Chapter 1
Least Deviation Problems

We begin this chapter by listing areas of science and technology where we come across problems relating to optimization of the uniform norm. After that we investigate least deviation problems using methods of convex analysis. We deduce a generalized alternation principle which completely characterizes solutions of such problems. In giving the definition of an extremal polynomial in the introduction we were motivated by this principle. We shall see that most solutions are polynomials with small value of the parameter g calculated by formula (5) in the introduction. Finally, we investigate the problem of finding the optimal stability polynomial: we show that it is solved by an extremal polynomial with $g \leq p - 1$.

1.1 Examples of Optimization

1.1.1 Inverting a Symmetric Matrix

Consider a typical computational algorithm such that optimizing it we arrive at a least deviation problem. We solve a system of linear equations $Au = f$ with a non-singular symmetric matrix A. When we have a large matrix, which occurs, for instance, in discretizing equations of mathematical physics, such direct methods as Gaussian elimination are too labour-consuming and cannot be applied, and so iterative algorithms are used for finding solutions. Consider a simple two-step iterative method

$$u_{j+1} := u_j - \alpha_j (Au_j - f), \quad j = 0, 1, 2, \dots,$$

in which the parameters α_j are determined by the condition that after n steps of the procedure the error $\varepsilon_n := u_n - u$ must be minimal. The error at the nth step can be expressed linearly in terms of the initial error:

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$$\varepsilon_n = P_n(A)\varepsilon_0, \quad P_n(t) := \prod_{j=0}^{n-1}(1-\alpha_j t),$$

and its Euclidean norm is no greater than the norm of the initial error times the deviation of the polynomial on the spectrum of the matrix: $\max_{t\in \mathrm{Sp}(A)} |P_n(t)|$. Calculating the spectrum is an even more laborious procedure than solving a system of linear equations, but often (for instance, for some physical reasons) we know a compact subset E of the real axis containing the spectrum of the matrix. In that case, for the parameters of the iterative procedure we take the reciprocal values of the zeros of a polynomial solving the following minimum problem: *among the polynomials in the space (2) satisfying* $P_n(0) = 1$ *find a polynomial that has the smallest uniform norm on the compact subset* E *of the real axis*.

1.1.2 Explicit Runge–Kutta Methods

The Runge–Kutta method has been used for the numerical integration of systems of ordinary differential equations for more than a century. Explicit schemes for solution of the Cauchy problem

$$\begin{cases} dy/dt = f(y,t), \\ y(0) = y_0, \end{cases} \qquad y(t) \in \mathbb{R}^m, \tag{1.1}$$

for ordinary differential equations have many advantages: they are simple to implement, can easily be parallelized, and take relatively little memory resources in comparison with implicit schemes. At the same time, for *stiff* problems (when a solution has a rapidly changing component) Courant's stability condition imposes too strong restrictions on the step h. For example, in the case of the simplest *Euler scheme* we consider a uniform time grid $t_j := jh$, $j = 0, 1, 2, \ldots$, and take the solution $y(t_{j+1})$ to be approximately

$$y_{j+1} := y_j + hf(y_j, t_j).$$

The local stability condition for the Euler scheme has the form $h < 2/\lambda$, where λ is the spectral radius of the current Jacobian matrix of the map $f(y,t)$. If the system (1.1) of equations to be solved is obtained by discretizing an evolution equation of mathematical physics in the space variables, then λ can be very large: for instance, the finite-difference Laplace operator is unbounded in the limit. Using variable time steps we can significantly (by a factor of millions) increase the average magnitude of step while keeping the method stable [93]. Applying the multistage Runge–Kutta method, with step h consisting of n smaller steps, to the simplest equation $dy/dt = \lambda y$, $\lambda > 0$, we obtain a relation between the values of an approximate solution at consecutive nodes: $y_{j+1} := R_n(\lambda h) y_j$, where $R_n(x)$ is a real polynomial of degree n, the *stability function* of the method. If a Runge–Kutta method has accuracy order p, then its stability function must approximate the exponential function with the same accuracy at the origin [71, 142]:

$$R_n(x) = 1 + x + x^2/2 + \cdots + x^p/p! + x^{p+1} P_{n-p-1}(x), \qquad \deg P_{n-p-1}(x) = n - p - 1. \tag{1.2}$$

Maximizing the *stability region* $\{x \colon |R_n(x)| \le 1\}$ of the Runge–Kutta method on the real axis, for instance, when solving parabolic-type equations by the method of lines, results in Problem B in the introduction [94, 103]. Zeros of classical Chebyshev and Zolotarëv polynomials are used in the DUMKA software package [71], which has proved to be effective in many non-linear evolution problems of mathematical physics [157].

1.1.3 Electrotechnics

The alternating current resistance of a passive (that is, consisting of passive elements: capacitors, inductors (coils), and resistors) bipolar circuit is a rational function of the current frequency. The following optimization problem arises in the design of digital or analogue multiband electrical filters.

Let E be a closed set of disjoint (frequency) intervals on the real axis. In each interval the transition function $F(\omega)$ is equal to 0 *(stopband) or* 1 *(passband). One must find $R(\omega)$, the best rational approximant of given degree n to the transition function in the uniform metric on E:*

$$||R - F||_E := max_{\omega \in E} |R(\omega) - F(\omega)| \qquad \longrightarrow \min .$$

When the set E contains just two intervals, the solution is known explicitly: it is a slight modification of a Zolotarëv fraction, the solution of Zolotarëv's third problem [162], which has a parametric representation in terms of elliptic functions:

$$R(u) = sn(K(\tau)|\tau); \qquad x(u) = sn(K(n\tau)|n\tau), \qquad u \in \mathbb{C}, \tag{1.3}$$

where $sn(\cdot|\cdot)$ is the elliptic sine function and $K(\cdot)$ is the complete elliptic integral with modulus τ. This function (its graph is given in Fig. 1.1) shares many common properties with Chebyshev polynomials; the latter occur as a certain limiting case of Zolotarëv fractions. W Cauer in 1930-ies used this function as a frequency response function for the synthesis of the so-called elliptic electrical filter with one stopband and one passband. These filters are widely used in contemporary electronic industry and display the sharpest transition between the passband and stopband.

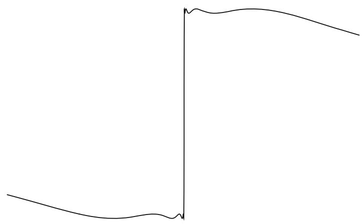

Fig. 1.1 The graph of the Zolotarëv fraction of degree $n = 8$

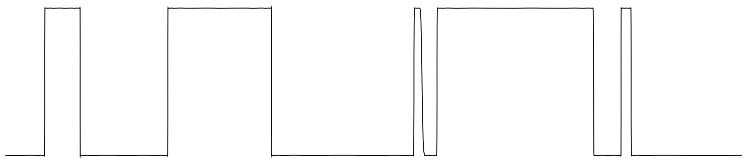

Fig. 1.2 The best rational approximant of degree $n = 88$ to a transition function

The solution of the above-stated problem with an arbitrary number of bands also has an analytic representation in terms of higher analogues of (1.3), see details in [26]. The best rational approximant of degree $n = 88$ to a certain transition function F is presented in Fig. 1.2 (computed by D.V.Yarmolich). For other optimization problems arising in electrical engineering see [14, 44, 45, 49, 154].

1.1.4 V. A. Markov's Problem

In 1892 V. A. Markov considered the following problem [102]:

Find a polynomial $P(x)$ of degree at most n and with uniform norm 1 on a finite interval E that delivers the maximum to a fixed linear form $\langle p^*|P\rangle$ of the coefficients of a polynomial.

Special cases of this problem, when $\langle p^*|P\rangle$ is one of the coefficients of $P(x)$, the value of $P(x)$, or the value of some derivative of $P(x)$ at a fixed point, were considered by A. A. Markov [101]. In its simplest form, for $n = 2$ A. A. Markov's problem was posed by D. I. Mendeleev in his treatise "Études des dissolutions aqueuses, fondée sur les changements de leurs poids spécifiques" (1887). The solution of the above problem reduces to Problem A of least deviation with one constraint $\langle p^*|P\rangle = 1$.

Lemma 1.1. *If $P(x)$ is a (not necessarily unique) solution of V. A. Markov's problem, then $\frac{P(x)}{\langle p^*|P\rangle}$ is a solution of Problem A with one constraint $\langle p^*|P\rangle = 1$. If $P(x)$ is a (not necessarily unique) solution of of problem A with a single constraint, then $P(x)/\|P\|_E$ is a solution of V. A. Markov's problem.*

1.1.5 Other Applications

The following problems provide further examples of problems of least deviation in the uniform metric:

1. Picking interpolation nodes for functions of prescribed smoothness defined on a compact subset of the real axis [31].
2. Examples of extremal polynomials are also provided by the Chebyshev splines of type $[n, 0; k_+, k_-]$, $n, k_\pm \geq 0$, that is, the polynomials of degree n with zeros of order $k_\pm$ at $x = \pm 1$ which have an $(n+1-k_+-k_-)$-*alternance* on $[-1, 1]$.
3. Optimizing one-dimensional quadrature formulae of Gauss type [31, 97].

Definition 1.1. A function $F(x)$ is said to have an l-alternance on the compact set E if there are l different points of E where $F(x)$ takes the values $\pm||F||_E$ and the values in the neighbouring points have opposite signs.

1.2 Analyzing Optimization Problems

The most common solutions of least deviation problems are polynomials whose normalizations are extremal in the sense of the definition in the introduction. The reason for this is explained by convex analysis [129, 146]. We now look at Problem A in the introduction:

Let E be a finite system of closed real intervals. Minimize the norm $\|P_n\|_E$ of a polynomial, provided that fixed linear constraints are imposed on its coefficients $c_0, c_1, \ldots, c_n$.

Assume that we look for the solution of Problem A on a fixed affine plane L_{n+1-r} of codimension r in the space of polynomials (2). Such an $(n+1-r)$-plane can be described as the result of a translation of the annihilator of an r-dimensional subspace L_r^* of the dual space. With each non-trivial polynomial $T(x)$ in (2) we associate a convex polyhedral cone in the dual space, which we define below. By the extremal points of the polynomial $T(x)$ with respect to E we shall mean the set

$$\mathrm{ext}_E(T) := \{x \in E : T(x) = \pm\|T\|_E\}.$$

We assign to each extremal point x a functional x^* on the space of polynomials: $\langle x^*|P\rangle := P(x) \cdot \mathrm{sign}\, T(x)$. The *conical hull* of these functionals

$$\mathrm{cone}\{x_1^*, x_2^*, \ldots, x_m^*\} := \left\{\sum_{s=1}^{m} \alpha_s x_s^* : \alpha_s \geq 0; \sum_{s=1}^{m} \alpha_s > 0\right\}, \quad m = \#\mathrm{ext}_E(T), \tag{1.4}$$

does not contain the origin because $\left\langle \sum_{s=1}^{m} \alpha_s x_s^* | T \right\rangle = \|T\|_E \sum_{s=1}^{m} \alpha_s > 0$; we associate it with the polynomial T. By the *dual cone* we shall mean the cone of polynomials which are positive at each functional in (1.4) (note that only the non-negativity is required in the standard definition).

Theorem 1.1. *A polynomial $T(x) \in L_{n+1-r}$ delivers a minimum in the least deviation Problem A if and only if the subspace L_r^* intersects the cone* (1.4) *associated with the polynomial.*

Remark 1.1. The cone in this statement, which is generated by all extremal points of the polynomial T can be replaced, in view of Carathéodory's principle, by a cone generated by at most $n+2-r$ extremal points. Thus adjusted, Theorem 1.1 becomes an interpretation of I. Singer criterion of extremality in [138] and [146].

Proof. A polynomial $T \in L_{n+1-r}$ fails to be a solution of Problem A if and only if the norms of the polynomials decrease in some direction issued from T and lying in the plane L_{n+1-r}. Such a direction can be defined by a polynomial $P(x)$ annihilating all functionals in L_r^* and taking values of the same sign as T at the extremal points of T. The following two assertions are therefore equivalent:

(i) $T(x)$ is a solution of Problem A and
(ii) The annihilator of L_r^* is disjoint from the cone dual to (1.4).
We can dualize (ii).

1. *If the lineal L_r^* intersects the cone* (1.4)*, then the annihilator of L_r^* is disjoint from the cone dual to* (1.4). For if their intersection is also non-empty, then there exist a functional p^* in the first intersection and a polynomial $P(x)$ in the second such that $0 = \langle p^*|P\rangle > 0$.
2. *If L_r^* is disjoint from the cone* (1.4)*, then the annihilator of L_r^* intersects the cone dual to* (1.4). Indeed, assume that L_r^* is disjoint from the cone (1.4). Now using induction we shall increase L_r^* to a hyperplane disjoint from the cone. This hyperplane annihilates a polynomial which is positive on the half-space bounded by the hyperplane and containing the cone. This polynomial belongs to both the dual cone and $(L_r^*)^\perp$, and so these two sets intersect. It remains to describe the increasing procedure for L_r^*.

At each step, if $r < n$, then we consider a two-dimensional subspace L_2^* linearly independent of L_r^*. Its intersection with the convex cone $L_r^* + \text{cone}\{x_1^*, x_2^*, \dots, x_m^*\}$ is a convex two-dimensional sector with opening less than π for it does not contain the origin. Hence L_2^* contains a one-dimensional subspace L_1^* disjoint from $L_r^* + \text{cone}\{x_1^*, x_2^*, \dots, x_m^*,\}$. Setting $L_{r+1}^* := L_1^* + L_r^*$ we complete the induction step. □

For a fixed subset E of the real axis various Problems A of least deviation differ in the position of the plane L_{n+1-r} and therefore can be indexed by points in the real projective Grassmannian $\text{Gr}(n+2, n+2-r)$ of dimension $r(n+2-r)$. We see that more problems can be posed than there exist solutions, so that the question of the rate of the occurrence of each polynomial in (2) among the solutions of least deviation problems suggests itself. The affine planes L_{n+1-r} incident to a fixed point in the space (2) will be indexed by the directing lineals $L_r^* \in \text{Gr}(n+1, r)$.

In the following statement we characterize the set of Problems A solved by a fixed polynomial $T(x)$.

Lemma 1.2. *The lineals L_r^* in the Grassmannian* $\mathrm{Gr}(n+1, r)$ *which intersect the fixed cone (1.4) form a closed subset of a Schubert cycle of codimension* $\max(0, n+2-r-\#\,\mathrm{ext}_E(T))$ *which has a non-empty (relative) interior.*

Proof. The lineals L_r^* intersecting the cone (1.4) form a closed subset of the Grassmannian because adding the vertex to the cone we make it closed. The set in question lies in some Schubert cycle which we describe below.

Functionals generating the cone (1.4) associated with a polynomial $T(x)$ are linearly independent if their number $m := \#\,\mathrm{ext}_E(T)$ does not exceed the dimension of the space (2). Assume that the linear span of (1.4) belongs to a filtration of the dual space:

$$0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{n+1}.$$

If a subspace L_r^* intersects the cone, then we obtain the first inequality in the following system (while the other inequalities hold by dimensional considerations):

$$\dim(L_r^* \cap \mathbb{R}^{\min(m,n+1)}) \geq 1; \quad \dim(L_r^* \cap \mathbb{R}^{n+s+1-r}) \geq s, \quad s = 1, 2, \ldots, r,$$

which means that L_r^* lies in the Schubert cycle whose Young diagram is an $(n+1-r) \times r$ rectangle without the (horizontal) row of length $\max(n+2-r-m, 0)$ in the lower right corner. We shall now indicate a subdomain of this Schubert cycle the elements of which intersect the cone (1.4).

In the proof of Theorem 1.1 we established the existence of a support hyperplane of the cone (1.4) containing its origin, but disjoint from the cone proper. Consider now an arbitrary subspace L_l^* of dimension $l := \min(m, n+2-r)-1$ that lies in the intersection of this hyperplane and the linear span of the cone. For a fixed point p^* in the relative interior of the cone there exists a neighbourhood of the origin $\mathscr{O} \subset L_l^*$ such that $p^* + \mathscr{O}$ lies in the cone. We consider now the set of pairs (y^*, L_{r-1}^*), where $y^* \in \mathscr{O}$ and L_{r-1}^* is a subspace of the support hyperplane such that $\dim(L_{r-1}^* \cap L_l^*) = 0$. Such subspaces L_{r-1}^* fill an open subset of the Grassmannian $\mathrm{Gr}(n, r-1)$, which contains at any rate a Schubert cycle of the maximum dimension $(r-1)\cdot(n+1-r)$. Each pair (y^*, L_{r-1}^*) defines an r-subspace spanned by L_{r-1}^* and the vector $p^* + y^*$ and intersecting the cone. By construction, distinct pairs define distinct r-subspaces. We have thus defined an embedding in the set of r-subspaces intersecting the cone (1.4) of a domain in the space of dimension $(r-1)\cdot(n+1-r)+l = r(n+1-r) - \max(n+2-r-m, 0)$ equal to the dimension of the Schubert cycle in the previous paragraph. □

Discussion. We see that the greater is the number of extremal points of a polynomial on E, the higher is the dimension of the space of Problems A solved by this polynomial. Of course, our arguments do not mean that a slight perturbation of the conditions of an arbitrary problem brings the number of extremal points of the solution T (which also is not necessarily unique) close to the expected quantity $n+2-r$. Although each polynomial with $\#\,\mathrm{ext}_E(T) < n+2-r$ is a solution of fewer problems, the number of such polynomials is much greater. A crude dimension evaluation shows that these two effects roughly counterbalance each other: in (2) the polynomials with $\#\,\mathrm{ext}_E(T) = m$ lie on submanifolds of codimension $m-1$ and each is a solution of an $((r-1)\cdot(n+1-r)+m-1)$-dimensional set of problems, which yields precisely the dimension of the Grassmannian $\mathrm{Gr}(n+2, n+2-r)$, the index set of least deviation problems.

1.3 Chebyshev Subspaces

Which least deviation problems automatically have extremal polynomials as solutions? Each extremal point of a polynomial T lying in the interior of E is critical, and the value $\pm\|T\|_E$ at this point has an even multiplicity. That is, we are interested in problems whose solutions have many extremal points, provided that the boundary of E consists of few points. For instance, the number of extremal points of a solution is at least $n+2-r$ if the polynomials satisfying the homogeneous constraints of the problem form a *Chebyshev* subspace.

Definition 1.2. Finite dimensional space of functions is said to be Chebyshev on E iff any function in this space has no more zeroes in E, than the dimension of the space minus one.

The main source of Chebyshev subspaces is provided by *divisor spaces* occurring in algebraic geometry. Let D be a divisor (a formal finite sum of points with integer multiplicities) in the Riemann sphere that is symmetric relative to the real axis and assume that $\mathsf{D} + n\cdot\infty \geq 0$. By the space of this divisor we shall mean the subspace of polynomials in (2) such that the multiplicities of their zeros (and poles: a pole has a negative multiplicity) at an arbitrary point in the Riemann sphere are no smaller than the multiplicity of this point in the divisor:

$$\mathscr{L}(-\mathsf{D}) := \{P \in \mathbb{R}[x] \colon (P) \geq \mathsf{D}\}. \tag{1.5}$$

The codimension of $\mathscr{L}(-\mathsf{D})$ in the space of polynomials (2) is equal to the minimum of $\deg \mathsf{D} + n$ and $n+1$. If the support of the divisor D is disjoint from the set E, then the divisor space is Chebyshev on E. The constraints in the corresponding least deviation problem fix the values of the solution T at the finite points in D (and the values of its first derivatives if the multiplicity of the point is higher than 1), and also fix several leading coefficients of T if the point at infinity has multiplicity higher than $(-n)$ in the divisor.

Theorem 1.2 (S. N. Bernstein [33]).

1. *If a lineal $(L_r^*)^{\perp}$ is Chebyshev on a set E, then the solution of the corresponding least deviation problem A has at least $n+2-r$ extremal points in E.*

2. *If the same lineal is Chebyshev on the convex hull of E, then the solution is unique and is characterized by the property of having an $(n+2-r)$-alternance on E (the definition of alternance was given is Sect. 1.1.5.)*

Proof. 1. If $T(x)$ is a solution of the least deviation problem, then by Theorem 1.1 for some extremal points x_s of this polynomial and *positive* weights α_s we obtain

$$\sum_{s=1}^{m} \alpha_s \cdot \operatorname{sign} T(x_s) \cdot P(x_s) = 0 \quad \text{for each } P(x) \in (L_r^*)^{\perp}. \tag{1.6}$$

Assume that the number m of extremal points in (1.6) is less than $n+2-r$. The dimension of $(L_r^*)^{\perp}$ is $n+1-r$, therefore there exists a polynomial $P(x) \in (L_r^*)^{\perp}$ vanishing at $n-r$ points: at $x_1, x_2, \dots, x_{m-1}$ and at some $n+1-r-m$ points in $E \setminus \operatorname{ext}_E(T)$. Since $(L_r^*)^{\perp}$ is a Chebyshev space, it follows that $P(x_m) \neq 0$, and therefore in (1.6) we have $\alpha_m = 0$, a contradiction.

2. Assume that the solution $T(x)$ has the same sign at two neighbouring extremal points, x_s and x_{s+1}. We consider a polynomial $P(x) \in (L_r^*)^{\perp}$ vanishing at the remaining $n-r$ extremal points (see the remark to Theorem 1.1). Then equality (1.6) takes the form $\alpha_s P(x_s) + \alpha_{s+1} P(x_{s+1}) = 0$. This means that $P(x)$ must also have a zero on the interval $[x_s, x_{s+1}]$, in contradiction with the Chebyshev property of the space $(L_r^*)^{\perp}$ on $\operatorname{conv} E$. Thus, each solution T has an $(n+2-r)$-alternance on E. Conversely, each polynomial $T(x) \in L_{n+1-r}$ having an $(n+2-r)$-alternance on E is a unique solution. If there exists another polynomial whose deviation on E does not exceed that of $T(x)$, then their difference belongs to $(L_r^*)^{\perp}$ and has at least $n+1-r$ zeros on $\operatorname{conv} E$, so that it is trivial. □

1.4 The Problem of Optimal Stability Polynomial

Problem B of finding an optimal stability polynomial can be reduced to the least deviation problem A with *Chebyshev constraints* that we have just considered.

Theorem 1.3 ([40, 127]). *The problem of optimal stability polynomial is uniquely solvable. A polynomial*

$$R_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!} + o(x^p) \tag{1.7}$$

and an interval $E = [-L, 0]$ on which $\|R_n\|_E$ has deviation 1 solve Problem B if and only if R_n has an $(n+1-p)$-alternance on $E \setminus \{0\}$.

Proof. As l increases, the closed ball $\{\|P\|_{[-l,0]} \leq 1\}$ in the space of polynomials (2) contracts (linearly, but anisotropically) and, in the limit as $l \to \infty$, contains only constant polynomials, which cannot satisfy the constraints (1.7) if $p > 1$.

Hence there exist (i) a largest interval $E := [-L, 0]$ and (ii) a polynomial $R_n(x)$ with deviation 1 on E satisfying the constraints.

1. We claim that at the same time $R_n(x)$ is a solution of Problem A with constraints (1.7) on the interval $E' := [-L, -\varepsilon]$; here ε is any positive quantity that is smaller than 1, $L/2$, and $1/\max|P''(x)|$, where the maximum is considered over the compact set $\{(P, x)\colon x \in [-L/2, 0];\ \|P\|_{[-L,-L/2]} \le 1;\ \deg P \le n\}$. Indeed, assume that there exists a polynomial $P(x)$ in the space (2) satisfying the $r = p + 1$ constraints (1.7) and with deviation less than 1 on E'. In view of the local increase of $P(x)$ in a neighbourhood of the origin and the smallness of ε, $\|P\|_E \le 1$. Since the value of $P(x)$ at the end-point $x = -L$ is less than 1 in absolute value, E can be increased while keeping the norm of $P(x)$ the same, which contradicts the maximality of E.
   The linear constraints in (1.7) mean that the $r = p + 1$ lower coefficients of the polynomial are fixed. A polynomial of degree at most n which satisfies the corresponding homogeneous constraints has a zero of order r at $x = 0$, so it has at most $n - r$ zeros on E', and therefore the corresponding subspace is Chebyshev. By Bernstein's Theorem 1.2 the least deviation polynomial $R_n(x)$ is unique and has an $(n + 1 - p)$-alternance on E'.
2. Conversely, let $R_n(x)$ be a polynomial of the form (1.7) with an $(n + p - 1)$-alternance on the half-open interval $[-L, 0)$ and with deviation 1 in this interval. By Theorem 1.2, $R_n(x)$ solves the least deviation problem with constraints (1.7) on the set $E' = [-L, -\varepsilon]$, where $\varepsilon > 0$ is sufficiently small. The optimal stability polynomial has deviation 1 on E' and satisfies the same constraints. Since the least deviation problem has a unique solution, R_n is the optimal stability polynomial. □

1.4.1 Properties of Optimal Stability Polynomials

The existence of an alternance enables us to find an estimate for the number of zeros of the optimal stability polynomial and its derivative in the stability region $E = [-L, 0]$. Their precise number is described by the following result.

Lemma 1.3. *The solution $R_n(x)$ of the optimization Problem* B *and its derivative $R_n'(x)$ have only simple zeros, which lie in E and $\mathbb{C} \setminus \mathbb{R}$ in the following amounts:*

The number of zeros of R_n	in E	in $\mathbb{C} \setminus \mathbb{R}$
p even	$n - p$	p
p odd	$n - p + 1$	$p - 1$

The number of zeros of dR_n/dx	in E	in $\mathbb{C} \setminus \mathbb{R}$
p even	$n - p + 1$	$p - 2$
p odd	$n - p$	$p - 1$

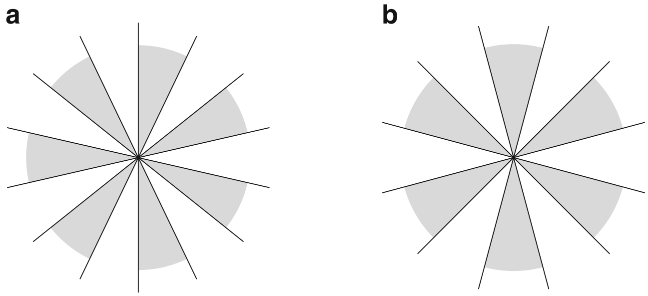

Fig. 1.3 Order stars in a neighbourhood of the origin for (**a**) even p and (**b**) odd p

Proof. Assume that a real polynomial approximates the exponential function at the origin with precise order p: $P_n(x) - \exp(x) \sim x^{p+1}$. Then it has at least $2[p/2]$ distinct complex zeros [1]. The proof of this statement is based on the analysis of the topology of the *order stars* [71].

We consider two open subsets of the complex plane: black and white, that are symmetric relative to the real axis. At points in the white subset the function $P_n(x)/\exp(x)$ is less than 1 in absolute value, in the black subset its absolute value is greater than 1. These two subsets, which are called the order stars, have the following easily verified properties [71]:

(a) The black and the white sets have precisely one unbounded component each.
(b) In the neighbourhood of the origin these subsets make up curvilinear sectors of angle $\pi/(p+1)$ and of alternating colours (see Fig. 1.3).
(c) Each bounded component of the white set contains a zero of the polynomial (use the maximum principle for the harmonic function $\log|P(x)\exp(-x)|$).
(d) The black set has no bounded components (they would contain poles of the polynomial) and is therefore connected.

Hence one concludes that an arbitrary component of the white set contains at most one sector: otherwise (d) fails. If a white component contains a sector lying strictly in the upper or the lower half-plane, then the entire component lies in this half-plane since the white subset is mirror-symmetric. In addition, such a component must be bounded: by (a) the unbounded component intersects both half-planes. We see from Fig. 1.3 that for even p there exist p white sectors disjoint from the real axis and for odd p there exist at least $p-1$ such sectors. Each of them lies in some bounded white component disjoint from the real axis and containing a zero of the polynomial $P_n(x)$ by (c). Correspondingly, our polynomial has at least $2[p/2]$ complex zeros and its derivative has at least $2[(p-1)/2]$ complex zeros.

One can say more about the position of the zeros of the optimal stability polynomial $R_n(x)$ and its derivative. Between two neighbouring points of the alternance there exists a zero of the polynomial, and each point of the alternance lying in the interior of E is a zero of its derivative. The interval between the origin and the extreme right point x_1 in the alternance contains either a zero of the polynomial (when $R_n(x_1) = -1$) or a zero of its derivative (when $R_n(x_1) = 1$). For even p we have already found $(n - p) + p = n$ distinct zeros of R_n and for odd p we have found $(n - p) + (p - 1) = n - 1$ distinct zeros of R_n'. Hence $R_n(x_1) = (-1)^p$ and the distribution of zeros is as required in the lemma. □

1.5 Problems and Exercises

1. Find the value of the parameter (5) for the Chebyshev spline of type $[n, 0; k_+, k_-]$ (see the definition in Sect. 1.1.5).

   Answer. $g = \max(k_+ - 1, 0) + \max(k_- 1, 0)$.
2. A *Shabat polynomial* is a polynomial with precisely two finite critical values; for definiteness let these be ± 1. Find the extremality order g (defined by (5)) of a Shabat polynomial $P(x)$ from the topology of the associated *graph* $P^{-1}([-1, 1])$.
3. Prove Lemma 1.1.
4. Prove the first theorem of V. A. Markov [102]:

   A polynomial $T(x)$ solves Problem A *with one linear constraint $\langle p^* | T(x)\rangle = 1$ if and only if the space (2) contains no polynomial $P(x)$ which (1) satisfies the homogeneous linear constraint $\langle p^* | P\rangle = 0$ and (2) takes values of the same sign as T at all extremal points of T: $P(x)/T(x) > 0$, $x \in \mathrm{ext}_E(T)$.*
5. Let $x_1 > x_2 > \cdots > x_m$ be the extremal points of a polynomial $T(x)$ on a closed interval E. Consider the Lagrange interpolation polynomials with these nodes: $\Phi(x) := \prod_{s=1}^{m} (x - x_s)$; $\Phi_s(x) := \Phi(x)/(x - x_s)$. Prove the second theorem of V. A. Markov [102]:

   A polynomial $T(x)$ solves Problem A *with one linear constraint $\langle p^* | T(x)\rangle = 1$ if and only if (1) for $m < n + 1$ and $s = 0, 1, \ldots, n - m$, $\langle p^* | x^s \Phi(x)\rangle = 0$, and (2) some quantities in the set $(-1)^s \langle p^* | \Phi_s\rangle T(x_s)$, $s = 1, \ldots, m$, are distinct from zero, but no two of them have distinct signs.*

   Hint. Assume that there exists a polynomial $P(x)$ mentioned in the first Markov theorem. Divide $P(x)$ by $\Phi(x)$ with a remainder and expand the remainder in the polynomials $\Phi_s(x)$. Now bearing in mind that the sign of $\Phi_s(x_s)$ is opposite to the sign of $(-1)^s$ we can derive a contradiction with conditions (1) and (2).
6. Prove a theorem due to Carathéodory: let X be a subset of $\mathbb{R}^n$; then any point in its convex hull is a convex combination of at most $n + 1$ points in X.
7. Prove that the cone (1.4) associated with a polynomial $T(x)$ has dimension equal to $m := \#\,\mathrm{ext}_E(T)$, provided that $m \le n + 1$.

   Hint. The values which functionals generating the cone take at the basis 1, x, x^2, $x^3 \ldots$ of the space of polynomials make up a Vandermonde matrix.

8. Give an example of a polynomial $T(x)$ of degree n and a set E such that the number $m := \#\mathrm{ext}_E(T)$ of extremal points is larger than $n+1$. Show that always $m \leq 2n$.
9. Let D be a divisor symmetric relative to the real axis such that $\mathsf{D} + n \cdot \infty \geq 0$. Prove that, if its support is disjoint from compact subset E of the real axis, then $\mathscr{L}(-\mathsf{D})$ is a Chebyshev space on E, that is, the number of zeros (with multiplicities) on E of a polynomial $P(x) \in \mathscr{L}(-\mathsf{D})$ is at least one less that the dimension of the space.
10. Prove properties (a–d) of order stars, which have been used in the proof of Lemma 1.3.
11. Use Vièta's formulae to prove that a real polynomial $P(x)$ approximating the exponential function at the origin $x = 0$ with order p (so that $P(x) - \exp(x) = o(x^p)$) has at least $[p/2]$ pairs of complex conjugate roots.
12. On the basis of Lemma 1.3 show that for odd p the optimal stability polynomial has a similar graph to the classical Chebyshev polynomial of degree $n - p + 1$. What is the qualitative graph of the optimal stability polynomials for even p?

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