

Chapter 2

Differential Electromagnetic Equations in Fractional Space

In this chapter a novel generalization of differential electromagnetic equations in fractional space is provided. Firstly, basic vector differential operators are generalized in fractional space and then using these fractional operators Maxwell's, Laplace's, Poisson's and Helmholtz's equations have been worked out in fractional space. The differential electromagnetic equations in fractional space, established in this chapter, provide a basis for application of the concept of fractional space in practical electromagnetic wave propagation and scattering problems in fractal media.

In Sect. 2.1 a review of already existing study to construct a generalized Laplacian operator using integration in D -dimensional fractional space is briefly described. In Sect. 2.2, fractional space generalization of the Del operator, written as ∇_D , and its related differential operators (i.e., gradient, divergence and curl) in vector calculus is obtained. In Sect. 2.3, a novel fractional space generalization of differential Maxwell's equations is presented. In Sect. 2.4, fractional space generalization of the Laplace and Poisson's equations is established in addition to fractional space generalization of potentials for static field. In Sect. 2.5, potentials for time varying fields in fractional space are derived. In Sect. 2.6, the Helmholtz's equation in fractional space is established. Finally, this chapter is summarized in Sect. 2.7.

2.1 Fractional Space Generalization of Laplacian Operator

In [1] a formalism is provided for integration on D -dimensional fractional space. According to this formalism, the integration of radially symmetric function $f(r)$ in a D -dimensional fractional space is given by [1]:

$$\int d\mathbf{x}_0 f(r(\mathbf{x}_0, \mathbf{x}_1)) = \int_0^\infty dr W(r) f(r) \quad (2.1)$$

where $r(\mathbf{x}_0, \mathbf{x}_1)$ is the distance between two points $\mathbf{x}_0$ and $\mathbf{x}_1$, and weight $W(r)$ given by

$$W(r) = \sigma(D)r^{D-1} \quad (2.2)$$

with

$$\sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \quad (2.3)$$

From this a single variable Laplacian operator is derived in D -dimensional fractional space as:

$$\nabla_D^2 f(r) = \left[\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right] f(r), \quad 0 < D \leq 1 \quad (2.4)$$

In Eq. 2.4 and throughout the discussion, the subscript D is used to emphasize the dimension of space in which this operator is defined. An extension of formalism in Eq. 2.1 to two variable integration yields an expression for a two-coordinate Laplacian operator in fractional space.

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{D-2}{y} \frac{\partial}{\partial y}, \quad 0 < D \leq 2 \quad (2.5)$$

In [2] the results from [1] are generalized to n orthogonal coordinates and Laplacian operator in D -dimensional fractional space in three-spatial coordinates is given as:

$$\begin{aligned} \nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x} \frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y} \frac{\partial}{\partial y} \\ + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z} \frac{\partial}{\partial z} \end{aligned} \quad (2.6)$$

where, three parameters ($0 < \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ and $0 < \alpha_3 \leq 1$) are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $D = \alpha_1 + \alpha_2 + \alpha_3$. It is obvious that for three dimensional space ($D = 3$), if we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in (2.6), the fractional Laplacian operator ∇_D^2 reduces to the classical Laplacian operator ∇^2 [3] in Euclidean space.

2.2 Fractional Space Generalization of Del Operator and Related Differential Operators

In this section we wish to develop a generalization of vector differential operators in fractional space using scalar Laplacian operator described in previous section.

2.2.1 Del Operator in Fractional Space

From Eq. 2.6, we consider single variable Laplacian operator in fractional space:

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{D-1}{x} \frac{\partial}{\partial x}, \quad 0 < D \leq 1 \quad (2.7)$$

We wish to find an expression for Del operator ∇_D in fractional space. As

$$\nabla_D = |\nabla_D| \cdot \hat{\nabla}_D \quad (2.8)$$

In single variable case we assume $\hat{\nabla}_D = \hat{x}$ also $|\nabla_D| = \sqrt{\nabla_D^2}$, because $\nabla_D \cdot \nabla_D = \nabla_D^2$, where ∇_D^2 is given in (2.7):

$$|\nabla_D| = \sqrt{\frac{\partial^2}{\partial x^2} + \frac{D-1}{x} \frac{\partial}{\partial x}} \quad (2.9)$$

Expansion of (2.9) using Binomial series expansion [3] for $|x| \gg 1$, ignoring terms involving second or higher degree of x in denominator, leads to the following form:

$$|\nabla_D| = \frac{\partial}{\partial x} + \frac{1}{2} \frac{D-1}{x} \quad (2.10)$$

From (2.8) and (2.10), Del operator in single variable x with fractional dimension D is given by:

$$\nabla_D = \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{D-1}{x} \right) \hat{x} \quad (2.11)$$

Extending above procedure to three variable case for $|x|, |y|, |z| \gg 1$ we get Del operator ∇_D in fractional space as follows:

$$\begin{aligned} \nabla_D = & \left(\frac{\partial}{\partial x} + \frac{1}{2} \frac{\alpha_1 - 1}{x} \right) \hat{x} + \left(\frac{\partial}{\partial y} + \frac{1}{2} \frac{\alpha_2 - 1}{y} \right) \hat{y} \\ & + \left(\frac{\partial}{\partial z} + \frac{1}{2} \frac{\alpha_3 - 1}{z} \right) \hat{z} \end{aligned} \quad (2.12)$$

where, parameters ($0 < \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ and $0 < \alpha_3 \leq 1$) are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $D = \alpha_1 + \alpha_2 + \alpha_3$. It is important to mention that Eq. 2.12 and all differential operators presented in later sections are valid in far-field region only i.e ($|x|, |y|, |z| \gg 1$) because of the first order approximation given by (2.10). Clearly, for three dimensional space ($D = 3$), if we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in (2.12), the fractional Del operator ∇_D reduces to the classical Del operator ∇ [3] in Euclidean space.

2.2.2 Gradient Operator in Fractional Space

The gradient of a scalar field ψ in fractional space is a vector that represents both the magnitude and the direction of maximum space rate of increase of ψ in fractional space. Using (2.12) the modified form of the gradient of scalar field ψ , written as $\text{grad}_D \psi$, in far-field region in the fractional space is given as:

$$\begin{aligned} \text{grad}_D \psi = \nabla_D \psi = & \left(\frac{\partial \psi}{\partial x} + \frac{1}{2} \frac{(\alpha_1 - 1)\psi}{x} \right) \hat{x} + \left(\frac{\partial \psi}{\partial y} + \frac{1}{2} \frac{(\alpha_2 - 1)\psi}{y} \right) \hat{y} \\ & + \left(\frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{(\alpha_3 - 1)\psi}{z} \right) \hat{z} \end{aligned} \quad (2.13)$$

2.2.3 Divergence Operator in Fractional Space

From (2.12) a generalized form of divergence of a vector $\mathbf{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$ at point $P(x_0, y_0, z_0)$ in far-field region in the fractional space is written as $\text{div}_D \mathbf{F}$ and is given by

$$\begin{aligned} \text{div}_D \mathbf{F} = \nabla_D \cdot \mathbf{F} = & \frac{\partial F_x}{\partial x} + \frac{1}{2} \frac{(\alpha_1 - 1)F_x}{x} + \frac{\partial F_y}{\partial y} + \frac{1}{2} \frac{(\alpha_2 - 1)F_y}{y} \\ & + \frac{\partial F_z}{\partial z} + \frac{1}{2} \frac{(\alpha_3 - 1)F_z}{z} \end{aligned} \quad (2.14)$$

2.2.4 Curl Operator in Fractional Space

The modified form of curl of a vector $\mathbf{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$ at point $P(x_0, y_0, z_0)$ in far-field region in the fractional space is written as $\text{curl}_D \mathbf{F}$ and using (2.12) it is given by

$$\begin{aligned} \text{curl}_D \mathbf{F} = \nabla_D \times \mathbf{F} = & \left[\left(\frac{\partial F_z}{\partial y} + \frac{1}{2} \frac{(\alpha_2 - 1)F_z}{y} \right) - \left(\frac{\partial F_y}{\partial z} + \frac{1}{2} \frac{(\alpha_3 - 1)F_y}{z} \right) \right] \hat{x} \\ & + \left[\left(\frac{\partial F_x}{\partial z} + \frac{1}{2} \frac{(\alpha_3 - 1)F_x}{z} \right) - \left(\frac{\partial F_z}{\partial x} + \frac{1}{2} \frac{(\alpha_1 - 1)F_z}{x} \right) \right] \hat{y} \\ & + \left[\left(\frac{\partial F_y}{\partial x} + \frac{1}{2} \frac{(\alpha_1 - 1)F_y}{x} \right) - \left(\frac{\partial F_x}{\partial y} + \frac{1}{2} \frac{(\alpha_2 - 1)F_x}{y} \right) \right] \hat{z} \end{aligned} \quad (2.15)$$

or

$$\text{curl}_D \mathbf{F} = \nabla_D \times \mathbf{F} = \left| \begin{array}{ccc} \frac{\partial}{\partial x} + \frac{\hat{x}}{2} \frac{\alpha_1 - 1}{F_x} & \frac{\partial}{\partial y} + \frac{\hat{y}}{2} \frac{\alpha_2 - 1}{F_y} & \frac{\partial}{\partial z} + \frac{\hat{z}}{2} \frac{\alpha_3 - 1}{F_z} \\ \frac{\partial}{\partial y} + \frac{\hat{y}}{2} \frac{\alpha_2 - 1}{F_y} & \frac{\partial}{\partial z} + \frac{\hat{z}}{2} \frac{\alpha_3 - 1}{F_z} & \frac{\partial}{\partial x} + \frac{\hat{x}}{2} \frac{\alpha_1 - 1}{F_x} \\ \frac{\partial}{\partial z} + \frac{\hat{z}}{2} \frac{\alpha_3 - 1}{F_z} & \frac{\partial}{\partial x} + \frac{\hat{x}}{2} \frac{\alpha_1 - 1}{F_x} & \frac{\partial}{\partial y} + \frac{\hat{y}}{2} \frac{\alpha_2 - 1}{F_y} \end{array} \right| \quad (2.16)$$

2.3 Fractional Space Generalization of Differential Maxwell's Equations

The Maxwell's equations are the fundamental equations describing the behavior of electric and magnetic fields. In classical electromagnetic theory following quantities are dealt with:

$\mathbf{E}$ = electric field intensity (V/m)

$\mathbf{B}$ = magnetic field intensity (A/m)

$\mathbf{D}$ = electric flux density (C/m^2)

$\mathbf{H}$ = magnetic flux density (W/m^2)

$\mathbf{J}$ = electric current density (A/m^2)

ρ_v = electric charge density (C/m^3)

with $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$, where μ and ε are permeability and permittivity of the medium, respectively.

All of these quantities are functions of space variables x, y, z and time t . The basic classical Maxwell's equations in differential form in Euclidean space are [4]:

$$\nabla \cdot \mathbf{D} = \rho_v \quad (2.17)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.18)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.19)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.20)$$

Also the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \quad (2.21)$$

is implicit in Maxwell's equations.

Now we wish to have a generalized form of Maxwell's equations in D -dimensional fractional space. From the results of [Sect. 3](#), we are now able to write differential form of Maxwell's equations in far-field region in the fractional space as follows:

$$\text{div}_D \mathbf{D} = \rho_v \quad (2.22)$$

$$\text{div}_D \mathbf{B} = 0 \quad (2.23)$$

$$\text{curl}_D \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.24)$$

$$\text{curl}_D \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.25)$$

and the continuity equation in fractional space as:

$$\text{div}_D \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \quad (2.26)$$

where, div_D and curl_D are defined in Eqs. 2.14–2.16. Equations 2.22–2.25 provide generalization of classical Maxwell's equations from integer dimensional Euclidean space to a non-integer dimensional fractional space. For $D = 3$, these fractional equations can be reduced to classical Maxwell's equations in Euclidean space.

In phasor form, assuming a time factor $e^{j\omega t}$, Maxwell's equations in fractional space are given by replacing $\frac{\partial}{\partial t}$ with $j\omega$ [4] as below:

$$\text{div}_D \mathbf{D}_s = \rho_{vs} \quad (2.27)$$

$$\text{div}_D \mathbf{B}_s = 0 \quad (2.28)$$

$$\text{curl}_D \mathbf{E}_s = -j\omega \mathbf{B}_s \quad (2.29)$$

$$\text{curl}_D \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s \quad (2.30)$$

and the phasor form of continuity equation in fractional space as:

$$\text{div}_D \mathbf{J}_s = -j\omega \rho_{vs} \quad (2.31)$$

where, $\mathbf{D}_s$, $\mathbf{B}_s$, $\mathbf{E}_s$, $\mathbf{H}_s$, $\mathbf{J}_s$, ρ_{vs} represent the phasor form of instantaneous quantities $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$ and ρ_v , respectively.

2.4 Fractional Space Generalization of Potentials for Static Fields, Poisson's and Laplace's Equations

From Maxwell's equations in previous section, it is shown that the behavior of electrostatic field in fractional space can be described by two differential equations:

$$\text{div}_D \mathbf{E} = \frac{\rho_v}{\varepsilon_0} \quad (2.32)$$

$$\text{curl}_D \mathbf{E} = 0 \quad (2.33)$$

where, ε_0 is permittivity of free space. Equation 2.33 being equivalent to the statement that $\mathbf{E}$ is the gradient of a scalar function, the scalar potential for electric field ψ . Because

$$\text{curl}_D (-\text{grad}_D \psi) = 0 \quad (2.34)$$

so,

$$\mathbf{E} = -\text{grad}_D \psi \quad (2.35)$$

A detailed proof of Eq. 2.34 is provided in Appendix A. Equations 2.32 and 2.35 can be combined into one partial differential equation for the single function $\psi(x, y, z)$ as follows:

$$\text{div}_D \text{grad}_D \psi = \frac{\rho_v}{\epsilon_0} \quad (2.36)$$

As $\text{div}_D \text{grad}_D \psi = \nabla_D^2 \psi$, so finally we get

$$\nabla_D^2 \psi = \frac{\rho_v}{\epsilon_0} \quad (2.37)$$

where ∇_D^2 is scalar Laplacian operator in fractional space given by (2.6). Equation 2.37 is called *Poisson's equation* in fractional space. In regions of space that lack a charge density, the scalar potential ψ satisfies the *Laplace's equation* given by:

$$\nabla_D^2 \psi = 0 \quad (2.38)$$

Equations 2.37–2.38 are important in solving practical electrostatic problems in fractional space.

From Maxwell's equations in last section, it is shown that the behavior of magnetostatic field in fractional space can be described by two differential equations:

$$\text{div}_D \mathbf{B} = 0 \quad (2.39)$$

$$\text{curl}_D \mathbf{H} = \mathbf{J} \quad (2.40)$$

From Eq. 2.40 we say that in problems concerned with finding the magnetic fields in a current free region, the curl_D of magnetic field $\mathbf{H}$ is zero. Any vector with zero curl_D may be represented as the grad_D of a scalar (see e.g., Eq. 2.34). Thus, the magnetic field for points in such regions can be expressed as

$$\mathbf{H} = -\text{grad}_D \psi_m \quad (2.41)$$

where, ψ_m (in amperes) is the magnetic scalar potential and the minus sign is taken to complete the analogy with electrostatic field in (2.35).

From (2.39), the divergence of $\mathbf{B}$ is zero everywhere, so using (2.39) and (2.41)

$$\text{div}_D (\mu \text{grad}_D \psi_m) = 0 \quad (2.42)$$

Thus for a homogenous medium in fractional space the magnetic scalar potential ψ_m satisfies the Laplace equation:

$$\nabla_D^2 \psi_m = 0 \quad (2.43)$$

From (2.39) we know that for magnetostatic field $\partial \text{div}_D \mathbf{B} = 0$. Also we know that

$$\partial \text{div}_D \text{curl}_D \mathbf{A} = 0 \quad (2.44)$$

In order to satisfy (2.39) and (2.44) simultaneously, we can define vector magnetic potential $\mathbf{A}$ (in webers/meter) such that

$$\mathbf{B} = \text{curl}_D \mathbf{A} \quad (2.45)$$

Now if we substitute (2.45) into (2.40) we get

$$\text{curl}_D \text{curl}_D \mathbf{A} = \mu \mathbf{J} \quad (2.46)$$

This may be considered as differential equation relating $\mathbf{A}$ to the current density $\mathbf{J}$. Using vector identity

$$\text{curl}_D \text{curl}_D \mathbf{A} = \text{grad}_D (\partial \text{div}_D \mathbf{A}) - \nabla_D^2 \mathbf{A} \quad (2.47)$$

with

$$\partial \text{div}_D \mathbf{A} = 0 \quad (2.48)$$

in (2.46) we get

$$\nabla_D^2 \mathbf{A} = -\mu \mathbf{J} \quad (2.49)$$

This is a vector equivalent of Poisson's equation in (2.37). It includes three component scalar equations which are exactly of the poisson form.

2.5 Fractional Space Generalization of Potentials for Time-Varying Fields

As we have seen, in Maxwell's equations fields are related to each other and sources as well. But sometimes it is helpful to introduce some intermediate functions, known as potential functions, which are directly related to sources and from which we can drive fields [4]. Such functions are found useful for static fields as well (see e.g., Eqs. 2.35, 2.41, 2.45).

From (2.45) we have $\mathbf{B} = \text{curl}_D \mathbf{A}$. This relation may now be substituted into Maxwell's equation (2.24) to get

$$\text{curl}_D \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right] = 0 \quad (2.50)$$

Equation 2.50 states that curl_D of a certain quantity is zero. But this condition allows a vector to be derived as a grad_D of a scalar ψ .

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\text{grad}_D \psi \quad (2.51)$$

$$\mathbf{E} = -\text{grad}_D \psi - \frac{\partial \mathbf{A}}{\partial t} \quad (2.52)$$

Equation 2.45 and 2.52 are the valid relationships between fields and potential functions $\mathbf{A}$ and ψ . We substitute (2.52) into (2.22), to obtain

$$-\nabla_D^2 \psi - \frac{\partial(\text{div}_D \mathbf{A})}{\partial t} = \frac{\rho_v}{\varepsilon} \quad (2.53)$$

Then by substituting (2.45) and (2.52) into (2.53), we get

$$\text{curl}_D \text{curl}_D \mathbf{A} = \mu \mathbf{J} + \mu \varepsilon \left[-\text{grad}_D \frac{\partial \psi}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right] \quad (2.54)$$

Using the vector identity (2.45) and choosing

$$\text{div}_D \mathbf{A} = -\mu \varepsilon \frac{\partial \psi}{\partial t} \quad (2.55)$$

Equation 2.53 and 2.54 can be reduced to

$$\nabla_D^2 \psi - \mu \varepsilon \frac{\partial^2 \psi}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \quad (2.56)$$

$$\nabla_D^2 \mathbf{A} - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \quad (2.57)$$

Thus the potential functions $\mathbf{A}$ and ψ , defined in terms of sources $\mathbf{J}$ and ρ_v by the Eqs. 2.56 and 2.57 in fractional space, may be used to drive electric and magnetic fields using (2.45) and (2.52).

2.6 Fractional Space Generalization of the Helmholtz's Equation

From Eqs. 2.24 and 2.25, using $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$, we finally obtain

$$\text{curl}_D \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (2.58)$$

$$\text{curl}_D \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad (2.59)$$

Taking curl_D of Eq. 2.58 on both sides and using (2.59) gives

$$\text{curl}_D \text{curl}_D \mathbf{E} = -\mu \frac{\partial}{\partial t} \left[\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right] \quad (2.60)$$

This result can be simplified using (2.47) and (2.26) in (2.60) as :

$$\nabla_D^2 \mathbf{E} = \frac{1}{\varepsilon} \text{grad}_D \rho_v + \mu \frac{\partial \mathbf{J}}{\partial t} + \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (2.61)$$

For source-free region ($\rho_v = 0, \mathbf{J} = 0$) (2.61) becomes

$$\nabla_D^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (2.62)$$

Equation 2.62 is the Helmholtz's equation, or wave equation, for $\mathbf{E}$ in fractional space. An identical equation for $\mathbf{H}$ in fractional space can also be derived in the same manner:

$$\nabla_D^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad (2.63)$$

2.7 Summary

A novel fractional space generalization of the differential electromagnetic equations, that is helpful in studying the behavior of electric and magnetic fields in fractal media, is provided. A new form of vector differential operator Del, written as ∇_D , and its related differential operators is formulated in fractional space. Using these modified vector differential operators, the classical Maxwell's electromagnetic equations have been worked out. The Laplace's, Poisson's and Helmholtz's equations in fractional space are derived by using modified vector differential operators. Also a new fractional space generalization of potentials for static and time-varying fields is presented. For all investigated cases, when integer dimensional space is considered, the classical results can be recovered. The provided fractional space generalization of differential electromagnetic equations is valid in far-field region only. The differential electromagnetic equations in fractional space, established in this work, provide a basis for application of the concept of fractional space in practical electromagnetic wave propagation and scattering phenomenon in far-field region in any fractal media.

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