

Chapter 2

Computation

2.1 Introduction

Very often, the ability to generate quantitative information on specific systems is central to the analysis and design in the investigation of science and engineering systems. Applied analysts and designers of complex systems rely heavily on numerical solutions to the mathematical models representing a given physical system under study. This leads to the task of computation, which must be accurate and efficient.

It is well known that there are many problems in the analysis of dynamic systems which are not amenable to exact solutions. Indeed, this is the usual case, and one seldom finds a real problem to be *exactly* solvable. Thus the only recourse is to develop methods to approximate the behavior of the system under study. Even this relatively modest goal is often found to be elusive.

The mathematical models are usually in terms of differential equations, and there is a great need to solve them in order to investigate the dynamic behavior of the systems. However, these equations are not, in general, *exactly* solvable, and therefore, it is necessary to resort to approximations. For example, the notion of a derivative of a function, which involves the concept of a mathematical limit, is replaced by a finite difference operation as an approximation. The differential equation is thus replaced by an equation involving finite differences. This latter model is solved with the expectation that its solution represents the solution of the original differential equation as the differential step size becomes smaller and smaller. While this expectation is not fulfilled with absolute rigor in all cases, this approach works well for the most part and forms the backbone of quantitative analysis of engineering systems. In particular, it does not matter whether an approximate problem is solved *exactly* or an exact problem is solved *approximately*.

As we go deeper into this task, we soon realize that the ease and success of computation is influenced strongly by the nature of the phenomenon being studied. In particular, the rate at which the system behavior unfolds has a crucial bearing on the ability to compute. There are many dynamic phenomena in which the system behavior is *nonuniform*, i.e., varies in different parts of the domain of interest.

For example, the velocity of a fluid changes rapidly near a solid boundary because of viscosity, resulting in a *boundary layer* and *slowly* in a region far away from the boundary. Another classic example is that of the motion of a smaller planet orbiting around a larger body in a highly elliptic orbit. The coordinates of the orbiting planet change rapidly near the periapsis (eg. the perigee) and slowly in the other parts of the orbit. In such cases, the computational step size needs to be small in regions of rapid changes and larger in other regions in order to describe the behavior accurately. Such a behavior is characterized as a *multiple scale* system or a *stiff* system. This type of behavior is also called an *asymptotic phenomenon* [1] and gives rise to incongruities and nonuniformities in a simplistic description of the phenomenon. Such systems require special methods and care in developing either numerical or analytical approximations to represent the system behavior accurately.

The philosophy and methodology discussed in this book are based on a general approach to the analysis of complex dynamic systems as embodied in the Method of Multiple Scales [2, 3]. In this approach, the independent variable, usually *time*, is extended into a higher dimensional space. Thus, we consider new time scales, i.e., *fast time*, *slow time*, etc., as generated from the original time variable, i.e., *real time*. The different aspects of the dynamic phenomena are now described with respect to each of the new time scales, resulting in a composite description of the complex dynamics. The primary advantage is that the description with respect to each new time scale is obtained more easily than the description in the original time. However, the success of the approach is predicated on approximation theory and, in particular, on the concept of asymptotic analysis. It is, therefore, necessary to present an introduction to the ideas and methods of asymptotic analysis.

Utilizing the great advances in high-speed computer technology, it is tempting to combine the ideas and techniques of asymptotic analysis with numerical methods to develop powerful methods for computation. With the advent of high-speed computers two aspects of systems analysis can be recognized. First, the exact analytical solutions, if available, will enable the analyst to predict the qualitative and quantitative features of the phenomenon being studied. On the other hand, exact solutions may be impossible, but accurate approximate solutions are often achievable. Thus, one may obtain numerical solutions to specific problems which may be analytically intractable.

Historically, there has been a great deal of interest in the computational aspects in physical applications. Many great mathematicians like Newton and Gauss were interested in computation and devoted considerable effort to the task of computation. Newton developed many numerical methods still in use today, and Gauss spent considerable time and effort in computing the trajectories of celestial bodies. In contrast, some of these tedious computations could be done readily by a personal computer today! This underscores the fact that until recently the lack of high-speed computational capability forced mathematicians and analysts to resort to clever approximation methods in order to analyze the complex dynamics of physical systems. These ingenious and highly efficient analytical methods, in conjunction with high-speed computers, lead to extremely useful approaches for solving complex problems. In general, approximations yield qualitative insight on such questions as

system stability, response, parameter sensitivity, etc. However, when one wishes to obtain a solution to a specific problem which is analytically intractable, then one usually resorts to the use of a computer. A great wealth of technical literature exists in both areas. In this work, an attempt is presented to combine asymptotic methods with numerical computation.

In the investigation of complex engineering systems the analyst and designer have found it necessary to employ sophisticated methods to solve the various problems. For example, in the field of flight vehicles, the development of high performance systems such as aircraft with large flight envelopes, vertical and short take-off and landing (V/STOL) vehicles and space vehicles requires elaborate and detailed study to meet the performance requirements. Physical intuition and experience as well as computer simulations usually provide answers to many of the questions. However, in order to glean physical insight and better understand the system behavior, one needs to *simplify* the mathematical representation of the system so that conventional methods are applicable. On the other hand, computers provide *specific* answers to *specific* questions.

Assuming that linear systems with constant coefficients can be solved exactly, the analyst and designer have relied on such models to investigate measures of system behavior and performance such as stability, response and design criteria. They have been successfully applied to many problems. Indeed, this approach is in common use, and is the first treatment of choice. The success of the simple approach rests on the validity of approximating a complex physical system by one which is tractable. However, as the systems become more complex in an effort to achieve greater performance capability, situations arise where simplistic representations are no longer adequate. More sophisticated methods then become necessary. Several such methods are available, and are constantly undergoing further development and refinement, but most of these are not well known. As the problems amenable to these techniques are widely prevalent, adequate dissemination and knowledge of such techniques would prove to be extremely useful.

In systems analysis, approximations are widely used, though implicitly in many cases. However, a systematic use of rigorous approximation theory offers much greater rewards in general, than *ad hoc* approaches. A systematic approach to approximation theory can be found in the subject of *asymptotics*, -the branch of mathematics that deals with the study of applied mathematical systems in *limiting* cases. A substantial body of knowledge exists in this field, in both the rigorous and applied senses.

Asymptotic methods have been successfully applied to many problems in many fields such as fluid mechanics, astrodynamics, and physics. However, the fields of dynamic systems analysis, control and computation have not largely made good use of these powerful techniques. It is the purpose of this work to develop a general concept and technique in applied mathematics and to discuss its wide applicability on various problems of analysis and computation in modern engineering systems. The subject of asymptotics traditionally concerns a class of phenomena in a somewhat unclear region of overlap between applied mathematics and physics. There is a vast literature, rich in techniques and results, both in the classical problems of mathematical physics and in the more applied problems of science and advanced engineering. The subject

is deep enough to satisfy the need for strict rigor of the most demanding analyst and yet is suitable for practical problem solving. In recent times, it is this latter feature that is assuming greater significance in the analysis of complex engineering systems.

We will now address some basic concepts in approximation theory.

2.2 Convergence and Computation

Asymptotic approximations usually involve power series. The following well-known classical examples may be considered in the context of computation.

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots \quad (2.1)$$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \quad (2.2)$$

$$\frac{1}{(1-t)} = 1 + t + t^2 + t^3 + \cdots \quad |t| < 1 \quad (2.3)$$

The sum of these series is finite and they converge to the function being represented. It is believed that the term “convergent series” was first introduced by J. Gregory in 1667. Rigorous general tests for convergence were not developed until much later, through the works of D’Alembert (1768), Cauchy (1821), Abel (1826), Raabe (1832), DeMorgan (1842), and Dirichlet (1862). These led to the practice of defining functions by series, still used today. In fact, solutions to differential equations are often developed in series form such as those using Frobenius’ method. The series representations suggest the possibility of using the truncated series to calculate the function approximately. While this approach works rather well with some series, it is notoriously inadequate with others. The practical success of such a scheme depends critically on the *speed* or rate of convergence. For the above examples, for small values of t ($0 < t < 1$), the convergence is quite rapid and the truncated series provides a good approximation of the function. For large values of t the truncated series does not lead to a good approximation and Eq. (2.3) is not even valid for $t > 1$. Further, consider the following representations.

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \quad (2.4)$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad (2.5)$$

$$\frac{1}{\sqrt{2}} = 1 - \frac{1}{2(1!)} + \frac{1(3)}{2^2(2!)} - \frac{1(3)(5)}{2^3(3!)} + \cdots \quad (2.6)$$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n^s}; \quad \text{Re}(s) > 1 \quad (2.7)$$

All these series are known to be convergent but their rate of convergence is so slow that they cannot be used directly to calculate the quantity approximately to good accuracy. For instance, Leibnitz's series for π as given by Eq. (2.4) requires 40 million terms for an accuracy of eight significant figures and the tenth partial sum is good only to one significant figure ! Similarly, the other series given above are computationally unsuitable. To compute $\ln 2$ with an error less than half the fifth significant figure, more than 100,000 terms are needed. The convergence of Eq. (2.6) is so slow that nearly 10^{10} terms are required to compute $1/\sqrt{2}$ to an accuracy of 10^{-5} . The famous Riemann Zeta function $\zeta(s)$, so useful in the analytic theory of numbers, converges for $\text{Re}(s) > 1$ and diverges for $\text{Re}(s) \leq 1$. Near $\text{Re}(s) = 1$, the convergence is extremely slow. To compute $\zeta(1.1)$ for instance, with an error less than 1%, more than 10^{20} terms are needed, an impossible task even with modern high speed computers!

In contrast, consider the example first discussed by Euler,

$$S(x) = 1 - 1!x + 2!x^2 - 3!x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n n! x^n \quad (2.8)$$

Clearly, the series diverges for all $x \neq 0$. Yet for small x (say of order 10^{-2}) the terms at first decrease rapidly. It can be shown that the first few terms of the series provide a very good approximation (for small x) of the function

$$f(x) = \int_0^{\infty} \frac{e^{-t}}{(1+xt)} \quad (2.9)$$

Similarly the function

$$Ei(x) = \int_{-\infty}^x e^t t^{-1} dt; \quad x < 0 \quad (2.10)$$

can be represented by the series

$$Ei(x) = e^x x^{-1} \left(1 + \frac{1}{x} + \frac{2!}{x^2} + \cdots + \frac{m!}{x^m} + \cdots \right) \quad (2.11)$$

Clearly this series diverges for *all* finite values of x as m increases.

Finally, consider one of the earliest and celebrated examples of the Stirling series for the factorial function $(n-1)! = \Gamma(n)$ which is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt; \quad \text{Re}(z) > 0 \quad (2.12)$$

Table 2.1 Comparison of Taylor and Stirling series

N	z = 1; 0!	z = 1; 0!	z = 2; 1!	z = 2; 1!	z = 4; 3!	z = 4; 3!
	Taylor Srs	Stirling Srs	Taylor Srs	Stirling Srs	Taylor Srs	Stirling Srs
1	1.0000000	0.9989817	0.5000000	0.9994814	0.2500000	5.9989717
2	0.631028	1.0021036	0.2307980	1.0003143	0.0755546	6.0002470
3	1.0053784	0.9997110	-1.0659140	0.9999927	-0.0347938	6.0000008
4	1.1372230	0.9994994	-0.6210392	0.9999789	-0.02532067	5.9999956
5	0.9561385	1.0002224	0.2689872	1.0000024	0.0076113	6.0000001
6	0.9963376	1.0002867	0.9819551	1.0000035	-0.0239235	6.0000002
7	1.0059817	0.9997401	-4.6896423	0.9999990	-0.0050138	5.9999999
	-	-	-	-	-	-
	-	-	-	-	-	-
∞	1.00000000		1.00000000		6.00000000	

This has the series representation

$$(n-1)! \sim \left(\sqrt{\frac{2\pi}{n}} \right) (n/e)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots \right) \quad (2.13)$$

Although this series diverges for all n , it provides a very accurate approximation of $(n-1)!$ for large values of n . Indeed, it is remarkable that even when n is not large the approximation is still very good. For example, the first term of the Stirling series for $3!$ yields 5.9989 in comparison with the exact answer of 6. And yet the series was developed as an approximation for $n \rightarrow \infty$. The extreme accuracy of the Stirling series even for small values of $n!$ such as $1!$ and $3!$ is illustrated in the following table generated as a numerical experiment. Table 2.1 shows a term-by-term comparison of the rates of convergence of the partial sums of the Taylor series for $\frac{1}{\Gamma(z)}$ and the Stirling series for $\Gamma(z)$ for $z = 1, 2, 4$. Note that the Taylor series ultimately converges to the correct value, but it requires a very large number of terms (actually infinite terms)! The Stirling series, however, is far more useful in numerical computations. The usefulness of such a series was noticed by Stirling and others. The first rigorous foundation of these “asymptotic series” and their interpretation is due to H. Poincaré in 1886.

As already noted, the series representation for π given by Leibnitz converges so slowly that forty million terms are needed to compute π to an accuracy of eight significant figures! However, the series developed by the Indian mathematical genius Srinivasa Ramanujan converges so rapidly that π can be computed to eight significant figures by the very *first term* of the series! Indeed, every successive term of Ramanujan’s series increases the computational accuracy by eight digits. This series is given by

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n!)(1103 + 26390n)}{(n!)^4 396^{4n}} \quad (2.14)$$

According to mathematicians, this is the fastest converging series for π . The incredible accuracy of the series depends crucially on the specific numbers in the series. For example, it has been proved that the number 1103 makes the series true to within one part in $10^{10,000,000}$ [4]. It has been said that Ramanujan-type algorithm for approximating π can be shown to be very close to the best possible. It was not known for a long time how Ramanujan came up with the correct numbers as he did not give any proofs. It has since been established that the numbers were obtained as singular solutions of modular equations of which he was a master. The interested reader can refer to the sources cited.

Ramanujan's is a remarkable story. He was born in a poor family in Southern India in 1887. He did not complete high school as he did not pass the subject of English, which was required for matriculation in British-ruled India. However, he was a mathematical genius of the highest order, and worked entirely on his own, as he did not receive a formal education in mathematics. Therefore, much of his profound work is rediscovery. He sent a humble letter containing some results he had derived, to G.H.Hardy, the world famous number theorist at Cambridge University in England. Upon seeing this, Hardy recognized genius and arranged for Ramanujan to travel to England. They collaborated from 1914-1919 and these were years of the First World War. As recounted by Hardy, during the war time conditions, life was hard and Ramanujan became seriously ill and was hospitalized. Hardy said that he visited Ramanujan at the hospital and remarked that the taxi in which he rode had an uninteresting number. They were two number theorists conversing, and Ramanujan asked what the number was. Hardy replied that it was 1729 for which he did not see any interesting properties. Ramanujan said it was one of the most interesting numbers. When Hardy asked why, Ramanujan said, "Don't you see, it is the smallest number that can be expressed as the sum of two cubes in two different ways". That is, $1729 = 9^3 + 10^3 = 12^3 + 1^3$. Soon after, Ramanujan returned to India and died in 1920 at the age of thirty three!

In 1985, Ramanujan's formula was used to compute π to 17.5 million digits, in 1986, to 29 million digits, in 1987 to 134 million digits, and so on. Using this formula in the year 2002, the Japanese computed π on a Hitachi supercomputer, to 1.24 trillion decimal places ! One may wonder about the need for determining π to such a large number of decimal places. In fact, calculating π to the 39th digit is sufficient to compute the radius of the known universe to an accuracy of the order of the radius of the hydrogen atom ! One whimsical reason to compute π is "because it is there", like the need to climb a mountain. A more serious reason might be that an accurate knowledge of π is helpful in solving inverse problems by using π as a benchmark.

In fact, Ramanujan has given fourteen other representations for computing π . One of these is the infinite continued fraction

$$\frac{\pi}{4} = \frac{1}{1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \dots}}}}} \quad (2.15)$$

Ramanujan was one of the few mathematicians, along with Gauss, who were skilled in continued fractions. According to the mathematician Berndt, “In this area, Ramanujan is probably unsurpassed in all of mathematical history” [4, 5].

Stirling [6] was one of the first to use asymptotics in giving a representation of the factorial function. MacLaurin and Euler in the eighteenth century were also among the earliest in employing asymptotic series. In the nineteenth century, Poincaré [7, 8] was pre-eminent in establishing the rigorous mathematical foundations of asymptotic analysis. In current practice, following Poincaré, the term *asymptotic series* has replaced the older *semi-convergent* series used by Stieltjes [9]. There has been a tremendous growth in the development of asymptotics in the twentieth century, with a vast literature. A large number of techniques have been developed and the subject abounds with applications in diverse fields of investigation. It is, therefore, impossible to do justice to the large number of contributors. The prominent contributions are cited in the references.

In the field of simulation and computation of the dynamics of engineering systems, the use of asymptotic approximations has resulted in great success not only in describing the phenomenology but also in efficiency in computation. This is especially true of *stiff* systems which are characterized by the existence of different rates of change in the dynamics. In such problems the conventional numerical and analytical methods of approximation encounter considerable difficulty. Special approaches are then necessary. It is precisely this class of problems that is amenable to asymptotic methods such as the Multiple Scales approach, which will be discussed later. There are many examples of asymptotic phenomena which cover a wide range of problems, from the very practical to the highly theoretical applied analysis. We may now consider some examples of problems in the analysis of dynamic systems which are amenable to approximation theory. In linear time-invariant (LTI) systems an important problem is that of widely separated eigenvalues. The motion of flight vehicles is a combination of the different aspects of rigid body modes, actuator dynamics, structural modes, etc., all of which are well separated in frequency. Such a frequency separation could itself be the basis of approximation in an asymptotic sense. It would show at what level and to what extent the different modes would begin to contribute to the total motion.

The earliest systematic approach using asymptotic analysis was motivated by the motion of heavenly bodies. Indeed, Poincaré’s *les Methodes Nouvelles de la Mecanique Celeste* [7] established the firm foundations of asymptotic analysis and demonstrated its success on problems in celestial mechanics. In astrodynamics, there are many dynamic phenomena which exhibit a mixture of rapid and slow variations. For example, the motion of an earth satellite under the influence of gravity of an oblate earth shows a fast variation of the coordinates around the orbit, while the orbit itself shows a slow rotation. Other perturbations such as gravity due to the moon and sun, atmospheric drag or geomagnetic field, occur at different rates. These effects must be included in computing the orbits accurately. In the motion of conventional aircraft one can see that symmetric motions are characterized by the fast *short period* mode and the slower *phugoid* mode. In aerodynamics, boundary layer theory of Prandtl

[1] is a well-known example of multiple scale behavior. Fast changes in the fluid velocity occur inside the boundary layer, while slow changes occur outside it.

Recognition of the existence of different rates in a dynamic system is often useful in computation. For example, in the computation of highly elliptic orbits of an earth satellite, the integration step size has to be small enough to describe fast changes near the perigee, while larger steps would suffice at other parts of the orbit, leading to a nonuniform step size situation.

These are but a few examples of the presence of mixed dynamics at different rates. Most dynamic phenomena exhibit such a mixture and have been so recognized in many fields such as chemical reactor kinetics, economic systems, and so on. In these systems, it would be highly desirable to make use of the presence of fast and slow variations in order to separate them. The rewards to be gained are many, including enhanced insight and computational facility in addition to the development of constructive solutions for many important problems of analysis and design in engineering. The Multiple Time Scales (MTS) method is naturally suited to exploit the different time scales inherent in the system dynamics. In Chap. 5 and later chapters, the MTS method is extended into the Generalized Multiple Scales (GMS) method. It has been developed and applied to many problems involving linear time-varying and nonlinear differential equations in advanced engineering systems (see references).

Another common problem is that involving linear time-varying (LTV) systems. For example, in the analysis of flight vehicle dynamics through variable flight conditions, the usual approach is to treat the time variation of the system parameters as “slow”, and study the LTV system as a sequence of LTI systems. This approach is widely prevalent among dynamicists. For example, the motion of re-entry vehicles in the earth’s atmosphere, the dynamics of VTOL vehicles during transitions from hover to forward flight, etc., have been so treated, in terms of a number of LTI systems. At steady flight, the equations of motion are indeed time-invariant, and useful results have been obtained in the study of airplane motion. However, such an approximation, while it can be useful, has a limited range of validity. Because the approximation is valid only for very short times, a number of such solutions must be patched. This opens up questions on the actual number of such patched solutions and the boundary conditions at the instants of patching, errors of the approximation, etc., besides the tedium of patching the different solutions. Then there is the rather fundamental question on the slowness of time variation,—i.e., “*how slow is slow?*” The conclusions on stability of an LTV system from its LTI or *frozen* approximation can be quite erroneous. It can be shown that the frozen approximation is a rather simplistic asymptotic representation. For slowly varying systems a much more accurate asymptotic theory can be developed, in which the approximations are uniformly valid over long intervals [2]. LTI systems arise as a special case of this general result. A substantial amount of work exists on this subject. However, these powerful techniques are not widely known and not well exploited in engineering.

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