Chapter 2
Rheological Models: Integral and Differential Representations

Viscoelastic relations may be expressed in both integral and differential forms. Integral forms are very general and appropriate for theoretical work. Differential forms are related to rheological models that provide a more direct physical interpretation of viscoelastic behavior. In this chapter we describe the most usual rheological models, deduce their differential equations and, by solving them, we find the corresponding integral representations. These relations will be set in a more computational friendly form in Chap. 3 and extended to three-dimensional situations in Chap. 4 and then used in analytical and computational solutions.

2.1 General Integral Relations

When the functional relation (1.5) in Chap. 1 is linear it has a simple and useful representation given by the Riesz theorem [1]: if the functional D is *linear* and *equi-continuous*, it can be written

$$\varepsilon(t) = \int_{\tau_0}^{t} D(t-\tau)d\sigma(\tau) \tag{2.1}$$

or

$$\varepsilon(t) = \int_{\tau_0}^{t} D(t-\tau)\dot{\sigma}(\tau)d\tau \tag{2.2}$$

Here, τ_0 should be chosen in a way that for $\tau < \tau_0$ the material is at rest, without stress and strain. From the relations above we see that $D(t-\tau)H(t-\tau_0)$ represents

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the strain corresponding to a creep test with $\sigma(t) = H(t - \tau_0)$. $D(t - \tau)$ *is the creep function* or *creep compliance* of dimension L^2/F.

Equation (2.1) is an integral of the Stieltjes type. For these integrals, when $\sigma(\tau)$ has steps $\Delta\sigma_i H(t - \tau_i)$ we have

$$\varepsilon(t) = \int_{\tau_0}^{t} D(t-\tau)d\sigma(\tau) + \sum_i \Delta\sigma_i D(t-\tau_i) \tag{2.3}$$

As long as $\sigma(t)$ is continuous and differentiable, $\dot{\sigma}(t)$ exists and the form (2.2) can be used. Notice that the integration is performed with relation to τ; t acts as a parameter and as the superior limit of integration, but it is a constant inside the integral. Thus, for differentiation in relation to t we have to use the Leibnitz formula (See Appendix A).

Alternative forms of the integral representation. Besides the relations (2.1) and (2.2) we may use the inverse relations

$$\sigma(t) = \int_{\tau_0}^{t} E(t-\tau)d\varepsilon(\tau) \tag{2.4}$$

and

$$\sigma(t) = \int_{\tau_0}^{t} E(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{2.5}$$

exchanging the roles of stress and strain. $E(t - \tau)$ is the *specific relaxation function*, i.e., the stress response to a unit step of strain. Integrating (2.2) by parts, we obtain

$$\varepsilon(t) = \frac{\sigma(t)}{E(0)} + \int_{\tau_0}^{t} d(t-\tau)\sigma(\tau)d\tau \tag{2.6}$$

where

$$\begin{aligned} d(t-\tau) &= -\frac{\partial}{\partial\tau}D(t-\tau) \\ E(0) &= 1/D(t-t) \end{aligned} \tag{2.7}$$

Sometimes, instantaneous and delayed components of the specific creep are separated

$$\begin{aligned} D(t-\tau) &= \frac{1}{E(0)} + C(t-\tau) \\ E(t-\tau) &= E(\infty) + R(t-\tau) \end{aligned} \tag{2.8}$$

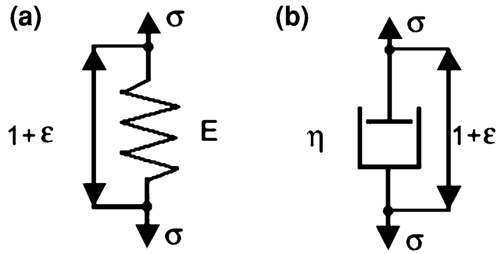

Fig. 2.1 **a** Hooke model (*spring*). **b** Newton model (*dashpot*)

2.2 Rheological Models

The behavior of viscoelastic materials under uniaxial loading may be represented by means of conceptual models composed of elastic and viscous elements which provide physical insight and have didactic value. Rheological models are described in most of the books on viscoelasticity such as Flugge [2], Christensen [3] and many others.

2.2.1 *The Basic Elements: Spring and Dashpot*

An ideal helicoidal *spring*, perfectly linear *elastic* and massless, represents Hooke model (see Fig. 2.1a):

$$\sigma(t) = E\varepsilon(t) \tag{2.9}$$

where E is the elasticity modulus with dimension [F/L^2]. Both length and cross-section are given unit values in order to identify force with stress and elongation with strain.

The *dashpot* (Fig. 2.1b) is an ideal *viscous* element that extends at a rate proportional to the applied stress, according to Newton equation

$$\dot{\varepsilon}(t) = \sigma(t)/\eta \tag{2.10}$$

where $\dot{\varepsilon} = \partial\varepsilon/\partial t$ is the *rate of strain* and η is the *viscosity coefficient*, with dimension [FT/L^2]. Combining springs and dashpots we obtain different models of viscoelastic behavior. The simplest viscoelastic models are those named after the scientists J. C. Maxwell and Lord Kelvin.

2.2.2 *Maxwell Model*

This model is the combination of a spring and a dashpot in series, Fig. 2.2a. For this system we may write the equations

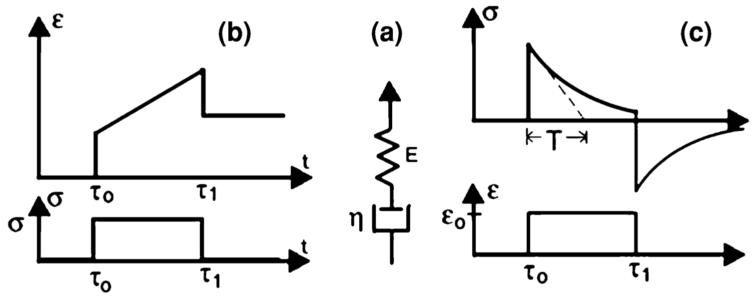

Fig. 2.2 Maxwell model: **a** rheological model, **b** creep test, **c** relaxation test

$$\begin{aligned} \varepsilon(t) &= \varepsilon_E(t) + \varepsilon_\eta(t) \\ \sigma_E(t) &= \sigma_\eta(t) = \sigma(t) \\ \sigma_E(t) &= E\varepsilon_E(t); \sigma_\eta(t) = \eta\dot{\varepsilon}_\eta(t) \end{aligned} \tag{2.11}$$

where the sub-indexes η and E indicate dashpot and spring respectively.

Differentiating the first Eq. 2.11 with respect to time t and using the constitutive relations for both spring and dashpot, we obtain

$$\dot{\varepsilon}(t) = \frac{\dot{\sigma}(t)}{E} + \frac{\sigma(t)}{\eta} \quad \varepsilon(t) = \sigma(t) = 0 \ \text{ for } t < \tau_0 \tag{2.12}$$

which is the differential equation for the Maxwell model. Solutions of (2.12) may be determined considering either stress or strain as the controlled variable. In the first case we have directly

$$\varepsilon(t) = \frac{\sigma(t)}{E} + \frac{1}{\eta}\int_{\tau_0}^{t} \sigma(\tau)d\tau \tag{2.13}$$

Integrating (2.13) by parts we obtain the alternative expression

$$\varepsilon(t) = \int_{\tau_0}^{t} \left(\frac{1}{E} + \frac{t-\tau}{\eta}\right)\dot{\sigma}(\tau)d\tau \tag{2.14}$$

Comparing this to (2.2) we see that

$$D(t-\tau) = \frac{1}{E} + \frac{t-\tau}{\eta}; \quad t \geq \tau \tag{2.15}$$

is the *creep function*. Since the strain response is unbounded for $t \to \infty$, one says that the Maxwell model exhibits unbounded creep and sometimes refers to it as Maxwell *fluid*. For a stress history $\sigma(t) = \sigma_0[H(t-\tau_0) - H(t-\tau_1)]$, with

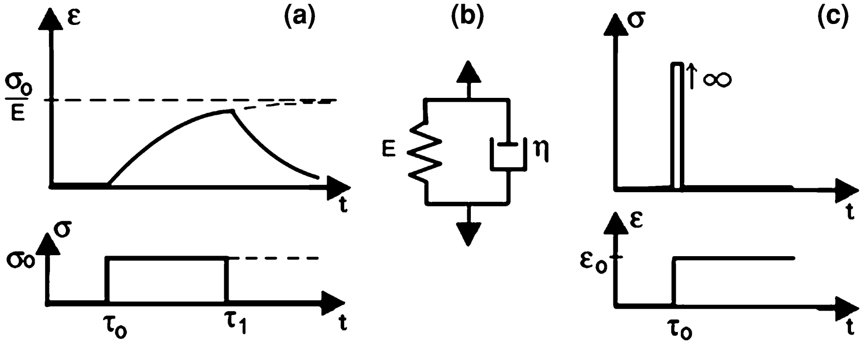

Fig. 2.3 Kelvin model: **a** creep test, **b** rheological model, **c** relaxation test

$\tau_0 < \tau_1$, as that shown in Fig. 2.2b, a residual deformation remains after unloading.

Considering now the strain history as given, we obtain from (2.12), using the general solution for first order differential equations in Appendix A,

$$\sigma(t) = E \int_{\tau_0}^{t} e^{-\frac{E}{\eta}(t-\tau)} \dot{\varepsilon}(\tau) d\tau \tag{2.16}$$

Then, comparing to (2.5) we see that

$$E(t - \tau_0) = E e^{-(t-\tau_0)/T}; \ T = \frac{\eta}{E}; \ t \geq \tau_0 \tag{2.17}$$

is the relaxation function that vanishes for $t \to \infty$. Relations using creep or relaxation functions, such as (2.14) and (2.16), are equivalent. A procedure to obtain one from the other is given in Sect. 2.4 and in Chap. 5.

The constant $T = \eta/E$ that appears in the exponential in (2.17) determines the rate of the relaxation process and is called *relaxation time*. The smaller the relaxation time, the faster the relaxation process, even though total relaxation takes theoretically an infinite time. For example, for $t - \tau_0 = 3T$ about 95% of the total relaxation is completed. Considering a loading–unloading history, such as $\varepsilon(t) = \varepsilon_0[H(t - \tau_0) - H(t - \tau_1)]$, with $\tau_0 < \tau_1$, the stress response changes signal (Fig. 2.2c).

2.2.3 *Kelvin Model*

This model combines a spring and a dashpot in parallel, Fig. 2.3b. From the relations

$$\begin{aligned}\sigma(t) &= \sigma_E(t) + \sigma_\eta(t) \\ \varepsilon_E(t) &= \varepsilon_\eta(t) = \varepsilon(t) \\ \sigma_E(t) &= E\varepsilon_E(t); \sigma_\eta(t) = \eta\dot{\varepsilon}(t)\end{aligned} \tag{2.18}$$

we can determine the differential equation

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t) \tag{2.19}$$

For a given strain history we have the stress directly from (2.19). A relaxation test is physically impossible with the Kelvin model because $\dot{\varepsilon}(t) = \varepsilon_0\delta(t)$ and the corresponding initial stress should be infinitely high.

For a given stress history $\sigma(t)$ the solution of (2.19) is

$$\varepsilon(t) = \frac{1}{\eta}\int_{\tau_0}^{t} \sigma(\tau)e^{-\frac{t-\tau}{\theta}}d\tau \quad ; \quad \theta = \frac{\eta}{E} \tag{2.20}$$

Comparing to (2.2) we see that

$$D(t-\tau_0) = \frac{1}{E}\left(1 - e^{-(t-\tau_0)/\theta}\right); \quad t \geq \tau_0 \tag{2.21}$$

is the creep function for the Kelvin model. For $t \to \infty$ we obtain $\varepsilon(\infty) = \sigma_0/E$ that corresponds to the *asymptotic elastic solution*, when all the stress is carried by the spring.

Again, we have equivalent differential and integral representations. Fig. 2.3 shows the results of creep and relaxation tests. The constant θ is called *retardation time* and is analogous in meaning to the relaxation time: an estimate of the time required for the creep process to approach completion.

2.3 Generalized Models

Maxwell and Kelvin models are adequate for qualitative and conceptual analyses, but generally poor for the quantitative representation of the behavior of real materials. In order to improve the representation we need to increase the number of parameters by combining a number of springs and dashpots. A systematic way to do that is to build generalized Maxwell and Kelvin models, shown in Fig. 2.4. The *generalized Maxwell model* is composed of $n + 1$ constituent elements in parallel, being n Maxwell models and an isolated spring (to warrant solid behavior) (see Fig. 2.4a).

The differential Eq. 2.12 for a generic Maxwell element r of a generalized Maxwell model may be written in the operational form

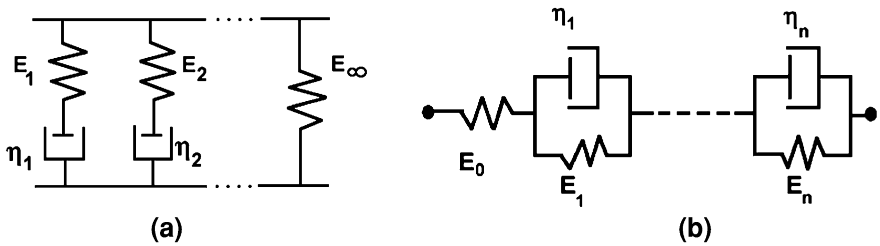

Fig. 2.4 Maxwell and Kelvin chains with instantaneous elasticity

$$\frac{\partial}{\partial t}\varepsilon(t) = \left(\frac{1}{E_r}\frac{\partial}{\partial t} + \frac{1}{\eta_r}\right)\sigma_r \tag{2.22}$$

where E_r, η_r and σ_r indicate the elastic constant, viscosity coefficients and stress of the *r-th* element, respectively. The symbol $\partial/\partial t$ is a differential operator that can be handled as an algebraic entity. For the generalized Maxwell model the strain is the same for all constituent elements and the total stress is given by the equation

$$\sigma(t) = \left(E_\infty + \sum_{r=1}^{n} \frac{\partial/\partial t}{\frac{\partial/\partial t}{E_r} + \frac{1}{\eta_r}}\right)\varepsilon(t) \tag{2.23}$$

From Fig. 2.4 and (2.17) it is clear that the relaxation function for the generalized Maxwell model is, for a generic value of τ

$$E(t-\tau) = E_\infty + \sum_{r=1}^{n} E_r e^{-\frac{t-\tau}{T_r}} \quad ; \quad T_r = \eta_r/E_r \tag{2.24}$$

The generalized Maxwell model provides an exponentially varying stress adding contributions with different relaxation times, one for each element in the chain. Thus, it is possible to fit experimental creep curves to any required degree of approximation if enough terms are used. To find the creep function for the generalized Maxwell model the differential Eq. 2.23 has to be solved, like in Example 1 below.

The *generalized Kelvin model* is composed of n Kelvin units in series plus an isolated spring. The stress at each unit is the same external stress $\sigma(t)$ while the total (observable) strain $\varepsilon(t)$ is the sum of the internal strains in each element. Writing (2.19) in the symbolic form for a generic Kelvin element r

$$\sigma_r(t) = \left(E_r + \eta_r {}^{\partial}\!/_{\partial t}\right)\varepsilon_r(t) \tag{2.25}$$

we have for the model in Fig. 2.4b

$$\varepsilon(t) = \left(\frac{1}{E_0} + \sum_{r=1}^{n} \frac{1}{E_r + \eta_r \partial/\partial t}\right)\sigma(t) \tag{2.26}$$

From Eq. 2.21 and Fig. 2.4b, it is easy to gather that the specific creep function for the generalized Kelvin model is, for a generic value of τ,

$$D(t-\tau) = \frac{1}{E_0} + \sum_{r=1}^{n} \frac{1}{E_r}\left[1 - e^{-\frac{t-\tau}{\theta_r}}\right] \quad ; \quad \theta_r = \eta_r / E_r \tag{2.27}$$

To find the relaxation function, the differential equation (2.26) has to be solved.

Example 1 Determine the differential equation of the *Zener model*, that is a particular case of the generalized Maxwell model composed by a Maxwell model with parameters $E_1 = E$, $\eta_1 = \eta$ in parallel with a spring of stiffness E_∞, Fig. 2.4a. Substituting these values into (2.23) we obtain

$$\sigma(t) = \left(E_\infty + \frac{\partial/\partial t}{\frac{\partial/\partial t}{E} + \frac{1}{\eta}}\right)\varepsilon(t) \tag{2.28}$$

Developing this symbolic equation we find

$$\sigma + \frac{\eta}{E}\dot{\sigma} = E_\infty \varepsilon + \frac{\eta(E_\infty + E)}{E}\dot{\varepsilon} \tag{2.29}$$

With $E_{z0} = E_\infty + E$, $\theta_z = \eta(E_\infty + E)/(E_\infty E) = \eta E_{z0}/[E_\infty(E_{z0}$ - $E_\infty)]$ and $T_z = \eta/E = \eta/(E_{z0} - E_\infty)$, we have the nice form

$$E_z(0)\left[\dot{\varepsilon}(t) + \frac{\varepsilon(t)}{\theta_z}\right] = \dot{\sigma}(t) + \frac{\sigma(t)}{T_z} \tag{2.30}$$

where $E_z(0) = E_{z0}$. Solving in ε we obtain, with the initial condition $\varepsilon(\tau_0) = \sigma(\tau_0)/E_{z0}$,

$$\varepsilon(t) = \frac{\sigma(t)}{E_z(\infty)} - \left[\frac{1}{E_z(\infty)} - \frac{1}{E_z(0)}\right]\int_0^t e^{-\frac{(t-\tau)}{\theta_z}}\dot{\sigma}(\tau)d\tau \tag{2.31}$$

being $E_z(\infty) = E_\infty$. The corresponding creep function is then

$$D(t-\tau) = \frac{1}{E_z(\infty)}\left[1 - \frac{E_z(0) - E_z(\infty)}{E_z(0)} e^{-\frac{(t-\tau)}{\theta_z}}\right] \tag{2.32}$$

Example 2 Determine the differential equation of the *standard solid* model which is a particular case of the generalized Kelvin model with a spring (E_0) and a Kelvin element ($E_1 = E$, $\eta_1 = \eta$) connected in series. Substituting these parameters in (2.26), we have

$$\varepsilon(t) = \left(\frac{1}{E_0} + \frac{1}{E + \eta\partial/\partial t}\right)\sigma(t) \tag{2.33}$$

Developing this equation, the following differential equation is obtained

$$\sigma + \frac{\eta}{E_0 + E}\dot{\sigma} = \frac{E_0 E}{E_0 + E}\varepsilon + \frac{E_0 \eta}{E_0 + E}\dot{\varepsilon} \tag{2.34}$$

Making $E_s(0) = E_0$, $E_s(\infty) = E_0 E/(E_0 + E)$, $\theta_s = \eta/E$ and $T_s = \eta/(E_0 + E)$, this differential equation can be written as

$$E_s(0)\left[\dot{\varepsilon}(t) + \frac{\varepsilon(t)}{\theta_s}\right] = \dot{\sigma}(t) + \frac{\sigma(t)}{T_s} \tag{2.35}$$

Comparing (2.35) to (2.30), we conclude that the standard and Zener models present similar differential equations. Then, the solution for each one of these models can be obtained from the solution of the other by a convenient change of parameters.

2.3.1 General Differential Representation

Equations 2.23 and 2.26 are differential equations with the general form

$$\sum_{i=0}^{h} p_i \frac{\partial^i \sigma}{\partial t^i} = \sum_{j=0}^{k} q_j \frac{\partial^j \varepsilon}{\partial t^j} \tag{2.36}$$

where p_i and q_j are material constants dependent on the viscoelastic model. Usually, without loss of generality, we assume $p_0 = 1$.

From (2.30), the constants for the Zener model are

$$p_0 = 1, p_1 = \frac{\eta}{E}, q_0 = E_\infty \text{ and } q_1 = \frac{\eta(E_\infty + E)}{E} \tag{2.37}$$

and, for the standard solid model (2.35),

$$p_0 = 1, p_1 = \frac{\eta}{E_0 + E}, q_0 = \frac{E_0 E}{E_0 + E} \text{ and } q_1 = \frac{E_0 \eta}{E_0 + E} \tag{2.38}$$

Generalized Kelvin and Maxwell models are equivalent, in the sense that it is always possible to find a generalized Maxwell model equivalent to a given generalized Kelvin one, as in Examples 1 and 2. In Chap. 5 it is shown how to go from a creep to a relaxation function. Then, with Eqs. (2.24) and (2.27) we can find the corresponding models.

2.4 Integral and Differential Operators

Viscoelastic relationships may also be indicated in the symbolic forms

$$\begin{aligned} \varepsilon &= D^* \sigma \\ \sigma &= E^* \varepsilon \end{aligned} \tag{2.39}$$

which have to be interpreted as alternatives to (2.1) and (2.4) respectively. As the linear operators E* and D* may be handled formally as algebraic quantities (with some care), this notation simplifies some calculations. The operational form is valid also for the differential representation. For example, the differential operator for the generalized Kelvin model is the expression inside the brackets in (2.26). With this notation viscoelastic and elastic equations have similar form. A more rigorous development and applications of the operational technique will be given in Chap. 5 through the use of *Laplace transform.*

Sometimes we need to invert the viscoelastic relations, i.e., to obtain the relaxation function corresponding to a given creep function and vice versa.

From (2.39), we have

$$\varepsilon = D^* E^* \varepsilon \tag{2.40}$$

Thus,

$$H(t - \tau_0) = D^* E(t - \tau) H(t - \tau_0) \tag{2.41}$$

In extended form, this is written [4, 5]

$$1 = D(t - \tau) E(t - t) + \int_{\tau_0}^{t} D(t - \tau) \dot{E}(\tau - \tau_0) d\tau \quad \text{for } t \geq \tau_0 \tag{2.42}$$

Equation (2.42) express the obvious fact that applying as a stress history the corresponding relaxation function, we obtain a constant unit deformation.

Example 3 Consider the relaxation function corresponding to the Zener model with $E(t) = E_1 + E_2 e^{-t/T}$; then $\dot{E}(t) = -E_2 e^{-t/T}/T$ and substituting into (2.42)

$$D(t - \tau_0)(E_1 + E_2) - \frac{E_2}{T} \int_{\tau_0}^{t} D(t - \tau) e^{-(t-\tau)/T} d\tau = 1 \tag{2.43}$$

Differentiating (2.43) in relation to t (Leibnitz rule)

$$\dot{D}(t - \tau_0)(E_1 + E_2) - \frac{E_2}{T} D(t - \tau_0) + \frac{E_2}{T^2} \int_{\tau_0}^{t} D(t - \tau) e^{-(t-\tau)/T} d\tau = 0 \tag{2.44}$$

Multiplying this equation by T and adding to (2.43) we eliminate the integral to obtain the differential equation

$$(E_1 + E_2) T \dot{D}(t - \tau_0) + E_1 D(t - \tau_0) = 1 \tag{2.45}$$

From which we obtain, with the initial condition $D(0) = 1/E(0) = 1/(E_1 + E_2)$

$$D(t - \tau_0) = \frac{1}{E_1} - \frac{E_2}{(E_1 + E_2)E_1} e^{-(t-\tau)/\theta} \tag{2.46}$$

with $\theta = (1 + E_2/E_1)T$.

The operational form is valid also for the differential representation. For example, the differential operator for the generalized Kelvin model is the expression inside the brackets in (2.26).

2.5 Thermodynamic Restrictions

The work done in deforming a viscoelastic body must be non-negative. Sufficient conditions that the relaxation function must satisfy are given in [6]. In reference to Eq. (2.5)

(1) $E(t)$ must be non-negative
(2) $E(t)$ must be a monotonically decreasing function with finite limit for $t \to \infty$.
(3) $E(t)$ must be convex downward.

Most of the functions that are usually used to approximate the relaxation function satisfy the conditions above.

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