Chapter 2
Theoretical Background

Abstract Elements of plasma physics and fluid turbulence are introduced in this chapter on the introductory level. These are theoretical building blocks for highlighting plasma turbulence in the solar system as well as astrophysical systems as a challenge in physics. Emphasis in plasma physics is on the fluid picture, called magnetohydrodynamics, and similarities and differences are discussed between magnetohydrodynamics and fluid turbulence.

2.1 Elements of Plasma Physics

2.1.1 Origin and Properties of Plasmas

When gas is heated, atoms are dissociated into ions (positively charged) and electrons (negatively charged). The ionized gas is electrically quasi-neutral but conducting due to high mobility of ions and electrons. It exhibits a variety of behaviors and collective motions, and is referred to as the plasma. This name was given by Langmuir in 1928 after the Greek term $\pi\lambda\acute{\alpha}\sigma\mu\alpha$, which means something molded or a moldable substance. Stellar interiors and atmospheres, gaseous nebulae, and the interstellar medium are mostly plasmas. They can be found in near-Earth space (Earth's ionosphere and magnetosphere) and in the solar system (interplanetary space and planetary magnetospheres), too. In our daily life, on the other hand, encounters with plasmas are rather limited: the flash of a lightning, the soft glow of the aurora borealis, the conducting gas inside a fluorescent tube or neon sign.

It was once thought that 99% of matter in the universe was in the plasma state, that is, matter is in the form of ionized gas, since conditions such as high temperature and low density are favorable for maintaining gas ionization. Also, there are sources of ionization other than high temperature. Modern cosmology and astrophysics, however, suggest that there must be a different kind of matter in the Universe other than baryonic or leptonic matters, referred to as the dark matter, in order to

Y. Narita, *Plasma Turbulence in the Solar System*, SpringerBriefs in Physics,
DOI: 10.1007/978-3-642-25667-7_2,

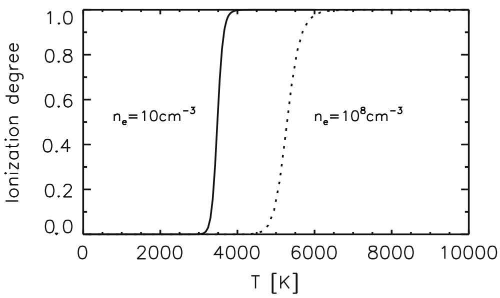

Fig. 2.1 Ionization degree for (atomic) hydrogen gas as a function of temperature. *Solid curve* is for the electron density typical of solar wind at 1 AU; *Dotted curve* is for density typical of solar corona near surface ($r = 1.1r_\odot$)

account for radial profile of galaxy rotation rate and distorted images of galaxy due to gravitational lensing (see recent developments in cosmology, for example, in [1]). Furthermore, the dark matter must be dominating the total mass density in the Universe in comparison to the baryonic matter by factor 6 or 7. Yet, the above statement of the plasma dominance is true for the baryon-lepton matters or "visible" matters such as protons, neutrons, and electrons. It is interesting that we are living in the only 1% (or less) of the baryonic Universe in which plasmas do not occur naturally [2].

2.1.1.1 Ionization Sources

There are a variety of ionization sources in space, but the primary source of ionization is high temperature. The ratio of the amount of ionization or number density for ionized atoms (n_{ion}) to that for neutral atoms (n_{atm}) expected in thermal equilibrium is given by the Saha equation:

$$\frac{n_{\mathrm{ion}}}{n_{\mathrm{atm}}} = \frac{1}{n_{\mathrm{e}}}\left(\frac{2\pi m_{\mathrm{e}} k_{\mathrm{B}} T}{h^2}\right)^{3/2} \mathrm{e}^{-U/k_{\mathrm{B}}T}, \tag{2.1}$$

where n_{e} denotes the electron number density, m_{e} the electron rest mass, k_{B} the Boltzmann constant, T the gas temperature, h the Planck constant, and U the ionization potential.

Figure 2.1 displays the profile of ionization degree, $n_{\mathrm{ion}}/(n_{\mathrm{ion}} + n_{\mathrm{atm}})$, as a function of temperature for hydrogen atoms in the ground state (ionization potential $U = 13.6\,\mathrm{eV}$). The ionization degree is shown for two different gas densities: $n_{\mathrm{e}} = 10\,\mathrm{cm}^{-3}$ (solid curve) which is typical of the solar wind at the Earth orbit (1 AU) and $n_{\mathrm{e}} = 10^8\,\mathrm{cm}^{-3}$ (dotted curve) typical of the solar corona at the radius $1.1r_\odot$ from the center of the sun (or 7×10^4 km above the solar surface).

As the temperature is raised, the ionization degree remains low until the ionization potential is only a few times of the thermal energy, $k_{\mathrm{B}}T$. Then the ionization degree rises abruptly, and the gas is in the plasma state. The gas then becomes fully ionized. In astrophysical systems (stars and interstellar medium) gas temperatures are usually of the order of millions of degrees, while the temperature on the Earth's atmosphere (on the ground) is about 300 K. Typical values of the electron temperatures in the solar wind and the corona are 300,000 K and 1,000,000 K, respectively, whereas the jumps in the ionization degree are about 3000 to 3500 K for the solar wind, and 5000 to 6000 K for the corona. The gas can remain fully ionized in the solar wind and corona, but not on the Earth.

The ionization degree depends on the electron density, as well. The ionization temperature becomes lower at lower density, and the slope changes slightly such that it is steeper at lower density and less steep at higher density. The reason for this is that a dilute gas (i.e., lower density) has a longer mean free path and can maintain ionization more easily, and collisions bringing ionized atoms into neutral become more frequent at higher density.

Besides the high temperature, there are other ionization sources. Photoionization is the process in which an incident photon ejects one or more electrons from an atom or molecule, particularly at shorter wavelengths such as UV and also X-ray. Thermal radiation (or blackbody radiation) from stars cover a broad range of wavelengths, and neutral gases near stars are subject to photoionization, even at the distance of 1 AU (cf. at Earth's ionosphere).

Collision with high-energy particles is another important ionization source. Galactic cosmic ray represents a homogeneous background of particles in interstellar and interplanetary space, mostly consisting of energetic protons reaching 10^{20} eV. When galactic cosmic ray hits a neutral atom, electrons are kicked out of the atom. Electron shower or precipitation from the Earth's magnetosphere also maintains ionization in the ionosphere, and causes aurora in the polar region.

Charge exchange is another process of ionization. Electrons can be exchanged from neutral atoms to ions when they meet, while the momenta of these species are maintained. This process creates energetic neutral atoms (which were initially energetic ions) and provides a very useful diagnostic and visualization tool for studying planetary magnetospheres.

In pulsar magnetospheres, the magnetic field originating from the neutron stars is so strong and, furthermore, the field is moving and co-rotating with the neutron stars so fast that pairs of electrons and positrons are created from the moving field. This creates a special kind of plasma, as there is no mass difference between two different particle species (cf. the proton mass is much larger than the electron mass, by factor 1836).

2.1.1.2 Fundamental Properties of Space Plasmas

Quasi-neutrality

In plasma, the electrostatic potential of individual ion is shielded by ambient, freely moving electrons. The potential curve does not fall as $1/r$ from the source (Coulomb potential), but falls faster as $\mathrm{e}^{-r/\lambda_\mathrm{D}}/r$ (Yukawa-type potential). Debye length λ_D characterizes the cutoff length scale of the potential and is a function of the electron temperature T_e and the number density n_e, $\lambda_\mathrm{D} = \sqrt{\varepsilon_0 k_\mathrm{B} T_\mathrm{e}/n_\mathrm{e} e^2}$, where ε_0 denotes the vacuum dielectric constant. Plasmas are quasi-neutral on spatial scales larger than Debye length (e.g., in the solar wind the Debye length is of order of 10 m), which justifies quasi-neutrality in plasma. Macroscopic properties such as the density, bulk velocity, and temperature often give a more convenient way of describing plasma dynamics. For the quasi-neutrality of plasma the energy density of electric field is found to be smaller than that of magnetic field (cf. Eq. 2.31).

Electrons are oscillating around ions due to the electrostatic attraction. This phenomenon gives the fundamental time scale, called the plasma oscillation, and its frequency is determined solely by the electron number density, $\omega_{\mathrm{pe}} = \sqrt{n_{\mathrm{e}} e^2 / m_{\mathrm{e}} \varepsilon_0}$. In the solar wind plasma, frequency ω_{pe} is of order of 10 kHz.

High Conductivity

Due to mobility of electrons and ions, plasma is electrically conducting. In a highly conducting medium (gas or fluid) the magnetic field is said to be "frozen-in" to the plasma, that is the field moves together with the medium. On the other hand, for small-scale or high-frequency fluctuating fields the response of the ions and electrons is frequency-dependent, and conductivity on the fluctuating electric field is also frequency-dependent. For this reason there are a number of wave modes or normal modes in plasma, some of which are associated with the bulk motion, and others with the individual ion motion or electron motion.

Collisionless gas

The huge spatial scale and the low density in the interplanetary or interstellar space make plasmas collisionless; the mean-free-path of the particles is comparable to the system size or even larger than that. This fact raises questions with regard to the physical processes of shock waves and turbulence, as they need dissipation mechanisms to heat the plasma and to convert the energy into the thermal one. In ordinary gas dynamics, in contrast, dissipation is provided by binary collisions between particles.

The collisionless property of the space plasma also makes it possible to sustain thermally non-equilibrium states for a longer time, i.e., a non-Maxwellian velocity distribution does not evolve into Maxwellian but evolves in a different way. Examples of non-Maxwellian distributions are counter-streaming beams upstream of the shock wave (the foreshock region, Fig. 1.4) and anisotropic velocity distribution between parallel and perpendicular directions to the magnetic field (temperature anisotropy) downstream of the shock wave.

The concept of temperature can be generalized and applied to a collisionless gas even though it is not in thermal equilibrium: The kinetic temperature is a measure of the spread of the velocity distribution. Because of the collisionless property, a plasma can have different temperatures at the same time; The kinetic temperature (or the spread of the velocity distribution) can be defined and measured in various directions parallel and perpendicular to the magnetic field. Furthermore, electrons and ions can have different temperatures.

Nonlinearity

Plasmas exhibit various kinds of nonlinear phenomena. Even in the one-fluid picture of plasma, called magnetohydrodynamics, there are different sources of nonlinearity originating in the fluid motion (advection), the Lorenz force acting on the plasma, and the coupling between the flow velocity and the magnetic field. Nonlinear effects are, for example, wave steepening or wave-wave interactions that lead the plasma to evolve into turbulence. Under some circumstances, on the other hand, the equations of plasma dynamics can be reduced into a simpler integrable form. In such a case nonlinear effects compete against other effects such as dispersion (which means the broadening of wave packet) and the plasma dynamics leads to formation of large-amplitude, stable solitary wave.

2.1.2 Magnetohydrodynamics

Three Pictures of Plasmas

Plasmas are collections of a very large number of electrically charged particles, and plasma dynamics can be treated and described in different ways. Roughly speaking, there are three approaches: (1) single particle approach; (2) kinetic approach; and (3) fluid approach.

In the single particle approach, the motion of individual particle is considered. Electric field, magnetic field, and current are computed by counting individual particles. The conceptual structure of the method is relatively straightforward: solving equations of motions for each particle under Coulomb force, Lorentz force, and other forces; and then determining electric and magnetic fields using Maxwell equations using the charge density and the electric current. Charged particles exhibit various dynamics: gyration around the magnetic field; drift motions of the guiding center (of gyrating particles) under the electric field, gravity, non-uniform magnetic field, and curved magnetic field lines.

In the kinetic approach, evolution of phase space density (spanned by spatial coordinate and velocity coordinate) or velocity distribution function is studied using Boltzmann-Vlasov equation, which comes from Liouville's theorem stating the conservation law of phase space density. Plasma dynamics is treated in a statistical fashion, and this approach can resolve wave-particle interactions (instabilities, wave excitation, wave damping).

In the fluid approach, plasma is treated as an electrically conducting, magnetized, fluid characterized by the macroscopic quantities such as density, bulk velocity, and temperature. One may consider plasmas as consisting of multi-fluids (electron fluid and ion fluid; or fluids for different particle species), or more simply as one fluid. In the latter case the treatment is called magnetohydrodynamics (MHD).

Fluid Picture

Macroscopic variables in the fluid picture are closely related to the velocity distribution of particles. Number density n or mass density ρ is given as the integrated velocity distribution over the velocity coordinate, i.e., total volume of the velocity distribution. The bulk velocity or the flow velocity, $\mathbf{u}$, is given as the first-order velocity moment of the distribution, i.e., the center of the distribution or the average of individual particle velocities. The thermal velocity and pressure are given as the second-order moment of the velocity distribution, i.e., the spread of the distribution.

MHD essentially consists of two equation sets: flow dynamics for a conducting medium and electromagnetism. These equations can be written as follows.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.2}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla \cdot \mathsf{P} + \mathbf{j} \times \mathbf{B} + \rho \nu \nabla^2 \mathbf{u} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{2.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.6}$$

where ν denotes the viscosity. P is the three-by-three pressure tensor, and each diagonal component represents the thermal pressure in different directions: for example, one parallel direction to the magnetic field and two perpendicular directions. One of the Maxwell equations is trivial under quasi-neutrality, $\nabla \cdot \mathbf{E} \simeq 0$. In MHD equations, the time derivative of the electric field is regarded to be so slow that the validity of the one-fluid approximation is not violated.

The above set of equations is not complete but has a closure problem associated with the thermal pressure. It cannot be determined within the above set of equations, and needs to be evaluated by other means. One possible way is to use the energy balance equation. Another possible way is to use the equation of state, relating the pressure to the density. Or one might assume the isothermal condition under which the thermal pressure is constant.

The electric current density can be evaluated either from the spatial variation of the magnetic field (Ampère's law) or from generalized Ohm's law. In a plasma Ohm's law becomes considerably more complicated, and contain different contributions to the electric current or to the electric field. Generalized Ohm's law is given as:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{\sigma}\mathbf{j} + \frac{1}{ne}\mathbf{j} \times \mathbf{B} - \frac{1}{ne}\nabla \cdot \mathsf{P}_e + \frac{m_e}{ne^2}\frac{\partial \mathbf{j}}{\partial t}. \tag{2.7}$$

Here σ denotes the conductivity. The first term on the right-hand-side represents the resistive, Ohmic term; the second term the Hall term; the third term the anisotropic electron pressure; and the last term the contribution of electron inertia to the current flow.

Incompressible MHD Equations

In the incompressible case, $\nabla \cdot \mathbf{u} = 0$, the MHD equations have a more symmetric structure between the magnetic field and the flow velocity. After arranging terms, the equations are expressed as follows.

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{b} - \nabla P_{\mathrm{tot}} \tag{2.8}$$

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2\right) \mathbf{b} = -(\mathbf{u} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{u} \tag{2.9}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.10}$$

$$\nabla \cdot \mathbf{b} = 0, \tag{2.11}$$

where the magnetic field is expressed in units of velocity, $\mathbf{b} = \mathbf{B}/\sqrt{\mu_0 \rho}$. The mass density ρ is assumed to be constant and has been omitted here. The symbol η denotes the diffusivity, $\eta = 1/\mu_0 \sigma$. P_{tot} is the total scalar pressure which is the sum of thermal and magnetic pressure, $P_{\mathrm{tot}} = p_{\mathrm{th}} + B^2/2\mu_0$. The fluid momentum equation (Eq. 2.8 which is the Navier–Stokes equation for incompressible fluid including the effect of the Lorentz force acting on the fluid element) and the induction equation (Eq. 2.9) have a very similar structure to each other in incompressible MHD. The difference of the two equations are (1) the pressure gradient term for the total pressure (thermal and magnetic pressures) appears in the fluid equation but not in the induction equation; and (2) the nonlinear terms are self-coupling (velocity to velocity, and magnetic field to magnetic field) in the fluid equation, whereas the induction equation exhibits cross-coupled nonlinear terms (velocity to magnetic field). Derivation of the induction equation (Eq. 2.9) makes use of the expression of electric field which is a sum of the convective field and the Ohmic field (cf., Eq. 2.7)

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{1}{\sigma}\mathbf{j}, \tag{2.12}$$

and Ampère's law (Eq. 2.4). There are four nonlinear terms in incompressible MHD: two of them are self-coupling terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$ (advection of the flow) and $(\mathbf{b} \cdot \nabla)\mathbf{b}$ (Lorentz force acting on the fluid) and the other two are cross-coupling terms $(\mathbf{u} \cdot \nabla)\mathbf{b}$ and $(\mathbf{b} \cdot \nabla)\mathbf{u}$ that stem from the convective electric field $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$.

It is interesting to see that the magnetic field and the flow velocity are almost interchangable. Indeed, exchanging the two variables $\mathbf{u}$ and $\mathbf{b}$ such that

$$\mathbf{u} = \pm \mathbf{b} \tag{2.13}$$

is one of special solutions (in the limit of constant total pressure, $\nu \to 0$, and $\eta \to 0$). This state is called the Alfvénic state and represents one of the minimum energy states in magnetohydrodynamics.

Magnetic Pressure and Tension

The Lorentz force acting on the fluid element $\mathbf{j} \times \mathbf{B}$ plays an important role in magnetohydrodynamics, introducing the concept of magnetic pressure and tension. On applying vector algebra to Ampère's law, the Lorentz force splits into two terms:

$$\mathbf{j} \times \mathbf{B} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \nabla \cdot (\mathbf{BB}). \tag{2.14}$$

The first term on the right-hand side suggests that the magnetic field has a pressure $p_{\text{mag}} = B^2/2\mu_0$ and its spatial gradient serves as a force. The magnetic pressure simply adds to the thermal pressure of the plasma. The second term is a consequence of the vector product of current and magnetic field and thus of the vector character of the magnetic field. It is the divergence of a magnetic stress tensor $\mathbf{BB}/\mu_0$. The curved magnetic field lines introduces a magnetic stress in the plasma, which contributes to tension and torsion in the conducting fluid.

The advection term, $\rho(\mathbf{u} \cdot \nabla)\mathbf{u}$, can also be expressed in the form of pressure gradient as $-\nabla \rho u^2/2$ when the flow is not helical, $\mathbf{u} \cdot \nabla \times \mathbf{u} = 0$. If we neglect the off-diagonal or stress terms in the pressure tensors, e.g., for cases where the plasma pressure is nearly isotropic and the magnetic field is approximately homogeneous, the pressure equilibrium can be expressed as

$$\nabla \left(\frac{\rho u^2}{2} + p_{\text{th}} + \frac{B^2}{2\mu_0} \right) = 0. \tag{2.15}$$

This means that in an equilibrium, isotropic, and quasi-neutral plasma the total pressure is a constant. Under these conditions one can define a plasma parameter *beta* as the ratio of thermal to magnetic pressure:

$$\beta = \frac{p_{th}}{p_B} = \frac{p_{th}}{B^2/2\mu_0}. \tag{2.16}$$

In anisotropic plasmas where the pressure has distinct parallel and perpendicular components the concept of plasma beta can be applied to these components separately. The parameter beta measures the relative importance of gas and magnetic field pressures. A plasma is called a low-beta plasma when $\beta \ll 1$ and a high-beta plasma for $\beta \sim 1$ or greater.

Frozen-in Magnetic Field

The induction equation shows that the magnetic field evolves in two different fashions: (1) the field is coupled to the plasma motion (or the flow velocity), referred

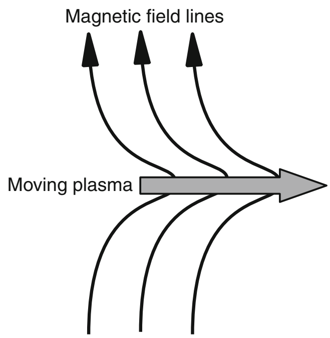

Fig. 2.2 Frozen-in magnetic field lines

to as the frozen-in magnetic field, and (2) the field strength is subject to diffusion and becomes weaker.

In order to study the transport of field lines and plasma more quantitatively, we may use the simplified version of Ohm's law $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$, the induction equation (Eq. 2.5) and the divergence-free condition of the field (Eq. 2.6). The time evolution of the magnetic field is then expressed as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{2.17}$$

In the case where the diffusivity η is negligible the magnetic field is said to be "frozen-in" into the plasma. The induction equation reduces into the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{2.18}$$

This is identical with the equation for the coupling of the vorticity ($\nabla \times \mathbf{u}$) in inviscid fluids, in which any vortex lines move together with the fluid. Equation (2.18) implies also that the magnetic field lines are constrained to move with the plasma. For example, if a patch of plasma populating in a bundle of field lines moves perpendicular to the field, the magnetic field lines will be deformed in the manner shown in Fig. 2.2.

It can also be shown that Eq. (2.18) implies that the total magnetic induction (or the field lines) encircled by a closed loop remains unchanged even if each point on this closed loop moves with a different local velocity. The concept of the frozen-in magnetic field can actually be identified by the plasma glued to the field. The equivalent expression is that the electric field is solely convective, determined by the flow velocity and the magnetic field:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0. \tag{2.19}$$

Equation (2.19) implies that in an infinitely conducting plasma there is no electric field in the frame co-moving with the plasma. Electric field can only result from the Lorentz transformation. Moreover, Eq. (2.19) contains another important point.

Since the cross-product between any velocity component parallel to the magnetic field and the field itself is zero, we can immediately see that any component of the electric field parallel to the magnetic field must vanish in an infinitely conducting plasma. An important consequence of the frozen-in magnetic field is that plasmas of different origins and attached to different magnetic field lines cannot be mixed under the frozen-in condition. Different plasmas may touch each other by forming a boundary or a current sheet but they are not mixed in the fluid picture.

In the case that the resistivity term dominates the evolution of magnetic field, Eq. (2.17) becomes a diffusion equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{2.20}$$

Under the influence of a finite resistance in the plasma, the magnetic field tends to diffuse across the plasma and to smooth out any local inhomogeneities. The characteristic time of magnetic field diffusion is found by replacing the vector derivative by the inverse of the characteristic gradient scale of the magnetic field, L. The local solution of the diffusion equation is then:

$$B = B_0 \mathrm{e}^{-t/\tau_d}, \tag{2.21}$$

where τ_d is the magnetic diffusion time given by

$$\tau_d = \frac{L^2}{\eta}. \tag{2.22}$$

Whenever $\eta \rightarrow 0$ or when the characteristic gradient length L is very large, the decay or diffusion time of the magnetic field becomes longer.

For an example consider the solar wind. Its density is of the order of $5\,\mathrm{cm}^{-3}$ and its electron temperature is about 50 eV. The estimated magnetic diffusion time is scaled as $\tau_d \simeq 0.3L^2$ for collisions between electrons and ions, where τ_d is given in units of seconds and L in meters. The time that the solar wind needs to flow with its typical velocity of 500 km/s across the Sun–Earth distance of 1.5×10^{11} m is $\tau_{\mathrm{sw}} = 3 \times 10^5$ s or 3.5 days. Thus, the magnetic field is permitted to diffuse across the solar wind over the short distance of only about $1.9\sqrt{\tau_{\mathrm{sw}}} \sim 10^3$ m during the travel across the Sun–Earth distance. The magnetic field is practically frozen-in to the solar wind and the field lines are carried by the flow.

The dominance of either the frozen-in field or the diffusion process can be measured by the magnetic Reynolds number. Rewrite the induction equation (Eq. 2.17) using the characteristic time and length scales:

$$\frac{B}{\tau} = \frac{UB}{L} + \frac{\eta B}{L^2}. \tag{2.23}$$

In this equation B is the average magnetic field strength and U represents the average plasma velocity perpendicular to the field, while τ denotes the characteristic time of magnetic field variations, and L is again the characteristic length over which the field varies. The ratio of the first to the second term yields the magnetic Reynolds number:

$$R_m = \frac{LU}{\eta}. \tag{2.24}$$

This is useful in deciding whether the magnetic field evolution is dominated by the diffusion process ($R_m \ll 1$) or the frozen-in process ($R_m \gg 1$). For example, the magnetic Reynolds number in the solar wind is estimated to be about $R_m \sim 7 \times 10^{16}$, reflecting our previous argument of the negligible magnetic diffusion in the solar wind. Note that only the perpendicular velocity enters the frozen-in convective motion. Any flow parallel to the magnetic field has no consequences. When $R_m \ll 1$ or even $R_m \sim 1$, the diffusion becomes important and may dominate the evolution of the magnetic field. In most large-scale and dilute plasmas in the solar system and astrophysical systems the frozen-in magnetic field is a reasonable assumption in a wide range of time scales.

Four Energy Forms

Plasmas exhibit various kinds of dynamical behaviors and processes. It is useful to consider which kind of energy is involved in each process. There are four different forms of energy densities when treating the plasma as a fluid.

(1) The kinetic energy density of the flow:

$$\mathcal{E}_{\mathrm{K}} = \frac{1}{2}\rho u^2 \tag{2.25}$$

(2) The thermal energy density or thermal pressure:

$$\mathcal{E}_{\mathrm{th}} = p_{\mathrm{th}} = nk_B T \tag{2.26}$$

where we applied the ideal gas law but other choices of equation of state (and therefore the plasma model) are also possible.

(3) The magnetic energy density:

$$\mathcal{E}_{\mathrm{M}} = \frac{B^2}{2\mu_0} \tag{2.27}$$

(4) The electric energy density:

$$\mathcal{E}_{\mathrm{E}} = \frac{\varepsilon_0 E^2}{2}. \tag{2.28}$$

It is worthwhile to note that the ideal gas law or the equation of state relating the pressure by the density and temperature closes the equation set of magnetohydrodynamics. The thermal pressure and temperature may be anisotropic between the directions parallel and perpendicular to the magnetic field. The kinetic and thermal energy densities represent the gas dynamical properties of the plasma.

Quasi-neutrality in the plasma suggests that the magnetic energy density is larger than that for the electric field. In fact, the Fourier transform of the induction equation yields the following expression:

$$\omega \delta \mathbf{B} = -\mathbf{k} \times \delta \mathbf{E}, \tag{2.29}$$

where ω and $\mathbf{k}$ denote the frequency and the wave vector of the wave or the field disturbance. The ratio of the electric to magnetic field amplitude is thus proportional to the propagation speed (or the phase speed) in the plasma,

$$\left| \frac{\delta E}{\delta B} \right| = \frac{\omega}{k} = v_{\mathrm{ph}}. \tag{2.30}$$

Under the non-relativistic treatment, where $v_{\mathrm{ph}} < c$ (the speed of light is expressed by c), the ratio of the electric to magnetic energy is much smaller than unity,

$$\frac{\mathscr{E}_{\mathrm{E}}}{\mathscr{E}_{\mathrm{M}}} = \frac{\varepsilon_0 E^2/2}{B^2/2\mu_0} = \frac{v_{\mathrm{ph}}^2}{c^2} \ll 1. \tag{2.31}$$

Validity of Magnetohydrodynamics

Magnetodydrodynamics is the one-fluid treatment of plasma and there is no distinction between the different particle species of the plasma. This approximation requires that time scale of variation of the plasma and fields must be longer than the characteristic time scale of the constituting particles, in particular ions. The characteristic frequency, ω, of any change must be smaller than the ion cyclotron frequency for the magnetohydrodynamic picture to be valid. This argument applies to the length scale, too. The magnetohydrodynamic picture describes plasma dynamics on the length scales larger than the ion gyro-radius. Magnetohydrodynamics is therefore restricted to slow time variations and large spatial scales. At such low frequencies the displacement current can safely be neglected in the Maxwell equations.

2.1.3 Fundamental Plasma Processes

Dynamics of plasma is diverse. Gas dynamical motion is coupled with electromagnetic field and the energy can be stored in different forms: kinetic energy, thermal energy, magnetic and electric field energy. Examples of plasma dynamics are waves and instabilities, turbulence, shock waves, and magnetic reconnection. Some of these processes stem from the gas dynamical properties such as shock waves, and others from the coupling between the plasma and the electromagnetic field.

2.1.3.1 Waves and Instabilities

In contrast to incompressible fluids, in which the perturbation mode is solely non-propagating eddies, MHD supports several types of waves or linear modes even in the incompressible limit. Consider the simplest case of a homogeneous plasma described by a constant thermal pressure and a constant density in a magnetic field. For sufficiently small perturbations we can linearize the MHD equations by splitting the pressure, density, and the magnetic field into the large-scale, background part and the fluctuating field. Furthermore, the fluctuating fields are assumed to be a plane wave characterized by the angular frequency ω and the wave vector $\mathbf{k}$ (Fourier transform), reducing the differential operators into algebraic expressions. The linearized MHD equations support three types of oscillation modes: the Alfvén mode, the fast and slow magnetosonic modes. The three MHD modes are non-dispersive waves, and the phase and group velocities are independent of frequencies. This reflects the fact that the one-fluid picture of the plasma, MHD, does not contain any intrinsic scales in the time nor spatial domain.

Linear MHD Modes

The Alfvén mode is characterized by the dispersion relation:

$$\omega^2 = k_{\parallel}^2 v_A^2, \tag{2.32}$$

where $k_{\parallel}$ denotes the parallel component of the wave vector with respect to the background magnetic field, and v_A denotes the Alfvén speed defined as $v_A = B_0/\sqrt{\mu_0 \rho_0}$. In this mode the plasma motion is incompressible and transverse to the background magnetic field. So is the magnetic field perturbation. The oscillation is maintained by the magnetic tension and the inertia of the plasma. The oscillating field lines may be analogously interpreted as an elastic string with bending shape (Fig. 2.3). The Alfvén wave propagates preferentially parallel or anti-parallel to the background magnetic field direction. In particular, the group speed of the wave is strictly aligned with the background field. The mode also exhibits correlation between the velocity fluctuation and the magnetic field fluctuation for anti-parallel propagation, and anti-correlation for parallel propagation.

The compressive waves are referred to as the fast and slow magnetosonic modes. The fast magnetosonic mode is characterized by the dispersion relation

$$\omega^2 = \frac{1}{2} k^2 \left[v_A^2 + c_s^2 + \sqrt{\left(v_A^2 + c_s^2\right)^2 - 4 v_A^2 c_s^2 \cos^2\theta} \right], \tag{2.33}$$

where c_s and θ denote the sound speed and the propagation angle with respect to the background magnetic field. A useful expression for the sound speed is $c_s = \sqrt{\gamma p_0/\rho_0}$, where γ is the polytropic index, p_0 the mean thermal pressure, and ρ_0 the mean mass density. The fast mode is compressible and the phase velocity is in the range $v_A^2 \leq v_{ph}^2 \leq v_A^2 + c_s^2$, being fastest for propagation perpendicular to the background magnetic field. The fluctuating field pattern of the fast mode is displayed

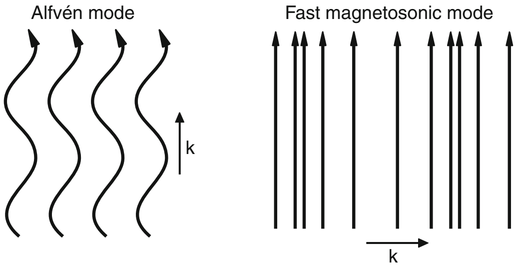

Fig. 2.3 Magnetic field lines for Alfvén and fast magnetosonic mode waves

in Fig. 2.3. The magnetic field and the plasma are compressed by the wave motion, such that the restoring force is large and hence the frequency and the propagation speed are high. It is worthwhile to note that this mode exists even if the sound speed is zero.

The slow magnetosonic wave, or slow mode, is characterized by the dispersion relation

$$\omega^2 = \frac{1}{2}k^2\left[v_A^2 + c_s^2 - \sqrt{\left(v_A^2 + c_s^2\right)^2 - 4v_A^2 c_s^2 \cos^2\theta}\right]. \tag{2.34}$$

The phase velocity is in the range $0 \leq v_{ph}^2 \leq c_s^2$. The magnetic field oscillation and the pressure variation are anti-correlated with each other such that the restoring force acting on the medium is weaker than that for the fast mode. For this reason the frequency and the propagation speed are the lowest among the three MHD wave modes.

Non-MHD waves

On smaller scales or at higher frequencies the one-fluid treatment of plasma is no more valid. Electrons and ions react differently to the wave electric field, and the dielectric property of the medium for fluctuating field becomes a complex function of frequency. Therefore, plasmas allow a large number of wave modes to exist at higher frequencies. Waves can be treated in different fashions at high frequencies: cold plasma approximation using single particle motion or the kinetic treatment using Boltzmann-Vlasov equation. In both treatments waves are dispersive, i.e., the dispersion relation is no longer linear such that the propagation speed is dependent on frequencies.

For example, in the cold plasma treatment, there are two wave modes with the propagation direction parallel or anti-parallel to the background magnetic field: R-mode and L-mode with for their right-handed and left-handed polarization sense temporal evolution of field rotation sense. Here, polarization means the temporal field rotation sense when looking in the direction of the background magnetic field. The R-mode can exist at frequencies up to the electron gyro-frequency at which the wave is subject to the cyclotron resonance with electrons. The L-mode, on the other hand, exists at frequencies up to the ion gyro-frequencies, which is much lower than that for electrons. The L-mode is also referred to as the ion cyclotron mode.

Instabilities

Plasmas also exhibit various kinds of instabilities such that the force acting on the disturbed medium does not restore disturbance back to the equilibrium state but helps the fluctuation even to grow exponentially. In that case, the solution of the linearized set of equations for plasma dynamics is characterized by frequencies given as a complex number. The imaginary part of the complex frequency is interpreted as either wave growth or damping, depending on the sign of the imaginary part. Solutions with growing waves are referred to as instabilities. There are two types of instabilities: macroscopic and microscopic. Macro-instability originates in an unstable gradient or inhomogeneity of plasma in the coordinate space. An example for this is the Kelvin-Helmholtz instability in a shear flow configuration, generating eddies in the velocity shear. Micro-instability, in contrast, originates in an unstable gradient in the velocity space or unstable shape of the velocity distribution. Examples are a beam configuration showing two peaks in the velocity distribution or temperature anisotropy showing different thermal spread between parallel and perpendicular directions to the background magnetic field.

Wave growth or damping implies that the energy is exchanged between the plasma and the electromagnetic field. Two mechanisms are particularly known: cyclotron resonance and Landau resonance. While the former represents electromagnetic wave-particle interaction through a circularly polarized electric field, the latter represents electrostatic wave-particle interaction through a longitudinal electric field (along the background magnetic field). Both mechanisms accelerate electrons and ions in the direction of electric field.

2.1.3.2 Collisionless Shock Waves

Shock waves in ordinary gas dynamics are formed when a supersonic flow encounters an obstacle or when the object moves in a gas at a supersonic speed. The former is a standing shock wave (called the bow shock) where the shock propagation speed cancels out with the incoming flow speed. Space and astrophysical plasmas also exhibit shock waves in various places: solar atmosphere, interplanetary shocks, planetary bow shocks, supernova explosion. The nearest plasma shock to us is Earth's bow shock standing in the solar wind at about 15–20 R_{E} in front of the Earth. Most shock waves in space plasma are associated with the properties of fast mode: the flow speed must exceed the fast magnetosonic speed for the shock wave to occur, and both the gas pressure and the magnetic pressure are enhanced across the shock wave.

One of the fundamental differences in shock waves between ordinary gas dynamics and space plasma is the dissipation mechanism. While the shock waves in ordinary gas dynamics are maintained by binary collisions for the dissipation mechanism,

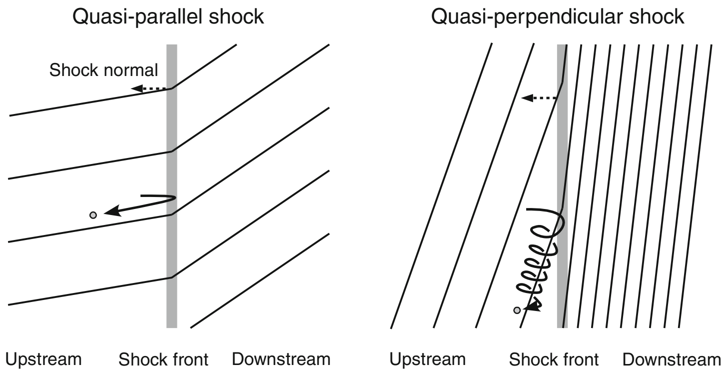

Fig. 2.4 Magnetic field lines across the quasi-parallel shock (*left*) and the quasi-perpendicular shock (*right*). At the quasi-parallel shock, charged particles are reflected at the shock front and stream backward along the field line. At the quasi-perpendicular shock, reflected particles cannot stream backward but gyrate around the magnetic field ahead of the shock front

those in space plasma cannot use binary collisions due to extremely large mean free paths. From in-situ observation of Earth's bow shock it was revealed that the plasma shock has a unique dissipation mechanism: particle reflection and acceleration, that is some particles in the incoming flow are accelerated and specularly reflected at the shock. Various mechanisms have been proposed for the reflection process: electric field at the shock wave repelling the particles; particle scattering in the shock-downstream region back to upstream; particle acceleration along the shock front. In any case, the phenomenon of particle reflection cannot be explained in the fluid picture of plasma, since not all particles are reflected but only a tiny portion of the incoming particle population.

Physical processes of the plasma shocks are further split into two distinct magnetic field geometries: quasi-parallel shock and quasi-perpendicular shock (Fig. 2.4). Here, the geometries refer to the angle between the upstream magnetic field direction and the shock normal direction.

At the quasi-parallel shock, the reflected particles escape into upstream along the magnetic field line and brake the incoming flow, warning the incoming flow about the existence of the shock wave. Hence the shock transition already starts with pre-thermalization in the upstream region, and this transition region is called the foreshock. Both ions and electrons are reflected and stream against the incoming flow. The foreshock plasma exhibits therefore a counter-streaming beam configuration: the incoming flow and the backstreaming beam. This forms an unstable velocity distribution, exciting waves. Across the shock the magnetic field lines are tilted more to the shock plane, which is the sense of field compression.

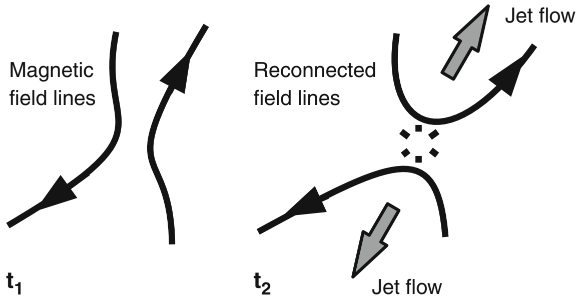

Fig. 2.5 Magnetic field lines before (t_1) and after reconnection t_2

At the quasi-perpendicular shock, the reflected particles gyrate around the upstream magnetic field direction and are swept by the flow back to the shock. The shock transition occurs therefore in a narrow layer consisting of the foot region ahead of the shock and the ramp on the scale of ion gyro-radius. Gyrating particles feel convective electric field (due to the flow) and are transported to the shock and further to the downstream region. For this reason the plasma is heated preferentially in the direction perpendicular to the magnetic field. The plasma downstream of the shock is characterized by temperature anisotropy (higher temperature in the perpendicular direction), which is subject to micro-instability, exciting the mirror mode. The magnetic field compression is more effective at the quasi-perpendicular shock.

2.1.3.3 Magnetic Reconnection and Dynamo

In plasmas energy conversion is possible between kinetic and magnetic energy. The magnetic reconnection is the process converting the energy from magnetic to kinetic, and the dynamo mechanism acts in the opposite sense from kinetic to magnetic. Magnetic reconnection occurs when the field configuration is anti-parallel across a thin current layer, and can be interpreted as the field lines cut and reconnected (Fig. 2.5). This process violates the frozen-in magnetic field and therefore requires local magnetic diffusion. The reconnected field lines have strong curvature and relaxes quickly by accelerating the plasma in two opposite directions as jet flow within a short time. A large amount of energy is released as the plasma jet flow (the conversion from the magnetic into the kinetic energy), and the magnetic reconnection is believed to be the very mechanism explaining explosion events in space plasma such as solar flares, coronal mass ejection, and geomagnetic substorms.

Dynamo is the process actively generating magnetic field in astrophysical bodies such as the Earth, the Sun, and other planets and stars. Without dynamo the magnetic field becomes weaker and decays due to the resistivity. Arguments for the need of the dynamo mechanism are: (1) The studies in palaeomagnetism revealed that Earth's magnetic field lasted longer than the decay time (the Earth outer core is not in the plasma state but in the liquid state, but it can be treated as an electrically conducting fluid and the use of magnetohydrodynamics is still valid) ; and (2) the solar magnetic field is oscillatory with the 11-year period. The essence in the dynamo mechanism is that the flow motion is preferable for twisting and stretching magnetic field lines such as helical flow motion due to buoyancy and Coriolis force, turbulence flow, and shear flow (or differential rotation).

2.2 Elements of Turbulence Theories

The set of equations of magnetohydrodynamics is highly nonlinear, yet the treatment of its turbulence can be found in fluid dynamics as a subset of magnetohydrodynamic equations (cf. similarity between the flow velocity and the magnetic field in incompressible MHD). The essence of fluid turbulence is reviewed as well as Kolmogorov's phenomenological model. Fluid turbulence is then compared with plasma turbulence.

2.2.1 Fluid Turbulence

In fluid dynamics two equations are primarily used: the continuity equation and the momentum balance equation representing conservation of mass and momentum, respectively. The continuity equation is the same as that in magnetohydrodynamics:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.35}$$

The momentum balance equation or Navier–Stokes equation (in a simplified form) is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \left(\frac{p}{\rho} \right) + \nu \nabla^2 \mathbf{u}, \tag{2.36}$$

where we assumed that the fluid is incompressible for simplicity satisfying the solenoidal condition:

$$\nabla \cdot \mathbf{u} = 0. \tag{2.37}$$

In other words, the mass density ρ is assumed to be constant. For an incompressible fluid the pressure p does not play an active role in dynamics, but it is *passive scalar* determined by the Poisson equation by taking divergence of Eq. (2.36). Navier–Stokes equation is nonlinear such that the flow velocity $\mathbf{u}$ is coupled to itself in the advection term. Although the Navier–Stokes equation appears to be relatively simple (there is only one nonlinear term in the incompressible fluid), its general solution is not known, and moreover, the existence and uniqueness of solution is not known, either. It is important to note that turbulence is not a property of fluid material but a property of flow whenever the fluid is set into motion and the advection term dominates in the momentum balance.

The convective derivative (time-dependent and advection term) requires that the Navier–Stokes equation or fluid equation be constructed in any frame of inertia as far as the frame of reference is transformed by Galilean transformation; its form is invariant under Galilean coordinate system transformation. Advection term is the driver of turbulent motion in the flow, and turbulence is characterized by the concepts of nonlinearity, unpredictability, and energy cascade and dissipation.

Nonliearity

The advection term may be interpreted as wave steepening. This can be seen by setting the flow velocity field as a plane wave, $u \propto \sin kx$. The advection term then yields $\mathbf{u} \cdot \nabla \mathbf{u} \sim k \sin(2kx)/k$, implying that the wave number is doubled from k into $2k$. An eddy with the vorticity $\nabla \times \mathbf{u}$ becomes enhanced by this term, thus the effect of advection can also be interpreted as vortex stretching.

The ratio of the magnitude of the advection to the dissipation term is called the Reynolds number Re,

$$\mathrm{Re} = \frac{|(\mathbf{u} \cdot \nabla)\mathbf{u}|}{|\nu \nabla^2 \mathbf{u}|} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu}, \tag{2.38}$$

where again U and L denote the characteristic speed and length of flow and we use replacement $\nabla \to 1/L$.

Transition from laminar (flow at a low Reynolds number) to turbulence (flow at a high Reynolds number) is illustrated in Fig. 2.6. Streamlines are smooth at low Reynolds numbers and become increasingly complicated with eddies on various scales at high Reynolds numbers.

Typical values of the kinematic viscosity of air is about $\nu = 0.1\,\mathrm{cm}^2\,\mathrm{s}^{-1}$. If we walk in the air on the ground, the Reynolds number is estimated to be of order $\mathrm{Re} \sim 10^4 (L = 10\,\mathrm{cm},\, U = 100\,\mathrm{cm\,s}^{-1})$. For a motor vehicle the Reynolds number is about $\mathrm{Re} \sim 10^6 (L = 10^2\,\mathrm{cm},\, U = 10^3\,\mathrm{cm\,s}^{-1})$. In geophysical and astrophysical systems both the length and the flow speed are large numbers, and the Reynolds number is also very large. In the outer core of the Earth, the solar convection zone, and galaxy of the Reynolds number is of order of 10^8, 10^{10}, and even 10^{11}, respectively.

Unpredictability

Turbulent flow is irregular and prediction of flow velocity is very difficult. It is chaos of fluid (chaos in the sense of chaotic behavior in a dynamical system). A slight difference at the initial time ends up with a totally different state. It is worthwhile to note that the momentum balance equation or Navier–Stokes equation is deterministic, and since the equation is given, it should be in principle possible to determine the future motion of the flow as we know what forces are acting on the fluid element and how string they are. However, it is virtually impossible to solve the equation exactly and to predict the flow velocity in future because an error or an uncertainty grows rapidly.

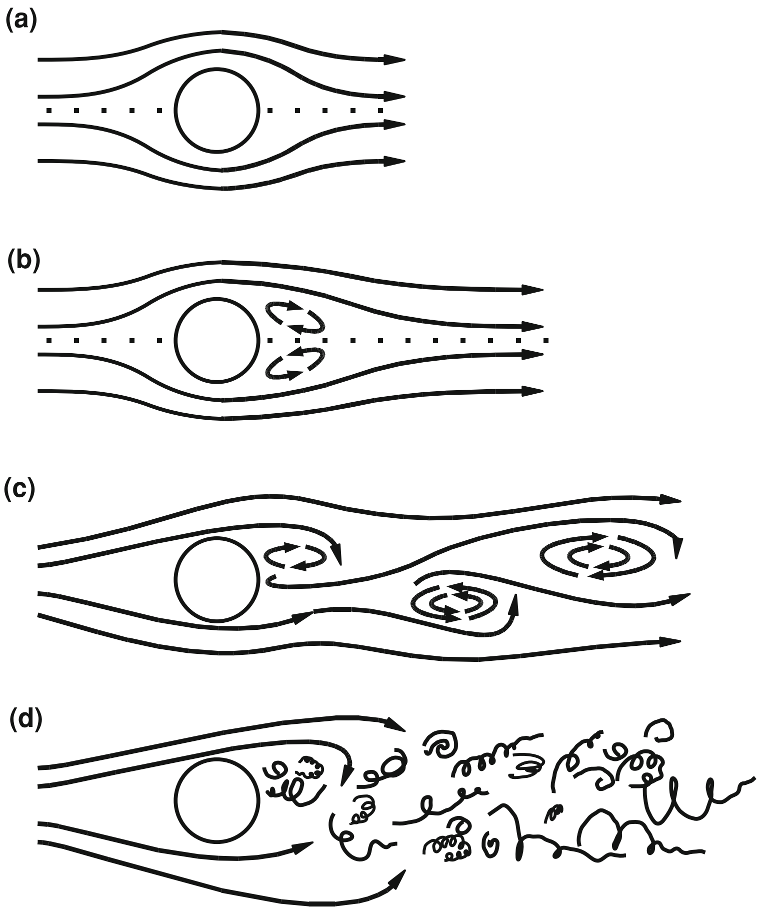

Fig. 2.6 Flow pattern around a cylinder at various Reynolds numbers: **a** 0.1; **b** 10; **c** 100; **d** 10000 (after Davidson, 2004, Turbulece: an introduction for scientists and engineers, fig. 1.4 p.8, by permission by Oxford University Press)

Energy cascade and dissipation

Nonlinear effects make it possible for different waves to interact with each other and to excite another wave mode. Wave steepening due to the advection term discussed above can be generalized such that the mode with the wave number k and the other mode with $2k$ are coupled to each other, and the wave–wave interaction proceeds successively in a cascade fashion. Turbulence is therefore characterized by the concept of energy cascade introduced in Chap. 1, in which large-scale eddies split into small-scale eddies, and further into eddies on even smaller scales. The energy flows from larger scales to smaller scales, and for this reason turbulent field does not show any characteristic size of eddies; different scales exist at once. For eddies on

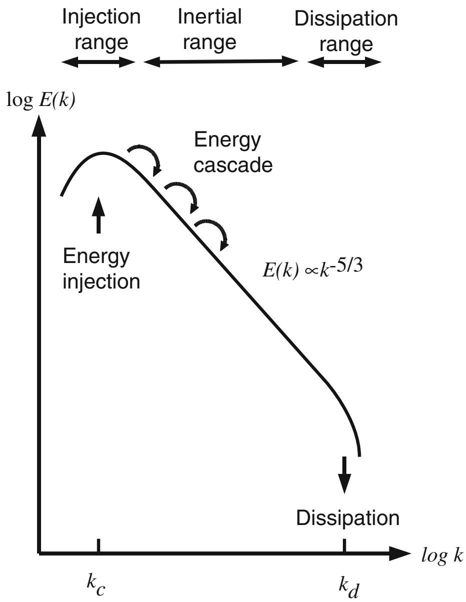

Fig. 2.7 Energy spectrum according to Kolmogorov's scaling law

larger scales the process of energy cascade serves as a sink, whereas for eddies on smaller scales it serves as a source. The concept of energy cascade can be formulated as a power-law energy spectrum (Fig. 2.7).

However, energy cascade cannot occur on arbitrary small scales. The reason for the existence of small-scale limit can be seen in the definition of the Reynolds number: on smaller scales the magnitude of the dissipation term becomes increasingly important. At sufficiently small wavelengths both terms (advection and dissipation) become of the same order in magnitude and the dissipation process will dominate and the kinetic energy of flow is converted to thermal energy. From the gas-dynamical point of view, this occurs on the scale of particle collisions, i.e., the mean free path. It is the dissipation process that heats the fluid on the smallest scale. In the stationary state of turbulence, energy is injected into the flow system on a large scale, which is transported to smaller scales by successive eddy splitting until the scale is reached where the curvature of the streamlines is so sharp that it is smoothed and the kinetic energy is converted into the thermal energy. If no energy is put into the flow system, energy cascade decays monotonously, and the flow becomes heated and smooth (freely decaying turbulence).

2.2.1.1 Phenomenological Model of Kolmogorov

The energy spectrum of turbulence was derived by Kolmogorov in 1941 [3, 4, 5]. His phenomenological derivation is based on the picture that physical process of turbulence is divided into three different ranges or spatial scales: injection range, inertial range, and dissipation range. They are divided from one spatial scale to another, in other words the separation is made in the wave number domain. Figure 2.7 displays the energy spectrum of turbulence in the wave number domain. On the largest scales or at lower wave numbers around k_c the energy is put or injected into the system by shear flow, inhomogeneity, instabilities, or external force. This range is called the injection range, and contains most of the fluctuation energy. In contrast, on the smallest scales or at higher wave numbers around k_d the dissipation process due to the finite viscosity in the fluid dominates and the fluctuation energy (or kinetic energy of flow) is converted into thermal energy. This range is called the dissipation range. The spectral energy drops sharply at the wave number for the dissipation scale (of order of the mean free path). Kolmogorov found that these two ranges are separated in a turbulence flow and connected by an intermediate range called the inertial range.

Kolmogorov found an analytical expression of the inertial-range spectrum. Its significance can be summarized as follows.

(1) The inertial range represents energy cascade or eddies splitting, transporting energy into smaller scales. It is the *inertia* of the turbulent flow in the wave number domain.
(2) The existence of the inertial range is universal in the sense that it is the property of the flow and independent from the type of fluid. It appears whenever the injection range and the dissipation range are well separated.
(3) The spectral curve must be power-law characterized by the exponent α, i.e., the energy spectrum is given as a function of the wave number, $E(k) \propto k^{-\alpha}$. This reflects the fact that the fluid equation has, in the limit of vanishing viscosity, symmetry with respect to the scaling transformation such that the length L and time T are scaled by λL and $\lambda^{1-h}T$, respectively, using the scaling factor λ and its exponent h. Scaling symmetry implies that turbulence does not exhibit any characteristic scale in fluctuation.
(4) The power-law exponent of the inertial-range spectrum is uniquely determined for isotropic, hydrodynamic turbulence, $\alpha = -5/3$.

Kolmogorov's phenomenological model is based on several assumptions: Fluctuation energy is transported by eddy splitting successively onto smaller scales; Equilibrium or balance holds in the energy transfer rate from the injection range (on the largest scale) down to the dissipation range (the smallest scale). The model uses two concepts in deriving the inertial-range spectrum: the energy transfer time and the energy transfer rate.

In the first step, the energy transfer time (or the eddy turnover time) is defined as

$$\tau_{\mathrm{ed}} = l/v, \tag{2.39}$$

which implies that an eddy characterized by the size l and the flow speed v breaks up into small-scale eddies after time l/v, i.e., after the fluid element along the eddy streamline finishes just one circulation. We divide the scales discretely and write the length and the velocity scales as $l_1, l_2, \ldots, l_n$ and $v_1, v_2, \ldots, v_n$. The energy transfer time at n-th generation of cascade is then:

$$\tau_n = \frac{l_n}{v_n}. \tag{2.40}$$

In the second step, the energy transfer rate is estimated. The energy (per unit mass) at n-th generation is given by

$$E_n = \frac{1}{2} v_n^2. \tag{2.41}$$

The energy transfer rate defined as the spectral energy flux at n-th generation of energy cascade is

$$\varepsilon_n = \frac{dE_n}{dt} \sim \frac{E_n}{\tau_n}. \tag{2.42}$$

Here we replaced the differentiation by division for the argument of scaling-law. It is modeled such that all the energy stored at the length l_n and the velocity v_n is transported to the next generation or to a smaller scale by eddy breakup within the turnover time τ_n. Note that energy spectrum represents the energy density in the wave number domain. Since E_n has the units of squared velocity, the energy spectrum $E(k_n)$ must have the units of squared velocity per wave number interval Δk_n,

$$E_n = E(k_n)\Delta k_n. \tag{2.43}$$

In the third step, we use the equipartition of the energy transfer rate for turbulence in the stationary state. It is constant throughout the injection range, the inertial range, and the dissipation range:

$$\varepsilon_{\text{inj}} = \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \cdots = \varepsilon_{\text{dis}} = \text{const}. \tag{2.44}$$

Here ε_{inj} denotes the energy input in the injection range, and ε_{dis} the energy dissipated by viscosity on the smallest scale.

In the fourth step, we derive the scaling-law for velocity and energy spectrum. The energy transfer rate can be expressed by the length l_n and the velocity v_n when combined with the energy transfer time τ_n, and furthermore it is assumed to be constant: $\varepsilon_n = v_n^3/l_n = \text{const}$. This suggests that the velocity is scaled using the length as $v = (\varepsilon l)^{1/3}$. The inertial-range energy spectrum can then be derived by comparing two different expressions for fluctuation energy. One is the energy scaled to the wave number:

$$E_n = \frac{1}{2} v_n^2 = \varepsilon^{2/3} k_n^{-2/3}. \tag{2.45}$$

The other is the expression using the spectral energy density:

$$E_n = E(k_n)\Delta k_n = C_K E(k_n) k_n, \tag{2.46}$$

where C_K denotes a coefficient that cannot be determined in the argument of scaling-law alone. The former equation uses the scaling law of velocity and the latter is a definition of the energy spectrum. Here we used the relation $\Delta k \propto k$, that is the wave number interval is equidistant on the logarithmic scale, $\log(\Delta k) = \text{const}$. Hence we obtain the energy spectrum, omitting the subscript n, as

$$E(k_n) = C_K \varepsilon^{2/3} k^{-5/3}. \tag{2.47}$$

This is the inertial-range spectrum. Various experiments and observations including wind tunnel experiments, water channel experiments, and atmosphere and ocean observations confirm that Kolmogorov's energy spectrum is valid in those measurements, and therefore realization of Kolmogorov's scaling or energy spectrum is one of the central topics in studying turbulence. The coefficient C_K is called the Kolmogorov constant. Various experiments of fluid turbulence suggests that $C_K \simeq 1.6$.

Kolmogorov's inertial-range spectrum can also be derived from dimensional analysis. Energy (per unit mass) is the squared amplitude and its spectrum has the dimension energy divided by wave number, $[E] = L^3 T^{-2}$. The wave number has the dimension $[k] = L^{-1}$. The energy transfer rate has the dimension $[\varepsilon] = [v^2/\tau] = L^2 T^{-3}$. If we use *Ansatz* with a dimensionless coefficient C as

$$E(k) = C\varepsilon^{\alpha} k^{\beta}, \tag{2.48}$$

we obtain $\alpha = 2/3$ and $\beta = -5/3$ and reproduce the energy spectrum of Kolmogorov.

While Kolmogorov's phenomenology successfully describes the energy spectrum of fluid turbulence, it should in principle be possible to derive this spectrum by solving the Navier–Stokes equation directly. Such a theoretical approach leads us to the so-called closure problem of turbulence. The advection term in the Navier–Stokes equation has the quadratic form with respect to the flow velocity $\mathbf{u} \cdot \nabla \mathbf{u}$ if explicitely written, where $\mathbf{u}$ is the flow velocity, and the statistical treatment of the Navier–Stokes equation yields the following problem. Computation of the second order moments such as energy ($\propto u^2$) and helicity ($\propto \mathbf{u} \cdot \nabla \times \mathbf{u}$) requires the knowledge of the third order moments because the advection term gives the third order moments (i.e., transport of the kinetic energy) when the equation is multiplied by the velocity $\mathbf{u}$, and solving the equation for the third order moments now requires the knowledge of the fourth order moments, again because of the advection term, and the problem is repeated at ever higher order moments. There appears always a higher order term to determine moments of any order. Therefore the statistical treatment of the fluid dynamics equation cannot be not closed without using any assumptions or approximations. It is worthwhile to note that a similar closure problem happens when one derives the set of fluid equations (continuity and momentum equations at the lowest orders) from the Liouville equation by means of the velocity moment. The reason is due to the transport term in the Liouville equation, and the fluid equation system can be closed when introducing, for example, the equation of state to relate the density (the zero-th order velocity moment) by the pressure (the second-order moment). The closure problem of turbulence is about the statistics of the random fluctuating bulk velocity, and it should not be confused with the closure problem of the Liouville equation (which is about the statistics of individual particle motions). One possible remedy about the turbulence closure problem is to assume some analytical form for expressing the second order moment and close the equation at the level of the second order, for example using the concept of mixing length. This family of theories is called the one-point closure theory. Using the mean-field decomposition method separating the velocity field into a mean field part and a fluctuating field part, the turbulence effect is expressed as the Reynolds stress tensor, a three-by-three correlation tensor of the fluctuating velocity field, and the essential task in the turbulence closure problem is to find a reasonable expression of this correlation tensor. The mean-field picture of turbulence also suggests that velocity shear or gradient on a large scale serves as the driver of turbulence, injecting the kinetic energy into the system, and then the injected energy is transferred from a large scale to a small scale by the action of the nonlinear effects of the flow.

Another family of the statistical theories of turbulence involves two-point closure of the correlation tensor, in which the moments of the Navier–Stokes equations are extended to higher orders until the fourth order moments appear, and then one uses a cumulant expansion so that the fourth order moments are replaced by products of the second order moments by assuming that the fluctuating field follows almost Gaussian distribution of probability (quasi-normal approximation). By using two more additional assumptions (eddy damping and Markovian process), it is possible to integrate the Navier–Stokes equation in the statistical sense and one obtains time evolution of energy and helicity spectra. Such a treatment with the three assumptions or approximations is referred to as the EDQNM theory, abbreviation of Eddy-Damped Quasi-Normal Markovian [7, 8, 9]. Further refinement in the two-point closure theory was initiated by Kraichnan, called Direct Interaction Approximation (DIA). In this method the higher order moments are evaluated using the method of renormalized Green functions [10]. The DIA method successfully reproduces Kolmogorov's energy spectrum as well as the Kolmogorov constant without introducing any arbitrary or adjustable parameters [11]. Although the DIA method demands a lot of calculation, it is regarded as the most refined theoretical approach in turbulence studies because it does not introduce any arbitrary or free parameter. Detailed calculations are also explained in Ref. [12] and [13].

The two-point closure theories are based on the assumption that statistics of fluctuating fields (i.e., probability distribution function) almost exhibits the Gaussian distribution. This assumption is useful because higher order moments are expressed by products of second order moments. In other words, second order moments (energy and helicity) determine dynamics and structure of turbulence. On the other hand, deviation from the Gaussian distribution is also an important feature in turbulence because it is the sign of non-vanishing higher order moments (or strictly speaking, higher order cumulants) and also the sign of wave–wave (or eddy-eddy) interactions. If the probability distribution function strictly followed the Gaussian distribution then fluctuations are composed of completely random and incoherent waves in the sense that the phases of the Fourier modes are randomly distributed and turbulence cannot continue energy cascade because the third order moment which is responsible for wave–wave interactions vanishes. Deviation from the Gaussian distribution can be recognized in real fluctuating fields as sparse, spiky signals. The deviation from the Gaussian distribution may be scale dependent such that one needs to establish a correction method in the energy cascade process to account for higher order structures or fine structures. Such spiky signals or localized fine structures are referred to as the intermittency, and it may be regarded as higher order correction of the statistical theory of turbulence. There are various kinds of intermittency models. Some deal with a log-normal distribution and others deal with a log-Poisson distribution.

Fluid turbulence is ultimately a heating process that occurs after energy cascade of eddies over many generations and decades of spatial scales. From the energy conservation viewpoint , the flow kinetic energy is transported from a large scale to a smaller scale by eddy splitting and then finally the kinetic energy is converted to the thermal energy in the dissipation range. Turbulence is the property of the flow and any fluid is subject to turbulence once it is set into motion under a high Reynolds number by some external force or inhomogeneities. The concept of viscosity, in contrast, is one of the properties of the fluid and is dependent on the molecular or other particle structure in the fluid. While gases in interplanetary and interstellar space can be treated as an application of fluid mechanics and it is natural to ask if these gases also exhibit turbulence behavior, one should keep in mind that these gases are ionized and electrically conducting, which gives gas dynamics more degrees of freedom and makes turbulence behavior more complex.

2.2.2 Plasma Turbulence

While fluid turbulence can be regarded as the energy conversion problem from kinetic to thermal energy, plasma turbulence is a more complex problem because four energy forms are involved: kinetic, thermal, electric, and magnetic energy. There are physical processes in plasma that convert energy from one type into another. For example, magnetic reconnection converts energy from magnetic to kinetic; wave damping converts energy from electric and magnetic into thermal energy. Energy cascade was solely carried by eddy splitting in ordinary fluid turbulence, since eddies are the only possible oscillatory motion in incompressible fluid mechanics, but in plasmas more complex wave–wave interaction and scattering processes may be the carrier of the energy cascade, in particular, Alfvén waves can exist for large amplitudes and therefore scattering of Alfvén waves are a good candidate of energy cascade in plasma turbulence. Dissipation is also a problem in plasma turbulence, since there is no particle collision. Instead of binary collisions between particles, wave damping or wave-particle interactions may work as the dissipation mechanism.

There are a variety of theoretical approaches to MHD turbulence. Some are phenomenological models and some follow the procedure of closure theories. One of the phenomenological models of MHD turbulence is based on the assumption of the

Alfvén wave scattering as the primary energy cascade process [14, 15] and predicts that the energy spectrum is expressed as the power-law to the wave number with the index $-3/2$. Other phenomenological models use energy cascade as competition between eddy splitting and Alfvén wave scattering [16, 17]. The energy spectrum in MHD turbulence depends on the Alfvén speed, the characteristic wave propagation speed in plasma. In contrast to the eddy splitting scenario of fluid turbulence, self-similarity of energy cascade in principle fails in MHD turbulence because of the Alfvén speed determined by the large-scale quantities (magnetic field strength and mass density) is involved.

Closure theories such as the EDQNM approximation can also be applied to the MHD turbulence problem. The EDQNM model suggests that five spectra are essential to characterize MHD turbulence: kinetic and magnetic energies; kinetic, magnetic, and cross helicities. It is possible, for example, to establish turbulence with a conserved magnetic helicity and vanishing cross helicity (called the helical MHD turbulence). It is also possible to establish turbulence with conserved cross helicity and vanishing magnetic helicity (called the dynamic alignment). MHD turbulence exhibits a higher degree of freedom than fluid turbulence and therefore makes the problem challenging. To date, no direct evidence is obtained in space plasma justifying the use of the phenomenological model or the closure theories of MHD turbulence. There are various reasons for this: theories are developed on several assumptions, and in particular, the assumption of isotropic turbulence is questionable because the existence of a large-scale or mean magnetic field introduces a special direction in plasma; previous in-situ observations using spacecraft have been performed in the one-dimensional, temporal domain because of single point measurements in space and therefore observation could not distinguish between temporal and spatial variations.

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Plasma Turbulence in the Solar System
Narita, Y.
2012, VIII, 102 p. 33 illus., Softcover
ISBN: 978-3-642-25666-0