generalised to the transport situation considered here in Ref. [38] which serves as the basis of the following discussion.

We now first explain the phase-space approach and use it afterwards in the way developed in Refs. [6, 38] for the calculation of the quantum transmission involving also higher-order semiclassical diagrams.[4]

2.3.3.1 Phase-Space Approach

Compared to the last subsection, we first switch the role of the reference orbit. Whereas we used there the crossing orbit as reference and calculated then the action difference and the crossing-angle distribution in terms of the crossing angle ϵ, we will consider here the orbit without the crossing that is close to itself in the encounter region, where the partner orbit crossed in configuration space itself in the last subsection. To the region, where the orbit is close to itself, we will refer as encounter region and to the parts of the orbit inside as encounter stretches. These are connected by so-called links. Imagining this encounter in phase space, we place

[4] We restrict for simplicity of presentation to two-dimensional systems with one constant Lyapunov exponent, generalisations can be found for example in [39, 40].

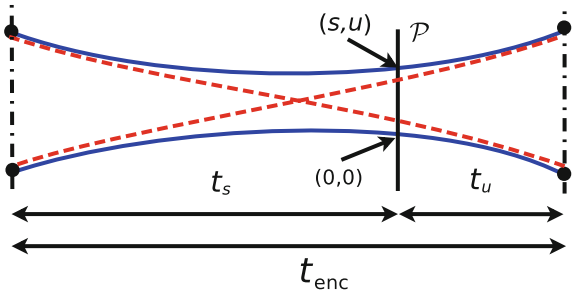

Fig. 2.4 An encounter with the ingredients considered in the phase-space approach: The duration of the encounter is given by t_{enc}, the sum of the times t_s and t_u traversed on the *left* and on the *right* of a Poincaré surface of section $\mathcal{P}$, respectively. The piercing points of the trajectory indicated by the fat (*blue*) line are given by $(0, 0)$ and (s, u). The partner trajectory is indicated by the *dashed* (*red*) line

a Poincaré surface of section $\mathcal{P}$ with its origin at $\tilde{x} = (\mathbf{r}, \mathbf{p})$ inside this region. The section consists of all points $\tilde{x} + \delta\tilde{x} = (\mathbf{r} + \delta\mathbf{r}, \mathbf{p} + \delta\mathbf{p})$ in the same energy shell as the reference point with the $\delta\mathbf{r}$ perpendicular to the momentum $\mathbf{p}$ of the trajectory. For the two-dimensional systems mostly considered here $\mathcal{P}$ is a two-dimensional surface, where every vector $\delta\tilde{x}$ can be expressed in terms of the stable direction $e_s\,(\tilde{x})$ and the unstable one $e_u\,(\tilde{x})$ [41]

$$\delta\tilde{x} = s e_s\,(\tilde{x}) + u e_u\,(\tilde{x}). \tag{2.53}$$

The expressions stable and unstable refer to the following: Consider two orbits, one starting at $\tilde{x}$ and the other one at $\tilde{x} + \delta\tilde{x}_0$, then the difference between the stable coordinates will decrease exponentially for positive times and increase for negative time exponentially in the limit of long time T, unstable coordinates behave just the other way round. The functional form of the exponentials can be written as $\mathrm{e}^{\lambda T}$ and $\mathrm{e}^{-\lambda T}$. Now we can come back to the trajectory with the encounter region, where we have put the Poincaré surface of section. The reference trajectory considered here, the fat (blue) line in Fig. 2.4, will pierce the Poincaré section twice: One of these points we will consider as the origin of the section the other piercing will take place at the distance (s, u), see Fig. 2.4. The coordinates of the piercing points of the partner trajectory, the dashed (red) one in Fig. 2.4 are determined in the following way: The unstable coordinate of the partner trajectory has to be the same as the one of that part of the first trajectory that the partner will follow for positive times. The stable coordinate is determined by the same requirement for negative times.

After this introduction into the determination of the (s, u)-coordinates, we are now ready to treat trajectory pairs that differ in encounters of arbitrary complexity. Following Refs. [6, 39, 40] and using the notation introduced there we will allow here that the two trajectories differ in arbitrarily many encounters involving an arbitrary number of stretches. In order to organise this encounter structure we introduce a vector $\mathbf{v} = (v_2, v_3, \ldots)$ with the component v_l determining the number of encounters

with l stretches involved. The overall number of encounters during an orbit will be denoted by $V = \sum_{l=2}^{\infty} v_l$, the overall number of encounter stretches by $L = \sum_{l=2}^{\infty} l v_l$. In an encounter of l stretches, we will get $l-1$ (s,u)-coordinates.

Now we can proceed by replacing the former expressions for the minimal loop time, the action difference and the crossing-angle distribution depending on the former small parameter ϵ, by the corresponding expressions depending on the new small parameters (s,u).

We start with the minimal loop time, to that one also refers as the duration of the encounter as the two parts of the trajectory are close, i.e. linearisable, to each other during that time, as we saw in the last subsection. Shifting the Poincaré surface of section through the encounter, the stable components will asymptotically decrease, the unstable ones will increase for increasing time. We then claim that both components have to be smaller than a classical constant c, giving the upper bound for the two trajectories to be linearisable. Its exact value will again, as in the last subsection, be unimportant for our final results. We then obtain the encounter duration t_{enc} as the sum of the times t_u, that the trajectory needs from $\mathcal{P}$ till the point where the first unstable component reaches c, and the time t_s, that the trajectory needs from $\mathcal{P}$ till the point where the last stable component falls below c, for an illustration see Fig. 2.4. Thus we get for $t_{\text{enc}} \to \infty$

$$t_{\text{enc}} = t_s + t_u \sim \frac{1}{\lambda} \ln \frac{c^2}{\max_i \{|s_i|\} \max_j \{|u_j|\}}. \tag{2.54}$$

Now we treat the action difference between the two trajectories. Expressing the actions of the paired trajectories as the line integral of the momentum along the trajectory as defined after Eq. (2.19), we can expand [3, 4] one action around the other by linearising the motion on the two trajectories around each other and express the result in terms of the (s,u)-coordinates. This yields for an l-encounter[5]

$$\Delta S \approx \sum_{j=1}^{l-1} u_j s_j. \tag{2.55}$$

The action difference of a trajectory pair is then obtained by adding the differences resulting from all encounters.

Finally we come to the crossing-angle distribution that will be replaced here by a weight function for the stable and unstable coordinates for a trajectory of time T. We first notice that the uniformity of the trajectory distribution implies in terms of our coordinates in $\mathcal{P}$ that a trajectory pierces through the section with the coordinates $(u, u+du)$ and $(s, s+ds)$ within the time interval $(t, t+dt)$ with the probability $ds du dt/\Omega(E)$. In general, we obtain for an l-encounter $ds^{l-1} du^{l-1} dt^{l-1} \, (1/\Omega(E))^{l-1}$. Integrating the product of the latter quantities for

[5] Strictly speaking [6] the (s,u)-coordinates used here and in the following calculation are for encounters involving more than two stretches *not* the same as the ones described before, but related to them via a linear and volume preserving transformation.

all encounters over all possible durations of the $L-V$ intra-encounter links, in a way that their durations are positive yields our weight function for a fixed position of $\mathcal{P}$. To take into account all possible positions of $\mathcal{P}$, we also integrate over all possible positions where it can be placed, and divide by t_{enc} to avoid overcounting of equivalent positions. Taking all link times positive, we obtain for the weight function[6]

$$w(\boldsymbol{u}, \boldsymbol{s}, T) \approx \frac{1}{\Omega^{L-V} \prod_{\alpha=1}^{V} t_{\text{enc}}^{\alpha}} \int_0^\infty dt_1 \ldots dt_L \Theta\left(T - \sum_{\alpha=1}^{V} l_\alpha t_{\text{enc}}^{\alpha} - \sum_{\alpha=1}^{L} t_\alpha\right)$$
$$= \frac{\left(T - \sum_{\alpha=1}^{V} l_\alpha t_{\text{enc}}^{\alpha}\right)^L}{L! \Omega^{L-V} \prod_{\alpha=1}^{V} t_{\text{enc}}^{\alpha}}. \tag{2.56}$$

with the Heaviside theta function $\Theta(x)$.

One additional problem arises, when treating trajectory pairs differing not only in one 2-encounter like in the last subsection: One can construct for one $\mathbf{v}$ different trajectory pairs, varying for example in the relative orientation or order in which the encounter stretches are traversed. We will count this number by a function $N(\mathbf{v})$ and describe briefly later in this section how it can be calculated.

2.3.3.2 Calculation of the Full Transmission

After the introduction into the phase-space approach we are now ready to calculate the contributions of orbit pairs differing in an arbitrary number of encounters of arbitrary size to the transmission. Taking the weight function, the action difference and the number of structures, we can transform the off-diagonal part of the summands in Eq. (2.31) into

$$\left|t_{\alpha,\beta}(k)\right|_{\text{off}}^2 = \frac{\pi\hbar}{2W_1 W_2} \sum_\gamma \sum_{\mathbf{v}} \left|D_\gamma\right|^2 N(\mathbf{v})$$
$$\times \left\langle \int_{-c}^{c} d\boldsymbol{s} d\boldsymbol{u} \exp\left(\frac{i}{\hbar} \Delta S\right) w(\boldsymbol{u}, \boldsymbol{s}, T) \right\rangle_{\Delta k} \tag{2.57}$$

with the average over a small k-window denoted by $\langle \ldots \rangle_{\Delta k}$. Inserting the formulas for the action difference (2.55), the weight function (2.56) and using the classical sum rule with the modification of the survival probability due to encounters, discussed in the last subsection, we can transform [38] the integral with respect to the length of the trajectory into one over the last link and obtain

[6] A derivation of this expression starting from a formula analogous to (2.39) and using the mixing property can be found in Appendix B of [39].

$$\left|t_{\alpha\beta}(k)\right|^2_{\text{off}} = \frac{2\pi\hbar}{\Omega}\sum_{\mathbf{v}} N(\mathbf{v})\left(\prod_{i=1}^{L+1}\int_0^\infty dt_i \exp\left(-\frac{t_i}{\tau_D}\right)\right)$$
$$\times\left\langle \int_{-c}^{c}\frac{ds\,d\boldsymbol{u}}{\Omega^{L-V}}\prod_{\alpha=1}^{V}\frac{\exp\left(-\frac{t^\alpha_{\text{enc}}}{\tau_D}+\frac{i}{\hbar}\sum_{j=1}^{l_\alpha-1}u_{\alpha j}s_{\alpha j}\right)}{t^\alpha_{\text{enc}}}\right\rangle_{\Delta k}$$
$$= \frac{1}{N_1+N_2}\sum_{n=1}^{\infty}\left(\frac{1}{N_1+N_2}\right)^n \sum_{\mathbf{v}}^{L-V=n}(-1)^V N(\mathbf{v}) \tag{2.58}$$

with the $L+1$ link times t_i. The (s,u)-integrals are calculated using the rule [6], that after expanding the exponential $e^{t_{\text{enc}}/\tau_D}$ into a Taylor series, only the t_{enc}-independent term contributes and yields for $\hbar \to 0$

$$\left\langle \frac{1}{\Omega}\int_{-c}^{c} ds\,du \exp\left(\frac{ius}{\hbar}\right)\right\rangle_{\Delta k} \sim \frac{1}{T_H} \tag{2.59}$$

with the Heisenberg time $T_H = \Omega/(2\pi\hbar) = 2\pi\hbar\bar{d}(E)$. Due to the last relation T_H can be, following the Heisenberg uncertainty principle, regarded as the time needed to resolve energy levels separated by distances of the order of the mean level spacing $\Delta(E) = 1/\bar{d}(E)$.

For the sum with respect to $\mathbf{v}$, one can derive recursion relations, yielding [6, 38]

$$\sum_{\mathbf{v}}^{L-V=n}(-1)^V N(\mathbf{v}) = \left(1-\frac{2}{\beta}\right)^n \tag{2.60}$$

with $\beta = 1$ and $\beta = 2$ for the case with and without time-reversal symmetry, respectively. Relations of this kind are derived from recursion relations obtained by expressing the connections inside and between encounters by permutations and considering the effect of shrinking one link in an arbitrarily complicated structure to zero.

We then obtain for $T(k)$ in the case with time-reversal symmetry [38]

$$T(k)^{\beta=1} \approx \frac{N_1N_2}{N_1+N_2} + \frac{N_1N_2}{N_1+N_2}\sum_{n=1}^{\infty}\left(\frac{-1}{N_1+N_2}\right)^n = \frac{N_1N_2}{N_1+N_2+1} \tag{2.61}$$

and in the case without time-reversal symmetry

$$T(k)^{\beta=2} \approx \frac{N_1N_2}{N_1+N_2}, \tag{2.62}$$

which agrees with the diagonal contribution, already obtained in the first subsection of this section. Both results are in agreement with RMT predictions [28] given in (2.36) and (2.37).

2.4 Spectral Statistics

After this introduction to transport through open systems we now want to analyse the spectral properties of chaotic systems. We first introduce the quantities usually studied in this context and afterwards show how the spectral form factor, for its definition see below, is calculated semiclassically within the phase-space approach.

In the context of spectral statistics often the autocorrelator of two spectral densities $d(E)$ (we will come to another quantity characterising the spectral properties of a chaotic system in Chap. 5) is considered, defined as

$$C(\epsilon) = \frac{1}{\bar{d}^2}\left\langle d\left(E + \frac{\epsilon}{2\pi\bar{d}}\right) d\left(E - \frac{\epsilon}{2\pi\bar{d}}\right)\right\rangle_{\Delta k} - 1. \tag{2.63}$$

The (-1) subtracts the contributions due to the mean spectral densities $\overline{d}$ contained in the first term. The function $C(\epsilon)$ can be expressed semiclassically using (2.23) and linearising the actions around the mean energy E, $S_\gamma\left(E + \epsilon/(2\pi\bar{d})\right) \approx S_\gamma(E) + \epsilon\hbar T_\gamma/T_H$, as

$$C_{\rm sc}(\epsilon) = \Re\frac{2}{T_H^2}\left\langle \sum_{\gamma,\gamma'} T_\gamma^{\rm prim} T_{\gamma'}^{\rm prim} C_\gamma C_{\gamma'} {\rm e}^{(i/\hbar)\left(S_\gamma - S_{\gamma'}\right) + (i\epsilon/T_H)\left(T_\gamma + T_{\gamma'}\right) - i(\pi/2)\left(\tau_\gamma - \tau_{\gamma'}\right)}\right\rangle_{\Delta k}. \tag{2.64}$$

We also define

$$R(\epsilon) = \frac{1}{4\pi^2\bar{d}^2}\left\langle {\rm Tr} G^{+{\rm osc}}\left(E + \frac{\epsilon}{2\pi\bar{d}}\right) {\rm Tr} G^{-{\rm osc}}\left(E - \frac{\epsilon}{2\pi\bar{d}}\right)\right\rangle_{\Delta k}, \tag{2.65}$$

expressed by using ${\rm Tr} G^-(E) = \left[{\rm Tr} G^+(E)\right]^*$ and Eq. (2.22) semiclassically as

$$R_{\rm sc}(\epsilon) = \frac{1}{T_H^2}\left\langle \sum_{\gamma,\gamma'} T_\gamma^{\rm prim} T_{\gamma'}^{\rm prim} C_\gamma C_{\gamma'} {\rm e}^{(i/\hbar)\left(S_\gamma - S_{\gamma'}\right) + (i\epsilon/T_H)\left(T_\gamma + T_{\gamma'}\right) - i(\pi/2)\left(\tau_\gamma - \tau_{\gamma'}\right)}\right\rangle_{\Delta k}. \tag{2.66}$$

Often also the corresponding Fourier transform, the spectral form factor is considered

$$K(\tau) = \frac{1}{\pi}\left\langle \int_{-\infty}^{\infty} d\epsilon {\rm e}^{-2i\epsilon\tau} R(\epsilon)\right\rangle_{\Delta k, \Delta\tau}, \tag{2.67}$$

that we want to study here.

The RMT results for the latter quantity are

$$\begin{aligned} K_{\mathrm{GUE}}(\tau) &= \tau, \quad \text{for } \tau \leq 1 \\ K_{\mathrm{GUE}}(\tau) &= 1, \quad \text{for } \tau > 1 \end{aligned} \tag{2.68}$$

in the case of the Gaussian unitary ensemble and

$$\begin{aligned} K_{\mathrm{GOE}}(\tau) &= 2\tau - \tau \ln(1+2\tau) = 2\tau - 2\tau^2 + 2\tau^3 + \mathcal{O}\left(\tau^4\right), \quad \text{for } \tau \leq 1 \\ K_{\mathrm{GOE}}(\tau) &= 2 - \tau \ln \frac{(2\tau+1)}{(2\tau-1)}, \quad \text{for } \tau > 1 \end{aligned} \tag{2.69}$$

in the case of the Gaussian orthogonal ensemble.

A semiclassical expression for $K(\tau)$ can be obtained by inserting (2.66) in Eq. (2.67) and performing the ϵ-integral

$$\begin{aligned} K_{\mathrm{sc}}(\tau) = \Bigg\langle \frac{1}{T_H} \sum_{\gamma,\gamma'} T_\gamma^{\mathrm{prim}} T_{\gamma'}^{\mathrm{prim}} C_\gamma C_{\gamma'} \mathrm{e}^{(i/\hbar)\left(S_\gamma - S_{\gamma'}\right) - i(\pi/2)\left(\tau_\gamma - \tau_{\gamma'}\right)} \\ \times \delta\left(\tau T_H - \frac{T_\gamma + T_{\gamma'}}{2}\right) \Bigg\rangle_{\Delta k, \Delta\tau}. \end{aligned} \tag{2.70}$$

As already in the last section we obtain a double sum over trajectories, however now periodic orbits are considered instead of lead-connecting paths like in the last section. We will now proceed with calculating the contributions to (2.70) surviving the averaging. In diagonal approximation we obtain the contribution $K_{\mathrm{diag}}(\tau)$

$$K_{\mathrm{diag}}(\tau) = \frac{1}{T_H} \sum_\gamma \left(T_\gamma^{\mathrm{prim}}\right)^2 \left|C_\gamma\right|^2 \delta\left(\tau T_H - T_\gamma\right). \tag{2.71}$$

The sum over orbits in the last equation is performed with the Hannay and Ozorio de Almeida sum rule [42]

$$\sum_\gamma \left(T_\gamma^{\mathrm{prim}}\right)^2 \left|C_\gamma\right|^2 \delta\left(\tau T_H - T_\gamma\right) \approx \tau T_H, \tag{2.72}$$

that can be derived in a similar way like (2.34) using ergodicity yielding finally [43]

$$K_{\mathrm{diag}}(\tau) = \tau \tag{2.73}$$

and twice the latter contribution in the presence of time-reversal symmetry, because then we can additionally pair each orbit with its time-reversed counterpart.

We want to show now that the semiclassically calculated off-diagonal contributions to $K(\tau)$ are in accordance with the results (2.68, 2.69) from RMT. We will as for the introduction to the phase-space approach follow [6, 39, 40] and use the notation introduced there. The calculation is performed within the phase-space approach. We introduced for the transmission in the last section

- the duration of an encounter $t_{\text{enc}}(\boldsymbol{u}, \boldsymbol{s})$,
- the weight function $w(\boldsymbol{u}, \boldsymbol{s}, T)$ measuring the density of encounters,
- the action difference of the orbit and its partner ΔS,
- the function $N(\mathbf{v})$ characterising the number of possible orbits for a vector $\mathbf{v}$.

Compared to the last section the only change in these quantities is that the weight function is altered due to the fact that here periodic orbits are considered. It is now given by

$$w(\boldsymbol{u}, \boldsymbol{s}, T) \approx \frac{T\left(T - \sum_{\alpha=1}^{V} l_\alpha t_{\text{enc}}^\alpha\right)^{L-1}}{(L-1)!\Omega^{L-V}\prod_{\alpha=1}^{V} t_{\text{enc}}^\alpha}. \tag{2.74}$$

The factor T not appearing in the case of open trajectories, Eq. (2.56), results here from the possibility that the final point of the first encounter stretch can lie everywhere on a *periodic* orbit. This was not possible on an *open* orbit. With these quantities we obtain for the off-diagonal contributions to the spectral form factor $K(\tau)$ after applying the sum rule (2.72) [6, 39, 40]

$$K_{\text{off}}(\tau) = \frac{2}{\beta}\tau\left\langle \sum_{\mathbf{v}} \frac{N(\mathbf{v})}{L}\int_{-c}^{c} d\boldsymbol{s}d\boldsymbol{u}\, w(\boldsymbol{u}, \boldsymbol{s}, \tau T_H)\, \mathrm{e}^{(i/\hbar)\Delta S}\right\rangle_{\Delta k, \Delta\tau} \tag{2.75}$$

with $\beta = 2$ in the unitary and $\beta = 1$ in the orthogonal case. The factor $\frac{1}{L}$ avoids overcounting of equivalent diagrams: In contrast to an open orbit for a periodic orbit each encounter stretch can be singled out to be the first one. All these contributions lead however to the same contribution to (2.75) that should only be counted once. As already mentioned before Eq. (2.59) only terms of $w(\boldsymbol{u}, \boldsymbol{s}, T)$ contribute to the $\boldsymbol{s}, \boldsymbol{u}$-integrals that are independent of t_{enc}. We consider in (2.74) only those terms, we thus replace

$$w(\boldsymbol{u}, \boldsymbol{s}, T) \to \left(\frac{T}{\Omega}\right)^{L-V} \frac{(-1)^V \prod_{\alpha=1}^{V} l_\alpha}{(L-V-1)!}, \tag{2.76}$$

where $\prod_{\alpha=1}^{V} l_\alpha$ can be transformed into $\prod_{l=2}^{\infty} l^{v_l}$ by noting that l_α depends only on the number l of encounter stretches in an encounter. This yields together with (2.59) for $K_{\text{off}}(\tau)$

$$K_{\text{off}}(\tau) = \frac{2}{\beta}\tau\sum_{\mathbf{v}} N(\mathbf{v})\,\tau^{L-V}\frac{(-1)^V\prod_{l=2}^{\infty} l^{v_l}}{L(L-V-1)!} \equiv \frac{2}{\beta(n-2)!}\sum_{n=2}^{\infty}\sum_{\mathbf{v}}^{L-V+1=n}\widetilde{N}(\mathbf{v})\,\tau^n \tag{2.77}$$

with

$$\widetilde{N}(\mathbf{v}) \equiv N(\mathbf{v})\frac{(-1)^V\prod_{l=2}^{\infty} l^{v_l}}{L}. \tag{2.78}$$

The function $K_{\text{off}}(\tau)$ can thus be expressed as

$$K_{\text{off}}(\tau) = \frac{2}{\beta} \sum_{n=2}^{\infty} K_n \tau^n \tag{2.79}$$

with

$$K_n \equiv \frac{1}{(n-2)!} \sum_{\mathbf{v}}^{L-V+1=n} \widetilde{N}(\mathbf{v}) \,. \tag{2.80}$$

The sum over $\mathbf{v}$ is like in the case of the transmission in the last section performed with the help of recursion relations for the contributions for different n, for a derivation see e.g. [39, 40], in the unitary case this yields

$$K_n = 0 \tag{2.81}$$

and in the orthogonal case

$$K_n = \frac{(-2)^{n-1}}{n-1}. \tag{2.82}$$

Both results are consistent with the RMT predictions (2.68) and (2.69) for $\tau \leq 1$. This is obvious for the unitary case, while in the orthogonal case it is obtained by comparing the τ-expansions of the semiclassical and RMT result for $\tau \leq 1$. This semiclassical theory we have described is up to now however not able to explain the correlations that lead to the behaviour of $K(\tau)$ for $\tau > 1$; we will come back to this point in Chap. 5.

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http://www.springer.com/978-3-642-24527-5

Semiclassical Approach to Mesoscopic Systems
Classical Trajectory Correlations and Wave Interference
Waltner, D.
2012, X, 182 p., Hardcover
ISBN: 978-3-642-24527-5