Chapter 1
Introduction

The classical problem of Plateau, although by far not the oldest problem in the Calculus of Variations, is certainly one of the best known. The mathematical formulation of the problem of finding a least area surface of the topological type of the disk spanning a closed contour goes back to Weierstrass. In particular, Weierstrass formulated the existence of the solution of the least area problem as a solution to a system of non-linear partial differential equations:

Set

$$B := \{w \in \mathbb{C} : |w| < 1\}$$

and

$$C := \{w \in \mathbb{C} : |w| = 1\} = \partial B.$$

A *closed Jordan curve Γ in* $\mathbb{R}$ is a subset of $\mathbb{R}^3$ which is homeomorphic to ∂B.

Given a closed Jordan curve Γ in $\mathbb{R}^3$ we say that $X : \overline{B} \to \mathbb{R}^3$ is a solution of Plateau's problem for the boundary contour Γ (or: a minimal surface spanned in Γ) if it fulfils the following three conditions:

(i) $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$;

(ii) The surface X satisfies in B the equations

$$\Delta X = 0 \tag{1.1}$$

$$|X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0; \tag{1.2}$$

(iii) The restriction $X|C$ of X to the boundary C of the parameter domain B is a homeomorphism of C onto Γ.

From the classical point of view, one of the difficulties in minimizing the area functional

$$A_B(X) = \int_B |X_u \wedge X_v| \, du \, dv$$

is that among all those surfaces X satisfying (iii) A is invariant under the action of the infinite dimensional diffeomorphism group of B. By replacing area by energy one reduces the symmetry group to the finite dimensional conformal group of

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the disk. Miraculously, the absolute minima of area and energy are the same. The Weierstrass equations (1.1) and (1.2) are then the variational equations of Dirichlet's energy.

The problem of the existence of a minimum of area spanning Γ remained open for a half a century until it was solved by Jesse Douglas (1931) and Tibor Radó (1930). For all his work on the Plateau problem, Douglas was awarded one of the first two Fields Medals of Mathematics (shared with Lars Ahlfors) at the International Congress of Mathematicians in Oslo in 1936.

Jesse Douglas (1897–1965)

Given the fact that the absolute minima of area and energy are the same, we can formulate the classical problem of Plateau as follows:

Given a closed Jordan curve Γ in $\mathbb{R}^3$, a mapping $X : B \to \mathbb{R}^3$ is said to be of class $\mathcal{C}(\Gamma)$ if $X \in H_2^1(B, \mathbb{R}^3)$, and if its trace $X|C$ can be represented by a weakly monotonic, continuous mapping $\varphi : C \to \Gamma$ of C onto Γ (i.e., every $L_2(C)$-representative of $X|C$ coincides with φ except for a subset of zero 1-dimensional Hausdorff measure).

Let

$$D(X) = D_B(X) := \frac{1}{2} \int_B (|X_u|^2 + |X_v|^2)\, du\, dv \tag{1.3}$$

be the Dirichlet integral of a mapping $X \in H_2^1(B, \mathbb{R}^3)$. Then we define the variational problem $\mathcal{P}(\Gamma)$ associated with Plateau's problem for the curve Γ as the following task:

Minimize Dirichlet's integral $D(X)$, *defined by* (1.3), *in the class* $\mathcal{C}(\Gamma)$.

In other words, setting

$$e(\Gamma) := \inf\{D(X) : X \in \mathcal{C}(\Gamma)\}, \tag{1.4}$$

we are to find a surface $X \in \mathcal{C}(\Gamma)$ such that

$$D(X) = e(\Gamma) \tag{1.5}$$

is satisfied.

In his solution, Douglas minimized an energy essentially equivalent to Dirichlet's energy, which later proved to be a very powerful method for dealing with minimal surfaces of arbitrary topological type and connectivity.

Almost from the beginning, the question arose as to whether the absolute minimizers were immersed or not. A point p where X is not immersed, i.e.

$$X_u(p) = X_v(p) = 0$$

is called a *branch point*. It follows easily that interior branch points are isolated. In 1932 Douglas [1] and in 1942 Courant [1] thought that they had found absolute minizers which had branch points. We should note here that from the early 1930s until his death in 1972 Courant worked on and popularized the field of minimal surfaces.

The example of Douglas was refuted in 1933 by Radó while Courant's example survived until the pioneering work of Robert Osserman in 1970, and then of Gulliver–Osserman and Royden in 1973.

In his now classic paper, Osserman constructed a discontinuous parameter transformation allowing a reparametrization of a minimal surface in a vicinity of an interior branch point, such that the area of the surface can be reduced. He had to distinguish between *true* and *false branch points* (the latter are those which have a neighbourhood whose image is still an embedded surface), but in his proof he overlooked some difficulties appearing for false branch points. In 1973, both H.W. Alt [1] and R. Gulliver [2] independently extended Osserman's line of argument to surfaces which are *absolute* minimizers of prescribed mean curvature with least energy and also treated the case of false branch points. The joint work of Gulliver, Osserman and Royden [1] in 1973 proved that *all* minimal surfaces bounded by rectifiable Jordan curves do not have any false branch points, even if they do not minimize the Dirichlet energy.

This difficult work has remained open mostly to experts in the field. For more historical comments, see the Scholia (Chap. 9).

In this book we give proof of the fact that in $\mathbb{R}^3$ any solution of Plateau's problem which is a *relative* minimizer of Dirichlet's integral D or, equivalently, the area functional A, is an immersion in the sense that it has no interior or (with mild assumptions) boundary branch points. This fact can easily be proved for planar boundaries (Dierkes, Hildebrandt and Sauvigny [1]), while the corresponding result in $\mathbb{R}^n$ is false for $n \geq 4$ according to a famous example of Federer. Therefore it remains to prove the assertion for a *nonplanar* boundary curve Γ in $\mathbb{R}^3$. The proof given here is based on the observation that one can compute any higher derivative of Dirichlet's integral in the direction of so-called *(interior) forced Jacobi fields*, using methods of complex analysis such as power series expansions and Cauchy's integral theorem as well as the residue theorem. These Jacobi fields lie in the kernel of the second variation of D; they also play a fundamental role in the index theory and the Morse theory of minimal surfaces. So, in a very strong sense, this book is about energy and the fact that it can be reduced in the presence of an interior or boundary branch point. This is in the spirit of Douglas' original approach to the Plateau problem. Since area is less than or equal to energy, reducing energy means that you can also reduce area. In this connection we must mention the work of Beeson [1].

Although the computations in this book are sometimes tedious, they are simple in principle. The main analytical idea is to find, using function theory, paths so that the calculation of higher order derivatives of Dirichlet's energy, through the use of Cauchy's integral theorem, along these paths reduces to a few manageable terms. In a sense, we are doing calculus on infinite dimensional manifolds. In order to convey to the reader a feeling for the methods to be applied, we begin by calculating the first five derivatives of Dirichlet's integral in the direction of special types of forced Jacobi fields, thereby establishing that a relative D-minimizing solution of Plateau's problem cannot have certain kinds of interior branch points. These introductory calculations will be carried out in Chap. 2 as a warm up for the general case, together with an outline of the variational procedure to be used in the sequel. These calculations are made transparent by shifting the branch point that is studied into the origin, and by bringing the minimal surface into a *normal form* with respect to the branch point $w = 0$ with an *order* n. Then also the *index* m of this branch point can be defined, with $m > n$. Furthermore, $w = 0$ is called an *exceptional branch point* if there is an integer $\kappa > 1$ such that $m + 1 = \kappa(n + 1)$. This notion is related to that of the false branch point, but it is a weaker notion. It will turn out that it is particularly difficult to exclude that a relative minimizer of D can have an exceptional branch point at $w = 0$. In fact, we are only able to exclude exceptional branch points for weak relative minimizers of A in $\mathcal{C}(\Gamma)$. However, we do present conditions under which a minimal surface with an exceptional branch point cannot be a relative minimizer of D. In the non-exceptional case, one can "always" reduce energy (and area), and surprisingly the monotonicity of a minimal surface on the boundary plays no role in being able to do so.

In Chap. 2 it is described how the variations $\hat{Z}(t)$ of a minimal surface $\hat{X}$ are constructed by using interior forced Jacobi fields. This leads to the (rather weak) notion of a *weak minimizer of* D. Any absolute or weak relative minimizer of D in $\mathcal{C}(\Gamma)$ will be a weak D-minimizer, and the aim is to investigate whether such minimizers can have $w = 0$ as an interior branch point. This possibility is excluded if one can find an integer $L \geq 3$ and a variation $\hat{Z}(t)$ of $\hat{X}$, $|t| \ll 1$, such that $E(t) := D(\hat{Z}(t))$ satisfies

$$E^{(j)}(0) = 0 \quad \text{for } 1 \leq j \leq L - 1, \qquad E^{(L)}(0) < 0.$$

It will turn out that the existence of such an L depends on the order n and the index m of the branch point $w = 0$.

In our first chapter, this idea is studied by investigating the third, fourth and fifth derivatives of $E(t)$ at $t = 0$. Here one meets fairly simple cases for testing the technique demonstrating its efficacy. Furthermore, the difficulties are exhibited that will come up generally.

The first case of a general nature is treated in Chap. 3. Assuming that $n+1$ is even and $m + 1$ is odd (whence $w = 0$ is non-exceptional) it will be seen that $E^{(m+1)}(0)$ can be made negative while $E^{(j)}(0) = 0$ for $1 \leq j \leq m$, and so $\hat{X}$ cannot be a weak minimizer of D.

The general situation is studied in Chaps. 4 to 7. In Chap. 4 is shown that $w = 0$ cannot be a non-exceptional branch point of a weak relative minimizer of D. We derive simple formulae for the first non-vanishing derivatives of Dirichlet's energy and show that they can be made negative. Such a result is no longer true for an exceptional branch point $w = 0$, apart from some special cases. In Chaps. 5, 6 and 7 it is proved that a weak relative minimizer of A in $\mathcal{C}(\Gamma)$ cannot have exceptional interior branch points if Γ is a smooth closed Jordan curve in $\mathbb{R}^3$.

In Chap. 8 we study boundary branch points of a minimal surface $\hat{X}$ with a smooth boundary contour. In particular we first show that $\hat{X}$ cannot be a weak relative minimizer of D if it has a boundary branch point whose order n and index m satisfy the condition $2m - 2 < 3n$ (Wienholtz's theorem).

We then will show that if the torsion and curvature of Γ are both non-zero, then a priori $2m + 2 \le 6(n + 1)$. As a consequence it follows that $\hat{X}$ is not a minimizer in the non-exceptional cases; i.e. $m + 1 \ne k(n + 1)$, $k = 2$ or 3. This is a partial resolution to boundary regularity for smooth contours. Considering only the Taylor expansion about a branch point, we then argue that the question of whether a minimal surface with an exceptional boundary branch point is or is not a minimum is not decidable.

In conclusion, if the boundary contour is C^∞ or more simply if a minimal surface $\hat{X}$ is C^∞ with a non-exceptional interior or boundary branch point, we can find a C^∞ surface Y which is C^∞ close to $\hat{X}$ having less energy and area. This is much stronger than what was previously known and indicates the power of using derivatives as opposed to cut and paste constructions.

In the Scholia (Chap. 9) we describe some of the history of the main results of this book. Finally, we note that some of the introductory material also appears in Dierkes, Hildebrandt and Tromba [1], but we include it for completeness. The author wishes to thank Stefan Hildebrandt for reworking the manuscript and for his encouragement, the Max Planck Institute in Leipzig for their support, Frau Birgit Dunkel for her excellent typing of the manuscript and finally my wife Inga without whose love and support this book could not have been written.

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