Chapter 2
Introduction to Continuum Mechanics

The mechanics of a deformable body treated here is based on Newton's laws of motion and the laws of thermodynamics. In this Chapter we present the fundamental concepts of continuum mechanics, and, for conciseness, the material is presented in Cartesian tensor formulation with the implicit assumption of Einstein's summation convention. Where this convention is exempted we shall denote the index thus: $(!\alpha)$.

2.1 Newtonian Mechanics

Newtonian mechanics consists of

1. The first law (The law of inertia),
2. The second law (The law of conservation of linear momentum), and
3. The third law (The law of action and reaction).

Newton's first law defines inertial frames of reference. In general it can be described as follows; "if no force acts on a body, it remains immobilized or in a state of constant motion". However, if this is true, the first law is obviously induced by the second law,[1] and the laws might be incomplete as a physical system. We must rephrase the first law as follows:

The First Law: If no force acts on a body, there exist frames of reference, referred to as *inertial frames*, in which we can observe that the body is either stationary or moves at a constant velocity.

[1] In the second law (2.2) if we set $\boldsymbol{f} = \boldsymbol{0}$ and solve the differential equation, we obtain $\boldsymbol{v} =$ constant since $m =$ constant, which suggests that the first law is included in the second law. This apparent contradiction results from the misinterpretation of the first law.

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We now understand that the first law assures the existence of inertial frames, and the second and third laws are valid in the inertial frames. This is the essence of Newtonian mechanics.

Let m and $\boldsymbol{v}$ be the mass and velocity of a body, respectively, then the momentum $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{2.1}$$

We now have the following:

The Second Law: In an inertial frame, if a force $\boldsymbol{f}$ acts on a body, the following law of conservation of linear momentum is valid:

$$\frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = \boldsymbol{f} \tag{2.2}$$

Since the mass m is conserved in Newtonian mechanics, (2.2) can be rewritten as

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{f}. \tag{2.3}$$

Let us consider a material point which is represented by two different frames as $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$ where $\boldsymbol{x}$, $\boldsymbol{x}^*$ denote positions and t, t^* denote time. Under the Galilean transformation given by

$$\boldsymbol{x}^* = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{V}t, \qquad t^* = t + a, \tag{2.4}$$

both frames form the equivalent inertial frames (Fig. 2.1). This is referred to as the Galilean principle of relativity. Here $\boldsymbol{Q}$ is a time-independent transformation tensor (cf. Appendix A.3), $\boldsymbol{V}$ is a constant vector and a is a scalar constant.

The Galilean transformation (2.4) gives, in fact, the condition which results in the law of conservation of linear momentum (2.2) in both the frames $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$. That is, differentiating (2.4) yields

$$\boldsymbol{v}^* = \frac{d\boldsymbol{x}^*}{dt^*} = \boldsymbol{Q}\frac{d\boldsymbol{x}}{dt} + \boldsymbol{V} = \boldsymbol{Q}\boldsymbol{v} + \boldsymbol{V}$$

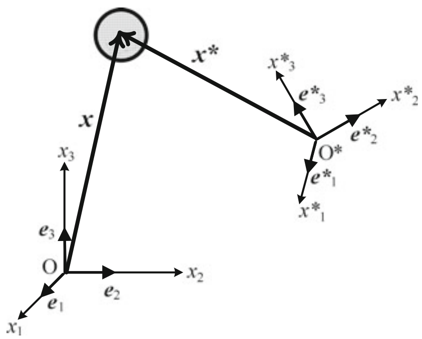

Fig. 2.1 Galilean transformation

and the force vector $\boldsymbol{f}$ is transformed in the same manner:

$$\boldsymbol{f}^* = \boldsymbol{Q}\,\boldsymbol{f}\,. \tag{2.5}$$

The mass conservation law is $m^* = m$, and (2.5) suggests that the law of conservation of linear momentum is satisfied in the inertial frames in the following manner:

$$\boldsymbol{f}^* = m^* \frac{d\boldsymbol{v}^*}{dt^*} = m \frac{d(\boldsymbol{Q}\,\boldsymbol{v} + \boldsymbol{V})}{dt} = \boldsymbol{Q} \left(m \frac{d\boldsymbol{v}}{dt} \right) = \boldsymbol{Q}\,\boldsymbol{f}\,. \tag{2.6}$$

Note that the force vector represented in the form of (2.5) is a fundamental hypothesis of Newtonian mechanics (i.e., the frame indifference of a force vector; see Sect. 2.2.2). The third law gives the interacting forces for a two-body problem, and this will not be treated here.

Note 2.1 (Inertial frame and the relativity principle). The first law (i.e., the law of inertia) gives a condition that there are multiple frames of equivalent inertia that are moving under a relative velocity $\boldsymbol{V}$. Therefore, the first law guarantees that the second law (i.e., the law of conservation of linear momentum) is realized in any inertial frame. This implies that the second law is not always appropriate in different frames of reference, which are moving under a general relative velocity. In fact, in an accelerated frame that includes rotational motion, such as on the surface of Earth, there is a centrifugal force and a Coriolis effect. In Sect. 2.2.2 we will treat a general law of change of frame, which is related to a constitutive theory that describes the material response (see Sect. 2.8).

If the relative velocity $\boldsymbol{V}$ of the Galilean transformation (2.4) approaches the speed of light c, the uniformity of time is not applicable, and the Newtonian framework is no longer valid. That is, the Galilean transformation is changed into the Lorentz transformation under invariance of Maxwell's electromagnetic equations, and the equation of motion is now described in relation to Einstein's theory of relativity. ■

2.2 Deformation Kinematics

When we consider the motion of a material body, constituent atoms and molecules are not directly taken into consideration since this will require an inordinate amount of analysis. Therefore we represent the real material by an equivalent shape of a subdomain of the n-dimensional real number space $\mathbb{R}^n$, and apply the Newtonian principles to this image. This procedure leads to *Continuum Mechanics*; the term 'continuum' is a result of the continuity properties of the n-dimensional real number space $\mathbb{R}^n$.

2.2.1 Motion and Configuration

Consider a material point X in the body $\mathfrak{B}$, and we have an image in the n-dimensional real number space $\mathbb{R}^n$ that is referred as a *configuration*.[2] We choose the configuration, $\boldsymbol{X}(X) \in \Omega_0 \subset \mathbb{R}^n$, at the time $t = t_0$ as a *reference configuration*, and treat the subsequent deformation and motion of the *current configuration*, $\boldsymbol{x} \in \Omega \subset \mathbb{R}^n$, at the time $t = t$ with respect to the reference configuration (Fig. 2.2).

Assume that a reference point $\boldsymbol{X} \in \Omega_0$ moves to the current point $\boldsymbol{x} \in \Omega$. The motion is represented as

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) \tag{2.7}$$

Note that the time change $\boldsymbol{x}(\boldsymbol{X}, t)$ of a specific point $\boldsymbol{X}$ gives a trajectory. Then the velocity of the material point is calculated by

$$\boldsymbol{v}(\boldsymbol{X}, t) = \frac{d\boldsymbol{x}}{dt}(\boldsymbol{X}, t) \tag{2.8}$$

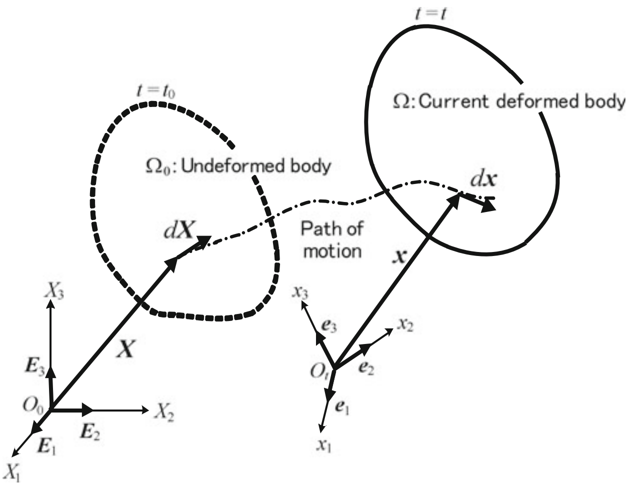

Fig. 2.2 Motion and deformation of a material body

[2] 'Configuration' is defined as an invertible continuous function that maps every material point $X \in \mathfrak{B}$ to a point z in a subset of the n-dimensional real number space $\mathbb{R}^n$. A time-dependent motion is considered; therefore the configuration is a function of the material point X and time t. The configuration at a given time t_0 is set as a reference configuration κ, and the point $\boldsymbol{X} \in \mathbb{R}^n$ corresponding to a material point X is written as $\boldsymbol{X} = \kappa(X)$, $X = \kappa^{-1}(\boldsymbol{X})$ where κ^{-1} is an inverse mapping of κ. The current configuration χ at time t maps X to $\boldsymbol{x} \in \mathbb{R}^n$ as $\boldsymbol{x} = \chi(X, t)$, $X = \chi^{-1}(\boldsymbol{X}, t)$. The composite function $\chi_\kappa = \chi \circ \kappa^{-1}$ is introduced as $\boldsymbol{x} = \chi(\kappa^{-1}(\boldsymbol{X}), t) = \chi_\kappa = \chi \circ \kappa^{-1}(\boldsymbol{X}, t) = \chi_\kappa(\boldsymbol{X}, t)$. The function χ_κ gives a mapping between the position vector $\boldsymbol{X}$ in the reference configuration and the position vector $\boldsymbol{x}$ in the current configuration. Since this formal procedure is complicated, the above simplified descriptions are employed.

We should pay attention to the fact that the velocity $\boldsymbol{v}$ has a simple differential form of (2.8) with respect to t, since $\boldsymbol{x}(\boldsymbol{X}, t)$ is a function both of $\boldsymbol{X}$ and of t but $\boldsymbol{X}$ is a fixed frame of reference.

Two material points can never occupy the same position, therefore (2.7) has a unique inverse:

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t) \tag{2.9}$$

This relation permits the definition of the velocity $\boldsymbol{v}(\boldsymbol{X}, t)$; this can be written as a function of $\boldsymbol{x}$ and t:

$$\boldsymbol{v}(\boldsymbol{X}(\boldsymbol{x}, t), t) = \boldsymbol{v}(\boldsymbol{x}, t). \tag{2.10}$$

We can generalize this procedure: If a function ϕ is represented in terms of $(\boldsymbol{X}, t)$, we call it the *Lagrangian description*, and if ϕ is given in terms of $(\boldsymbol{x}, t)$, it is the *Eulerian description*. The choice of the form is arbitrary but will be influenced by any advantage of a problem formulation in either description. For example, in solid mechanics, the Lagrangian description is commonly used, while in fluid mechanics the Eulerian description is popular. This is because in solid mechanics we can attach labels (e.g., visualize 'strain gauges' at various points) on the surface of a solid body, and each material point can be easily traced from the reference state to the current state. On the other hand for a fluid we measure the velocity $\boldsymbol{v}$ or pressure p at the current position $\boldsymbol{x}$, therefore the Eulerian description better represents the fluid (note that for a fluid it is difficult to know the exact reference point $\boldsymbol{X}$ corresponding to all the current points $\boldsymbol{x}$).

The coordinate system with the basis $\{\boldsymbol{E}_I\}$ $(I = 1, 2, 3)$ for the undeformed body Ω_0 is usually different from the coordinate system with the basis $\{\boldsymbol{e}_i\}$ $(i = 1, 2, 3)$ that describes the deformed body Ω in the current configuration (Fig. 2.2):

$$\boldsymbol{X} = X_I \boldsymbol{E}_I, \qquad \boldsymbol{x} = x_i \boldsymbol{e}_i. \tag{2.11}$$

$\{\boldsymbol{E}_I\}$, $\{\boldsymbol{e}_i\}$ represent the orthogonal coordinate systems (i.e., Cartesian); we will employ the same orthogonal coordinate systems $\{\boldsymbol{E}_I\} = \{\boldsymbol{e}_i\}$ for simplicity unless otherwise mentioned. Then, the position vectors $\boldsymbol{X}$ and $\boldsymbol{x}$ can be written as

$$\boldsymbol{X} = X_i \boldsymbol{e}_i, \qquad \boldsymbol{x} = x_i \boldsymbol{e}_i$$

Let $\phi(\boldsymbol{X}, t)$ be a Lagrangian function. Since $\boldsymbol{X}$ is time-independent, the time differentiation $\dot{\phi}$ is simply given by

$$\dot{\phi}(\boldsymbol{X}, t) = \frac{d\phi}{dt}. \tag{2.12}$$

Here $\dot{\phi} = d\phi/dt$ is referred to as the *material time derivative* of $\phi(\boldsymbol{X}, t)$. On the other hand, for an Eulerian function $\phi(\boldsymbol{x}, t)$ the total differential $d\phi(\boldsymbol{x}, t)$ can be written as

$$d\phi(\boldsymbol{x}, t) = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x_i} dx_i,$$

and $v_i = dx_i/dt$, and therefore we have the following relationship between the material time derivative and Eulerian time derivative

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \nabla \phi. \tag{2.13}$$

Here the symbol ∇ (called 'nabla') implies[3]

$$\nabla = \boldsymbol{e}_i \frac{\partial}{\partial x_i}. \tag{2.14}$$

The second term of the r.h.s. of (2.13) gives a convective term.

The velocity $\boldsymbol{v}$ is written either in the Lagrangian form or in the Eulerian form; therefore the acceleration $\boldsymbol{a}$ can be represented in either form:

$$\boldsymbol{a}(\boldsymbol{X}, t) = \frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{a}(\boldsymbol{x}, t) \tag{2.15}$$

In indicial notation (2.15) is given by

$$a_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial v_j}. \tag{2.16}$$

Note 2.2 (Differentiation of vector and tensor valued functions). Let $\boldsymbol{u} = u_i \boldsymbol{e}_i$ and $\boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ be vector and tensor valued functions. It is understood that the gradients $\mathrm{grad}\,\boldsymbol{u}$, $\mathrm{grad}\,\boldsymbol{T}$ can be of two forms:

$$\mathrm{grad}\,\boldsymbol{u} = \begin{cases} \boldsymbol{u} \otimes \nabla = \dfrac{\partial u_i}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \\ \nabla \otimes \boldsymbol{u} = \dfrac{\partial u_i}{\partial x_j} \boldsymbol{e}_j \otimes \boldsymbol{e}_i, \end{cases} \qquad \mathrm{grad}\,\boldsymbol{T} = \begin{cases} \boldsymbol{T} \otimes \nabla = \dfrac{\partial T_{ij}}{\partial x_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k, \\ \nabla \otimes \boldsymbol{T} = \dfrac{\partial T_{jk}}{\partial x_i} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k. \end{cases}$$

The former are called the *right form*, and the latter are the *left form*.

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[3] If the gradient is used with respect to Eulerian coordinates with the basis $\{\boldsymbol{e}_i\}$, it is denoted as (2.14). If we explicitly explain the gradient with respect to the Eulerian system, it is denoted as

$$\mathrm{grad} = \boldsymbol{\nabla}_x = \boldsymbol{e}_i \frac{\partial}{\partial x_i}.$$

If the gradient is operated with respect to Lagrangian coordinates $\{\boldsymbol{E}_I\}$, it is represented as

$$\mathrm{Grad} = \boldsymbol{\nabla}_X = \boldsymbol{E}_I \frac{\partial}{\partial X_I}.$$

Right and left forms of divergence $\operatorname{div}\boldsymbol{u}$, $\operatorname{div}\boldsymbol{T}$ and rotation $\operatorname{rot}\boldsymbol{u}$, $\operatorname{rot}\boldsymbol{T}$ are given by

$$\operatorname{div}\boldsymbol{u} = \begin{cases} \boldsymbol{u}\cdot\nabla = \dfrac{\partial u_i}{\partial x_i}, \\ \nabla\cdot\boldsymbol{u} = \dfrac{\partial u_i}{\partial x_i}, \end{cases} \qquad \operatorname{div}\boldsymbol{T} = \begin{cases} \boldsymbol{T}\cdot\nabla = \dfrac{\partial T_{ij}}{\partial x_j}\boldsymbol{e}_i, \\ \nabla\cdot\boldsymbol{T} = \dfrac{\partial T_{ji}}{\partial x_j}\boldsymbol{e}_i, \end{cases}$$

$$\operatorname{rot}\boldsymbol{u} = \begin{cases} \boldsymbol{u}\wedge\nabla = e_{ijk}\dfrac{\partial u_i}{\partial x_j}\boldsymbol{e}_k, \\ \nabla\wedge\boldsymbol{u} = e_{jik}\dfrac{\partial u_i}{\partial x_j}\boldsymbol{e}_k, \end{cases} \qquad \operatorname{rot}\boldsymbol{T} = \begin{cases} \boldsymbol{T}\wedge\nabla = e_{jkl}\dfrac{\partial T_{ik}}{\partial x_l}\boldsymbol{E}_I\otimes\boldsymbol{e}_j, \\ \nabla\wedge\boldsymbol{T} = e_{ikl}\dfrac{\partial T_{lj}}{\partial x_k}\boldsymbol{E}_I\otimes\boldsymbol{e}_j, \end{cases}$$

In this volume we will predominantly use the right form, and we symbolically write the right forms in the same manner as the left form: e.g.

$$\operatorname{grad}\boldsymbol{u} = \frac{\partial u_i}{\partial x_j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j, \qquad \operatorname{grad}\boldsymbol{T} = \frac{\partial T_{ij}}{\partial x_k}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k,$$

$$\operatorname{div}\boldsymbol{u} = \frac{\partial u_i}{\partial x_i}, \qquad \operatorname{div}\boldsymbol{T} = \frac{\partial T_{ij}}{\partial x_j}\boldsymbol{e}_i,$$

$$\operatorname{rot}\boldsymbol{u} = e_{ijk}\frac{\partial u_i}{\partial x_j}\boldsymbol{e}_k, \qquad \operatorname{rot}\boldsymbol{T} = e_{jkl}\frac{\partial T_{ik}}{\partial x_l}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$$

In addition, if the right-divergence form of the second-order tensor $\boldsymbol{T} = T_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$ is as given above, the divergence theorem (B.6) given in Appendix B.2 can be written as

$$\int_G \operatorname{div}\boldsymbol{T}\,dv = \int_G \nabla\cdot\boldsymbol{T}\,dv = \int_{\partial G}\boldsymbol{T}\boldsymbol{n}\,ds.$$

If $\boldsymbol{u}$ and $\boldsymbol{T}$ are given as functions of the reference basis such that $\boldsymbol{u} = u_I(\boldsymbol{X})\boldsymbol{E}_I$, $\boldsymbol{T} = T_{IJ}(\boldsymbol{X})\boldsymbol{E}_I\otimes\boldsymbol{E}_J$, we have

$$\operatorname{Grad}\boldsymbol{u} = \frac{\partial u_I}{\partial X_J}\boldsymbol{E}_I\otimes\boldsymbol{E}_J, \qquad \operatorname{Grad}\boldsymbol{T} = \frac{\partial T_{IJ}}{\partial X_K}\boldsymbol{E}_I\otimes\boldsymbol{E}_J\otimes\boldsymbol{E}_K,$$

$$\operatorname{Div}\boldsymbol{u} = \frac{\partial u_I}{\partial X_I}, \qquad \operatorname{Div}\boldsymbol{T} = \frac{\partial T_{IJ}}{\partial X_J}\boldsymbol{E}_I,$$

$$\operatorname{Rot}\boldsymbol{u} = e_{IJK}\frac{\partial u_I}{\partial X_J}\boldsymbol{E}_K, \qquad \operatorname{Rot}\boldsymbol{T} = e_{JKL}\frac{\partial T_{IK}}{\partial X_L}\boldsymbol{E}_I\otimes\boldsymbol{E}_J. \qquad \blacksquare$$

2.2.2 Changes of Frame and Frame Indifference ♣

Let us consider two different points $\boldsymbol{x}$ and $\boldsymbol{x}_0$ in the current body, and introduce a two-point vector $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_0$ ($\boldsymbol{u}$ does not imply displacement). Then, it is easily observed that the length $|\boldsymbol{u}| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2}$ is an invariant; i.e., it has the same value in any coordinate system. We call this property of $|\boldsymbol{u}|$ the *principle of frame indifference*. The concept of frame indifference is a fundamental requirement for a constitutive theory between, for example, stress and strain (see Sect. 2.8).

The frame indifference of the two-point vector $\boldsymbol{u}$ is proved as follows: Let us introduce two coordinate systems, System 1 and System 2, which have bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}_i^*\}$, respectively (Fig. 2.3). In the framework of Newtonian mechanics, the two-point vector $\boldsymbol{x} - \boldsymbol{x}_0$ and time t of System 1 can be related to $\boldsymbol{x}^* - \boldsymbol{x}_0^*$ and t^* of System 2 by

$$\boldsymbol{x}^* - \boldsymbol{x}_0^* = \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0) \tag{2.17}$$

$$t^* = t - a \tag{2.18}$$

where $\boldsymbol{Q} = Q_{ij}\boldsymbol{e}_i^* \otimes \boldsymbol{e}_j$ is the following coordinate transformation tensor:

$$\boldsymbol{e}_i^* = \boldsymbol{Q}\boldsymbol{e}_i\,. \tag{2.19}$$

The tensor $\boldsymbol{Q}$ is orthonormal:

$$\boldsymbol{Q}\,\boldsymbol{Q}^T = \boldsymbol{I}^*, \qquad \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I} \tag{2.20}$$

where $\boldsymbol{I}$ is the unit tensor defined on System 1, and $\boldsymbol{I}^*$ is that defined on System 2. Note that a in (2.18) is a constant.[4] It is understood that the length $|\boldsymbol{x} – \boldsymbol{x}_0|$ is invariant under the change of frame since from (2.17) to (2.20) we have

$$|\boldsymbol{x}^* - \boldsymbol{x}_0^*|^2 = (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \boldsymbol{Q}^T\boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0) = |\boldsymbol{x} - \boldsymbol{x}_0|^2. \tag{2.21}$$

In general, a frame indifferent scalar function f is defined by

$$f^*(\boldsymbol{x}^*) = f(\boldsymbol{x}). \tag{2.22}$$

The frame indifferent vector function $\boldsymbol{u}$ is given by

$$\boldsymbol{u}^*(\boldsymbol{x}^*) = \boldsymbol{Q}\,\boldsymbol{u}(\boldsymbol{x}), \qquad \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{Q}^T\boldsymbol{u}^*(\boldsymbol{x}^*). \tag{2.23}$$

[4]Here we deal with a general case in which two coordinate systems may not be inertial systems. If both are inertial systems, $\boldsymbol{Q}$ is time-independent as given by (2.4). Therefore, we have

$$\boldsymbol{x}^* = \boldsymbol{Q}\,\boldsymbol{x} + \boldsymbol{V}t, \quad \boldsymbol{x}_0^* = \boldsymbol{Q}\,\boldsymbol{x}_0 + \boldsymbol{V}t \quad \Rightarrow \quad \boldsymbol{u} = \boldsymbol{x}^* - \boldsymbol{x}_0^* = \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0),$$

which shows that the two-point vector $\boldsymbol{u}$ is frame indifferent.

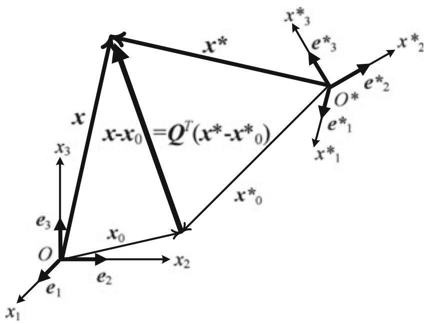


Fig. 2.3 Coordinate transformation and frame indifferent vector

A frame indifferent second-order tensor function $\boldsymbol{T}$ is characterized by the frame indifference of a transformed vector $\boldsymbol{v} = \boldsymbol{T}\boldsymbol{u}$. That is, let $\boldsymbol{u}$ be a frame indifferent vector ($\boldsymbol{u}^* = \boldsymbol{Q}\,\boldsymbol{u}$), and $\boldsymbol{v}$ and $\boldsymbol{v}^*$ be the vectors corresponding to $\boldsymbol{u}$ and $\boldsymbol{u}^*$ transformed by $\boldsymbol{T}$ and $\boldsymbol{T}^*$ such that

$$\boldsymbol{v} = \boldsymbol{T}\,\boldsymbol{u}, \qquad \boldsymbol{v}^* = \boldsymbol{T}^*\,\boldsymbol{u}^*.$$

Then, if $\boldsymbol{v}^* = \boldsymbol{Q}\,\boldsymbol{v}$, $\boldsymbol{T}$ is said to be frame indifferent. Now we have

$$\boldsymbol{v}^* = \boldsymbol{Q}^*\,\boldsymbol{v} = \boldsymbol{T}^*\,\boldsymbol{u}^* = \boldsymbol{T}^*\,\boldsymbol{Q}\,\boldsymbol{u} \qquad \Rightarrow \qquad \boldsymbol{v} = \boldsymbol{Q}^T\boldsymbol{T}^*\,\boldsymbol{Q}\,\boldsymbol{u}.$$

Therefore the requirement for frame indifference of a second-order tensor $\boldsymbol{T}$ is defined by

$$\boldsymbol{T} = \boldsymbol{Q}^T\boldsymbol{T}^*\,\boldsymbol{Q}, \qquad \boldsymbol{T}^* = \boldsymbol{Q}\,\boldsymbol{T}\,\boldsymbol{Q}^T. \tag{2.24}$$

2.2.3 Motion in a Non-inertial System ♣

We consider an equation of motion for the case where System 1 is inertial while System 2 is non-inertial. Since the two-point vector $\boldsymbol{x} - \boldsymbol{x}_0$ should be frame indifferent, satisfying (2.17) and (2.18), a position vector $\boldsymbol{x}^*(t)$ in System 2 is given by

$$\boldsymbol{x}^*(t) = \boldsymbol{x}_0^*(t) + \boldsymbol{Q}(t)(\boldsymbol{x} - \boldsymbol{x}_0) \tag{2.25}$$

where we can regard $\boldsymbol{x}_0$ as the origin of the coordinate system O, and $\boldsymbol{x}_0^*(t)$ as the origin of the coordinate system O^* (see Fig. 2.3). Taking a material time derivative of (2.25) and substituting the inverse relation of (2.17) gives the velocity:

$$\boldsymbol{v}^* = \frac{d\boldsymbol{x}_0^*}{dt} + \boldsymbol{\Omega}(\boldsymbol{x}^* - \boldsymbol{x}_0^*) + \boldsymbol{Q}\boldsymbol{v}, \tag{2.26}$$

$$\boldsymbol{\Omega} = \frac{d\boldsymbol{Q}}{dt}\boldsymbol{Q}^T \tag{2.27}$$

The result (2.26) shows that the velocity vector $\boldsymbol{v}$ is not frame indifferent. Note that by differentiating $(2.20)_1$ the second-order tensor $\boldsymbol{\Omega}$ is understood to be anti-symmetric:

$$\frac{d\boldsymbol{Q}}{dt}\boldsymbol{Q}^T + \boldsymbol{Q}\frac{d\boldsymbol{Q}^T}{dt} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{\Omega} = \frac{d\boldsymbol{Q}}{dt}\boldsymbol{Q}^T = -\boldsymbol{Q}\frac{d\boldsymbol{Q}^T}{dt} = -\boldsymbol{\Omega}^T. \tag{2.28}$$

The acceleration vector is given by a material time derivative of (2.26):

$$\boldsymbol{a}^* = \frac{d\boldsymbol{v}^*}{dt} = \frac{d^2\boldsymbol{x}_0^*}{dt^2} + 2\boldsymbol{\Omega}\left(\boldsymbol{v}^* - \frac{d\boldsymbol{x}_0^*}{dt}\right) + \left(\frac{d\boldsymbol{\Omega}}{dt} - \boldsymbol{\Omega}^2\right)(\boldsymbol{x}^* - \boldsymbol{x}_0^*) + \boldsymbol{Q}\frac{d\boldsymbol{v}}{dt} \tag{2.29}$$

This suggests that the acceleration is not frame indifferent.

Now we define a rotation vector $\boldsymbol{\omega}$ with respect to the original coordinate system by

$$\omega_i = -\frac{1}{2}e_{ijk}\Omega_{kj} = \frac{1}{2}e_{ijk}\Omega_{jk} \tag{2.30}$$

then for any vector $\boldsymbol{a}$ we have

$$\boldsymbol{\Omega}\boldsymbol{a} = \boldsymbol{\omega} \wedge \boldsymbol{a} \tag{2.31}$$

Thus, (2.29) can be written as

$$\frac{d\boldsymbol{v}^*}{dt} = \frac{d^2\boldsymbol{x}_0^*}{dt^2} + \frac{d\boldsymbol{\Omega}}{dt}\boldsymbol{Q}(\boldsymbol{x}-\boldsymbol{x}_0) + \boldsymbol{\omega}\wedge[\boldsymbol{\omega}\wedge\boldsymbol{Q}(\boldsymbol{x}-\boldsymbol{x}_0)] + 2\boldsymbol{\omega}\wedge\boldsymbol{Q}\boldsymbol{v} + \boldsymbol{Q}\frac{d\boldsymbol{v}}{dt} \tag{2.32}$$

The third term of the r.h.s. of this equation gives a centrifugal force and the fourth is the Coriolis force.

As shown by (2.5), a fundamental hypothesis of Newtonian mechanics is that the force $\boldsymbol{f}$ is frame indifferent ($\boldsymbol{f}^* = \boldsymbol{Q}\,\boldsymbol{f}$). Therefore, referring to (2.32) and (2.27), the equation of motion in System 2, which is non-inertial, can be written as

$$m^*\frac{d\boldsymbol{v}^*}{dt^*} = \boldsymbol{f}^* + \boldsymbol{f}^{\mathrm{ai}}, \tag{2.33}$$

$$\begin{aligned}\boldsymbol{f}^{\mathrm{ai}} = m\Bigg[&\frac{d^2\boldsymbol{x}_0^*}{dt^2} + \frac{d\boldsymbol{\Omega}}{dt}(\boldsymbol{x}^* - \boldsymbol{x}_0^*) + \boldsymbol{\omega}\wedge\boldsymbol{\omega}\wedge(\boldsymbol{x}^* - \boldsymbol{x}_0^*)\\ &+2\boldsymbol{\omega}\wedge\left(\boldsymbol{v}^* - \frac{d\boldsymbol{x}_0^*}{dt} - \boldsymbol{\Omega}(\boldsymbol{x}^* - \boldsymbol{x}_0^*)\right)\Bigg]\end{aligned} \tag{2.34}$$

where $\boldsymbol{f}^{\mathrm{ai}}$ is an apparent inertial force observed in the non-inertial system.

2.2.4 Deformation Gradient, Strain and Strain Rate

We distinguish between the coordinate system $\{\boldsymbol{E}_I\}$ of the Lagrangian description and the coordinate system $\{\boldsymbol{e}_i\}$ of the Eulerian description in order to understand the relationship between both systems.

As shown in Fig. 2.2, an increment vector $d\boldsymbol{x} = dx_i \boldsymbol{e}_i$ of a point $\boldsymbol{x} \in \Omega$ in the deformed body Ω is related to the increment $d\boldsymbol{X} = dX_I \boldsymbol{E}_I$ of the corresponding point $\boldsymbol{X} \in \Omega_0$ in the undeformed body Ω_0 by

$$d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X} \tag{2.35}$$

where the second-order tensor $\boldsymbol{F}$ is the *deformation gradient*;

$$\boldsymbol{F} = \mathrm{Grad}\,\boldsymbol{x} = \nabla_X \boldsymbol{x} = F_{iI}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_I, \tag{2.36}$$

$$F_{iI} = \frac{\partial x_i}{\partial X_I}, \qquad \mathrm{Grad} = \nabla_X = \boldsymbol{E}_I \frac{\partial}{\partial X_I} \tag{2.37}$$

which relates the infinitesimal line segment $d\boldsymbol{X} \in \Omega_0$ to the corresponding segment $d\boldsymbol{x} \in \Omega$. As observed in (2.36), the deformation gradient $\boldsymbol{F}$ plays a role in the coordinate transformation from $\boldsymbol{E}_I$ to $\boldsymbol{e}_i$. Since no material point vanishes, there is an inverse relation of $\boldsymbol{F}$:

$$\boldsymbol{F}^{-1} = \mathrm{grad}\,\boldsymbol{X} = \nabla_x \boldsymbol{X} = F^{-1}_{Ii}\, \boldsymbol{E}_I \otimes \boldsymbol{e}_i, \tag{2.38}$$

$$F^{-1}_{Ii} = \frac{\partial X_I}{\partial x_i}, \qquad \mathrm{grad} = \nabla_x = \boldsymbol{e}_i \frac{\partial}{\partial x_i}. \tag{2.39}$$

If the right form (see Note 2.2) is employed, the related forms of deformation gradient can be written as follows:

$$\boldsymbol{F} = \frac{\partial x_i}{\partial X_I} \boldsymbol{e}_i \otimes \boldsymbol{E}_I, \qquad \boldsymbol{F}^T = \frac{\partial x_i}{\partial X_I} \boldsymbol{E}_I \otimes \boldsymbol{e}_i, \tag{2.40}$$

$$\boldsymbol{F}^{-1} = \frac{\partial X_I}{\partial x_i} \boldsymbol{E}_I \otimes \boldsymbol{e}_i, \qquad \boldsymbol{F}^{-T} = \frac{\partial X_I}{\partial x_i} \boldsymbol{e}_i \otimes \boldsymbol{E}_I. \tag{2.41}$$

Here $\boldsymbol{F}^{-T}$ implies $(\boldsymbol{F}^{-1})^T$.

Now we need to measure the extent of deformation of an elemental length located at a material point. To do so, we compare the length $|d\boldsymbol{x}|$ with its original length $|d\boldsymbol{X}|$ (see Fig. 2.2) by comparing the difference of both lengths as a squared measure:

$$|\, d\boldsymbol{x}\,|^2 - |\, d\boldsymbol{X}\,|^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} - d\boldsymbol{X} \cdot d\boldsymbol{X}$$

Substituting the deformation gradient yields

$$d\boldsymbol{x} \cdot d\boldsymbol{x} = d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X}, \qquad d\boldsymbol{X} \cdot d\boldsymbol{X} = d\boldsymbol{x} \cdot \boldsymbol{B}^{-1} d\boldsymbol{x}$$

where

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = F_{kI} F_{kJ} \, \boldsymbol{E}_I \otimes \boldsymbol{E}_J, \qquad \boldsymbol{B} = \boldsymbol{F} \boldsymbol{F}^T = F_{iI} F_{jI} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.42}$$

$\boldsymbol{C}$ is referred to as the *right Cauchy-Green tensor* and $\boldsymbol{B}$ the *left Cauchy-Green tensor*. Then the deformation measure can be written as

$$|\, d\boldsymbol{x} \,|^2 - |\, d\boldsymbol{X} \,|^2 = d\boldsymbol{X} \cdot 2\boldsymbol{E} \, d\boldsymbol{X} = d\boldsymbol{x} \cdot 2\boldsymbol{e} \, d\boldsymbol{x} \tag{2.43}$$

where we set

$$\boldsymbol{E} = \frac{1}{2} (\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2} (C_{IJ} - \delta_{IJ}) \, \boldsymbol{E}_I \otimes \boldsymbol{E}_J, \tag{2.44}$$

$$\boldsymbol{e} = \frac{1}{2} \left(\boldsymbol{i} - \boldsymbol{B}^{-1}\right) = \frac{1}{2} \left(\delta_{ij} - B_{ij}^{-1}\right) \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.45}$$

Note that $\boldsymbol{I} = \delta_{IJ} \, \boldsymbol{E}_I \otimes \boldsymbol{E}_J$ and $\boldsymbol{i} = \delta_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ are the unit tensors in the reference and current configurations, respectively. $\boldsymbol{E}$ is referred to as the *Lagrangian strain* or *Green strain*, and $\boldsymbol{e}$ the *Eulerian strain* or *Almansi strain* (see, e.g., Malvern 1969; Spencer 2004).

Since the deformation gradient $\boldsymbol{F}$ is invertible and positive definite ($\det \boldsymbol{F} > 0$), we can introduce the following polar decomposition:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R} \tag{2.46}$$

where

$$\boldsymbol{R} = R_{iI} \, \boldsymbol{e}_i \otimes \boldsymbol{E}_I, \qquad \boldsymbol{U} = U_{IJ} \, \boldsymbol{E}_I \otimes \boldsymbol{E}_J, \qquad \boldsymbol{V} = V_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.47}$$

$\boldsymbol{R}$ is referred to as the *rotation tensor*, $\boldsymbol{U}$ the *right stretch tensor*, $\boldsymbol{V}$ the *left stretch tensor*. $\boldsymbol{R}$ is orthonormal ($\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}$, $\boldsymbol{R} \boldsymbol{R}^T = \boldsymbol{i}$), which gives the rotation of $\boldsymbol{C}$ and $\boldsymbol{B}^{-1}$ to their principal axes. Under the polar decomposition we have

$$\boldsymbol{C} = \boldsymbol{U}^2, \qquad \boldsymbol{B} = \boldsymbol{V}^2 \tag{2.48}$$

As understood from (2.48), $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric and positive definite.

Note 2.3 (Small strain theory). In most textbooks on elasticity theory a displacement vector is defined as $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$. However we know that $\boldsymbol{x} = x_i \boldsymbol{e}_i$, $\boldsymbol{X} = X_I \boldsymbol{E}_I$ and the transformation of both bases are locally defined by the deformation gradient $\boldsymbol{F}$; therefore it is difficult to introduce the globally defined displacement vector $\boldsymbol{u}$ unless a common rectangular Cartesian coordinate system is used.

Now let us use a common basis $\boldsymbol{e}_i$ and introduce an incremental form as

$$d\boldsymbol{u} = d\boldsymbol{x} - d\boldsymbol{X} = (\boldsymbol{F} - \boldsymbol{i}) \, d\boldsymbol{X} = \boldsymbol{H} \, d\boldsymbol{X} \tag{2.49}$$

where

$$\boldsymbol{H} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \boldsymbol{F} - \boldsymbol{i} = (F_{ij} - \delta_{ij})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.50}$$

Then the Green strain is given by

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H}^T \boldsymbol{H}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.51}$$

Since the third term of the r.h.s. is second-order infinitesimal, the small strain tensor is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.52}$$

where we identify the coordinates X_i with x_i (Little 1973; Davis and Selvadurai 1996; Barber 2002).

The strain with components given by (2.52) is referred to as the tensorial strain, while the strain in which the shearing components are changed into

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \qquad \gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}, \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

which can be denoted by the vector

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \gamma_{xy},\ \gamma_{yz},\ \gamma_{zx}\right]^T \tag{2.53}$$

that is referred to as the engineering strain. ■

Note 2.4 (Generalized strain measure (Hill 1978)♣). Since the right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ is symmetric and the components are real numbers, there are three real eigenvalues that are set as λ_i^2 $(i = 1, 2, 3)$ and the corresponding eigenvectors are given by $\boldsymbol{N}_i$; then we have

$$\left(\boldsymbol{F}^T \boldsymbol{F}\right) \boldsymbol{N}_i - \lambda_i^2 \boldsymbol{N}_i = \boldsymbol{0}, \qquad \boldsymbol{N}_i \cdot \boldsymbol{N}_j = \delta_{ij} \qquad (i\ :\ \text{not summed}). \tag{2.54}$$

Let us set a material fiber along $\boldsymbol{N}_i$ as $d\boldsymbol{X}_i$, therefore we can write

$$d\boldsymbol{X}_i = dX_i\, \boldsymbol{N}_i \quad \Rightarrow \quad \left(\boldsymbol{F}^T \boldsymbol{F}\right) d\boldsymbol{X}_i = \lambda_i^2\, d\boldsymbol{X}_i \qquad (i\ :\ \text{not summed}) \tag{2.55}$$

where λ_i is referred to as the principal stretch , and $\boldsymbol{N}_i$ $(i = 1, 2, 3)$ form Lagrangian triads. Referring $(2.48)_1$, the right stretch tensor $\boldsymbol{U}$ can be written as

$$\boldsymbol{U} = \sum_i \lambda_i\, \boldsymbol{N}_i \otimes \boldsymbol{N}_i. \tag{2.56}$$

The rotation tensor $\boldsymbol{R}$ given by $(2.47)_1$ transforms the Lagrangian triads $\boldsymbol{N}_i$ into the Eulerian triads $\boldsymbol{n}_i$:

$$\boldsymbol{n}_i = \boldsymbol{R}\,\boldsymbol{N}_i\,, \qquad \boldsymbol{R} = \sum_i \boldsymbol{n}_i \otimes \boldsymbol{N}_i\,. \tag{2.57}$$

The left stretch tensor $\boldsymbol{V}$ given by $(2.47)_3$ can now be written as

$$\boldsymbol{V} = \sum_i \lambda_i\,\boldsymbol{n}_i \otimes \boldsymbol{n}_i\,. \tag{2.58}$$

Following Hill (1978) the generalized Lagrangian strain measure is defined by

$$\mathfrak{E} = \sum_i f(\lambda_i)\,\boldsymbol{N}_i \otimes \boldsymbol{N}_i\,. \tag{2.59}$$

Here $f(\lambda_i)$ is a scale function which satisfies the conditions

$$f(1) = 0, \qquad f'(1) = 1.$$

Consider the following example:

$$f(z) = \frac{z^{2n} - 1}{2n}. \tag{2.60}$$

We can introduce the following family of n-th order Lagrangian strain measures:

$$\boldsymbol{E}(n) = \sum_i \frac{(\lambda_i)^{2n} - 1}{2n}\,\boldsymbol{N}_i \otimes \boldsymbol{N}_i\,. \tag{2.61}$$

The Green strain given by (2.44) corresponds to $\boldsymbol{E}(1)$. Furthermore from (2.60) we have

$$\lim_{n\to 0} \frac{(\lambda_i)^{2n} - 1}{2n} = \ln \lambda_i\,.$$

As $n \to 0$, the logarithmic strain $\boldsymbol{E}(0)$ is given by

$$\boldsymbol{E}(0) = \ln\,\boldsymbol{U} = \sum_i \ln(\lambda_i)\,\boldsymbol{N}_i \otimes \boldsymbol{N}_i\,. \tag{2.62}$$

The generalized Eulerian strain measure is given by

$$\mathfrak{e} = \sum_i f(\lambda_i)\,\boldsymbol{n}_i \otimes \boldsymbol{n}_i = \boldsymbol{R}\,\mathfrak{E}\,\boldsymbol{R}^T, \tag{2.63}$$

and the family of n-th order Eulerian strain measures is introduced by

$$\boldsymbol{e}(n) = \sum_i \frac{(\lambda_i)^{2n} - 1}{2n} \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \qquad \boldsymbol{e}(0) = \ln \boldsymbol{V} = \sum_i \ln(\lambda_i)\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i. \quad (2.64)$$

It should be noted that $\boldsymbol{E}(-1) \neq \boldsymbol{e}(1)$. ■

Recall the definition of the deformation gradient $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$; its material time-derivative defines the following *velocity gradient*:

$$\overline{d\boldsymbol{x}}^{\,\cdot} = \dot{\boldsymbol{F}}\, d\boldsymbol{X} = \boldsymbol{L}\, d\boldsymbol{x} \qquad \Rightarrow \qquad \boldsymbol{L} \equiv \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}. \quad (2.65)$$

Since the direct forms of $\dot{\boldsymbol{F}}$, $\boldsymbol{F}^{-1}$ are given by

$$\dot{\boldsymbol{F}} = \dot{F}_{iI}\, \boldsymbol{e}_i \otimes \boldsymbol{E}_I = \frac{\partial v_i}{\partial X_I} \boldsymbol{e}_i \otimes \boldsymbol{E}_I, \qquad \boldsymbol{F}^{-1} = \frac{\partial X_I}{\partial x_i} \boldsymbol{E}_I \otimes \boldsymbol{e}_i \quad (2.66)$$

$\boldsymbol{L}$ can be written as

$$\boldsymbol{L} = \text{grad}\, \boldsymbol{v} = \frac{\partial v_i}{\partial x_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_k \quad (2.67)$$

On the other hand, $\boldsymbol{F}\,\boldsymbol{F}^{-1} = \boldsymbol{i}$, therefore taking its material time differential with (2.65) $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$ yields the inverse of $\dot{\boldsymbol{F}}$:

$$\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1}\boldsymbol{L}. \quad (2.68)$$

The velocity gradient $\boldsymbol{L}$ is decomposed into its symmetric part $\boldsymbol{D}$, called the *stretch tensor* or *rate-of-deformation tensor*, and its anti-symmetric part $\boldsymbol{W}$, called the *spin tensor*:

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = \boldsymbol{D} + \boldsymbol{W}, \quad (2.69)$$

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right), \qquad \boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{L} - \boldsymbol{L}^T\right). \quad (2.70)$$

The material time differentiation of the Green strain $\boldsymbol{E}$ given by (2.44) is

$$\dot{\boldsymbol{E}} = \frac{1}{2}\left(\dot{\boldsymbol{F}}^T\boldsymbol{F} + \boldsymbol{F}^T\dot{\boldsymbol{F}}\right) = \boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F} \qquad \Rightarrow \qquad \boldsymbol{D} = \boldsymbol{F}^{-T}\dot{\boldsymbol{E}}\boldsymbol{F}^{-1} \quad (2.71)$$

where (2.65) is used.

Note 2.5 (Embedded coordinates ♣). In solids we can trace each material point $\boldsymbol{X}$ by attaching labels on its surface during deformation. Then, as shown in Fig. 2.4, it is easy to introduce a coordinate system in which the label values of the coordinates are not changed ($x^i = X^i$) but the reference basis $\boldsymbol{G}_i$ (before deformation) is changed into the current basis $\boldsymbol{g}_i$ (after deformation). This is referred to as the *embedded coordinate system*.

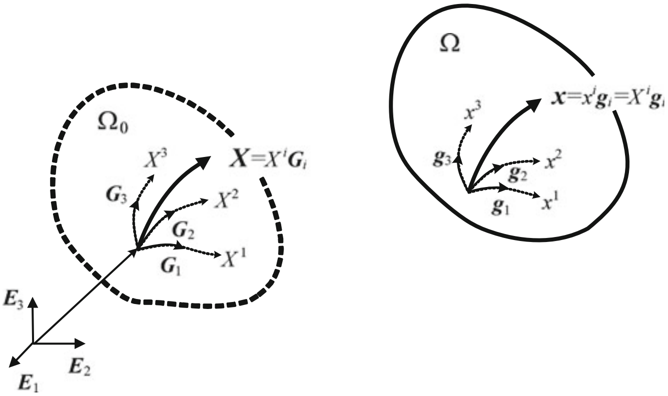


Fig. 2.4 Embedded coordinate system

In the embedded coordinate system, the transformation rule of the base vectors is given by

$$\boldsymbol{g}_i(t) = \boldsymbol{F}(t)\boldsymbol{G}_i \qquad \Rightarrow \qquad \boldsymbol{F}(t) = \boldsymbol{g}_i(t) \otimes \boldsymbol{G}^i, \tag{2.72}$$

and the following relationships are obtained:

$$\boldsymbol{F}^{-1} = \boldsymbol{G}_i \otimes \boldsymbol{g}^i, \qquad \boldsymbol{G}_i = \boldsymbol{F}^{-1}\boldsymbol{g}_i, \tag{2.73}$$

$$\boldsymbol{F}^{-T} = \boldsymbol{g}^i \otimes \boldsymbol{G}_i, \qquad \boldsymbol{g}^i = \boldsymbol{F}^{-T}\boldsymbol{G}^i. \tag{2.74}$$

The unit tensor in the current deformed body is written as $\boldsymbol{i} = \delta_i^{\,j}\boldsymbol{g}^i \otimes \boldsymbol{g}_j = \delta^i{}_j\,\boldsymbol{g}_i \otimes \boldsymbol{g}^j = g_{ij}\,\boldsymbol{g}^i \otimes \boldsymbol{g}^j = g^{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j$, and the right Cauchy-Green tensor $\boldsymbol{C}$ is given by

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = (\boldsymbol{G}^i \otimes \boldsymbol{g}_i)(\boldsymbol{g}_j \otimes \boldsymbol{G}^j) = g_{ij}\,\boldsymbol{G}^i \otimes \boldsymbol{G}^j. \tag{2.75}$$

Let an arbitrary second-order tensor $\boldsymbol{K}$ in the current body be written as

$$\boldsymbol{K} = K^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = K^i{}_j\,\boldsymbol{g}_i \otimes \boldsymbol{g}^j = K_i^{\,j}\,\boldsymbol{g}^i \otimes \boldsymbol{g}_j = K_{ij}\,\boldsymbol{g}^i \otimes \boldsymbol{g}^j, \tag{2.76}$$

then the second-order tensors that have the same components in the undeformed body are given by

$$\begin{aligned} \boldsymbol{K}^{(I)} &= K^{ij}\,\boldsymbol{G}_i \otimes \boldsymbol{G}_j &&= \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F}^{-T} \\ \boldsymbol{K}^{(II)} &= K^i{}_j\,\boldsymbol{G}_i \otimes \boldsymbol{G}^j &&= \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F} \\ \boldsymbol{K}^{(III)} &= K_i^{\,j}\,\boldsymbol{G}^i \otimes \boldsymbol{G}_j &&= \boldsymbol{F}^T\boldsymbol{K}\boldsymbol{F}^{-T} \\ \boldsymbol{K}^{(IV)} &= K_{ij}\,\boldsymbol{G}^i \otimes \boldsymbol{G}^j &&= \boldsymbol{F}^T\boldsymbol{K}\boldsymbol{F}. \end{aligned} \tag{2.77}$$

Material time differentiation of (2.72) yields $\dot{\boldsymbol{F}} = \dot{\boldsymbol{g}}_i(t) \otimes \boldsymbol{G}^i$ (note $\boldsymbol{G}^i =$ constant). As shown in (2.65), $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$, therefore the velocity gradient $\boldsymbol{L}$ in the embedded coordinates is

$$\dot{\boldsymbol{g}}_i(t) = \boldsymbol{L}\boldsymbol{g}_i(t) \qquad \Rightarrow \qquad \boldsymbol{L}(t) = \dot{\boldsymbol{g}}_i(t) \otimes \boldsymbol{g}^i(t). \tag{2.78}$$

The stretch tensor $\boldsymbol{D}$ and spin tensor $\boldsymbol{W}$ are now given by

$$\boldsymbol{D} = \frac{1}{2}\left(\dot{\boldsymbol{g}}_i \otimes \boldsymbol{g}^i + \boldsymbol{g}^i \otimes \dot{\boldsymbol{g}}_i\right) = \frac{1}{2}\left(\dot{g}_{ij}\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j\right), \tag{2.79}$$

$$\boldsymbol{W} = \frac{1}{2}\left(\dot{\boldsymbol{g}}_i \otimes \boldsymbol{g}^i - \boldsymbol{g}^i \otimes \dot{\boldsymbol{g}}_i\right) \tag{2.80}$$

■

2.2.5 Transport Theorems and Jump Condition

A volume element dV in the undeformed body Ω_0 is related to the volume element dv in the deformed body Ω through the determinant, J, of the deformation gradient $\boldsymbol{F}$ by

$$dv = J\, dV, \tag{2.81}$$

$$J = \det \boldsymbol{F} = \left|\frac{\partial x_i}{\partial X_I}\right| = e_{IJK} \frac{\partial x_1}{\partial X_I} \frac{\partial x_2}{\partial X_J} \frac{\partial x_3}{\partial X_K} \tag{2.82}$$

where J is referred to as the *Jacobian*. If we recall that

$$e_{IJK} \frac{\partial x_1}{\partial X_I} \frac{\partial x_1}{\partial X_J} \frac{\partial x_2}{\partial X_K} = 0, \quad e_{IJK} \frac{\partial x_2}{\partial X_I} \frac{\partial x_2}{\partial X_J} \frac{\partial x_3}{\partial X_K} = 0, \quad \cdots$$

the material time derivative of the Jacobian is given by

$$\dot{J} = J \frac{\partial v_k}{\partial x_k} \qquad \Rightarrow \qquad \dot{J} = J\, \nabla \cdot \boldsymbol{v} = J \operatorname{tr} \boldsymbol{L} = J \operatorname{tr} \boldsymbol{D} \tag{2.83}$$

Under this relation the material time derivative of the integral of an arbitrary function ϕ can be calculated as

$$\frac{d}{dt}\int_{\Omega} \phi\, dv = \frac{d}{dt}\int_{\Omega_0} \phi\, J\, dV = \int_{\Omega}\left(\frac{d\phi}{dt} + \phi \frac{\partial v_i}{\partial x_i}\right) dv$$

We now have the following *Reynolds' transport theorem*:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi\, dv &= \int_\Omega \left(\frac{d\phi}{dt} + \phi \nabla\cdot \boldsymbol{v}\right) dv = \int_\Omega \left(\frac{d\phi}{dt} + \phi \,\mathrm{tr}\, \boldsymbol{L}\right) dv \\ &= \int_\Omega \left(\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi \boldsymbol{v})\right) dv = \int_\Omega \frac{\partial \phi}{\partial t} dv + \int_{\partial\Omega} \phi \boldsymbol{v}\cdot\boldsymbol{n}\, ds. \qquad (2.84)\end{aligned}$$

Next we give the transport theorem for a surface integral: The determinant of a (3×3)-matrix $\boldsymbol{A}$ is given by

$$e_{rst} \det \boldsymbol{A} = e_{ijk} A_{ir} A_{js} A_{kt}.$$

Therefore (2.81) can be written as

$$e_{ijk}\, J^{-1} = e_{IJK} \frac{\partial X_I}{\partial x_i}\frac{\partial X_J}{\partial x_j}\frac{\partial X_K}{\partial x_k}, \qquad e_{IJK}\, J = e_{ijk} \frac{\partial x_i}{\partial X_I}\frac{\partial x_j}{\partial X_J}\frac{\partial x_k}{\partial X_K}$$

A surface element $d\boldsymbol{S} = d\boldsymbol{X} \wedge \delta\boldsymbol{X}$ consists of line elements $d\boldsymbol{X}$ and $\delta\boldsymbol{X}$ in the undeformed body and the corresponding surface element $d\boldsymbol{s} = d\boldsymbol{x} \wedge \delta\boldsymbol{x}$ consists of line elements $d\boldsymbol{x}$ and $\delta\boldsymbol{x}$ in the deformed body

$$d\boldsymbol{S} = \boldsymbol{N}\, dS = d\boldsymbol{X} \wedge \delta\boldsymbol{X}, \qquad d\boldsymbol{s} = \boldsymbol{n}\, ds = d\boldsymbol{x} \wedge \delta\boldsymbol{x} \qquad (2.85)$$

where $\boldsymbol{N}$ and $\boldsymbol{n}$ are outward normals of $d\boldsymbol{S}$ and $d\boldsymbol{s}$, respectively (Fig. 2.5). If $d\boldsymbol{S}$ is deformed into $d\boldsymbol{s}$, $(2.85)_1$ implies

$$N_I\, dS = e_{IJK}\, dX_J\, dX_K = e_{IJK} \frac{\partial X_J}{\partial x_j}\frac{\partial X_K}{\partial x_k} dx_j\, dx_k,$$

and we have

$$\frac{\partial X_I}{\partial x_i} N_I\, dS = e_{IJK} \frac{\partial X_I}{\partial x_i}\frac{\partial X_J}{\partial x_j}\frac{\partial X_K}{\partial x_k} dx_j dx_k = J^{-1} n_i\, ds \;\Rightarrow\; n_i\, ds = J \frac{\partial X_I}{\partial x_i} N_I\, dS$$

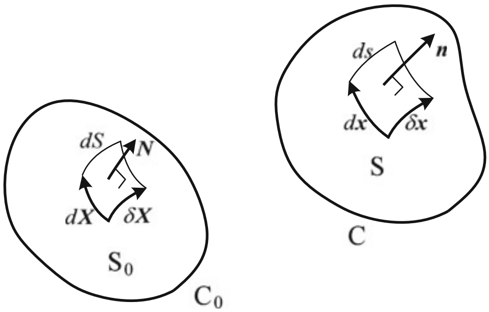


Fig. 2.5 Surface elements in the reference body C_0 and the current body C

Its vector form is given by

$$d\boldsymbol{s} = \boldsymbol{n}\, ds = J\, \boldsymbol{F}^{-T} \boldsymbol{N}\, dS = J\, \boldsymbol{F}^{-T} d\boldsymbol{S} \tag{2.86}$$

where $\boldsymbol{F}^{-T} = (\boldsymbol{F}^{-1})^T$. Equation 2.86 is referred to as *Nanson's formula*.

The time-differentiation of $\boldsymbol{F}\ \boldsymbol{F}^{-1} = \boldsymbol{I}$ together with (2.65) gives

$$\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1}\boldsymbol{L},$$

therefore the time-derivative of (2.86) can be written as

$$\frac{d}{dt}(d\boldsymbol{s}) = (\dot{\boldsymbol{F}}^{-1})^T J \boldsymbol{N}\, dS + (\boldsymbol{F}^{-1})^T \dot{J} \boldsymbol{N}\, dS = \left[(\operatorname{tr} \boldsymbol{L})\boldsymbol{I} - \boldsymbol{L}^T\right] d\boldsymbol{s}.$$

Thus the *transport theorem for the surface integral* of a scalar-valued function is given by

$$\frac{d}{dt}\int_S \phi\, d\boldsymbol{s} = \int_S \left[\frac{d\phi}{dt} + \phi \operatorname{tr} \boldsymbol{L} - \phi \boldsymbol{L}^T\right] d\boldsymbol{s}. \tag{2.87}$$

For a vector function $\boldsymbol{q}$ we have $\boldsymbol{q} \cdot \boldsymbol{L}^T d\boldsymbol{s} = \boldsymbol{L}\boldsymbol{q} \cdot d\boldsymbol{s}$ and

$$\frac{d\boldsymbol{q}}{dt} + \boldsymbol{q} \operatorname{tr} \boldsymbol{L} - \boldsymbol{L}\boldsymbol{q} = \frac{\partial \boldsymbol{q}}{\partial t} + \nabla \wedge (\boldsymbol{q} \wedge \boldsymbol{v}) + \boldsymbol{v}\,(\nabla \cdot \boldsymbol{q}).$$

Thus the transport theorem (2.87) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_S \boldsymbol{q} \cdot d\boldsymbol{s} &= \int_S \left[\frac{\partial \boldsymbol{q}}{\partial t} + \nabla \wedge (\boldsymbol{q} \wedge \boldsymbol{v}) + \boldsymbol{v}\,(\nabla \cdot \boldsymbol{q})\right] \cdot d\boldsymbol{s} \\ &= \int_S \left[\frac{\partial \boldsymbol{q}}{\partial t} + (\nabla \cdot \boldsymbol{q})\,\boldsymbol{v}\right] \cdot d\boldsymbol{s} + \int_C (\boldsymbol{v} \wedge \boldsymbol{q}) \cdot d\boldsymbol{x}\end{aligned} \tag{2.88}$$

where $\boldsymbol{C}$ is a line surrounding the surface S.

Next, using (2.65) the time-derivative of the line element $d\boldsymbol{x}$ is given as

$$\frac{d}{dt}(d\boldsymbol{x}) = \dot{\boldsymbol{F}}\, d\boldsymbol{X} = \boldsymbol{L}\, d\boldsymbol{x}.$$

Thus the *transport theorem for the line integral* is derived as

$$\frac{d}{dt}\int_C \phi\, d\boldsymbol{x} = \int_C \left(\frac{d\phi}{dt} + \phi\, \boldsymbol{L}\right) d\boldsymbol{x}. \tag{2.89}$$

For a vector function $\boldsymbol{q}$ this can be written as

$$\frac{d}{dt}\int_C \boldsymbol{q}\cdot d\boldsymbol{x} = \int_C \left(\frac{d\boldsymbol{q}}{dt} + \boldsymbol{L}^T\boldsymbol{q}\right)\cdot d\boldsymbol{x}. \tag{2.90}$$

We next consider transport theorems involving a singular surface Σ that separates a body Ω into Ω^+ and Ω^- (Fig. 2.6). For example, a singular surface corresponds to the front of a shock wave causing a sonic boom or a migrating frozen front during ground freezing. Let the velocity of the singular surface Σ be $\boldsymbol{V}$, $\partial\Omega^+$ be the surface of Ω^+ except for Σ, $\partial\Omega^-$ be the surface of Ω^- except for Σ, and $\boldsymbol{n}$ be a unit normal on Σ directed to Ω^+.

Let ϕ be any function. ϕ^+ implies the value of ϕ on Σ approached from Ω^+, and, similarly, ϕ^- is the value approached from Ω^-. By applying the transport theorem (2.84) at each domain we have

$$\frac{d}{dt}\int_{\Omega+}\phi\, dv = \int_{\Omega+}\frac{\partial\phi}{\partial t}dv + \int_{\partial\Omega+}\phi\boldsymbol{v}\cdot\boldsymbol{n}\, ds - \int_\Sigma \phi^+\boldsymbol{V}\cdot\boldsymbol{n}\, ds$$
$$\frac{d}{dt}\int_{\Omega-}\phi\, dv = \int_{\Omega-}\frac{\partial\phi}{\partial t}dv + \int_{\partial\Omega-}\phi\boldsymbol{v}\cdot\boldsymbol{n}\, ds + \int_\Sigma \phi^-\boldsymbol{V}\cdot\boldsymbol{n}\, ds$$

where $\boldsymbol{v}$ is the velocity of each material point. Adding both equations under $V_n = \boldsymbol{V}\cdot\boldsymbol{n}$ yields the following *Reynolds' transport theorem with a singular surface*:

$$\frac{d}{dt}\int_\Omega \phi\, dv = \int_\Omega \frac{\partial\phi}{\partial t}dv + \int_{\partial\Omega}\phi\boldsymbol{v}\cdot\boldsymbol{n}\, ds - \int_\Sigma [\![\phi]\!] V_n\, ds \tag{2.91}$$

where

$$[\![\phi]\!] = \phi^+ - \phi^- \tag{2.92}$$

implies a *jump* of the function ϕ on Σ.

We will show in Sects. 2.3–2.5 that physical conservation laws can be written in the following form:

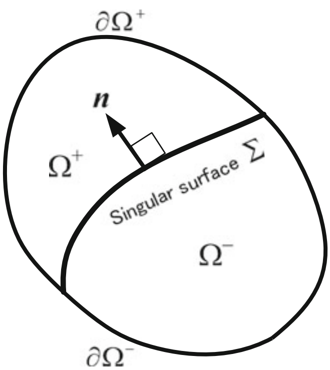


Fig. 2.6 Singular surface

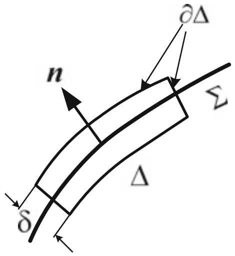


Fig. 2.7 Infinitesimal domain involving a singular surface

$$\frac{d}{dt}\int_{\Omega} \rho\varphi \, dv = \int_{\Omega} \rho s \, dv + \int_{\partial\Omega} q \, ds \tag{2.93}$$

where s is a source per mass of the field variable φ and q is a flux flowing into the body Ω through the surface $\partial\Omega$. We then apply the transport theorem (2.91) for an infinitesimal domain Δ with its boundary $\partial\Delta$ (Fig. 2.7) using the conservation law (2.93):

$$\int_{\Delta} \frac{\partial(\rho\varphi)}{\partial t} dv + \int_{\partial\Delta} \rho\varphi \, \boldsymbol{v}\cdot\boldsymbol{n} \, ds - \int_{\Sigma} [\![\rho\varphi]\!] V_n \, ds = \int_{\Delta} \rho s \, dv + \int_{\partial\Delta} q \, ds.$$

If the thickness δ of the domain Δ approaches zero, all the volumetric terms vanish:

$$\int_{\Sigma} [\![\rho\varphi\,(\boldsymbol{V}-\boldsymbol{v})\cdot\boldsymbol{n} + q]\!] \, ds = 0.$$

We can conclude that, for the conservation law (2.93) involving the singular surface Σ, we have the following singular surface equation:

$$[\![\rho\varphi\,(\boldsymbol{V}-\boldsymbol{v})\cdot\boldsymbol{n} + q]\!] = 0. \tag{2.94}$$

2.3 Mass Conservation Law

We refer again to Fig. 2.2. If there is no mass flux, the total mass $\mathcal{M}$ of the undeformed body Ω_0 is conserved in the deformed body Ω:

$$\mathcal{M} = \int_{\Omega_0} \rho_0 \, dV = \int_{\Omega} \rho \, dv \tag{2.95}$$

where ρ_0 and ρ are the mass densities before and after deformation, respectively. Substituting (2.81) into (2.95) yields

$$\rho_0 - \rho J = 0 \quad \Rightarrow \quad J = \frac{\rho_0}{\rho} \tag{2.96}$$

The time differential form of (2.95) using the Reynolds' transport theorem (2.84) gives

$$\frac{d\mathcal{M}}{dt} = \frac{d}{dt}\int_{\Omega} \rho\, dv = \int_{\Omega} \frac{\partial \rho}{\partial t} dv + \int_{\partial\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{n}\, ds \;=\; 0,$$

or in the local form we have the following *mass conservation law*:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \boldsymbol{v} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \;=\; 0. \tag{2.97}$$

Equation 2.97 is sometimes referred to as the *continuity equation*. Then using (2.81), (2.83) and (2.97) we can see that

$$\dot{dv} = -\frac{\dot{\rho}}{\rho} dv = \operatorname{tr} \boldsymbol{D}\, dV \quad \Rightarrow \quad \nabla \cdot \boldsymbol{v} = \operatorname{tr} \boldsymbol{D} = -\frac{\dot{\rho}}{\rho} = -\frac{d(\ln \rho)}{dt}. \tag{2.98}$$

If the material is incompressible, ρ is constant, and the mass conservation law (2.97) can be written as

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_i}{\partial x_i} = 0, \tag{2.99}$$

which gives the *incompressibility condition*.

The mass conservation law (2.97) gives an alternative form of the Reynolds' transport theorem (2.84) as

$$\frac{d}{dt}\int_{\Omega} \rho \phi\, dv = \int_{\Omega} \rho \frac{d\phi}{dt} dv. \tag{2.100}$$

2.4 Law of Conservation of Linear Momentum and Stress

Newton's second law states that in an inertial frame the rate of linear momentum is equal to the applied force. Here, by applying the second law to a continuum region, we define the Cauchy stress, and derive the equation of motion.

2.4.1 Eulerian Descriptions

Linear momentum of the deformed body Ω is given by

$$\mathcal{L} = \int_{\Omega} \rho \boldsymbol{v}\, dv.$$

If an external force per unit area $\boldsymbol{t}$, called the *traction* or *stress vector*, acts on a boundary $\partial\Omega$, with the body force per unit volume $\boldsymbol{b}$ acting in the volume, the total force is

$$\mathcal{F} = \int_{\partial\Omega} \boldsymbol{t}\, ds + \int_{\Omega} \rho \boldsymbol{b}\, dv.$$

Since Newton's second law is given by $\dot{\mathcal{L}} = \mathcal{F}$,

$$\frac{d}{dt}\int_{\Omega} \rho \boldsymbol{v}\, dv = \int_{\partial\Omega} \boldsymbol{t}\, ds + \int_{\Omega} \rho \boldsymbol{b}\, dv. \tag{2.101}$$

An example of the body force $\boldsymbol{b} = (b_1, b_2, b_3)$ is the gravitational force. If it acts in the negative z-direction, we have $\rho\boldsymbol{b} = (0, 0, -\gamma)$ where $\gamma = \rho g$ (ρ is the mass density, g the acceleration due to gravity and γ the unit weight).

We consider a surface S within the body Ω. One part of Ω bisected by S is denoted by Ω^+ and the other part is denoted by Ω^- (Fig. 2.8). The outward normal vector $\boldsymbol{n}$ is set on the surface S observed from Ω^+. Let an infinitesimal rectangular parallelepiped be located on the surface S with the thickness δ. A surface of the $\boldsymbol{n}$ side of the parallelepiped is referred to as ΔS^+, the opposite side is ΔS^- and other lateral surfaces are ΔS^δ. The surface area of ΔS^+ and ΔS^- is ΔS, and the total area of the lateral surfaces is ΔS^δ. Tractions, i.e., forces per unit area, acting on ΔS^+, ΔS^- and ΔS^δ are $\boldsymbol{t}^+$, $\boldsymbol{t}^-$ and $\boldsymbol{t}^\delta$, respectively. Applying Newton's second law (2.101) to the parallelepiped gives

$$\rho \frac{d\boldsymbol{v}}{dt} \delta \Delta S = \boldsymbol{t}^+ \Delta S + \boldsymbol{t}^- \Delta S + \boldsymbol{t}^\delta \Delta S^\delta + \rho\, \boldsymbol{b}\, \delta \Delta S,$$

and $\delta \to 0$ under $\Delta S =$ constant. The terms in the above equation that are dependent on the volume and the lateral surface all vanish, with the result

$$\boldsymbol{t}^- = -\boldsymbol{t}^+. \tag{2.102}$$

This relation is known as *Cauchy's lemma*.

We next consider an infinitesimal tetrahedron OABC in the body Ω (Fig. 2.9). Let the area of ABC be ds, the outward unit vector on ABC be $\boldsymbol{n}$, the traction acting on ABC be $\boldsymbol{t}$, while the distance between ABC and O is h. In addition, let

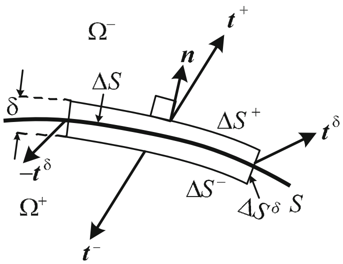
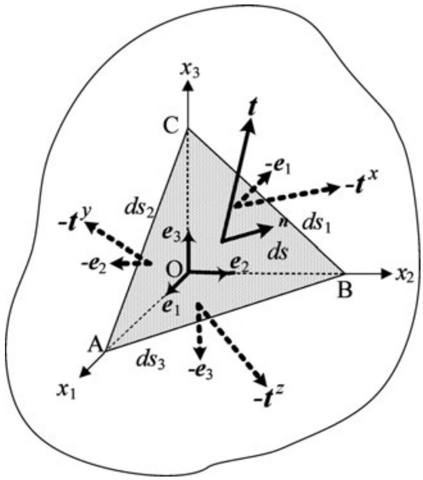


Fig. 2.8 Stress vector defined on an internal surface

Fig. 2.9 Tetrahedron defining the stress



the areas of OBC, OCA and OAB be ds_1, ds_2 and ds_3, respectively, so that we have

$$ds_i = ds\,\cos(\boldsymbol{n}, x_i) = n_i\,ds \quad (i = 1, 2, 3).$$

The volume of the tetrahedron is given as $dv = h\,ds/3$. Let the traction acting on the x^+-surface be $\boldsymbol{t}^x = \boldsymbol{t}^1$. Similarly, $\boldsymbol{t}^y = \boldsymbol{t}^2$ and $\boldsymbol{t}^z = \boldsymbol{t}^3$, which are tractions acting on the y^+- and z^+-surface, respectively. The outward unit normal on OBC is $-\boldsymbol{e}_1$, therefore it is the $\boldsymbol{x}^-$-surface. Then using Cauchy's lemma (2.102) the traction acting on this surface is given by

$$-\boldsymbol{t}^x ds_1 = (-t_1^x, -t_2^x, -t_3^x)^T n_1 ds.$$

Similar results can be obtained for the other surfaces. Now let the body force acting in this tetrahedron be $\boldsymbol{b}$, and the linear momentum be $\rho\boldsymbol{v}$. Then the law of conservation of linear momentum for the tetrahedron states that

$$-\boldsymbol{t}^x n_1 ds - \boldsymbol{t}^y n_2 ds - \boldsymbol{t}^z n_3 ds + \boldsymbol{t}\,ds + \rho\boldsymbol{b}\frac{1}{3}h\,ds = \rho\dot{\boldsymbol{v}}\frac{1}{3}h\,ds.$$

As $h \to 0$, we obtain

$$\boldsymbol{t} = \boldsymbol{t}^x n_1 + \boldsymbol{t}^y n_2 + \boldsymbol{t}^z n_3.$$

Alternatively,

$$\boldsymbol{t} = \boldsymbol{\sigma}^T\boldsymbol{n} = \sigma_{ji}\,n_i\,\boldsymbol{e}_i, \qquad \boldsymbol{\sigma} = \sigma_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.103}$$

where $\sigma_{ij} = t_j^i$. The second-order tensor $\boldsymbol{\sigma}$ is referred to as the *Cauchy stress* or simply *stress*. From (2.103) we understand that the stress tensor $\boldsymbol{\sigma}$ gives a transformation law that maps the unit outward normal $\boldsymbol{n}$ to the traction $\boldsymbol{t}$ acting on that surface.

Returning to the law of conservation of linear momentum (2.101) and using the mass conservation law, we have

$$\frac{d}{dt}\int_{\Omega} \rho \boldsymbol{v}\, dv = \int_{\Omega} \rho \frac{d\boldsymbol{v}}{dt}\, dv.$$

Therefore, by substituting (2.103) into the first term of the r.h.s. of (2.101) and applying the divergence theorem, we have

$$\int_{\partial\Omega} \boldsymbol{t}\, ds = \int_{\partial\Omega} \boldsymbol{\sigma}^T \boldsymbol{n}\, ds = \int_{\Omega} \operatorname{div} \boldsymbol{\sigma}^T\, dv$$

Then we can obtain the following *Eulerian form of the equation of motion* defined in the current deformed body:

$$\rho \frac{d\boldsymbol{v}}{dt}(\boldsymbol{x}, t) = \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = \operatorname{div} \boldsymbol{\sigma}^T(\boldsymbol{x}, t) + \rho\, \boldsymbol{b}(\boldsymbol{x}, t) \tag{2.104}$$

where

$$\operatorname{div} \boldsymbol{\sigma}^T = \nabla \cdot \boldsymbol{\sigma}^T = \frac{\partial \sigma_{ji}}{\partial x_j} \boldsymbol{e}_i.$$

The component form of (2.104) is given by

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho\, b_i \tag{2.105}$$

For a static equilibrium problem the partial differential equation system in Eulerian form together with the boundary conditions is given by

$$\nabla \cdot \boldsymbol{\sigma}^T + \rho\, \boldsymbol{b} = \boldsymbol{0} \tag{2.106}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}} \qquad \text{on} \quad \partial\Omega_u, \tag{2.107}$$

$$\boldsymbol{\sigma}^T \boldsymbol{n}(\boldsymbol{x}) = \bar{\boldsymbol{t}} \qquad \text{on} \quad \partial\Omega_t \tag{2.108}$$

2.4.2 Lagrangian Descriptions ♣

We introduce a relationship between the Cauchy stress $\boldsymbol{\sigma}$ defined in the deformed body with its basis $\{\boldsymbol{e}_i\}$ and the *first Piola-Kirchhoff stress* $\boldsymbol{\Pi}$ defined in the undeformed body with its basis $\{\boldsymbol{E}_I\}$ as follows:

$$\boldsymbol{t}\, ds = \boldsymbol{\sigma}^T \boldsymbol{n}\, ds = \boldsymbol{t}^0\, dS = \boldsymbol{\Pi}^T \boldsymbol{N}\, dS \tag{2.109}$$

where $\boldsymbol{n}$ and $\boldsymbol{N}$ are unit outward normals defined on ds for the deformed body and on dS for the undeformed body. The vector $\boldsymbol{t}$ is a traction defined on ds, and $\boldsymbol{t}^0$ is the shifted vector of $\boldsymbol{t}$ on dS (see Fig. 2.10). By substituting Nanson's relation (2.86), such that $\boldsymbol{n}\,ds = J\,\boldsymbol{F}^{-T}\boldsymbol{N}\,dS$, into (2.109), the first Piola-Kirchhoff stress can be written as

$$\boldsymbol{\Pi} = \Pi_{Ii}\boldsymbol{E}_I \otimes \boldsymbol{e}_i = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}, \qquad \Pi_{Ii} = J\frac{\partial X_I}{\partial x_j}\sigma_{ji} \tag{2.110}$$

The transpose of the first Piola-Kirchhoff stress $\boldsymbol{S} = \boldsymbol{\Pi}^T$ is known as the nominal stress:

$$\boldsymbol{S} = S_{iI}\boldsymbol{e}_i \otimes \boldsymbol{E}_I = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}, \qquad S_{iI} = J\sigma_{ij}\frac{\partial X_I}{\partial x_j} \tag{2.111}$$

Note that in some books, e.g., Kitagawa (1987) pp. 33, the first Piola-Kirchhoff stress and the nominal stress are defined in an opposite sense.

Since the first Piola-Kirchhoff stress $\boldsymbol{\Pi}$ is not symmetric as understood by (2.110), we introduce a symmetrized tensor $\boldsymbol{T}$, called the *second Piola-Kirchhoff stress*, and the Euler stress $\boldsymbol{\tau}$, which is the transformed tensor of $\boldsymbol{T}$, into the deformed body using the rotation tensor $\boldsymbol{R}$:

$$\boldsymbol{T} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{\Pi}\,\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\boldsymbol{S}, \tag{2.112}$$

$$\boldsymbol{\tau} = \boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^T. \tag{2.113}$$

Referring to (2.109), Newton's equation of motion (2.101) can be expressed by the Lagrangian description as

$$\int_{\Omega_0} \rho\,\frac{d\boldsymbol{v}}{dt}(\boldsymbol{X}, t)\,J\,dV = \int_{\partial\Omega_0} \boldsymbol{\Pi}^T\boldsymbol{N}(\boldsymbol{X}, t)\,dS + \int_{\Omega_0} \rho\,\boldsymbol{b}(\boldsymbol{X}, t)\,J\,dV.$$

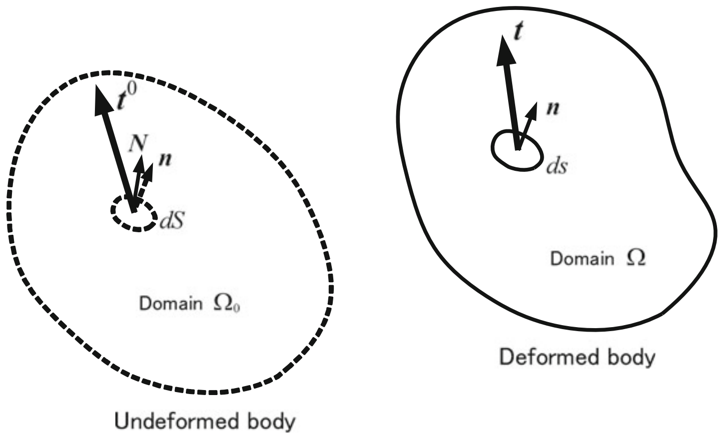


Fig. 2.10 Traction vectors represented for undeformed and deformed bodies

Recalling $J = \rho_0/\rho$ and applying the divergence theorem to the first term of the r.h.s. yields the following *Lagrangian form of the equation of motion*:

$$\rho_0 \frac{d\boldsymbol{v}}{dt}(\boldsymbol{X}, t) = \mathrm{Div}\, \boldsymbol{\Pi}^T(\boldsymbol{X}, t) + \rho_0 \boldsymbol{b}(\boldsymbol{X}, t) \tag{2.114}$$

where we have

$$\mathrm{Div}\, \boldsymbol{\Pi}^T = (\Pi_{Ii} \boldsymbol{e}_i \otimes \boldsymbol{E}_I)\left(\boldsymbol{e}_j \frac{\partial}{\partial X_J}\right) = \frac{\partial \Pi_{Ii}}{\partial X_I} \boldsymbol{e}_i .$$

Then a component form of (2.114) is given by

$$\rho_0 \frac{dv_i}{dt} = \frac{\partial \Pi_{Ii}}{\partial X_I} + \rho_0\, b_i . \tag{2.115}$$

For a static equilibrium problem, the system of partial differential equations in Lagrangian form together with the boundary conditions is given by

$$\mathrm{Div}\, \boldsymbol{\Pi}^T(\boldsymbol{X}) + \rho_0 \boldsymbol{b}(\boldsymbol{X}) = \boldsymbol{0} \tag{2.116}$$

$$\boldsymbol{u}(\boldsymbol{X}) = \bar{\boldsymbol{u}}^0 \qquad \text{on} \quad \partial\Omega_u, \tag{2.117}$$

$$\boldsymbol{\Pi}^T \boldsymbol{N}(\boldsymbol{X}) = \bar{\boldsymbol{t}}^0 \qquad \text{on} \quad \partial\Omega_t \tag{2.118}$$

2.5 Conservation of Moment of Linear Momentum and Symmetry of Stress

Let $\boldsymbol{x}$ be a position vector in the deformed body Ω, then the total moment of linear momentum of the body with respect to the origin O is calculated by

$$\mathcal{H} = \int_\Omega \boldsymbol{x} \wedge \rho \boldsymbol{v}\, dv,$$

and the total torque $\mathcal{T}$ due to an external force $\boldsymbol{t}$ and a body force $\boldsymbol{b}$ is given by

$$\mathcal{T} = \int_{\partial\Omega} \boldsymbol{x} \wedge \boldsymbol{t}\, ds + \int_\Omega \boldsymbol{x} \wedge \rho\, \boldsymbol{b}\, dv.$$

The *conservation law for the moment of linear momentum* states that $\dot{\mathcal{H}} = \mathcal{T}$; therefore we have

$$\frac{d}{dt} \int_\Omega \boldsymbol{x} \wedge \rho \boldsymbol{v}\, dv = \int_{\partial\Omega} \boldsymbol{x} \wedge \boldsymbol{t}\, ds + \int_\Omega \boldsymbol{x} \wedge \rho\, \boldsymbol{b}\, dv. \tag{2.119}$$

Applying the transport theorem (2.84) to the l.h.s. of (2.119) and noting $d\boldsymbol{x}/dt \wedge \boldsymbol{v} = \boldsymbol{v} \wedge \boldsymbol{v} = \boldsymbol{0}$ yields

$$e_{ijk}x_j\ \rho\frac{dv_k}{dt} = e_{ijk}\frac{\partial x_j\sigma_{lk}}{\partial x_l} + e_{ijk}x_j\ \rho b_k$$
$$\Rightarrow e_{ijk}x_j\left(\rho\frac{dv_k}{dt} - \frac{\partial\sigma_{lk}}{\partial x_l} - b_k\right) - e_{ijk}\sigma_{jk} = 0.$$

The terms in ( ) of this equation vanish because of the equation of motion (2.104), and eventually the following result is obtained:

$$e_{ijk}\sigma_{jk} = 0.$$

This implies

$$\sigma_{jk} = \sigma_{kj} \quad \Rightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2.120}$$

That is, if we have the conservation law of moment of linear momentum and assume no point-wise source term of moment, the Cauchy stress is symmetric. A further exposition of the symmetry property of the Cauchy stress is given by Selvadurai (2000b).

2.6 Incremental Forms of the Equation of Equilibrium ♣

For nonlinear problems such as elasto-plastic materials it is necessary to use a formulation based on an incremental form of the equation of equilibrium. We can introduce either the *total Lagrangian form* or the *updated Lagrangian form*. In the former case the incremental form is expressed in Lagrangian terms, while in the latter case the incremental form is given in an Eulerian description.

2.6.1 Total Lagrangian Form

The total Lagrangian form of the equation of equilibrium is obtained by differentiating (2.116) directly. Thus the partial differential equation system together with the boundary conditions is given by

$$\mathrm{Div}\,\dot{\boldsymbol{\Pi}}^T(\boldsymbol{X}) + \rho_0\dot{\boldsymbol{b}}(\boldsymbol{X}) = \boldsymbol{0}, \tag{2.121}$$

$$\boldsymbol{v}(\boldsymbol{X}) = \bar{\boldsymbol{v}}^0 \qquad \text{on} \quad \partial\Omega_u, \tag{2.122}$$

$$\dot{\boldsymbol{\Pi}}^T\boldsymbol{N}(\boldsymbol{X}) = \bar{\dot{\boldsymbol{t}}}^0 \qquad \text{on} \quad \partial\Omega_t. \tag{2.123}$$

The component form can be written as

$$\frac{\partial \dot{\Pi}_{Ii}(\boldsymbol{X})}{\partial X_I} + \rho_0 \dot{b}_i(\boldsymbol{X}) = 0, \tag{2.124}$$

$$v_i(\boldsymbol{X}) = \bar{v}_i^0 \qquad \text{on} \quad \partial\Omega_u^0, \tag{2.125}$$

$$\dot{\Pi}_{Ii} N_I(\boldsymbol{X}) = \bar{\dot{t}}_i^0 \qquad \text{on} \quad \partial\Omega_t^0. \tag{2.126}$$

2.6.2 Updated Lagrangian Form

By integrating each term of Nanson's relation $\boldsymbol{n}\, ds = J\, \boldsymbol{F}^{-T} \boldsymbol{N}\, dS$ we have

$$\int_{\partial\Omega} \boldsymbol{n}\, ds = \int_{\partial\Omega_0} J\, \boldsymbol{F}^{-T} \boldsymbol{N}\, dS = \int_{\Omega_0} \mathrm{Div}(J\, \boldsymbol{F}^{-T})\, dV = \boldsymbol{0},$$
$$\int_{\partial\Omega_0} \boldsymbol{N}\, dS = \int_{\partial\Omega} J^{-1}\, \boldsymbol{F}^{T} \boldsymbol{n}\, ds = \int_{\Omega} \mathrm{div}(J^{-1}\, \boldsymbol{F}^{T})\, dv = \boldsymbol{0}.$$

These give the following equations:

$$\mathrm{Div}(J\, \boldsymbol{F}^{-T}) = \frac{\partial}{\partial X_I}\left(J\, F_{Ii}^{-1}\right) \boldsymbol{e}_i = \boldsymbol{0}, \quad \mathrm{div}(J^{-1}\, \boldsymbol{F}^{T}) = \frac{\partial}{\partial x_i}\left(J^{-1}\, F_{iI}\right) \boldsymbol{E}_I = \boldsymbol{0}. \tag{2.127}$$

Next, (2.110) implies that

$$J \boldsymbol{\sigma} = \boldsymbol{F} \boldsymbol{\Pi},$$

which can be time-differentiated to yield the *nominal stress rate* $\overset{\circ}{\boldsymbol{\Pi}}$ defined by

$$\overset{\circ}{\boldsymbol{\Pi}} \equiv J^{-1} \boldsymbol{F} \dot{\boldsymbol{\Pi}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\, \mathrm{tr}\, \boldsymbol{D}. \tag{2.128}$$

Equation 2.128 implies that $\overset{\circ}{\boldsymbol{\Pi}}$ is an image of $\dot{\boldsymbol{\Pi}}$ in the deformed body mapped from the undeformed body by $J^{-1}\boldsymbol{F}$. A component form of (2.128) is given by

$$\overset{\circ}{\Pi}_{ij} = J^{-1} F_{iI} \dot{\Pi}_{Ij} \tag{2.129}$$

We take the divergence of (2.129) using $(2.127)_2$ and obtain

$$\frac{\partial \overset{\circ}{\Pi}_{ji}}{\partial x_j} = J^{-1} F_{jI} \frac{\partial \dot{\Pi}_{Ii}}{\partial x_j} = J^{-1} \frac{\partial x_j}{\partial X_I} \frac{\partial \dot{\Pi}_{Ii}}{\partial x_j} = J^{-1} \frac{\partial \dot{\Pi}_{Ii}}{\partial X_I}.$$

Since $J = \rho_0/\rho$, (2.124) for the deformed body can be given as

$$\frac{\partial \overset{\circ}{\Pi}_{ji}}{\partial x_j} + \rho \dot{b}_i(\boldsymbol{x}) = 0. \tag{2.130}$$

Thus the partial differential equation system of the updated Lagrangian form together with the boundary conditions is given by

$$\operatorname{div} \overset{\circ}{\boldsymbol{\Pi}}{}^T(\boldsymbol{x}) + \rho \dot{\boldsymbol{b}}(\boldsymbol{x}) = \boldsymbol{0}, \tag{2.131}$$

$$\boldsymbol{v}(\boldsymbol{x}) = \bar{\boldsymbol{v}} \qquad \text{on} \quad \partial\Omega_u, \tag{2.132}$$

$$\overset{\circ}{\boldsymbol{\Pi}}{}^T \boldsymbol{n}(\boldsymbol{x}) = \bar{\dot{\boldsymbol{t}}} \qquad \text{on} \quad \partial\Omega_t. \tag{2.133}$$

It must be noted that, as understood from (2.131), "the updated Lagrangian form is expressed in Eulerian terms".

2.7 Specific Description of the Equation of Motion ♣

As we shall discuss in Chap. 3, specific descriptions of stress and stress increments are preferable when formulating energy theorems. Hence we will now rewrite the equations of motion in the specific forms.

2.7.1 *Eulerian Equation of Motion*

Let us define the normalized measure of Cauchy stress $\boldsymbol{\sigma}^{\ddagger}$ by

$$\boldsymbol{\sigma}^{\ddagger}(\boldsymbol{x}, t) = \frac{1}{\rho}\boldsymbol{\sigma}(\boldsymbol{x}, t). \tag{2.134}$$

Then the Eulerian equation of motion (2.104) can be written as follows:

$$\rho \frac{d\boldsymbol{v}}{dt} = \operatorname{div}(\rho \boldsymbol{\sigma}^{\ddagger})^T + \rho\, \boldsymbol{b}. \tag{2.135}$$

It is interesting to note that the dimension of Cauchy stress $\boldsymbol{\sigma}$ is M/LT^2 (N/m^2 = Pa in MKS) while for $\boldsymbol{\sigma}^{\ddagger}$ it is L^2/T^2 (J/kg in MKS), which has the dimension of energy per unit mass.

2.7.2 Lagrangian Equation of Motion

The specific first Piola-Kirchhoff stress $\boldsymbol{\Pi}^{\ddagger}$, specific second Piola-Kirchhoff stress $\boldsymbol{T}^{\ddagger}$ and specific Euler stress $\boldsymbol{\tau}^{\ddagger}$ are defined by

$$\boldsymbol{\Pi}^{\ddagger} = \frac{1}{\rho}\boldsymbol{\Pi}, \qquad \boldsymbol{T}^{\ddagger} = \frac{1}{\rho_0}\boldsymbol{T}, \qquad \boldsymbol{\tau}^{\ddagger} = \boldsymbol{R}\boldsymbol{T}^{\ddagger}\boldsymbol{R}^{T} \tag{2.136}$$

Since $J = \rho_0/\rho$, and using (2.110) and (2.112),

$$\boldsymbol{\Pi}^{\ddagger} = \boldsymbol{F}^{-1}\boldsymbol{\sigma}^{\ddagger}, \tag{2.137}$$

$$\boldsymbol{T}^{\ddagger} = \boldsymbol{F}^{-1}\boldsymbol{\sigma}^{\ddagger}\boldsymbol{F}^{-T}. \tag{2.138}$$

Then the Lagrangian equation of motion (2.114) can be written as

$$\rho_0 \frac{d\boldsymbol{v}}{dt} = \mathrm{Div}\,(\rho_0 \boldsymbol{\Pi}^{\ddagger})^{T} + \rho_0 \boldsymbol{b}. \tag{2.139}$$

2.7.3 Incremental Form of the Total Lagrangian Equation of Motion

Differentiating (2.139) with respect to time yields

$$\rho_0 \frac{d\dot{\boldsymbol{v}}(\boldsymbol{X},t)}{dt} = \mathrm{Div}\left(\rho_0 \dot{\boldsymbol{\Pi}}^{\ddagger}(\boldsymbol{X},t)\right)^{T} + \rho_0 \dot{\boldsymbol{b}}(\boldsymbol{X},t) \tag{2.140}$$

where $\dot{\boldsymbol{v}} = d\boldsymbol{v}/dt = \boldsymbol{a}$ ($\boldsymbol{a}$ is the acceleration), and the l.h.s. of (2.140) is the rate of acceleration, which can be regarded as an increment of the velocity. Equation 2.140 is equivalent to (2.121) if $\boldsymbol{v} = \boldsymbol{0}$. The component form of (2.140) is given by

$$\rho_0 \frac{d\dot{v}_i}{dt} = \frac{\partial}{\partial X_I}\left(\rho_0 \dot{\Pi}^{\ddagger}_{Ii}\right) + \rho_0 \dot{b}. \tag{2.141}$$

2.7.4 Incremental Form of the Updated Lagrangian Equation of Motion

From (2.137), $\boldsymbol{\sigma}^{\ddagger} = \boldsymbol{F}\boldsymbol{\Pi}^{\ddagger}$ and differentiating this yields

$$\dot{\boldsymbol{\sigma}}^{\ddagger} = \boldsymbol{L}\boldsymbol{\sigma}^{\ddagger} + \boldsymbol{F}\dot{\boldsymbol{\Pi}}^{\ddagger}.$$

Thus we can define the specific nominal stress rate by

$$\overset{\circ}{\boldsymbol{\Pi}}^{\ddagger} \equiv \boldsymbol{F}\,\dot{\boldsymbol{\Pi}}^{\ddagger} = \dot{\boldsymbol{\sigma}}^{\ddagger} - \boldsymbol{L}\boldsymbol{\sigma}^{\ddagger} \tag{2.142}$$

On the other hand we have

$$\frac{\partial}{\partial X_I} = F_{jI}\frac{\partial}{\partial x_j}.$$

Then (2.140) can be expressed as

$$\rho_0 \frac{d\dot{\boldsymbol{v}}(\boldsymbol{x},t)}{dt} = \operatorname{div}\left(\rho_0 \overset{\circ}{\boldsymbol{\Pi}}^{\ddagger}(\boldsymbol{x},t)\right)^T + \rho_0 \dot{\boldsymbol{b}}(\boldsymbol{x},t) \tag{2.143}$$

The l.h.s. of (2.143) can be written in the normal Eulerian description as

$$\rho_0 \frac{d\dot{\boldsymbol{v}}(\boldsymbol{x},t)}{dt} = \rho_0\left(\frac{\partial \dot{\boldsymbol{v}}}{\partial t} + \boldsymbol{v}\cdot \operatorname{div}\dot{\boldsymbol{v}}\right). \tag{2.144}$$

The component form of (2.143) is given as follows:

$$\rho_0\left(\frac{\partial \dot{v}_i}{\partial t} + v_j\frac{\partial \dot{v}_i}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\rho_0 \overset{\circ}{\Pi}{}^{\ddagger}_{ji}\right) + \rho_0 \dot{b}_i. \tag{2.145}$$

Equation 2.143 is equivalent to (2.131) if $\boldsymbol{v}=\boldsymbol{0}$; however, it is interesting to note that the body force $\dot{\boldsymbol{b}}$ is modified by ρ, while each term of (2.143) is modified by ρ_0.

2.8 Response of Materials: Constitutive Theory

The governing equations that control material responses are given by the mass conservation law (2.97) and the equation of motion (2.104) if no energy conservation is considered. Note that the Cauchy stress is symmetric under the conservation law of moment of linear momentum. Furthermore, if the change of mass density is small (or it may be constant), the equation to be solved is given by (2.104). The unknowns in this equation are the velocity $\boldsymbol{v}$ (or displacement $\boldsymbol{u}$ in the small strain theory) and the stress $\boldsymbol{\sigma}$, i.e. giving a total of nine, that is, three for $\boldsymbol{v}$ (or $\boldsymbol{u}$) and six for $\boldsymbol{\sigma}$. However, the equation of motion (2.104) consists of three components, therefore it cannot be solved, suggesting that we must introduce a relationship between $\boldsymbol{v}$ (or $\boldsymbol{u}$) and $\boldsymbol{\sigma}$. The framework that provides this relationship is referred to as a constitutive theory.

2.8.1 Fundamental Principles of Material Response

The constitutive law is fundamentally determined for each material, and it gives an empirical rule. To establish constitutive laws the following physical conditions are required:

Principle of determinism: The stress is determined by the history of the motion undergone by the body.
Principle of local action: The stress at a point is not influenced by far-field motions.
Principle of frame indifference: The response of a material must be described under the frame indifference (see Sect. 2.2.2).

The principle of frame indifference is sometimes called *objectivity*.

Let a stress $\boldsymbol{\sigma}$ be described as a function of a material point $\boldsymbol{x}$ in the deformed body at time t:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x}, t). \tag{2.146}$$

As shown by (2.18), a coordinate transformation of the point $\boldsymbol{x}$ between two different coordinate systems defined in the deformed body Ω is written as

$$\boldsymbol{x}^* = \boldsymbol{x}_0^* + \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0), \qquad t^* = t - a, \tag{2.147}$$

and the frame indifference of the stress $\boldsymbol{\sigma}$ is therefore

$$\boldsymbol{\sigma}^*(\boldsymbol{x}^*, t^*) = \boldsymbol{Q}(t)\, \boldsymbol{\sigma}(\boldsymbol{x}, t)\, \boldsymbol{Q}^T(t). \tag{2.148}$$

Because (2.147) implies that $\boldsymbol{F}^* d\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{F}\, d\boldsymbol{X}$, the deformation gradient $\boldsymbol{F}$ is transformed as

$$\boldsymbol{F}^* = \boldsymbol{Q}\, \boldsymbol{F}, \qquad \boldsymbol{Q} = Q_{ij} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_j. \tag{2.149}$$

Thus $\boldsymbol{F}$ is not frame indifferent. Some tensors introduced in Sect. 2.2.4 are verified as follows:

$$\boldsymbol{C}^* = \boldsymbol{F}^{*T} \boldsymbol{F}^* = (\boldsymbol{Q}\boldsymbol{F})^T (\boldsymbol{Q}\boldsymbol{F}) = \boldsymbol{F}^T \boldsymbol{F} = \boldsymbol{C}, \tag{2.150}$$

$$\boldsymbol{B}^* = \boldsymbol{F}^* \boldsymbol{F}^{*T} = (\boldsymbol{Q}\boldsymbol{F})(\boldsymbol{Q}\boldsymbol{F})^T = \boldsymbol{Q}\, \boldsymbol{B}\, \boldsymbol{Q}^T, \tag{2.151}$$

$$\boldsymbol{E}^* = \boldsymbol{F}^* \boldsymbol{F}^{*T} - \boldsymbol{I}^* = (\boldsymbol{Q}\boldsymbol{F})^T (\boldsymbol{Q}\boldsymbol{F}) - \boldsymbol{I} = \boldsymbol{E}, \tag{2.152}$$

$$\boldsymbol{L}^* = \dot{\boldsymbol{F}}^* (\boldsymbol{F}^*)^{-1} = (\boldsymbol{Q}\dot{\boldsymbol{F}} + \dot{\boldsymbol{F}}\boldsymbol{Q})(\boldsymbol{Q}\boldsymbol{F})^{-1} = \boldsymbol{Q}\, \boldsymbol{L}\, \boldsymbol{Q}^T + \boldsymbol{\Omega}, \tag{2.153}$$

$$\boldsymbol{D}^* = \frac{1}{2}\left(\boldsymbol{L}^* + \boldsymbol{L}^{*T}\right) = \boldsymbol{Q}\, \boldsymbol{D}\, \boldsymbol{Q}^T, \tag{2.154}$$

$$\boldsymbol{W}^* = \frac{1}{2}\left(\boldsymbol{L}^* - \boldsymbol{L}^{*T}\right) = \boldsymbol{Q}\, \boldsymbol{W}\, \boldsymbol{Q}^T + \boldsymbol{\Omega}. \tag{2.155}$$

Note that the right Cauchy-Green tensor $\boldsymbol{C}$ and the Green strain $\boldsymbol{E}$ are not frame indifferent, but frame invariant.

2.8.2 Convected Derivative, Corotational Derivative and Frame Indifference ♣

If we consider time-differentiation of vectors and tensors, not only the components but also the basis must be differentiated. Thus even if the original vectors and tensors are frame indifferent, their time-derivatives are not in general frame indifferent. In order to avoid this difficulty several time-differential forms are considered.

Embedded coordinates are used (recalling Note 2.5) with the covariant basis $\{\boldsymbol{g}_i\}$ and contravariant basis $\{\boldsymbol{g}^i\}$ in the deformed body (i.e., Eulerian description), so that a vector $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = v^i\, \boldsymbol{g}_i = v_i\, \boldsymbol{g}^i. \tag{2.156}$$

Its material time derivative is given by

$$\dot{\boldsymbol{v}} = \dot{v}^i\, \boldsymbol{g}_i + v^i\, \dot{\boldsymbol{g}}_i = \dot{v}_i\, \boldsymbol{g}^i + v_i\, \dot{\boldsymbol{g}}^i. \tag{2.157}$$

Recalling (2.78), the time-derivative of the covariant basis is $\dot{\boldsymbol{g}}_i = \boldsymbol{L}\boldsymbol{g}_i$. On the other hand, since $\boldsymbol{g}^i = \boldsymbol{F}^{-T}\boldsymbol{G}^i$ due to (2.74) and $\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1}\boldsymbol{L}$ due to (2.68), the time-derivative of the contravariant basis is given by

$$\dot{\boldsymbol{g}}^i = -\boldsymbol{L}^T \boldsymbol{g}^i. \tag{2.158}$$

Now we can define the *convected derivatives* by

$$\frac{\delta^c \boldsymbol{v}}{\delta t} = \dot{v}^i\, \boldsymbol{g}_i, \qquad \frac{\delta_c \boldsymbol{v}}{\delta t} = \dot{v}_i\, \boldsymbol{g}^i. \tag{2.159}$$

Then from (2.158) we have

$$\frac{\delta^c \boldsymbol{v}}{\delta t} = \dot{\boldsymbol{v}} - \boldsymbol{L}\boldsymbol{v} = \boldsymbol{F}\frac{d(\boldsymbol{F}^{-1}\boldsymbol{v})}{dt}, \tag{2.160}$$

$$\frac{\delta_c \boldsymbol{v}}{\delta t} = \dot{\boldsymbol{v}} + \boldsymbol{L}^T\boldsymbol{v} = \boldsymbol{F}^{-T}\frac{d(\boldsymbol{F}^T\boldsymbol{v})}{dt}. \tag{2.161}$$

As observed in (2.159), $\delta^c\boldsymbol{v}/\delta t$, $\delta_c\boldsymbol{v}/\delta t$ denotes a part of $\boldsymbol{v}$ excluding the change of basis, which corresponds to the change of $\boldsymbol{v}$ when the observer is moving along the same coordinate system of the deformed body. $\delta^c\boldsymbol{v}/\delta t$ is known as the *contravariant derivative* or *upper convected rate*, and $\delta_c\boldsymbol{v}/\delta t$ is known as the *covariant derivative* or *lower convected rate*.[5]

[5] The convected derivatives of a vector $\boldsymbol{v}$ are sometimes written as $\delta^c\boldsymbol{v}/\delta t = \overset{\triangleleft}{\boldsymbol{v}}$, $\delta_c\boldsymbol{v}/\delta t = \overset{\triangleright}{\boldsymbol{v}}$. For the second-order tensor $\boldsymbol{T}$ these are $\delta^{cc}\boldsymbol{T}/\delta t = \overset{\triangleleft}{\boldsymbol{T}}$, $\delta_{cc}\boldsymbol{T}/\delta t = \overset{\triangleright}{\boldsymbol{T}}$.

For a second-order tensor $\boldsymbol{T}$ four convected derivatives are introduced as follows:

$$\frac{\delta^{cc}\boldsymbol{T}}{\delta t} = \dot{T}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j, \qquad \frac{\delta^{c}_{\cdot c}\boldsymbol{T}}{\delta t} = \dot{T}^{i}_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j,$$
$$\frac{\delta^{\cdot c}_{c}\boldsymbol{T}}{\delta t} = \dot{T}^{\cdot j}_{i}\boldsymbol{g}^i \otimes \boldsymbol{g}_j, \qquad \frac{\delta_{cc}\boldsymbol{T}}{\delta t} = \dot{T}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j. \tag{2.162}$$

Applying (2.78) and (2.158) and noting that $\boldsymbol{L}^T = \boldsymbol{g}^i \otimes \dot{\boldsymbol{g}}_j$ yields

$$\begin{aligned}
T^{ij}\dot{\boldsymbol{g}}_i \otimes \boldsymbol{g}_j &= \boldsymbol{L}\,T^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \boldsymbol{L}\boldsymbol{T},\\
T^{ij}\boldsymbol{g}_i \otimes \dot{\boldsymbol{g}}_j &= T^{ij}\boldsymbol{g}_i \otimes \boldsymbol{L}\boldsymbol{g}_j = (T^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j)\boldsymbol{L}^T = \boldsymbol{T}\boldsymbol{L}^T,\\
T^{i}_{\cdot j}\dot{\boldsymbol{g}}_i \otimes \boldsymbol{g}^j &= \boldsymbol{L}\,T^{i}_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{g}^j = \boldsymbol{L}\boldsymbol{T},\\
T^{i}_{\cdot j}\boldsymbol{g}_i \otimes \dot{\boldsymbol{g}}^j &= -T^{i}_{\cdot j}\boldsymbol{g}_i \otimes \boldsymbol{L}^T\boldsymbol{g}^j = -\boldsymbol{T}\boldsymbol{L},\\
T^{\cdot j}_{i}\dot{\boldsymbol{g}}^i \otimes \boldsymbol{g}_j &= -\boldsymbol{L}^T T^{\cdot j}_{i}\boldsymbol{g}^i \otimes \boldsymbol{g}_j = -\boldsymbol{L}^T\boldsymbol{T},\\
T^{\cdot j}_{i}\boldsymbol{g}^i \otimes \dot{\boldsymbol{g}}_j &= T^{\cdot j}_{i}\boldsymbol{g}_i \otimes \boldsymbol{L}\boldsymbol{g}^j = \boldsymbol{T}\boldsymbol{L}^T,\\
T_{ij}\dot{\boldsymbol{g}}^i \otimes \boldsymbol{g}^j &= -\boldsymbol{L}^T T_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j = -\boldsymbol{L}^T\boldsymbol{T},\\
T_{ij}\boldsymbol{g}^i \otimes \dot{\boldsymbol{g}}^j &= -T_{ij}\boldsymbol{g}^i \otimes \boldsymbol{L}^T\boldsymbol{g}^j = -\boldsymbol{T}\boldsymbol{L}.
\end{aligned}$$

Thus (2.162) can be written as

$$\frac{\delta^{cc}\boldsymbol{T}}{\delta t} = \dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T = \boldsymbol{F}\frac{d(\boldsymbol{F}^{-1}\boldsymbol{T}\boldsymbol{F}^{-T})}{dt}\boldsymbol{F}^T, \tag{2.163}$$

$$\frac{\delta^{c}_{\cdot c}\boldsymbol{T}}{\delta t} = \dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{L} = \boldsymbol{F}\frac{d(\boldsymbol{F}^{-1}\boldsymbol{T}\boldsymbol{F})}{dt}\boldsymbol{F}^{-1}, \tag{2.164}$$

$$\frac{\delta^{\cdot c}_{c}\boldsymbol{T}}{\delta t} = \dot{\boldsymbol{T}} + \boldsymbol{L}^T\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T = \boldsymbol{F}^{-T}\frac{d(\boldsymbol{F}^T\boldsymbol{T}\boldsymbol{F}^{-T})}{dt}\boldsymbol{F}^T, \tag{2.165}$$

$$\frac{\delta_{cc}\boldsymbol{T}}{\delta t} = \dot{\boldsymbol{T}} + \boldsymbol{L}^T\boldsymbol{T} + \boldsymbol{T}\boldsymbol{L} = \boldsymbol{F}^{-T}\frac{d(\boldsymbol{F}^T\boldsymbol{T}\boldsymbol{F})}{dt}\boldsymbol{F}^{-1}. \tag{2.166}$$

If an orthonormal coordinate transformation tensor $\boldsymbol{Q}$ ($\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$) is used instead of the deformation gradient $\boldsymbol{F}$, the concept of the convected derivative can be extended. That is, let $\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T$ be an antisymmetric rotation tensor generated by $\boldsymbol{Q}$ as shown in (2.27), then the *corotational derivative* of a second-order tensor $\boldsymbol{T}$ due to $\boldsymbol{Q}$ is defined by

$$\frac{\mathfrak{D}_Q\boldsymbol{T}}{\mathfrak{D}t} = \dot{\boldsymbol{T}} + \boldsymbol{T}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{T} = \boldsymbol{Q}\frac{d(\boldsymbol{Q}^T\boldsymbol{T}\boldsymbol{Q})}{dt}\boldsymbol{Q}^T. \tag{2.167}$$

Here $\mathfrak{D}_Q \boldsymbol{T}/\mathfrak{D}t$ represents an objective part of the time derivative of the second-order tensor $\boldsymbol{T}$ (the proof is similar to (2.170) as shown below). For example if the rotation tensor $\boldsymbol{R}$ and the spin tensor $\boldsymbol{W}$ are used instead of $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$, we can introduce the *Zaremba-Jaumann rate* as follows:

$$\frac{\mathfrak{D}\boldsymbol{T}}{\mathfrak{D}t} = \dot{\boldsymbol{T}} + \boldsymbol{T}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{T} = \boldsymbol{R}\frac{d(\boldsymbol{R}^T\boldsymbol{T}\boldsymbol{R})}{dt}\boldsymbol{R}^T. \tag{2.168}$$

The material time derivative of a vector-valued or tensor-valued function is not always objective as described above even if the original function is objective. It can be said that the convected derivative and corotational derivative are introduced to ensure objectivity of the time-derivative. For example we have

$$\frac{\delta_c \boldsymbol{v}^*}{\delta t} = \frac{d\boldsymbol{v}^*}{dt^*} + \boldsymbol{L}^{*T}\boldsymbol{v}^* = \frac{d(\boldsymbol{Q}\boldsymbol{v})}{dt} + (\boldsymbol{Q}\,\boldsymbol{L}^T\,\boldsymbol{Q}^T - \boldsymbol{\Omega})\,(\boldsymbol{Q}\boldsymbol{v}) = \boldsymbol{Q}\frac{\delta_c \boldsymbol{v}}{\delta t} \tag{2.169}$$

$$\begin{aligned}\frac{\delta_{cc}\boldsymbol{T}^*}{\delta t} &= \frac{d\boldsymbol{T}^*}{dt^*} + \boldsymbol{L}^{*T}\boldsymbol{T}^* + \boldsymbol{T}^*\boldsymbol{L}^* \\ &= \frac{d(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)}{dt} + (\boldsymbol{Q}\,\boldsymbol{L}^T\,\boldsymbol{Q}^T - \boldsymbol{\Omega})\,\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T(\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T + \boldsymbol{\Omega}) \\ &= \boldsymbol{Q}\frac{\delta_{cc}\boldsymbol{T}}{\delta t}\boldsymbol{Q}^T. \end{aligned}\tag{2.170}$$

Note that the time derivative $\dot{\boldsymbol{\sigma}}$ of Cauchy stress $\boldsymbol{\sigma}$ is not objective, but the Zaremba-Jaumann rate $\mathfrak{D}\boldsymbol{\sigma}/\mathfrak{D}t$ is objective.

2.8.2.1 Spin of Eulerian Triads $\boldsymbol{\Omega}^E$

Recall that the Eulerian triads $\{\boldsymbol{n}_i\}$ was introduced in (2.57). Let us define the spin $\boldsymbol{\Omega}^E$ of $\{\boldsymbol{n}_i\}$ by

$$\dot{\boldsymbol{n}}_i = \boldsymbol{\Omega}^E\,\boldsymbol{n}_i \quad \Rightarrow \quad \boldsymbol{\Omega}^E = \dot{\boldsymbol{n}}_i \otimes \boldsymbol{n}_i = \overline{\Omega}^E_{ij}\,\boldsymbol{n}_i \otimes \boldsymbol{n}_i, \quad \overline{\Omega}^E_{ij} = \boldsymbol{n}_i \cdot \dot{\boldsymbol{n}}_i. \tag{2.171}$$

Since $\boldsymbol{n}_i \otimes \boldsymbol{n}_i = \boldsymbol{i}$ ($\boldsymbol{i}$ is the unit tensor for the deformed body), the time-differential yields

$$\boldsymbol{\Omega}^E = \dot{\boldsymbol{n}}_i \otimes \boldsymbol{n}_i = -\boldsymbol{n}_i \otimes \dot{\boldsymbol{n}}_i = -(\boldsymbol{\Omega}^E)^T.$$

This shows that $\boldsymbol{\Omega}^E$ is antisymmetric.

We can define the following Lagrangian tensor $\boldsymbol{\Omega}^{ER}$, which is the pull-back of $\boldsymbol{\Omega}^E$ to the undeformed body by the rotation tensor $\boldsymbol{R} = \boldsymbol{n}_i \otimes \boldsymbol{N}_i$:

$$\boldsymbol{\Omega}^{ER} = \boldsymbol{R}^T\boldsymbol{\Omega}^E\boldsymbol{R} = \overline{\Omega}^E_{ij}\,\boldsymbol{N}_i \otimes \boldsymbol{N}_i \quad \Rightarrow \quad \boldsymbol{R}^T\dot{\boldsymbol{n}}_i = \boldsymbol{\Omega}^{ER}\boldsymbol{N}_i \tag{2.172}$$

2.8.2.2 Spin of Lagrangian Triads $\boldsymbol{\Omega}^L$

Recall that the Lagrangian triads $\{\boldsymbol{N}_i\}$ was introduced in (2.55). Let us also define the spin $\boldsymbol{\Omega}^L$ of $\{\boldsymbol{N}_i\}$ by

$$\dot{\boldsymbol{N}}_i = \boldsymbol{\Omega}^L \boldsymbol{N}_i \quad \Rightarrow \quad \boldsymbol{\Omega}^L = \dot{\boldsymbol{N}}_i \otimes \boldsymbol{N}_i = \overline{\Omega}^L_{ij} \boldsymbol{N}_i \otimes \boldsymbol{N}_i, \quad \overline{\Omega}^L_{ij} = \boldsymbol{N}_i \cdot \dot{\boldsymbol{N}}_i. \tag{2.173}$$

It is obvious that $\boldsymbol{\Omega}^L$ is antisymmetric. We can define the following Eulerian tensor $\boldsymbol{\Omega}^{RL}$, which is the push-forward of $\boldsymbol{\Omega}^L$ to the deformed body by the rotation tensor $\boldsymbol{R} = \boldsymbol{n}_i \otimes \boldsymbol{N}_i$:

$$\boldsymbol{\Omega}^{RL} = \boldsymbol{R}\boldsymbol{\Omega}^L\boldsymbol{R}^T = \overline{\Omega}^L_{ij} \boldsymbol{n}_i \otimes \boldsymbol{n}_i \quad \Rightarrow \quad \boldsymbol{R}\dot{\boldsymbol{N}}_i = \boldsymbol{\Omega}^{RL}\boldsymbol{n}_i. \tag{2.174}$$

2.8.2.3 Eulerian Spin $\boldsymbol{\omega}^R$ and Lagrangian Spin $\boldsymbol{\omega}^{RR}$

Time-differentiating $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{i}$ yields $\dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = \boldsymbol{0}$, therefore the Eulerian spin $\boldsymbol{\omega}^R$ and Lagrangian spin $\boldsymbol{\omega}^{RR}$, which is the pull-back of the Eulerian spin into the undeformed body, can be defined by

$$\boldsymbol{\omega}^R = \dot{\boldsymbol{R}}\boldsymbol{R}^T = \omega^R_{ij} \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \qquad \boldsymbol{\omega}^{RR} = \boldsymbol{R}^T\boldsymbol{\omega}^R\boldsymbol{R} = \boldsymbol{R}^T\dot{\boldsymbol{R}} = \omega^R_{ij} \boldsymbol{N}_i \otimes \boldsymbol{N}_i. \tag{2.175}$$

On the other hand, because of (2.171) and the relation $\boldsymbol{n}_i = \boldsymbol{R}\boldsymbol{N}_i$, we have

$$\begin{aligned} \overline{\Omega}^E_{ij} &= \boldsymbol{n}_i \cdot \dot{\boldsymbol{n}}_j = \boldsymbol{n}_i \cdot (\dot{\boldsymbol{R}}\boldsymbol{N}_i + \boldsymbol{R}\dot{\boldsymbol{N}}_i) = \boldsymbol{n}_i \cdot (\dot{\boldsymbol{R}}\boldsymbol{R}^T)\boldsymbol{n}_j + \boldsymbol{n}_i \cdot \boldsymbol{R}\boldsymbol{\Omega}^L\boldsymbol{N}_j \\ &= \boldsymbol{n}_i \cdot \boldsymbol{\omega}^R\boldsymbol{n}_j + \boldsymbol{n}_i \cdot \boldsymbol{\Omega}^{RL}\boldsymbol{n}_j. \end{aligned}$$

Thus the component form of the Eulerian spin is

$$\omega^R_{ij} = \overline{\Omega}^E_{ij} - \overline{\Omega}^L_{ij}. \tag{2.176}$$

The direct notations are given by

$$\boldsymbol{\omega}^R = \boldsymbol{\Omega}^E - \boldsymbol{\Omega}^{RL}, \quad \boldsymbol{\omega}^{RR} = \boldsymbol{\Omega}^{ER} - \boldsymbol{\Omega}^L. \tag{2.177}$$

2.8.2.4 Corotational Derivatives

Since the deformation gradient $\boldsymbol{F}$ can be written in terms of the polar decomposition as defined in (2.46), its time-differentiation gives

$$\dot{\boldsymbol{F}} = \dot{\boldsymbol{R}}\boldsymbol{U} + \boldsymbol{R}\dot{\boldsymbol{U}}.$$

By using (2.65) we obtain the following corotational derivative $\overset{\circ}{\boldsymbol{V}}$ due to $\boldsymbol{\omega}^R$:

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = \boldsymbol{W} + \boldsymbol{D} = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{U}^{-1}\boldsymbol{R}^T) = \boldsymbol{\omega}^R + \overset{\circ}{\boldsymbol{V}}\boldsymbol{V}^{-1} \tag{2.178}$$

$$\overset{\circ}{\boldsymbol{V}} = \boldsymbol{R}\dot{\boldsymbol{U}}\boldsymbol{R}^T = \frac{\mathfrak{D}_R\boldsymbol{V}}{\mathfrak{D}t} = \dot{\boldsymbol{V}} + \boldsymbol{V}\boldsymbol{\omega}^R - \boldsymbol{\omega}^R\boldsymbol{V}. \tag{2.179}$$

It should be noted that the term $\overset{\circ}{\boldsymbol{V}}\boldsymbol{V}^{-1}$ includes an antisymmetric part. Both $\boldsymbol{U}$ and $\dot{\boldsymbol{U}}$ are symmetric, therefore we have

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{R}\left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} + \boldsymbol{U}^{-1}\dot{\boldsymbol{U}}\right)\boldsymbol{R}^T, \qquad \boldsymbol{W} = \boldsymbol{\omega}^R + \frac{1}{2}\boldsymbol{R}\left(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1} - \boldsymbol{U}^{-1}\dot{\boldsymbol{U}}\right)\boldsymbol{R}^T. \tag{2.180}$$

On the other hand, by recalling that $\boldsymbol{V} = \sum \lambda_i\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$, $\boldsymbol{\Omega}^E = \dot{\boldsymbol{n}}_i \otimes \boldsymbol{n}_i$, the time-derivative $\dot{\boldsymbol{V}}$ can be given as

$$\dot{\boldsymbol{V}} = \sum\left(\dot{\lambda}_i\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \lambda_i\, \dot{\boldsymbol{n}}_i \otimes \boldsymbol{n}_i + \lambda_i\, \boldsymbol{n}_i \otimes \dot{\boldsymbol{n}}_i\right) = \sum \dot{\lambda}_i\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i + \boldsymbol{\Omega}^E\boldsymbol{V} - \boldsymbol{V}\boldsymbol{\Omega}^E.$$

Thus we can define the following corotational derivative $\overset{\triangledown}{\boldsymbol{V}}$ due to $\boldsymbol{\Omega}^E$:

$$\overset{\triangledown}{\boldsymbol{V}} = \frac{\mathfrak{D}_E\boldsymbol{V}}{\mathfrak{D}t} = \dot{\boldsymbol{V}} + \boldsymbol{V}\boldsymbol{\Omega}^E - \boldsymbol{\Omega}^E\boldsymbol{V} = \sum \dot{\lambda}_i\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i. \tag{2.181}$$

Equation 2.179 gives the relationship between both corotational derivatives as

$$\overset{\circ}{\boldsymbol{V}} = \overset{\triangledown}{\boldsymbol{V}} + \boldsymbol{\Omega}^{RL}\boldsymbol{V} - \boldsymbol{V}\boldsymbol{\Omega}^{RL}. \tag{2.182}$$

It should be emphasized that from (2.179), $\overset{\circ}{\boldsymbol{V}} = \mathfrak{D}_R\boldsymbol{V}/\mathfrak{D}t$ gives the corotational derivative of $\boldsymbol{V}$ due to $\boldsymbol{\omega}^R$, while from (2.181), $\overset{\triangledown}{\boldsymbol{V}} = \mathfrak{D}_E\boldsymbol{V}/\mathfrak{D}t$ gives the corotational derivative of $\boldsymbol{V}$ due to $\boldsymbol{\Omega}^E$.

2.8.3 Invariants of Stress and Strain and Isotropic Elastic Solids

2.8.3.1 Invariants and Spherical Decomposition of a Second-Order Real-Valued Symmetric Tensor

A second-order real-valued symmetric tensor $\boldsymbol{T}$ gives three real eigenvalues λ which are determined by the characteristic equation:

$$\det(\boldsymbol{T} - \lambda\boldsymbol{I}) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0, \tag{2.183}$$

$$I_1 = \operatorname{tr}\boldsymbol{T}, \quad I_2 = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}\boldsymbol{T}^2\right], \quad I_3 = \det\boldsymbol{T} \tag{2.184}$$

where $I_1,\ I_2,\ I_3$ are the first, second and third principal invariants, respectively.

The mean or volumetric tensor $\overline{\boldsymbol{T}}$ and the deviatoric tensor $\boldsymbol{T}'$ are defined by

$$\overline{\boldsymbol{T}} = \frac{1}{3}(\operatorname{tr}\boldsymbol{T})\boldsymbol{I}, \qquad \boldsymbol{T}' = \boldsymbol{T} - \overline{\boldsymbol{T}}. \tag{2.185}$$

Since the first invariant of the deviatoric tensor $\boldsymbol{T}'$ is zero ($J_1 = \operatorname{tr}\boldsymbol{T}' \equiv 0$), its second and third invariants are

$$J_2 = \frac{1}{2}T'_{ik}T'_{ki} = \frac{1}{2}\operatorname{tr}(\boldsymbol{T}')^2, \qquad J_3 = \det\boldsymbol{T}'. \tag{2.186}$$

Let us define the k-th moment $\bar{I}_k$ of a tensor $\boldsymbol{T}$ by

$$\bar{I}_k = \operatorname{tr}\boldsymbol{T}^k. \tag{2.187}$$

Note 2.6 (Cayley-Hamilton Theorem). The well-known Cayley-Hamilton theorem states that

$$C(\boldsymbol{T}) = -\boldsymbol{T}^3 + I_1\boldsymbol{T}^2 - I_2\boldsymbol{T} + I_3\boldsymbol{I} = \boldsymbol{0} \tag{2.188}$$

which is similar to the characteristic equation (2.183).

Proof. Let us introduce an orthonormal basis $\{\boldsymbol{e}_i\}$ ($i = 1, 2, 3$), and let the coefficient matrix of $\boldsymbol{T}$ be T_{lk} ($\boldsymbol{T} = T_{lk}\,\boldsymbol{e}_l \otimes \boldsymbol{e}_k$), which gives

$$\boldsymbol{T}\boldsymbol{e}_k = T_{lk}\,\boldsymbol{e}_l. \tag{2.189}$$

We define a tensor $\boldsymbol{B}_{lk}$ with tensorial components given by

$$\boldsymbol{B}_{lk} = T_{lk}\boldsymbol{I} - \delta_{lk}\boldsymbol{T}$$

(note that $\boldsymbol{I} = \delta_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$); thus (2.189) is equivalent to

$$\boldsymbol{B}_{lk}\boldsymbol{e}_l = \boldsymbol{0}. \tag{2.190}$$

We should recall that the adjoint A^*_{ij} of a regular matrix A_{ij} is given by $A^*_{km}\,A_{lk} = (\det\boldsymbol{A})\,\delta_{ml}$, and if we multiply the adjoint $\boldsymbol{B}^*_{km}$ (with tensorial components) by (2.190), we obtain

$$\boldsymbol{B}^*_{km}\boldsymbol{B}_{lk}\boldsymbol{e}_l = \det(\boldsymbol{B}_{ij})\,\boldsymbol{\delta}_{ml}\boldsymbol{e}_l = \boldsymbol{0}, \quad \Rightarrow \quad C(\boldsymbol{T})\,\boldsymbol{e}_l = \det(T_{lk}\boldsymbol{I} - \delta_{lk}\boldsymbol{T})\,\boldsymbol{e}_l = \boldsymbol{0} \tag{2.191}$$

$$\boldsymbol{\delta}_{ml} = \begin{cases} \boldsymbol{I} & \text{if } m = l \\ \boldsymbol{0} & \text{if } m \neq l. \end{cases}$$

By multiplying v_l with this result and setting $\boldsymbol{v} = v_l \boldsymbol{e}_l$, we have $C(\boldsymbol{T})\,\boldsymbol{v} = \boldsymbol{0}$ for an arbitrary $\boldsymbol{v}$, which implies that we have (2.188) ■

Operating the trace on (2.188) and using the definition of the k-th moment given by (2.187) yields $I_3 = \bar{I}_3 - I_1\,\bar{I}_2 + I_2\,I_1$. Since $I_1 = \bar{I}_1$, the relationships between the invariants and moments are given by

$$I_1 = \bar{I}_1, \qquad I_2 = \frac{1}{2}\left[(\bar{I}_1)^2 - \bar{I}_2\right], \qquad I_3 = \frac{1}{3}\bar{I}_3 - \frac{1}{2}\bar{I}_1\bar{I}_2 + \frac{1}{6}(\bar{I}_1)^3. \quad (2.192)$$

Thus the third invariant J_3 of the deviatoric tensor $\boldsymbol{T}'$ is given by the third moment $\bar{J}_3$:

$$J_3 = \frac{1}{3}T'_{ik}T'_{kl}T'_{li} = \frac{1}{3}\text{tr}\,(\boldsymbol{T}')^3 = \frac{1}{3}\bar{J}_3. \quad (2.193)$$

Let us introduce the *norm*[6] of a second-order tensor $\boldsymbol{T}$ by

$$||\boldsymbol{T}|| = (\boldsymbol{T} : \boldsymbol{T})^{1/2} = T_{ij}\,T_{ij}.$$

Since $||\boldsymbol{I}|| = \sqrt{3}$, the 'signed magnitude' of the volumetric tensor $\overline{\boldsymbol{T}}$ is calculated as

$$\overline{T} = \frac{1}{\sqrt{3}}T_{kk}. \quad (2.194)$$

If the 'basis tensor' of $\overline{\boldsymbol{T}}$ is introduced by

$$\boldsymbol{n}^{(1)} = n_{ij}^{(1)}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \qquad n_{ij}^{(1)} = \frac{\partial \overline{T}}{\partial T_{ij}} = \frac{T_{ij}}{\overline{T}} = \frac{\delta_{ij}}{\sqrt{3}}, \quad (2.195)$$

we have

$$\overline{\boldsymbol{T}} = \overline{T}\,\boldsymbol{n}^{(1)}. \quad (2.196)$$

For the deviatoric tensor $\boldsymbol{T}'$ we introduce the norm T' and the 'basis tensor' $\boldsymbol{n}^{(2)}$ by

$$T' = (\boldsymbol{T}' : \boldsymbol{T}')^{1/2} = (T_{ij}\,T_{ij} - \overline{T}^2)^{1/2}, \quad (2.197)$$

[6] The inner product of the second-order tensors $\boldsymbol{A}$, $\boldsymbol{B}$ is introduced by $\boldsymbol{A} : \boldsymbol{B} = \text{tr}\,(\boldsymbol{A}^T\boldsymbol{B}) = A_{ij}\,B_{ij}$. $\boldsymbol{A}$ and $\boldsymbol{B}$ are *orthogonal* if $\boldsymbol{A} : \boldsymbol{B} = 0$.

$$\boldsymbol{n}^{(2)} = n_{ij}^{(2)}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j\,, \qquad n_{ij}^{(2)} = \frac{\partial T'}{\partial T'_{ij}} = \frac{T'_{ij}}{T'} \tag{2.198}$$

$$\Rightarrow \quad \boldsymbol{T}' = T'\boldsymbol{n}^{(2)}. \tag{2.199}$$

Lode's angle T_θ of $\boldsymbol{T}$ and the Lode parameter y_T can be introduced by

$$y_T = \cos\,(3T_\theta) = \frac{3\sqrt{3}J_3}{2\,(J_2)^{3/2}} = \sqrt{6}\,\mathrm{tr}\,(\boldsymbol{n}^{(2)})^3. \tag{2.200}$$

If the tensor of Lode's angle is defined by

$$\boldsymbol{T}_\theta = T_\theta\,\boldsymbol{n}^{(3)}, \tag{2.201}$$

its 'basis tensor' $\boldsymbol{n}^{(3)} = n_{ij}^{(3)}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be calculated as

$$\begin{aligned} n_{ij}^{(3)} &= T'\frac{\partial T_\theta}{\partial T_{ij}} = T'\frac{\partial T_\theta}{\partial y_T}\frac{\partial y_T}{\partial T_{ij}} = \frac{\sqrt{6}}{\sin\,(3\tau_\theta)}\left[n_{ij}^{(2)}\mathrm{tr}\,(\boldsymbol{n}^{(2)})^3 - n_{ik}^{(2)}n_{kj}^{(2)} + \frac{1}{\sqrt{3}}n_{ij}^{(1)}\right] \\ \Rightarrow \quad \boldsymbol{n}^{(3)} &= \frac{\sqrt{6}}{\sin\,(3\tau_\theta)}\left[\boldsymbol{n}^{(2)}\mathrm{tr}\,(\boldsymbol{n}^{(2)})^3 - (\boldsymbol{n}^{(2)})^2 + \frac{1}{\sqrt{3}}\boldsymbol{n}^{(1)}\right] \end{aligned} \tag{2.202}$$

where we used the relationship $\mathrm{tr}\,(\boldsymbol{n}^{(2)})^2 = 1$.

The basis tensors $\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(2)}$, $\boldsymbol{n}^{(3)}$ are mutually orthogonal in the sense of

$$\boldsymbol{n}^{(\alpha)} : \boldsymbol{n}^{(\beta)} = \delta_{\alpha\beta}. \tag{2.203}$$

Thus the second-order real-valued symmetric tensor $\boldsymbol{T}$ is written in orthogonal components by

$$\boldsymbol{T} = T_{(\alpha)}\,\boldsymbol{n}^{(\alpha)} \qquad (\alpha:\ \text{summed}) \tag{2.204}$$

where we set

$$T_{(1)} = \overline{T}, \qquad T_{(2)} = T', \qquad T_{(3)} = T_\theta.$$

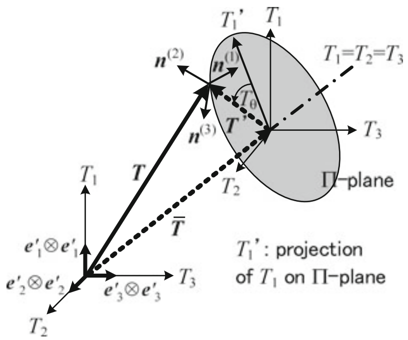
The result (2.204) is referred to as the *spherical decomposition* of $\boldsymbol{T}$.

2.8.3.2 Geometrical Interpretation of Spherical Decomposition in the Principal Space

Let the eigenvalue representation of the second-order real-valued symmetric tensor $\boldsymbol{T}$ be given by

$$\boldsymbol{T} = \sum_{i=1}^{3} T_i \boldsymbol{e}'_i \otimes \boldsymbol{e}'_i.$$

Fig. 2.11 Deviatoric, volumetric and Lode's components in the principal space



Then the volumetric, deviatoric and Lode's components $\overline{\boldsymbol{T}}$, $\boldsymbol{T}'$, T_θ are as shown in Fig. 2.11 together with the base tensors $\boldsymbol{n}^{(\alpha)}$ $(\alpha = 1, 2, 3)$ (since the tensors in the principal space are termed 'vectors', we will use this designation). $\overline{\boldsymbol{T}}$ is the projection of $\boldsymbol{T}$ on the diagonal axis $T_1 = T_2 = T_3$ (which is referred to as the hydrostatic axis for the stress). The difference vector $\boldsymbol{T} - \overline{\boldsymbol{T}}$ gives the deviatoric vector $\boldsymbol{T}'$. The orthogonal plane to $\overline{T}$ including the vector $\boldsymbol{T}'$ is referred to as the Π-*plane*. . If the projected axis of T_1 on the Π-plane is T_1', the angle between T_1' and $\boldsymbol{T}'$ gives the Lode's angle T_θ.

This implies that, by spherical decomposition, cylindrical polar coordinates are introduced in terms of the 'hydrostatic' axis.

2.8.3.3 Spherical Decompositions of Stress and Strain and the Response of an Isotropic Elastic Solid

The stress $\boldsymbol{\sigma}$ is a second-order real-valued symmetric tensor, and the spherical decomposition is given as follows:

$\bar{\boldsymbol{\sigma}} = I_1^\sigma \boldsymbol{I}/3$	Volumetric stress
$I_1^\sigma = \mathrm{tr}\,(\boldsymbol{\sigma})$	First invariant of stress
$\bar{\sigma} = I_1^\sigma/\sqrt{3}$	Magnitude of volumetric stress
$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}$	Deviatoric stress
$\sigma' = \|\boldsymbol{\sigma}'\| = (\boldsymbol{\sigma}' : \boldsymbol{\sigma}')^{1/2}$	Magnitude of deviatoric stress
$\sigma_\theta = \frac{1}{3}\cos^{-1}\{3\sqrt{3}J_3^\sigma/2(J_2^\sigma)^{3/2}\}$	Lode's angle for stress
$J_2^\sigma = \boldsymbol{\sigma}' : \boldsymbol{\sigma}'/2$	Second invariant of deviatoric stress
$J_3^\sigma = \det\,(\boldsymbol{\sigma}')$	Third invariant of deviatoric stress

For the strain $\boldsymbol{\varepsilon}$ we can introduce the spherical decomposition as follows:

$\bar{\boldsymbol{\varepsilon}} = I_1^\varepsilon \boldsymbol{I}/3$	Volumetric strain
$I_1^\varepsilon = \mathrm{tr}\,(\boldsymbol{\varepsilon})$	First invariant of strain
$\bar{\varepsilon} = I_1^\varepsilon/\sqrt{3}$	Magnitude of volumetric strain
$\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \bar{\boldsymbol{\varepsilon}}$	Deviatoric strain

$\varepsilon' = \|\boldsymbol{\varepsilon}'\| = (\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}')^{1/2}$	Magnitude of deviatoric strain
$\varepsilon_\theta = \frac{1}{3}\cos^{-1}\{3\sqrt{3}J_3^\varepsilon/2(J_2^\varepsilon)^{3/2}\}$	Lode's angle for strain
$J_2^\varepsilon = \boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}'/2$	Second invariant of deviatoric strain
$J_3^\varepsilon = \det(\boldsymbol{\varepsilon}')$	Third invariant of deviatoric strain

If a material body is an isotropic solid, the stress and strain are decomposed by using the same basis $\boldsymbol{n}^{(\alpha)}$:

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}' + \boldsymbol{\sigma}_\theta, \qquad \bar{\boldsymbol{\sigma}} = \bar{\sigma}\,\boldsymbol{n}^{(1)}, \qquad \boldsymbol{\sigma}' = \sigma'\boldsymbol{n}^{(2)}, \qquad \boldsymbol{\sigma}_\theta = \sigma'\sigma_\theta\,\boldsymbol{n}^{(3)}, \tag{2.205}$$

$$\boldsymbol{\varepsilon} = \bar{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}_\theta, \qquad \bar{\boldsymbol{\varepsilon}} = \bar{\varepsilon}\,\boldsymbol{n}^{(1)}, \qquad \boldsymbol{\varepsilon}' = \varepsilon'\boldsymbol{n}^{(2)}, \qquad \boldsymbol{\varepsilon}_\theta = \varepsilon'\varepsilon_\theta\,\boldsymbol{n}^{(3)}, \tag{2.206}$$

and the linear elastic response is written in terms of the volumetric and deviatoric components independently (cf. Note 2.7). Thus, referring to Fig. 2.11, the response gives a state that is symmetric about the hydrostatic axis as follows:

$$\bar{\boldsymbol{\sigma}} = 3\lambda\bar{\boldsymbol{\varepsilon}}, \qquad \boldsymbol{\sigma}' = 2\mu\boldsymbol{\varepsilon}'. \tag{2.207}$$

The coefficients $\lambda,\ \mu$ are called Lamé's constants. Then the response of the linear elastic solid, called the Hookean solid, is written as

$$\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}. \tag{2.208}$$

Young's modulus E and Poisson's ratio ν are related to Lamé's constants $\lambda,\ \mu$, the shear modulus G and bulk modulus K as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} = G, \quad K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}. \tag{2.209}$$

Note 2.7 (Lode's angle and the response of isotropic solids). If the elastic response of solids is written using Hooke's law as

$$\boldsymbol{\sigma} = \boldsymbol{D}^e\,\boldsymbol{\varepsilon}, \tag{2.210}$$

the most general form of the fourth order tensor $\boldsymbol{D}^e$ for isotropic materials is given by

$$D^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{2.211}$$

(cf. Malvern 1969, pp. 277). Since $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are symmetric ($\sigma_{ij} = \sigma_{ji}$, $\varepsilon_{ij} = \varepsilon_{ji}$), we have the condition $\nu = 0$, which causes no change in the Lode's angle component for isotropic solids, and the axi-symmetric response with respect to the hydrostatic axis. Another result is that there exist two independent elastic constants, although the number of eigenvalues of stress and strain is three.

We also note that a tensor function defined as

$$D^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} \tag{2.212}$$

is also isotropic (Little 1973; Spencer 2004). The constitutive equation (2.210) now becomes

$$\sigma = \lambda\delta_{ij}\varepsilon_{kk} + \mu\varepsilon_{ij} + \nu\varepsilon_{ij}. \tag{2.213}$$

Since $\varepsilon_{ij} = \varepsilon_{ji}$, no generality is lost by setting $\mu = \nu$ such that $\sigma = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$. ■

The inverse relation of (2.208) is

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\,\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}. \tag{2.214}$$

For two dimensional problems we can consider two idealized states: the plane strain state where $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$ and the plane stress state in which $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. Under these conditions, Hooke's law is rewritten for the vector forms of stress and strain as

$$\boldsymbol{\sigma} = \boldsymbol{D}^e\,\boldsymbol{\varepsilon}, \qquad \boldsymbol{\sigma} = [\sigma_{xx}\ \sigma_{yy}\ \sigma_{xy}]^T, \quad \boldsymbol{\varepsilon} = [\varepsilon_{xx}\ \varepsilon_{yy}\ \gamma_{xy}]^T \tag{2.215}$$

$$\text{Plane strain :} \qquad \boldsymbol{D}^e = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix} \tag{2.216}$$

$$\text{Plane stress :} \qquad \boldsymbol{D}^e = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}. \tag{2.217}$$

Here we have used the engineering shear strain $\gamma_{xy} = 2\varepsilon_{xy}$. The representation of stress and strain given by (2.215) is referred to as the contracted form.

If the material body involves an initial stress $\boldsymbol{\sigma}_0$ and/or initial strain $\boldsymbol{\varepsilon}_0$, Hooke's law is transformed to

$$\boldsymbol{\sigma} = \boldsymbol{D}^e(\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}_0 \tag{2.218}$$

If the initial strain is caused by a temperature difference $T - T_0$, we have $\boldsymbol{\varepsilon}_0 = \alpha(T - T_0)\boldsymbol{i}$ for an isotropic material body, therefore the above equation becomes

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\,\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} + \alpha(T - T_0)\,\delta_{ij} \tag{2.219}$$

where T_0 is the reference temperature and α is the thermal expansion coefficient.

Substituting Hooke's law (2.208) into the equation of motion (2.104) yields the following *Navier's equation* where the unknown variable is the displacement $\boldsymbol{u}$:

$$\rho \frac{d^2 u_i}{dt^2} = (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho b_i . \tag{2.220}$$

Note 2.8 (Solid and fluid). The term "solid" is used for the material body where the response is between the stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\varepsilon}$ or between the stress increment $d\boldsymbol{\sigma}$ and the strain increment $d\boldsymbol{\varepsilon}$. The term "fluid" is used for the material body where the response is between the stress $\boldsymbol{\sigma}$ and the strain rate $\dot{\boldsymbol{\varepsilon}}$ (or the stretch tensor $\boldsymbol{D}$). For a fluid we have to introduce a time-integration constant, which is referred to as the pressure p. ■

2.8.4 Newtonian Fluid

For simplicity, we describe the equations without mass density ρ. Since the specific stress $\boldsymbol{\sigma}^{\ddagger}(\boldsymbol{x}, t)$ is given in terms of an Eulerian description, we treat here the simplest response for that description. To satisfy the principle of determinism and the principle of local action mentioned in the previous section, the stress $\boldsymbol{\sigma}^{\ddagger}(\boldsymbol{x}, t)$ can be written in terms of $\boldsymbol{v}$ and $\nabla \boldsymbol{v}$:

$$\boldsymbol{\sigma}^{\ddagger} = \boldsymbol{\sigma}^{\ddagger}(\boldsymbol{v}, \nabla \boldsymbol{v}). \tag{2.221}$$

The frame indifference of the stress $\boldsymbol{\sigma}^{\ddagger}(\boldsymbol{x}, t)$ is a natural conclusion of Newtonian mechanics in that the force vector is frame indifferent. Since the stretch tensor $\boldsymbol{D}$ is frame invariant by (2.154), we use $\boldsymbol{D}$ instead of $\nabla \boldsymbol{v}$. From (2.27) we have

$$\boldsymbol{v}^* = \frac{d\boldsymbol{x}_0^*}{dt} + \frac{d\boldsymbol{Q}}{dt}(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{Q}\boldsymbol{v}.$$

Therefore the frame indifference requires the following condition:

$$\boldsymbol{\sigma}^{\ddagger *}(\boldsymbol{v}^*,\, \boldsymbol{D}^*) = \boldsymbol{Q}\boldsymbol{\sigma}^{\ddagger}\left(\dot{\boldsymbol{x}}_0^* + \dot{\boldsymbol{Q}}(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{Q}\boldsymbol{v},\; \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T\right)\boldsymbol{Q}^T$$

We define $\boldsymbol{x}_0^*$ as

$$\dot{\boldsymbol{x}}_0^* = -\dot{\boldsymbol{Q}}(\boldsymbol{x} - \boldsymbol{x}_0) - \boldsymbol{Q}\boldsymbol{v}$$

Then we can see that if we have

$$\boldsymbol{Q}\boldsymbol{\sigma}^{\ddagger}(\boldsymbol{D})\boldsymbol{Q}^T = \boldsymbol{\sigma}^{\ddagger}(\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T) \quad \Rightarrow \quad \boldsymbol{\sigma}^{\ddagger} = \boldsymbol{\sigma}^{\ddagger}(\boldsymbol{D}) \tag{2.222}$$

the fundamental principles mentioned in the previous section are satisfied. Since $\boldsymbol{D}$ is symmetric and non-negative definite, the most general form (Truesdell and Noll 1965, pp. 32; Malvern 1969, pp. 194) can be given by

$$\boldsymbol{\sigma} = \rho \boldsymbol{\sigma}^{\ddagger} = \phi_0 \boldsymbol{i} + \phi_1 \boldsymbol{D} + \phi_2 \boldsymbol{D}^2 \tag{2.223}$$

where ϕ_i $(i = 0, 1, 2)$ are functions of the invariants I_i^D $(i = 1, 2, 3)$ of $\boldsymbol{D}$:

$$\phi_i = \phi_i(I_1^D, I_2^D, I_3^D). \tag{2.224}$$

The invariants I_i^D are calculated by the following characteristic equation for specifying the eigenvalue ζ:

$$\det(\boldsymbol{D} - \zeta \boldsymbol{i}) = -\zeta^3 + I_1^D \zeta^2 - I_2^D \zeta^1 + I_3^D = 0, \tag{2.225}$$

$$I_1^D = \operatorname{tr} \boldsymbol{D} = \nabla \cdot \boldsymbol{v},$$

$$I_2^D = \frac{1}{2}\left[(I_1^D)^2 - \hat{I}_2^D\right], \qquad \hat{I}_2^D = \operatorname{tr}(\boldsymbol{D}^2),$$

$$I_3^D = \det \boldsymbol{D}$$

We omit the third term of the r.h.s. of (2.223) so as to linearize it:

$$\boldsymbol{\sigma} = \rho \boldsymbol{\sigma}^{\ddagger} = (-p + \lambda \operatorname{tr} \boldsymbol{D})\, \boldsymbol{i} + 2\mu \boldsymbol{D} \tag{2.226}$$

where p is the pressure and λ, μ are viscosities (μ is the shearing viscosity, and $\kappa = \lambda + 2\mu/3$ is the bulk viscosity: described below). The pressure p appears in this equation because $\boldsymbol{v}$ (and also $\boldsymbol{D}$) is a material time derivative of the position vector $\boldsymbol{x}$ of a material point in the deformed body, which needs an integration constant; this corresponds to the pressure. Note that usually the "pressure" is set positive for compression, therefore a negative sign of p appears in (2.226). Materials that behave as (2.226) are referred to as *Newtonian fluids*.

Let us resolve the stress $\boldsymbol{\sigma}$ and stretch tensor $\boldsymbol{D}$ into direct sums of volumetric and deviatoric components, respectively:

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}', \tag{2.227}$$

$$\bar{\boldsymbol{\sigma}} = \frac{1}{3}(\operatorname{tr} \boldsymbol{\sigma})\, \boldsymbol{i}, \qquad \boldsymbol{\sigma}' = \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}, \tag{2.228}$$

$$\boldsymbol{D} = \overline{\boldsymbol{D}} + \boldsymbol{D}', \tag{2.229}$$

$$\overline{\boldsymbol{D}} = \frac{1}{3}(\operatorname{tr} \boldsymbol{D})\, \boldsymbol{i}, \qquad \boldsymbol{D}' = \boldsymbol{D} - \overline{\boldsymbol{D}}. \tag{2.230}$$

$\bar{\boldsymbol{\sigma}}$ and $\overline{\boldsymbol{D}}$ are the volumetric components of each tensor, and $\boldsymbol{\sigma}'$ and $\boldsymbol{D}'$ are the deviatoric (or shearing) components. The volumetric component is orthogonal to the deviatoric one in the following sense:

$$\bar{\boldsymbol{\sigma}} : \boldsymbol{\sigma}' = \operatorname{tr}\left(\bar{\boldsymbol{\sigma}}^T \boldsymbol{\sigma}'\right) = 0, \qquad \overline{\boldsymbol{D}} : \boldsymbol{D}' = \operatorname{tr}\left(\overline{\boldsymbol{D}}^T \boldsymbol{D}'\right) = 0. \tag{2.231}$$

Since $(\nabla \cdot \boldsymbol{v})\boldsymbol{I} = 3\overline{\boldsymbol{D}}$, we can rewrite (2.226) as

$$\bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}' = -p\boldsymbol{i} + 3\left(\lambda + \frac{2}{3}\mu\right)\overline{\boldsymbol{D}} + 2\mu\boldsymbol{D}'$$

Recalling the orthogonality of the volumetric and deviatoric components, each component will give an independent response:

$$\bar{\boldsymbol{\sigma}} = -p\boldsymbol{i} + 3\kappa\overline{\boldsymbol{D}}, \qquad \boldsymbol{\sigma}' = 2\mu\boldsymbol{D}' \tag{2.232}$$

This is a direct result of the response of an isotropic linear fluid. In this equation the constant

$$\kappa = \lambda + \frac{2}{3}\mu \tag{2.233}$$

gives the bulk (i.e., volumetric) viscosity and μ is the shearing viscosity.

Thus the most fundamental constitutive law for a fluid is understood to be given as a Newtonian fluid defining a linear relationship between the stress $\boldsymbol{\sigma}$ and the stretch tensor $\boldsymbol{D}$ (recall that the stretch tensor $\boldsymbol{D}$ is equal to the strain rate for the solid with small strain). The constitutive law is also called *Stokes' law*, and can be rewritten as

$$\sigma_{ij} = -p\,\delta_{ij} + \lambda\, D_{kk}\,\delta_{ij} + 2\mu\, D_{ij} \tag{2.234}$$

Substituting Stokes' law (2.234) into the equation of motion (2.104) under the Eulerian description yields the following equation of motion for the unknown velocity $\boldsymbol{v}$:

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + (\lambda + \mu)\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \rho b_i \tag{2.235}$$

These are the *Navier-Stokes equations*. If the body force can be set as $\boldsymbol{b} = -\nabla\phi$ by a potential ϕ, we define

$$p^* = p + \rho\phi, \tag{2.236}$$

and the Navier-Stokes equations can be written as

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p^*}{\partial x_i} + (\lambda + \mu)\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}. \tag{2.237}$$

If the fluid is incompressible, the condition (2.99) applies and we have

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\partial x_j}+\rho b_i\,. \tag{2.238}$$

2.9 Small Strain Viscoelasticity Theory

The one-dimensional viscoelastic response is schematically illustrated in Fig. 2.12. Note that the 'stress relaxation' is a phenomenon that appears under a constant strain condition, while 'creep' is one that appears under a constant stress condition. The response shown is represented by a model based on an excitation-response theory together with a data management procedure. Note that we assume an isotropic material response.

2.9.1 Boltzmann Integral and Excitation-response Theory

The viscoelastic response is commonly described by using a form of *Boltzmann's hereditary integral*, referred to as the *excitation-response theory* (Gurtin and Sternberg 1962; Yamamoto 1972; Christensen 2003).

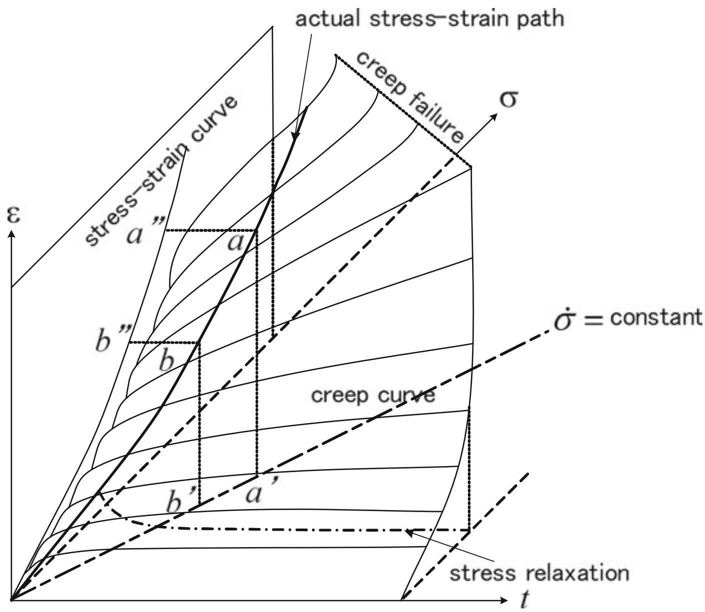


Fig. 2.12 Viscoelastic response

Let us consider a step input function

$$x(t) = \begin{cases} 0 & \text{for } t < 0, \\ x_0 = \text{constant} & \text{for } t > 0. \end{cases} \tag{2.239}$$

The corresponding response to this input can be written as

$$y(t) = \phi(t)\, x_0 \tag{2.240}$$

where $\phi(t)$ is referred to as the *after-effect function* (Fig. 2.13a) which satisfies the condition

$$\phi(t) = 0 \qquad \text{for } t < 0.$$

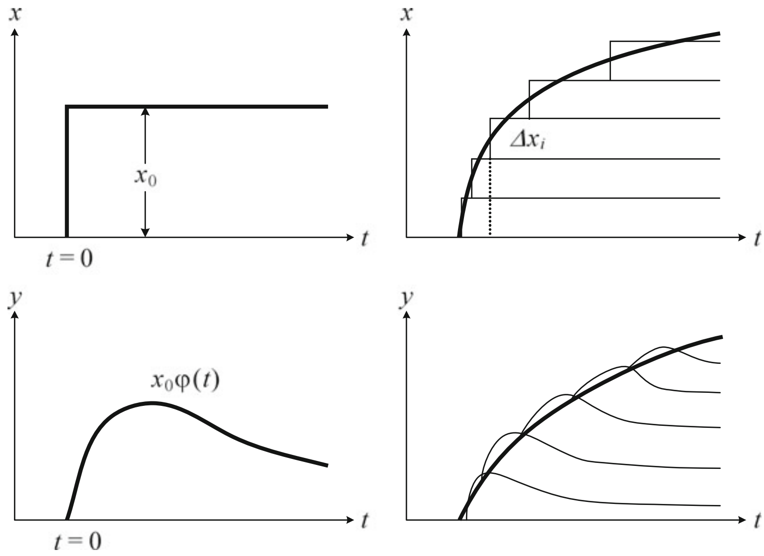
If the input $x(t)$ is given by a collection of step functions as shown in Fig. 2.13b, the response is written as

$$y(t) = \sum_i \phi(t - t_i)\Delta x_i . \tag{2.241}$$

Then for a general form of the input function $x(t)$, we have

$$y(t) = \int_{-\infty}^{t} \phi(t - s)\frac{dx(s)}{ds} ds. \tag{2.242}$$

This is referred to as *Boltzmann's superposition principle*.



(a) After–effect function (b) Superposition principle

Fig. 2.13 Boltzmann's hereditary integral

Integrating (2.242) by parts yields

$$y(t) = \phi(0+)\,x(t) + \int_{-\infty}^{t} \frac{d\phi(t-s)}{ds} x(s)\,ds \quad \equiv \int_{-\infty}^{t} \mu(t-s)\,x(s)\,ds \tag{2.243}$$

where

$$\phi(0+) = \lim_{t\to+0} \phi(t)$$

and

$$\mu(t-s) = \frac{d\phi(t-s)}{ds} + \delta(s)\,\phi(s) \tag{2.244}$$

is referred to as the *response function*.[7]

2.9.2 Stress Relaxation and the Relaxation Spectra: Generalized Maxwell Model

First we consider a simple uniaxial response. If a strain ε is given by

$$\varepsilon(t) = \begin{cases} 0 & \text{for } t < 0, \\ \varepsilon_0 & \text{for } t > 0, \end{cases} \tag{2.245}$$

we write the relaxation stress as

$$\sigma(t) = E(t)\varepsilon_0, \qquad E(t) = 0 \quad \text{for } t < 0 \tag{2.246}$$

where the after-effect function $E(t)$ is referred to as the *relaxation function*. Following Boltzmann's principle presented previously, if an input $\varepsilon(t)$ is given, the response can be written as

$$\sigma(t) = \int_{-\infty}^{t} E(t-s)\frac{d\varepsilon(s)}{ds} ds = \int_{-\infty}^{t} \Lambda(t-s)\,\varepsilon(s)\,ds. \tag{2.247}$$

The response of a viscoelastic material is generally represented as a model containing a combination of elastic, viscous and pure stress-relaxation properties. Then the relaxation function can be written as

$$E(t) = E_0 + \eta_\infty\,\delta(t) + \overline{E}(t) \tag{2.248}$$

[7] The definition of the δ-function is given by $\int_\Omega dy\,\delta(y-x)f(y) = f(x)$.

where $\overline{E}(t)$ is a smooth, monotonically decreasing function such that

$$\overline{E}(0) = \kappa < +\infty, \qquad \overline{E}(+\infty) = 0,$$

and can be represented by a Laplace transformation:

$$\overline{E}(t) = \int_0^\infty N(s) \exp(-st)\, ds. \tag{2.249}$$

In order to provide a discrete approximation of this function, the variable s is changed into τ by $\tau = 1/s$. Then (2.249) can be written as

$$\overline{E}(t) = \int_0^\infty H(\tau) \exp(-t/\tau)\, d(\ln \tau), \tag{2.250}$$

$$H(\tau) = \frac{1}{\tau} N\left(\frac{1}{\tau}\right), \tag{2.251}$$

and $H(\tau)$ is referred to as the *relaxation spectrum*.

Substituting (2.250) into (2.247) and changing the order of integration, we have

$$\sigma(t) = \sigma_0(t) + \sigma_\infty(t) + \int_0^\infty \sigma(t,\tau)\, d(\ln \tau), \tag{2.252}$$

$$\sigma(t,\tau) = H(\tau) \int_{-\infty}^t \exp\left(-(t-s)/\tau\right) \frac{d\varepsilon(s)}{ds} ds, \tag{2.253}$$

$$\sigma_0(t) = E_0\, \varepsilon(t), \qquad \sigma_\infty(t) = \eta_\infty \frac{d\varepsilon(t)}{dt}. \tag{2.254}$$

Differentiating (2.253)[8] we obtain

$$H(\tau) \frac{d\varepsilon(\tau)}{dt} = \frac{d}{dt}\sigma(t,\tau) + \frac{1}{\tau}\sigma(t,\tau). \tag{2.255}$$

---

[8] Leibnitz rule: If we have an integral of a continuous function f such as

$$\phi(x) = \int_{h_0(x)}^{h_1(x)} f(x,\xi)\, d\xi,$$

and if $h_1(x)$ and $h_0(x)$ are continuous on $R = \{\, (x,\xi) :\ a \le x \le b,\ c \le \xi \le d \,\}$, then

$$\frac{d\phi(x)}{dx} = f\left(x, h_1(x)\right) \frac{dh_1(x)}{dx} - f\left(x, h_0(x)\right) \frac{dh_0(x)}{dx} + \int_{h_0(x)}^{h_1(x)} \frac{\partial f(x,\xi)}{\partial x}\, d\xi$$

(see, e.g., Protter and Morrey 1977, pp. 284).

A mechanical representation of a simple Maxwell model is shown in Fig. 2.14a and can be indicated in the same form as (2.255): i.e.

$$E\frac{d\varepsilon}{dt} = \frac{d\sigma}{dt} + \frac{1}{\tau}\sigma$$

where E and η are the elastic and viscous constants, respectively, and $\tau = \eta/E$ is referred to as the *relaxation time*. Thus we understand that (2.252) gives a synthesis of each spectral response corresponding to τ, and this suggests that, for a discrete case, the generalized Maxwell model can be represented by Fig. 2.15a.

If the material is isotropic, the response can be written in both deviatoric and volumetric forms:

$$\boldsymbol{s}(t) = 2\int_{-\infty}^{t} G(t-s)\frac{d\boldsymbol{e}(s)}{ds}ds, \qquad \bar{\boldsymbol{\sigma}}(t) = 3\int_{-\infty}^{t} K(t-s)\frac{d\bar{\boldsymbol{\varepsilon}}(s)}{ds}ds \quad (2.256)$$

where

$$G(t) = G_0 + \eta_\infty^s\,\delta(t) + \overline{G}(t), \qquad K(t) = K_0 + \eta_\infty^v\,\delta(t) + \overline{K}(t), \quad (2.257)$$

$$\overline{G}(t) = \int_0^\infty \Phi^s(\tau)\exp(-t/\tau)\,d(\ln\tau), \qquad \overline{K}(t) = \int_0^\infty \Phi^v(\tau)\exp(-t/\tau)\,d(\ln\tau). \quad (2.258)$$

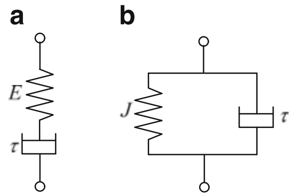


Fig. 2.14 (**a**) Maxwell model, (**b**) Kelvin-Voigt model

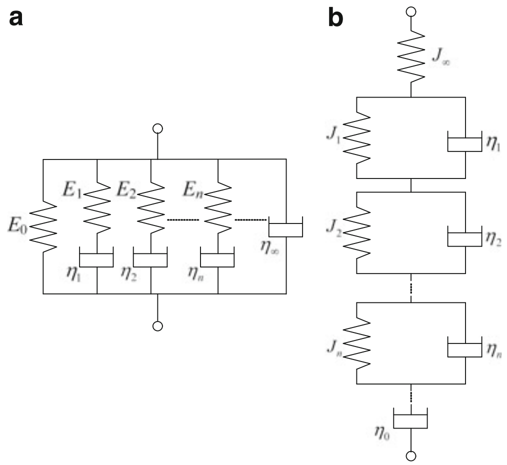


Fig. 2.15 (**a**) Generalized Maxwell model, (**b**) generalized Kelvin-Voigt model

Since $\boldsymbol{\sigma}(t) = \boldsymbol{s}(t) + \bar{\boldsymbol{\sigma}}(t)$, the total response can be represented as

$$\boldsymbol{\sigma}(t) = \int_{-\infty}^{t} \left[2G(t-s)\frac{d\boldsymbol{e}(s)}{ds} + 3K(t-s)\frac{d\bar{\boldsymbol{\varepsilon}}(s)}{ds} \right] ds = \int_{-\infty}^{t} \boldsymbol{D}(t-s)\frac{d\boldsymbol{\varepsilon}(s)}{ds}\, ds. \tag{2.259}$$

2.9.3 Creep and the Retardation Spectra: Generalized Kelvin-Voigt Model

Let us consider a simple uniaxial creep response. If a creep stress σ is given by

$$\sigma(t) = \begin{cases} 0 & \text{for } t < 0, \\ \sigma_0 & \text{for } t > 0, \end{cases} \tag{2.260}$$

and the corresponding strain response is written as

$$\varepsilon(t) = J(t)\,\sigma_0, \qquad J(t) = 0 \quad \text{for } t < 0, \tag{2.261}$$

then for a general input $\sigma(t)$ the strain response can be represented as

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-s)\frac{d\sigma(s)}{ds} ds = \int_{-\infty}^{t} \Gamma(t-s)\,\sigma(s)\, ds \tag{2.262}$$

where $J(t)$ is referred to as *creep function*.

In the same way as we did for stress relaxation, we combine the instantaneous elastic, viscous and pure stress-relaxation properties, and write the response as

$$J(t) = J_\infty + \frac{t}{\eta_0} + \overline{J}(t) \tag{2.263}$$

where $J_\infty = 1/E_\infty$. Since $\bar{J}(t)$ is a smooth, monotone decreasing function, we have

$$\overline{J}(0) = 0, \qquad \overline{J}(+\infty) = \rho < +\infty.$$

Therefore it can be represented by a Laplace transformation:

$$\overline{J}(t) = \int_{0}^{\infty} M(s)\,(1 - \exp(-st))\; ds = \int_{0}^{\infty} L(\tau)\,(1 - \exp(-t/\tau))\; d(\ln\tau), \tag{2.264}$$

$$L(\tau) = \frac{1}{\tau} M\left(\frac{1}{\tau}\right). \tag{2.265}$$

$L(\tau)$ is referred to as the *retardation spectrum*.

Substituting (2.264) into (2.262) and changing the order of integration, we have

$$\varepsilon(t) = \varepsilon_\infty(t) + \varepsilon_0(t) + \int_0^\infty \varepsilon(t,\tau)\, d(\ln\tau) \tag{2.266}$$

$$\varepsilon(t,\tau) = L(\tau)\int_0^\infty (1-\exp(-(t-s)/\tau))\,\frac{d\sigma(s)}{ds}ds, \tag{2.267}$$

$$\varepsilon_\infty(t) = J_\infty\,\sigma, \qquad \varepsilon_0(t) = \frac{1}{\eta_0}\int_{-\infty}^t \sigma(s)\,ds. \tag{2.268}$$

Differentiating (2.267), we obtain

$$L(\tau)\sigma(t) = \tau\frac{d}{dt}\varepsilon(t,\tau) + \varepsilon(t,\tau) \tag{2.269}$$

The response of a simple one-unit Kelvin-Voigt model shown by Fig. 2.14b can be written in a form similar to (2.269) as

$$J\sigma = \tau\frac{d\varepsilon}{dt} + \varepsilon$$

where $J = 1/E$ and η are the elastic compliance and viscous constant, respectively, and $\tau = \eta/E$ is referred to as the *retardation time*. Thus we can see that (2.266) gives a synthesis of each spectral response corresponding to τ, and this suggests that for a discrete case the generalized Kelvin-Voigt model can be represented by Fig. 2.15b.

If the material is isotropic, the response can be given separately for the deviatoric and volumetric deformations as follows:

$$\boldsymbol{e}(t) = \frac{1}{2}\int_{-\infty}^t B(t-s)\frac{d\boldsymbol{s}(s)}{ds}ds, \qquad \bar{\boldsymbol{\varepsilon}}(t) = \frac{1}{3}\int_{-\infty}^t C(t-s)\frac{d\bar{\boldsymbol{\sigma}}(s)}{ds}ds \tag{2.270}$$

where

$$B(t) = B_\infty + \frac{t}{\eta_0^s} + \overline{B}(t), \qquad C(t) = C_\infty + \frac{t}{\eta_0^v} + \overline{C}(t) \tag{2.271}$$

$$\overline{B}(t) = \int_0^\infty \Psi^s(\tau)\,(1-\exp(-t/\tau))\;d(\ln\tau),$$
$$\overline{C}(t) = \int_0^\infty \Psi^v(\tau)\,(1-\exp(-t/\tau))\;d(\ln\tau). \tag{2.272}$$

Since $\boldsymbol{\varepsilon}(t) = \boldsymbol{e}(t) + \bar{\boldsymbol{\varepsilon}}(t)$, the total response is thus represented as

$$\boldsymbol{\varepsilon}(t) = \int_{-\infty}^t \left[\frac{1}{2}B(t-s)\frac{d\boldsymbol{s}(s)}{ds} + \frac{1}{3}C(t-s)\frac{d\bar{\boldsymbol{\sigma}}(s)}{ds}\right]ds = \int_{-\infty}^t \boldsymbol{C}(t-s)\frac{d\boldsymbol{\sigma}(s)}{ds}\,ds. \tag{2.273}$$

2.9.4 Relaxation and Retardation Spectra and Their Asymptotic Expansion

The k-th derivative of (2.249) yields

$$\overline{E}^{(k)}(t) = \frac{d^k \overline{E}(t)}{dt^k} = (-1)^k \int_0^\infty N(s) s^k \exp(-st)\, ds$$

where $\overline{E}^{(k)}$ implies the k-time differentiation of $\overline{E}$. Since $s^k \exp(-st)$ shows the peak value at $s = k/t$ and the value increases with k, we can replace it by the δ-function so that we have

$$\int_0^\infty s^k \exp(-st)\, ds = \frac{k!}{t^{k+1}}.$$

Therefore we set

$$\overline{E}^{(k)}(t) = (-1)^k \frac{k!}{t^{k+1}} \int_0^\infty N(s)\delta(s - k/t)\, ds = (-1)^k \frac{k!}{t^{k+1}} N(k/t).$$

Now we change the variable as $t = k/s$, and obtain

$$\overline{E}^{(k)}(k/s) = (-1)^k s^{k+1} \frac{k!}{k^{k+1}} N(s).$$

This gives a function $N(s)$, and, when $k \to \infty$, we have the following asymptotic form:

$$N(s) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{s}\right)^{k+1} \overline{E}^{(k)}(k/s).$$

Since $\overline{E}$ is a function of k/s, and using (2.251) we have a function $H(\tau)$ instead of $N(s)$[9]:

$$H(\tau) = \frac{1}{\tau} N\left(\frac{1}{\tau}\right) = \lim_{k\to\infty} \frac{(-1)^k}{(k-1)!} (k\tau)^k\, \overline{E}^{(k)}(k\tau). \tag{2.274}$$

Equation 2.274 shows that the relaxation spectrum $H(\tau)$ can be approximated by

$$H_1(\tau) = -\tau \frac{\partial \overline{E}(\tau)}{\partial \tau}, \quad H_2(\tau/2) = \tau^2 \frac{\partial^2 \overline{E}(\tau)}{\partial \tau^2}, \quad H_3(\tau/3) = -\frac{\tau^3}{2} \frac{\partial^3 \overline{E}(\tau)}{\partial \tau^3}, \quad \cdots$$

When using actual experimental data, the differentiation frequently involves some numerical errors, therefore we can use either H_2 or, more often, H_1 for the spectral approximation. The practical procedures will be given in a later section.

[9] Note that the differentiation implies $\overline{E}^{(k)}(k\tau) = \dfrac{d^k \overline{E}(k\tau)}{d(k\tau)^k}$. Others are the same.

The most important advantage of this procedure is that we can determine the spectral points that govern the corresponding viscoelastic response, which gives their coefficients, specified by the least squares method.

The retardation spectrum $L(\tau)$ of (2.264) is written as

$$L(\tau) = \lim_{k\to\infty} \frac{(-1)^{k+1}}{(k-1)!} (k\tau)^k \bar{J}^{(k)}(k\tau). \tag{2.275}$$

For an isotropic material the relaxation spectra are given by

$$\Phi^s(\tau) = \lim_{k\to\infty} \frac{(-1)^k}{(k-1)!} (k\tau)^k \overline{G}^{(k)}(k\tau), \quad \Phi^v(\tau) = \lim_{k\to\infty} \frac{(-1)^k}{(k-1)!} (k\tau)^k \overline{K}^{(k)}(k\tau), \tag{2.276}$$

and the retardation spectra are

$$\Psi^s(\tau) = \lim_{k\to\infty} \frac{(-1)^k}{(k-1)!} (k\tau)^k \overline{B}^{(k)}(k\tau), \quad \Psi^v(\tau) = \lim_{k\to\infty} \frac{(-1)^k}{(k-1)!} (k\tau)^k \overline{C}^{(k)}(k\tau). \tag{2.277}$$

2.9.5 Experiments for Determining Viscous Properties

Several experiments are used to determine the viscoelastic properties of a material: creep tests (stress = constant), relaxation tests (strain = constant) and a constant stress rate test. Figure 2.12 shows the loading paths of each test for the one-dimensional case.

We first determine the creep function $J(t)$ based on the results of a creep test: If the creep stress is given by (2.260), the response is given by (2.261), therefore $\varepsilon(t) = J(t)\sigma_0$. Let us assume that the relaxation spectrum is represented by the first approximation of (2.275), i.e.,

$$L(\tau) = -\tau \frac{\partial \overline{J}(\tau)}{\partial \tau}.$$

By substituting (2.261) into this result we have

$$L(\tau) = -\frac{1}{\sigma_0}\tau \frac{\partial \varepsilon(\tau)}{\partial \tau}. \tag{2.278}$$

We can then plot the values of (2.278) calculated by the creep curve shown, for example, in Fig. 2.16. In this figure we specify discrete values $\tau_1, \tau_2, \ldots$. Since the discrete representation of (2.264) is given by

$$\overline{J}(t) = \sum_i J_i \left(1 - \exp(-t/\tau_i)\right), \qquad J_i = L(\tau_i)\, \Delta \ln \tau_i, \tag{2.279}$$

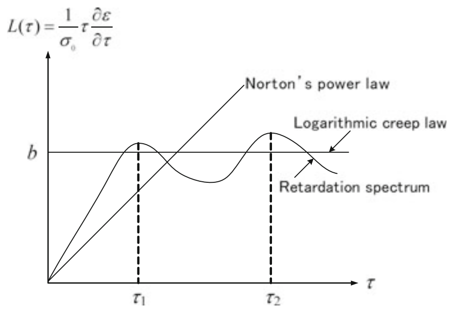


Fig. 2.16 Implication of retardation spectrum

we can substitute the above τ_i's into (2.279), and, using (2.261) and (2.263), we obtain the following:

$$\varepsilon(t) = \left[J_\infty + \frac{t}{\eta_0} + \sum_i J_i \left(1 - \exp(-t/\tau_i)\right) \right] \sigma_0. \tag{2.280}$$

We now need a procedure to determine η_0. From (2.280) the slope at the final elapsed time t^* is given by

$$\frac{1}{\sigma_0} \frac{d\varepsilon}{dt}\bigg|_{t=t^*} = \sum \frac{J_i}{\tau_i} \exp(-t^*/\tau_i) + \frac{1}{\eta_0}. \tag{2.281}$$

Thus if $\tau_i \ll t^*$, we can specify the slope $1/\eta_0$. If $\tau_i \approx t^*$, the reader is referred to Akagi (1980) for details of the method to specify η_0.

If τ_i $(i = 1, 2, \ldots)$ and η_0 are known, J_∞ and J_i $(i = 1, 2, \ldots)$ can be specified by a linear least squares method. That is, setting the error estimate as

$$\Pi = \frac{1}{2}\sigma_0^2 \left[J_\infty + \frac{t}{\eta_0} + \sum_i J_i \left(1 - \exp(-t/\tau_i)\right) \right]^2, \tag{2.282}$$

J_∞ and J_i can be calculated by solving the following simultaneous equations:

$$\frac{\partial \Pi}{\partial J_\infty} = 0, \qquad \frac{\partial \Pi}{\partial J_i} = 0, \qquad i = 1, 2, \ldots \tag{2.283}$$

For an isotropic material, the creep functions $B(t)$ and $C(t)$ are determined using the same procedure. For an axisymmetric triaxial stress state, let σ_1 be the axial stress, $\sigma_3 (< 0)$ (with $\sigma_2 = \sigma_3$) the confining pressure, ε_1 the measured axial strain and ε_3 the measured lateral strain. Then setting

$$\begin{aligned} s_{11} &= \frac{2}{3}(\sigma_1 - \sigma_3) = -2s_{22} = -2s_{33} \equiv q_0, \qquad s_{23} = s_{31} = s_{13} = 0, \\ \bar{\sigma} &= \frac{1}{\sqrt{3}}(\sigma_1 + 2\sigma_3) \equiv p_0, \end{aligned}$$

$$e_{11} = \frac{2}{3}(\varepsilon_1 - \varepsilon_3) = -2e_{22} = -2e_{33} \equiv \varepsilon^s, \qquad e_{23} = e_{31} = e_{13} = 0,$$
$$\bar{\varepsilon} = \frac{1}{\sqrt{3}}(\varepsilon_1 + 2\varepsilon_3) \equiv \varepsilon^v,$$

we have the following deviatoric and volumetric creep responses:

$$\varepsilon^s(t) = \frac{1}{2}B(t)\,q_0, \qquad \varepsilon^v(t) = \frac{1}{3}C(t)\,p_0. \tag{2.284}$$

Due to (2.277), the first approximations of the relaxation spectra of each component are given by

$$\Psi^s(\tau^s) = -\frac{2}{q_0}\tau^s\frac{\partial\varepsilon^s(\tau^s)}{\partial\tau^s}, \qquad \Psi^v(\tau^v) = -\frac{3}{p_0}\tau^v\frac{\partial\varepsilon^v(\tau^v)}{\partial\tau^v}. \tag{2.285}$$

Therefore the discrete relaxation times $\tau_1^s, \tau_2^s, \ldots,\ \tau_1^v, \tau_2^v, \ldots$ can be obtained by creep curves (similar to Fig. 2.16). This gives the discrete forms of (2.272) as

$$\begin{aligned}\overline{B}(t) &= \sum_i B_i\left(1 - \exp(-t/\tau_i^s)\right), \qquad B_i = \Psi^s(\tau_i^s)\,\Delta\ln\tau_i^s\\ \overline{C}(t) &= \sum_i C_i\left(1 - \exp(-t/\tau_i^v)\right), \qquad C_i = \Psi^v(\tau_i^v)\,\Delta\ln\tau_i^v\end{aligned} \tag{2.286}$$

and we obtain

$$\begin{aligned}\varepsilon^s(t) &= \frac{1}{2}\left[B_\infty + \frac{t}{\eta_0^s} + \sum_i B_i\left(1 - \exp(-t/\tau_i^s)\right)\right]q_0\\ \varepsilon^v(t) &= \frac{1}{3}\left[C_\infty + \frac{t}{\eta_0^v} + \sum_i C_i\left(1 - \exp(-t/\tau_i^v)\right)\right]p_0.\end{aligned} \tag{2.287}$$

Under the condition $\tau_i^s,\ \tau_i^v \ll t^*$, η_0^s and η_0^v can be specified by using the slopes of the creep curves at the final time stage t^* by

$$\frac{2}{q_0}\left.\frac{d\varepsilon^s}{dt}\right|_{t=t^*} \doteq \frac{1}{\eta_0^s}, \qquad \frac{3}{p_0}\left.\frac{d\varepsilon^v}{dt}\right|_{t=t^*} \doteq \frac{1}{\eta_0^v}. \tag{2.288}$$

The coefficients B_∞ and B_i $(i = 1, 2, \ldots)$, C_∞ and C_i $(i = 1, 2, \ldots)$ can be calculated by the least squares method where the error estimates are given by

$$\Pi^s = \frac{1}{8}q_0^2\left[B_\infty + \frac{t}{\eta_0^s} + \sum_i B_i\left(1 - \exp(-t/\tau_i^s)\right)\right]^2,$$

$$\Pi^v = \frac{1}{18}p_0^2\left[C_\infty + \frac{t}{\eta_0^v} + \sum_i C_i\left(1 - \exp(-t/\tau_i^v)\right)\right]^2,$$

which gives the simultaneous equations

$$\frac{\partial \Pi^s}{\partial B_\infty} = 0, \qquad \frac{\partial \Pi^s}{\partial B_i} = 0, \tag{2.289}$$

$$\frac{\partial \Pi^v}{\partial C_\infty} = 0, \qquad \frac{\partial \Pi^v}{\partial C_i} = 0. \tag{2.290}$$

Finally, we can see the relationship between the response given by the hereditary integral form and that given by conventional creep laws such as the logarithmic form

$$\varepsilon(t) = a + b \ln t, \tag{2.291}$$

and Norton's power law

$$\frac{\varepsilon}{t} = a\,\sigma^n. \tag{2.292}$$

Differentiating (2.291) yields

$$t\frac{d\varepsilon}{dt} = b.$$

Therefore the logarithmic creep law employs an averaged relaxation spectrum for all elapsed time (i.e., only one time-dependent mechanism is assumed) as shown in Fig. 2.16, which may cause difficulty under real, complex situations such as the long term behavior of rock. On the other hand the power law (2.292) gives

$$t\frac{d\varepsilon}{dt} = t\,(a\sigma_0^n)$$

for a creep stress σ_0, which implies a linear distribution of the spectra (Fig. 2.16).

2.10 Small Strain Plasticity: Flow Theory

In this book we are considering porous materials. Therefore, the stress treated here must be an effective stress $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\boldsymbol{I}$ where $\boldsymbol{\sigma}$ is the total stress, and p is the pore fluid pressure. Note that in this section we are using the sign convention for stresses adopted in continuum mechanics, therefore the tension stress/strain is considered positive, and the pore fluid pressure is positive, since $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\boldsymbol{I}$ (details are described in Chap. 6). In this Section we denote the stress as $\boldsymbol{\sigma}$ instead of the effective stress $\boldsymbol{\sigma}'$ for simplicity. Readers can see that all results in this section also work for the effective stress. It should be noted that in this section the deviatoric stress is denoted as $\boldsymbol{s}$ whereas in other expositions the deviatoric stress is written as $\boldsymbol{\sigma}'$. Similarly, the deviatoric strain is denoted as $\boldsymbol{e}$.

Symbols and notations used in this section are given below:

$\boldsymbol{\sigma}$	(Total) stress tensor (tension: +)
$I_1 = \mathrm{tr}\,(\boldsymbol{\sigma}) = \sigma_{ii}$	First invariant of stress
$I_2 = \{(\mathrm{tr}\,\boldsymbol{\sigma})^2 - \mathrm{tr}\,(\boldsymbol{\sigma}^2)\}/2$	Second invariant of stress
$I_3 = \det(\boldsymbol{\sigma})$	Third invariant of stress
$\bar{\boldsymbol{\sigma}} = \mathrm{tr}\,(\boldsymbol{\sigma})\boldsymbol{I}/3$	Volumetric stress tensor (tension, +)
$\bar{\sigma} = \mathrm{tr}\,(\boldsymbol{\sigma})/\sqrt{3}$	Magnitude of volumetric stress (tension, +)
$\boldsymbol{s} = \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}$	Deviatoric stress tensor
$s = \lvert\boldsymbol{s}\rvert = (s_{ij}\, s_{ij})^{1/2}$	Norm of deviatoric stress
$\sigma_\theta = \frac{1}{3}\cos^{-1}[3\sqrt{3}\, J_3^\sigma/\{2(J_2^\sigma)^{3/2}\}]$	Lode's angle for stress
$J_2 = s_{ij}s_{ij}/2 = 2s^2$	Second invariant of deviatoric stress
$J_3 = \det(\boldsymbol{s})$	Third invariant of deviatoric stress
$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p$	Strain increment tensor
$d\boldsymbol{\varepsilon}^e$	Elastic strain increment tensor
$d\boldsymbol{\varepsilon}^p$	Plastic strain increment tensor
$d\bar{\boldsymbol{\varepsilon}}^p = \mathrm{tr}\,(d\boldsymbol{\varepsilon}^p)\boldsymbol{I}/3$	Volumetric plastic strain increment tensor
$d\bar{\varepsilon}^p = \mathrm{tr}\,(d\boldsymbol{\varepsilon}^p)/\sqrt{3}$	Magnitude of volumetric plastic strain increment
$d\boldsymbol{e}^p = d\boldsymbol{\varepsilon}^p - d\bar{\boldsymbol{\varepsilon}}^p$	Deviatoric plastic strain increment tensor
$de^p = \lvert d\boldsymbol{e}^p\rvert = (de_{ij}^p\, de_{ij}^p)^{1/2}$	Norm of deviatoric plastic strain increment
κ	Hardening parameter

The result of a simple tension experiment for a metal is schematically shown in Fig. 2.17 with axes of axial stress σ_1 and axial strain ε_1 or deviatoric stress s and deviatoric strain e. In metals the volumetric plastic strain can generally be ignored ($\bar{\varepsilon}^p = 0$); therefore we can treat the behavior as a uniaxial response. On the other hand, the shearing behavior of geomaterials is inevitably accompanied by volume changes that are plastic, therefore we have to modify the original flow theory developed for metallic materials (Kachanov 2005; Lubliner 1990). Note that in small strain plasticity we assume that the plastic increment $d\boldsymbol{\varepsilon}$ can be decomposed into incremental elastic and plastic components:

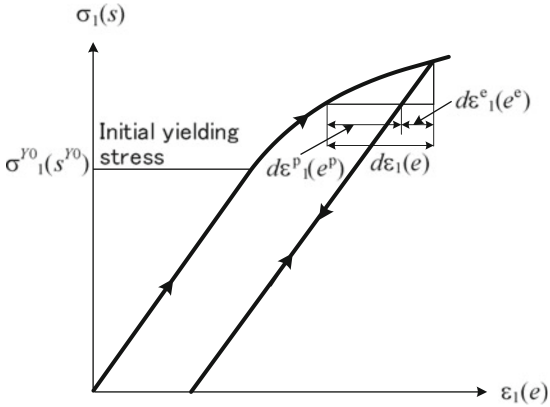
$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p. \tag{2.293}$$

2.10.1 Yield Function and Hardening Law

Let us denote the yield condition by

$$f(\boldsymbol{\sigma}, \kappa) = 0 \tag{2.294}$$

Fig. 2.17 Uniaxial stress-strain relationship



which includes the initial and subsequent yield surfaces (Fig. 2.17). Here κ is the hardening parameter which denotes a history of past stress and strain, and the increment can be denoted by

$$d\kappa = dW^p = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p \qquad \text{: work hardening rule} \tag{2.295}$$

$$d\kappa = |d\boldsymbol{\varepsilon}^p| \qquad \text{: strain hardening rule} \tag{2.296}$$

where dW^p is an increment of plastic work. Note that the initial yielding condition is given by $f(\boldsymbol{\sigma}, \kappa = 0) = 0$.

The yield function $f(\boldsymbol{\sigma}, \kappa)$ is classified into several hardening models depending on the history of loading:

$$f(\boldsymbol{\sigma}, \kappa) = f_1(\boldsymbol{\sigma}) - K(\kappa) \qquad \text{: isotropic hardening model} \tag{2.297}$$

$$f(\boldsymbol{\sigma}, \kappa) = f_1(\boldsymbol{\sigma} - \boldsymbol{\alpha}(\kappa)) \qquad \text{: kinematic hardening model} \tag{2.298}$$

$$d\boldsymbol{\alpha} = c\, d\boldsymbol{\varepsilon}^p \qquad \text{Prager model}$$

$$d\boldsymbol{\alpha} = c(\boldsymbol{\sigma} - \boldsymbol{\alpha})|d\boldsymbol{\varepsilon}^p| \qquad \text{Ziegler model}$$

$$f(\boldsymbol{\sigma}, \kappa) = f_1(\boldsymbol{\sigma} - \boldsymbol{\alpha}(\kappa)) - f_2(\kappa) \qquad \text{: anisotropic hardening model} \tag{2.299}$$

We illustrate the isotropic and kinematic hardening models for the one-dimensional problem in Fig. 2.18a, b, and for the two dimensional problem in Fig. 2.19. Note that the behavior shown in Fig. 2.18b is known as the Bauschinger effect. In geotechnical engineering practice the isotropic hardening model is widely used because, except in earthquake situations, it is rare that the loading direction is completely reversed.

The stress-dependent part of the yield function has been widely investigated using experimental methods. We summarize the results for the case of the isotropic

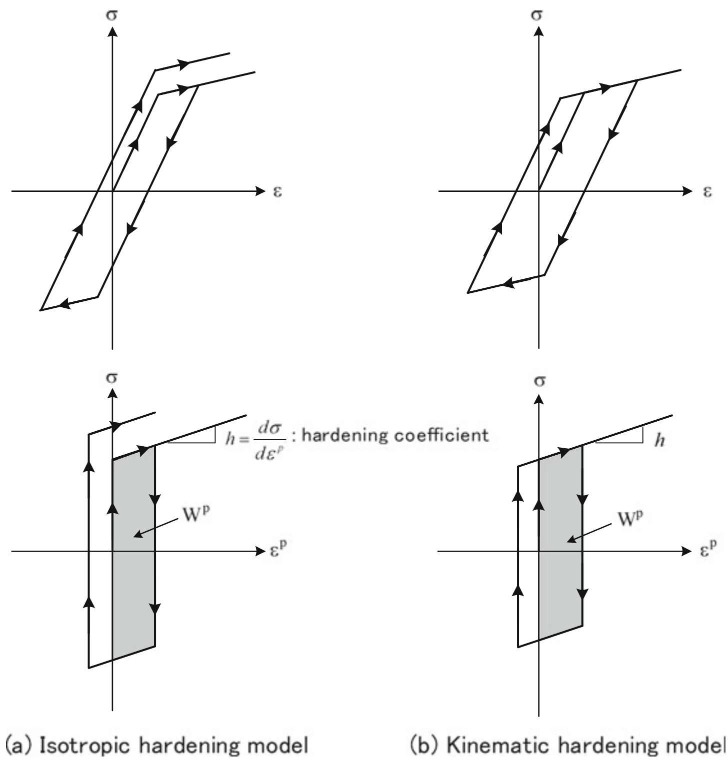


Fig. 2.18 Axial stress-strain relations and hardening models

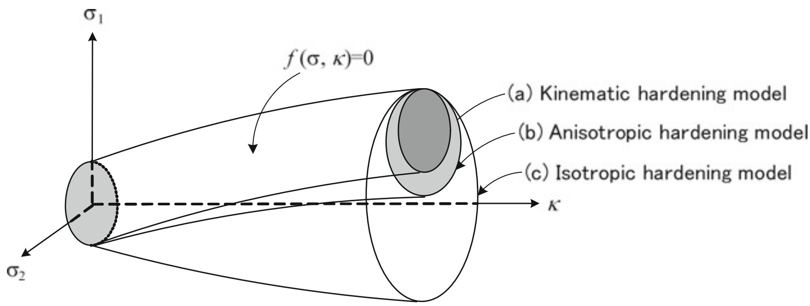


Fig. 2.19 Hardening model for a 2D problem

hardening (2.297) (see also Desai and Siriwardane 1984; Darve 1990; Davis and Selvadurai 2002; Pietruszczak 2010):

$$f_1(\boldsymbol{\sigma}) = \sqrt{J_2} \qquad : \text{von Mises}$$
$$f_1(\boldsymbol{\sigma}) = \sqrt{J_2} + \alpha I_1 \qquad : \text{Drucker-Prager}$$

$f_1(\boldsymbol{\sigma}) = |\sigma_1 - \sigma_3|$: Tresca

$\sigma_1 \geq \sigma_2 \geq \sigma_3$: principal stresses

$f_1(\boldsymbol{\sigma}) = \pm\frac{1}{2}(\sigma_1 - \sigma_3) + \frac{1}{2}(\sigma_1 + \sigma_3) \cdot \sin\phi - C\cos\phi$: Mohr-Coulomb

ϕ: internal friction angle, C: cohesion

$f_1(\boldsymbol{\sigma}) = (I_1)^3/I_3$: Lade-Duncan

$f_1(\boldsymbol{\sigma}) = (I_1 I_2)/I_3$: Matsuoka-Nakai

$f_1(\boldsymbol{\sigma}) = MD\ln\left(p'/p'_0\right) + D\,(q/p')$: Roscoe (Cam clay model)

(For the Cam clay model we use the effective stress $\boldsymbol{\sigma}'$.)

$p' = -\frac{1}{3}\sigma'_{ii}, \quad q = -(\sigma_1 - \sigma_3) = -(\sigma'_1 - \sigma'_3)$ ($\sigma_i,\ \sigma'_i$: positive for tension)

Under biaxial conditions, $p' = -\frac{1}{3}(\sigma'_1 + 2\sigma'_3)$

p'_0: preload

M: slope of critical state line

$D = (\lambda - \kappa)/M(1+e)$

$e = n/(1-n)$: void ratio

λ: slope of normal consolidation line

κ: swelling index (slope of unloading-reloading line)

The original Cam clay model employs isotropic hardening with strain hardening such as

$$K(\kappa) = \int d\kappa, \qquad d\kappa = |d\varepsilon_v^p| = \sqrt{d\varepsilon_v^p d\varepsilon_v^p}$$

(Schofield and Wroth 1968). Here $e = n/(1-n)$ is the void ratio, n is the porosity and $d\varepsilon_v^p = -\,d\varepsilon_{ii}^p$ is $\sqrt{3}$ times the volumetric plastic strain (positive for compression); it is related to the void ratio e and plastic increment of the void ratio de^p by

$$d\varepsilon_v^p = -\frac{de^p}{1+e}.$$

2.10.2 Prager's Consistency Condition

Subsequent yielding occurs at the stress $\boldsymbol{\sigma} + d\boldsymbol{\sigma}$ and plastic state $\kappa + d\kappa$ after the initial yielding if the yielding condition

$$f(\boldsymbol{\sigma} + d\boldsymbol{\sigma},\ \kappa + d\kappa) = 0$$

is satisfied. Using this with (2.294) results in

$$df = \frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f}{\partial \kappa} d\kappa = 0. \tag{2.300}$$

This is *Prager's consistency condition*. Substituting the work hardening law (2.295) into (2.300) gives

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f}{\partial \kappa}\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p = 0 \tag{2.301}$$

If we apply the strain hardening law (2.296), the consistency condition (2.300) remains unchanged.

2.10.3 Flow Rule and Incremental Constitutive Law

Let g be a scalar function such that the plastic strain increment $d\boldsymbol{\varepsilon}^p$ can be obtained as follows:

$$d\boldsymbol{\varepsilon}^p = \lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} \qquad \text{or} \qquad d\varepsilon_{ij}^p = \lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{2.302}$$

This is referred to as the *flow rule of plasticity*. The parameter λ is determined by the hardening law (mentioned later).

The flow rule (2.302) implies that the direction of the plastic strain increment $d\boldsymbol{\varepsilon}^p$ is normal to the surface g = constant, and coincides with the stress $\boldsymbol{\sigma}$. For isotropic materials this can be described as follows. We introduce the unit tensors (see Sect. 2.8.3) as

$$\boldsymbol{n}^{(1)} = \frac{\partial \bar{\sigma}}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} = \frac{\bar{\boldsymbol{\sigma}}}{\bar{\sigma}}, \quad \boldsymbol{n}^{(2)} = \frac{\partial s}{\partial \boldsymbol{s}} = \frac{\partial s}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{s}}{s}, \quad \boldsymbol{n}^{(3)} = \frac{1}{\sigma_\theta}\frac{\partial \sigma_\theta}{\partial \boldsymbol{\sigma}} \tag{2.303}$$

These are orthonormal as described in Sect. 2.8.3, which gives

$$d\boldsymbol{\varepsilon}^p = \lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} = \lambda \left(\frac{\partial g}{\partial \bar{\sigma}} \boldsymbol{n}^{(1)} + \frac{\partial g}{\partial s} \boldsymbol{n}^{(2)} + \sigma_\theta \frac{\partial g}{\partial \sigma_\theta} \boldsymbol{n}^{(3)} \right).$$

Therefore the direction of $d\boldsymbol{\varepsilon}^p$ coincides with the direction of the global stress $\boldsymbol{\sigma}$ if the material is isotropic.

Note 2.9 (Drucker's stability postulate). If the response of an elasto-plastic body is stable, the plastic work, W^p, must be non-negative:

$$W^p = \int (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) : d\boldsymbol{\varepsilon}^p \geq 0. \tag{2.304}$$

This is referred to as *Drucker's stability postulate*. Here $\boldsymbol{\sigma}_0$ is an arbitrary stress which satisfies

$$f(\boldsymbol{\sigma}_0, \kappa) < 0.$$

The condition (2.304) is written in the local form as

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \cdot d\boldsymbol{\varepsilon}^p \geq 0, \tag{2.305}$$

or by setting $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + d\boldsymbol{\sigma}$ we can write

$$d\boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^p \geq 0. \tag{2.306}$$

Substituting the flow rule (2.302) into the inequality (2.305) gives

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \cdot \lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} \geq 0.$$

Since $\boldsymbol{\sigma}_0$ is arbitrary, the above condition is satisfied if $g = f$ (the associated flow rule) and f is convex (Fig. 2.20). This gives a strong restriction for elasto-plastic materials, especially for granular media, since most of the experimental data show that if we apply the associated flow rule with a yield function, such as the Coulomb or Drucker-Prager type, the dilatancy (i.e., the volume change due to shearing) is over-estimated. ■

The incremental constitutive equation for applying the flow rule can be obtained as follows: Substituting the flow rule (2.302) into the consistency condition, we obtain

$$\lambda = \frac{1}{h} \frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma}, \tag{2.307}$$

$$h = \begin{cases} -\dfrac{\partial f}{\partial W^p} \boldsymbol{\sigma} : \dfrac{\partial g}{\partial \boldsymbol{\sigma}} & : \text{work hardening} \\ \\ -\dfrac{\partial f}{\partial \kappa} \left| \dfrac{\partial g}{\partial \boldsymbol{\sigma}} \right| & : \text{strain hardening} \end{cases} \tag{2.308}$$

where h is the hardening parameter, which will be determined later. Equation 2.307 is again substituted into the flow rule to give

$$d\boldsymbol{\varepsilon}^p = \frac{1}{h} \frac{\partial g}{\partial \boldsymbol{\sigma}} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} \right) = \frac{1}{h} \left(\frac{\partial g}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) d\boldsymbol{\sigma} \tag{2.309}$$

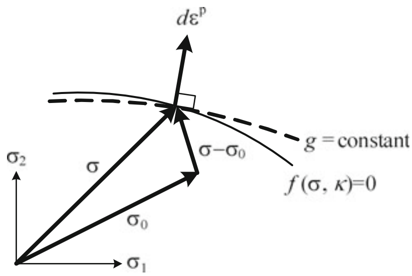


Fig. 2.20 Drucker's stability condition

where $\otimes$ denotes the tensor product, and if $\partial g/\partial\boldsymbol{\sigma}$ and $\partial f/\partial\boldsymbol{\sigma}$ are represented as vectors, it becomes

$$\frac{\partial g}{\partial\boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial\boldsymbol{\sigma}} = \left\{\frac{\partial g}{\partial\boldsymbol{\sigma}}\right\}\left\{\frac{\partial f}{\partial\boldsymbol{\sigma}}\right\}^T. \tag{2.310}$$

Equation 2.309 can be written in the indicial notation:

$$d\varepsilon_{ij}^p = \frac{1}{h}\frac{\partial g}{\partial\sigma_{ij}}\frac{\partial f}{\partial\sigma_{kl}}d\sigma_{kl}. \tag{2.311}$$

Let $\boldsymbol{C}^p$ be the plastic compliance tensor (cf. (2.309)) given by

$$\boldsymbol{C}^p = \frac{1}{h}\frac{\partial g}{\partial\boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial\boldsymbol{\sigma}} \qquad \text{or} \qquad C_{ij}^p = \frac{1}{h}\frac{\partial g}{\partial\sigma_{ij}}\frac{\partial f}{\partial\sigma_{kl}}, \tag{2.312}$$

then we have

$$d\boldsymbol{\varepsilon}^p = \boldsymbol{C}^p d\boldsymbol{\sigma} \qquad \text{or} \qquad d\varepsilon_{ij}^p = C_{ijkl}^p d\sigma_{kl}. \tag{2.313}$$

Equation 2.309 (or 2.313) is referred to as *Melan's formula*.

Since the plastic compliance tensor $\boldsymbol{C}^p$ of (2.312), determined by the flow rule, is represented by a product of two second-order tensors, the determinant is identically zero ($\det \boldsymbol{C}^p = 0$, if we set the second-order tensors as vectors as mentioned in (2.310)). Since it is not possible to obtain the inverse of $\boldsymbol{C}^p$ directly, we use the properties of the elastic compliance $\boldsymbol{C}^e$, which has the inverse, along with the direct sum of the strain increment given by (2.293). That is,

$$\begin{aligned} d\boldsymbol{\varepsilon} &= d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p = (\boldsymbol{C}^e + \boldsymbol{C}^p)\, d\boldsymbol{\sigma}, \\ \Rightarrow \quad d\boldsymbol{\sigma} &= \boldsymbol{D}^{ep} d\boldsymbol{\varepsilon}, \qquad \boldsymbol{D}^{ep} = (\boldsymbol{C}^e + \boldsymbol{C}^p)^{-1}. \end{aligned} \tag{2.314}$$

$\boldsymbol{D}^{ep}$ is determined explicitly as follows. Let

$$d\boldsymbol{\sigma} = \boldsymbol{D}^e d\boldsymbol{\varepsilon}^e = \boldsymbol{D}^e(d\boldsymbol{\varepsilon} - d\boldsymbol{\varepsilon}^p) = \boldsymbol{D}^e d\boldsymbol{\varepsilon} - \boldsymbol{D}^e\frac{1}{h}\frac{\partial g}{\partial\boldsymbol{\sigma}}\left(\frac{\partial f}{\partial\boldsymbol{\sigma}} : d\boldsymbol{\sigma}\right). \tag{2.315}$$

Taking the inner-product with $\partial f/\partial\boldsymbol{\sigma}$, the above gives

$$\frac{\partial f}{\partial\boldsymbol{\sigma}} : d\boldsymbol{\sigma} = \frac{h}{H}\frac{\partial f}{\partial\boldsymbol{\sigma}} : \boldsymbol{D}^e d\boldsymbol{\varepsilon}, \qquad H = h + \frac{\partial f}{\partial\boldsymbol{\sigma}} : \left(D^e\frac{\partial g}{\partial\boldsymbol{\sigma}}\right). \tag{2.316}$$

Using the symmetry property of $\boldsymbol{D}^e$ and substituting (2.316) into (2.315), we obtain

$$d\boldsymbol{\sigma} = \boldsymbol{D}^{ep} d\boldsymbol{\varepsilon}, \tag{2.317}$$

where

$$\boldsymbol{D}^{ep} = \boldsymbol{D}^e - \frac{1}{H}\left(\boldsymbol{D}^e\frac{\partial g}{\partial\boldsymbol{\sigma}}\right) \otimes \left(\boldsymbol{D}^e\frac{\partial f}{\partial\boldsymbol{\sigma}}\right). \tag{2.318}$$

The indicial form of $\boldsymbol{D}^{ep}$ is given by

$$D^{ep}_{ijkl} = D^{e}_{ijkl} - \frac{1}{H}\left(D^{e}_{ijmn}\frac{\partial g}{\partial \sigma_{mn}}\right)\left(D^{e}_{klst}\frac{\partial f}{\partial \sigma_{st}}\right). \tag{2.319}$$

We now need to determine the hardening parameter h. For simplicity we use the isotropic hardening model (2.307), and from (2.308) we have

$$h = \begin{cases} \dfrac{\partial K}{\partial W^p}\boldsymbol{\sigma} : \dfrac{\partial g}{\partial \boldsymbol{\sigma}} & : \text{work-hardening,} \\[2ex] \dfrac{\partial K}{\partial \kappa}\left|\dfrac{\partial g}{\partial \boldsymbol{\sigma}}\right| & : \text{strain-hardening.} \end{cases} \tag{2.320}$$

A function $\phi(\boldsymbol{x})$ is said to be m-th order homogeneous of $\boldsymbol{x}$ if, for any scalar t,

$$\phi(t\boldsymbol{x}) = t^m\phi(\boldsymbol{x}).$$

Then we have the following Euler's theorem (cf. Note 3.8, p. 108):

$$\boldsymbol{x}\cdot\frac{\partial \phi}{\partial \boldsymbol{x}} = m\,\phi(\boldsymbol{x})$$

It should be noted that the von Mises ($f_1 = \sqrt{J_2}$) and Drucker-Prager ($f_1 = \sqrt{J_2} + \alpha I_1$) yield functions are first order homogeneous functions; however

$$f_1 = J_2 + \alpha I_1$$

is not a homogeneous function.

Let us apply the associated flow rule ($g = f$) and assume that f is an m-th order homogeneous function, so that the work-hardening rule (2.320) can be written as

$$h = m\frac{\partial K}{\partial W^p} f_1. \tag{2.321}$$

If we define the equivalent stress by

$$f_1(\boldsymbol{\sigma}) = \sigma_e, \tag{2.322}$$

and determine the equivalent plastic strain increment by

$$dW^p = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p \equiv \sigma_e\, d\varepsilon^p_e, \tag{2.323}$$

the hardening parameter can be specified by

$$h = m\frac{\partial \sigma_e}{\partial \varepsilon^p_e}. \tag{2.324}$$

Let us now consider the physical implications of the equivalent stress and strain. For metallic materials, which show no volumetric plastic straining, we have

$$\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p = s\, de^p + \bar{\sigma} d\bar{\varepsilon}^p = s\, de^p.$$

where de^p is the deviatoric plastic strain increment and $d\bar{\varepsilon}^p$ is the volumetric plastic strain increment, and therefore (2.323) is meaningful. However, for porous materials that show a considerable amount of volumetric plastic strain the concept of equivalent stress and strain defined by (2.323) is not appropriate. Note that in the metallic plasticity theory, in order to utilize the result of simple tension we sometimes define

$$\sigma_e = \sqrt{3J_2} \quad (= \sigma_{11}),$$

and introduce

$$d\varepsilon_e^p = \sqrt{\frac{2}{3} d\boldsymbol{e}^p : d\boldsymbol{e}^p}.$$

This, of course, causes an adjustment of the coefficients.

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