Chapter 2
D-Branes and Orientifolds

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2.1 The Free Boson with Boundaries

2.1.1 Boundary Conditions

We start by discussing the Boundary Conformal Field Theory of the free boson theory in order to illustrate the appearance of boundaries from a Lagrangian and geometrical point of view.[1]

Conditions for the Fields

The two-dimensional action for a free boson $X(\tau, \sigma)$ is given by

$$\mathscr{S} = \frac{1}{4\pi} \int d\sigma \; d\tau \; \Big(\big(\partial_\sigma X\big)^2 + \big(\partial_\tau X\big)^2 \Big). \tag{2.1}$$

Note that we fixed the overall normalisation constant and we slightly changed our notation such that $\tau \in (-\infty, +\infty)$ denotes the two-dimensional time coordinate and $\sigma \in [0, \pi]$ is the coordinate parametrising the distance between the boundaries.

The variation of the action (2.1) is obtained in the usual way, but now with the boundary terms taken into account. More specifically, we compute the variation as follows

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[1] This lecture is based on one chapter in the lecture notes [1]. Some relevant general references can be found in there. For a good guide through the vast literature we refer to the review article [2].

M. Baumgartl et al. (eds.), *Strings and Fundamental Physics*,
Lecture Notes in Physics 851, DOI: 10.1007/978-3-642-25947-0_2,

$$\begin{aligned}\delta_X\,\mathscr{S} &= \frac{1}{\pi}\int d\sigma\; d\tau\,\Big(\big(\partial_\sigma X\big)\,\big(\partial_\sigma \delta X\big) + \big(\partial_\tau X\big)\,\big(\partial_\tau \delta X\big)\,\Big)\\ &= \frac{1}{\pi}\int d\sigma\; d\tau\,\Big(-\big(\partial_\sigma^2+\partial_\tau^2\big)X\cdot\delta X + \partial_\tau\big(\partial_\tau X\cdot\delta X\big) + \partial_\sigma\big(\partial_\sigma X\cdot\delta X\big)\Big).\end{aligned} \tag{2.2}$$

The equation of motion is obtained by requiring this expression to vanish for all variations δX. The vanishing of the first term in the last line leads to $\Box X = 0$ which we already obtained previously. The remaining two terms can be written as follows

$$\begin{aligned}&\frac{1}{\pi}\int d\sigma\; d\tau\,\Big(\,\partial_\tau\big(\partial_\tau X\cdot\delta X\big) + \partial_\sigma\big(\partial_\sigma X\cdot\delta X\big)\Big)\\ &= \frac{1}{\pi}\int d\sigma\; d\tau\;\nabla\cdot\Big(\nabla X\,\delta X\,\Big)\\ &= \frac{1}{\pi}\int_{\mathscr{B}} dl_{\mathscr{B}}\,\big(\nabla X\cdot\mathbf{n}\big)\,\delta X\end{aligned}$$

where we introduced $\nabla = (\partial_\tau, \partial_\sigma)^T$ and used Stokes theorem to rewrite the integral $\int d\sigma d\tau$ as an integral over the boundary $\mathscr{B}$. Furthermore, $dl_{\mathscr{B}}$ denotes the line element along the boundary and $\mathbf{n}$ is a unit vector normal to $\mathscr{B}$. In our case, the boundary is specified by $\sigma = 0$ and $\sigma = \pi$ so that $\mathbf{n} = (0, \pm 1)^T$ as well as $dl_{\mathscr{B}} = d\tau$. The vanishing of the last two terms in (2.2) can therefore be expressed as

$$0 = \frac{1}{\pi}\int d\tau\,\big(\partial_\sigma X\big)\,\delta X\,\Big|_{\sigma=0}^{\sigma=\pi}.$$

This equation allows for two different solutions and hence for two different boundary conditions. The first possibility is a Neumann boundary condition given by $\partial_\sigma X|_{\sigma=0,\pi} = 0$. The second possibility is a Dirichlet condition $\delta X|_{\sigma=0,\pi} = 0$ for all τ which implies $\partial_\tau X|_{\sigma=0,\pi} = 0$. In summary, the two different boundary condition for the free boson theory read as follows

$$\boxed{\begin{array}{rll} \partial_\sigma X|_{\sigma=0,\pi} &= 0 & \text{Neumann condition,}\\ \delta X|_{\sigma=0,\pi} &= 0 = \partial_\tau X|_{\sigma=0,\pi} & \text{Dirichlet condition.}\end{array}} \tag{2.3}$$

Remark

Let us remark that in string theory, a hypersurface in space-time where open strings can end is called a D-brane. In order to explain this point, let us consider a theory of N free bosons $X^\mu(\tau,\sigma)$ with $\mu = 0,\ldots,N-1$ which describe the motion of a string in an N-dimensional space-time. We organise the fields in the following way

$$\Bigg(\underbrace{X^0, X^1, \ldots, X^{r-1}}_{\text{Neumann conditions}},\ \underbrace{X^r, \ldots, X^{N-1}}_{\text{Dirichlet conditions}}\Bigg),$$

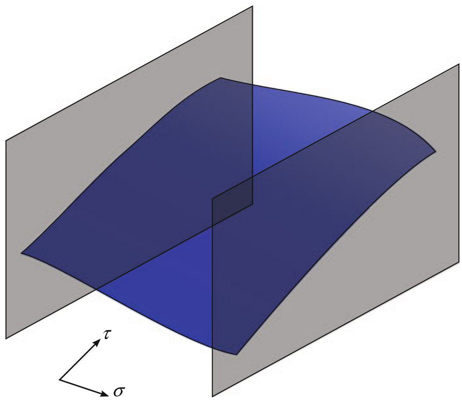

Fig. 2.1 Two-dimensional surface with boundaries which can be interpreted as an open string world-sheet stretched between two D-branes

where r denotes the number of bosons with Neumann boundary conditions leaving $(N - r)$ bosons with Dirichlet conditions.

Let us now focus on one endpoint of the open string, say at $\sigma = 0$. A Dirichlet boundary condition for X^μ reads $\delta X^\mu|_{\sigma=0} = 0$ which means that the endpoint of the open string is fixed to a particular value $x_0^\mu = \text{const}$. However, in case of Neumann boundary conditions, there is no restriction on the position of the string endpoint which can therefore take any value. Clearly, since the string moves in time, there are Neumann conditions for the time coordinate X^0. Then, the r-dimensional hypersurface in space-time described by $X^\mu = x_0^\mu = \text{const.}$ for $\mu = r, \ldots, N-1$ is called a D$(r-1)$-brane where D stands for Dirichlet.

As an example, take $N = 3$ and consider Fig. 2.1 where we see a world-sheet of an open string stretched between two D1-branes.

Conditions for the Laurent Modes

Above, we have considered the BCFT in terms of the real variables (τ, σ) which was convenient in order to arrive at (2.3). However, for more advanced studies a description in terms of complex variables is very useful. Similarly as before, a mapping from the infinite strip described by the real variables (τ, σ) to the complex upper half plane H^+ is achieved by $z = \exp(\tau + i\sigma)$. Note in particular, as illustrated in Fig. 2.2, the boundary $\sigma = 0, \pi$ is mapped to the real axis $z = \bar{z}$.

Having this map in mind, we can express the boundary conditions (2.3) for the field $X(\sigma, \tau)$ in terms of the corresponding Laurent modes. Recalling that $j(z) = i\,\partial X(z, \bar{z})$, we find

$$\partial_\sigma X = i\big(\partial - \bar{\partial}\big)X = j(z) - \bar{j}(\bar{z}) = \sum_{n\in\mathbb{Z}} \big(j_n\, z^{-n-1} - \bar{j}_n\, \bar{z}^{-n-1} \big),$$

$$i \cdot \partial_\tau X = i\big(\partial + \bar{\partial}\big)X = j(z) + \bar{j}(\bar{z}) = \sum_{n\in\mathbb{Z}} \big(j_n\, z^{-n-1} + \bar{j}_n\, \bar{z}^{-n-1} \big),$$

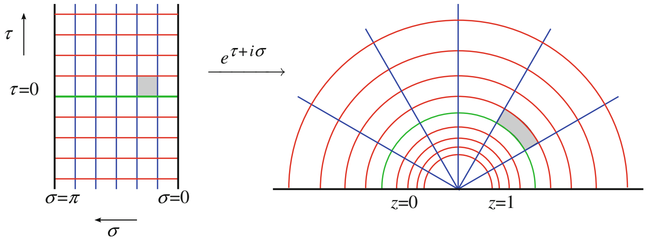

Fig. 2.2 Illustration of the map $z = \exp(\tau + i\sigma)$ from the infinite strip to the complex upper half plane H^+

where we used $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$ and $\overline{\partial} = \frac{1}{2}(\partial_0 + i\partial_1)$. For transforming the right-hand side of these equations as $z \mapsto e^w$ with $w = \tau + i\sigma$, we employ that $j(z)$ is a primary field of conformal dimension $h = 1$. In particular, recalling that conformal transformations act on primary fields of dimension $(h, \overline{h})$ as

$$\phi(z, \overline{z}) \mapsto \phi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi\left(f(z), \bar{f}(\bar{z})\right), \tag{2.4}$$

we have $j(z) = \left(\frac{\partial z}{\partial w}\right)^1 j(w) = z\, j(w)$ leading to

$$\begin{aligned} \partial_\sigma X &= \sum_{n\in\mathbb{Z}} \left(j_n\, e^{-n(\tau+i\sigma)} - \bar{j}_n\, e^{-n(\tau-i\sigma)} \right), \\ i \cdot \partial_\tau X &= \sum_{n\in\mathbb{Z}} \left(j_n\, e^{-n(\tau+i\sigma)} + \bar{j}_n\, e^{-n(\tau-i\sigma)} \right). \end{aligned} \tag{2.5}$$

The Neumann as well as the Dirichlet boundary conditions at $\sigma = 0$ are then easily obtained as

$$\begin{aligned} \partial_\sigma X \,\big|_{\sigma=0} &= \sum_{n\in\mathbb{Z}} (j_n - \bar{j}_n)\, e^{-n\,\tau} = 0, \\ \partial_\tau X \,\big|_{\sigma=0} &= \sum_{n\in\mathbb{Z}} (j_n + \bar{j}_n)\, e^{-n\,\tau} = 0. \end{aligned}$$

Since for generic τ the summands above are linearly independent, these two equations are respectively solved by $j_n \pm \bar{j}_n = 0$ for all n. In summary, we note that boundaries introduce relations between the chiral and anti-chiral modes of the conformal fields which read

$$\boxed{\begin{array}{lll} j_n - \bar{j}_n = 0 & & \text{Neumann condition,} \\ j_n + \bar{j}_n = 0, & (\pi_0 = 0) & \text{Dirichlet condition.} \end{array}} \tag{2.6}$$

From a string theory point of view, (2.6) implies that an open string has only half the degrees of freedom of a closed string.

A computation of the center of mass for the open string gives

$$\pi_0 = \frac{1}{2}\, j_0 = \frac{1}{2}\, \bar{j}_0. \tag{2.7}$$

In view of (2.6), we thus see that there are no restrictions on π_0 for Neumann boundary conditions and so the endpoints of the string are free to move along the D-brane. For Dirichlet conditions on the other hand, we have $\pi_0 = 0$ implying that the endpoints are fixed.

Combined Boundary Condition

In the previous paragraph, we have considered the boundary at $\sigma = 0$. Let us now turn to the other boundary at $\sigma = \pi$. Performing the same steps as before, we see that Neumann–Neumann as well as Dirichlet–Dirichlet conditions are characterised by the constraints found in (2.6).

However, mixed boundary conditions, e.g. Neumann-Dirichlet, require a modification. In particular, $j_n - \bar{j}_n = 0$ at $\sigma = 0$ and $j_n + \bar{j}_n e^{-2in\sigma} = 0$ at $\sigma = \pi$ can only be solved for $n \in \mathbb{Z} + \frac{1}{2}$. All possible combinations of boundary conditions are then summarised as

$$\begin{array}{lll} j_n - \bar{j}_n = 0, & n \in \mathbb{Z} & \text{Neumann–Neumann,} \\ j_n - \bar{j}_n = 0, & n \in \mathbb{Z} + \dfrac{1}{2} & \text{Neumann–Dirichlet,} \\ j_n + \bar{j}_n = 0, & n \in \mathbb{Z} + \dfrac{1}{2} & \text{Dirichlet–Neumann,} \\ j_n + \bar{j}_n = 0, & n \in \mathbb{Z} & \text{Dirichlet–Dirichlet.} \end{array}$$

Solutions to the Boundary Condition

Next, let us determine the solutions to the boundary conditions stated above. First, we integrate equations (2.5) to obtain $X(\tau, \sigma)$ in the closed sector

$$X\big(\tau, \sigma\big) = x_0 - i\,\big(\tau + i\sigma\big)\, j_0 - i\,\big(\tau - i\sigma\big)\, \bar{j}_0 + \sum_{n \neq 0} \frac{i}{n} \Big(j_n e^{-n(\tau + i\sigma)} + \bar{j}_n e^{-n(\tau - i\sigma)} \Big) \tag{2.8}$$

where x_0 is an integration constant. We then implement the boundary conditions to project onto the open sector. For the Neumann–Neumann case we find

$$X^{(\mathrm{N,N})}\big(\tau, \sigma\big) = x_0 - 2i\tau j_0 + 2i \sum_{n \neq 0} \frac{j_n}{n}\, e^{-n\,\tau} \cos\big(n\sigma\big),$$

and for the Dirichlet–Dirichlet case we obtain along the same lines

$$X^{(\mathrm{D,D})}\big(\tau,\sigma\big) = x_0 + 2\sigma j_0 + 2\sum_{n\neq 0}\frac{j_n}{n}\,e^{-n\,\tau}\,\sin\big(n\sigma\big).$$

Having arrived at this solution, we can become more concrete about the Dirichlet–Dirichlet boundary conditions. We impose that $X(\tau,\sigma=0)=x_0^a$ and $X(\tau,\sigma=\pi)=x_0^b$, which means that the endpoints of the string are fixed at positions x_0^a and x_0^b. Using the explicit solution for $X^{(\mathrm{D,D})}(\tau,\sigma)$, we obtain the relation

$$\boxed{j_0 = \frac{x_0^b - x_0^a}{2\pi}}\,. \tag{2.9}$$

Finally, for completeness, the solutions for the case of mixed Neumann–Dirichlet boundary conditions read as follows

$$X^{(\mathrm{N,D})}\big(\tau,\sigma\big) = x_0 + 2\,i\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{j_n}{n}\,e^{-n\,\tau}\cos\big(n\sigma\big),$$
$$X^{(\mathrm{D,N})}\big(\tau,\sigma\big) = x_0 + 2\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{j_n}{n}\,e^{-n\,\tau}\sin\big(n\sigma\big)\ .$$

Conformal Symmetry

Let us remark that equations (2.6) apply to the Laurent modes of the two $U(1)$ currents $j(z)$ and $\bar{j}(\bar{z})$ of the free boson theory leaving only a diagonal $U(1)$ symmetry. However, in addition there is always the conformal symmetry generated by the energy-momentum tensor. Since boundaries in general break certain symmetries, we expect also restrictions on the Laurent modes of energy-momentum tensor.

Indeed, recalling that $T(z)$ and $\overline{T}(\bar{z})$ can be expressed in terms of the currents $j(z)$ and $\bar{j}(\bar{z})$ in the following way

$$T(z) = \frac{1}{2}\,N\big(jj\big)(z), \qquad \overline{T}(\bar{z}) = \frac{1}{2}\,N\big(\bar{j}\,\bar{j}\big)(\bar{z}),$$

we find that the Neumann as well as the Dirichlet boundary conditions (2.6) imply for $L_n = \frac{1}{2}N(jj)_n$ that

$$\boxed{L_n - \overline{L}_n = 0}\,. \tag{2.10}$$

Let us emphasise that this condition can be expressed as $T(z)=\overline{T}(\bar{z})$ which in particular means the central charges of the holomorphic and anti-holomorphic theories have to be equal, i.e. $c=\bar{c}$. For string theory, this observation has the immediate implication that boundaries, that is D-branes, can only be defined for the Type II Superstring Theories, as opposed to the heterotic string theories.

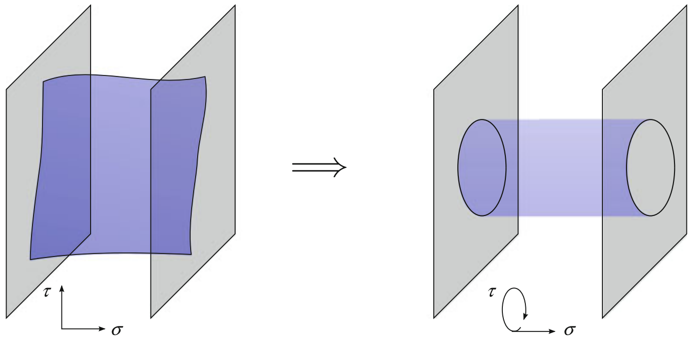

Fig. 2.3 Illustration how the cylinder partition function is obtained from the infinite strip by cutting out a finite piece and identifying the ends

2.1.2 Partition Function

Definition

Let us now consider the one-loop partition function for BCFTs. To do so, we first review the construction for the case without boundaries and then compare to the present situation.

- The one-loop partition function for CFTs without boundaries is defined as follows. We start from a theory defined on the infinite cylinder described by (τ, σ) where σ is periodic and $\tau \in (-\infty, +\infty)$. Next, we impose periodicity conditions also on the time coordinate τ yielding the topology of a torus. The partition function is then determined as

$$\mathscr{Z}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathscr{H}}\big(q^{L_0 - \frac{c}{24}} \overline{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\big). \tag{2.11}$$

- In the present case, the space coordinate σ is not periodic and thus we start from a theory defined on the infinite strip given by $\sigma \in [0, \pi]$ and $\tau \in (-\infty, +\infty)$. For the definition of the one-loop partition function, we again make the time coordinate τ periodic leaving us with the topology of a cylinder instead of a torus. This is illustrated in Fig. 2.3.
- Similarly to the modular parameter of the torus, there is a modular parameter t with $0 \leq t < \infty$ parametrising different cylinders. The inequivalent cylinders are described by $\{(\tau, \sigma) : 0 \leq \sigma \leq \pi, 0 \leq \tau \leq t\}$.

For the partition function, we need to determine the operator generating translations in time circling the cylinder once along the τ direction. Because boundaries lead to an identification of the left- and right-moving sector as required by (2.10), we see that this operator is the Hamiltonian say in the open sector

$$H_{\text{open}} = (L_{\text{cyl.}})_0 = L_0 - \frac{c}{24},$$

which we inferred from the closed sector Hamiltonian $H_{\text{closed}} = (L_{\text{cyl.}})_0 + (\overline{L}_{\text{cyl.}})_0$. In analogy to the case of the torus partition function, we then define the cylinder partition function as $\mathscr{Z} = \text{Tr}\exp(-2\pi t H_{\text{open}})$ which can be brought into the following form

$$\mathscr{Z}^{\mathscr{C}}\big(t\big) = \text{Tr}_{\mathscr{H}_{\mathscr{B}}}\Big(q^{L_0 - \frac{c}{24}} \Big) \qquad \text{where} \qquad q = e^{-2\pi t}.$$

Here, the superscript $\mathscr{C}$ on $\mathscr{Z}$ indicates the cylinder partition function and $\mathscr{H}_{\mathscr{B}}$ denotes the Hilbert space of all states satisfying one of the boundary conditions (2.6). Clearly, from a string theory point of view, this is just the Hilbert space of an open string.

Free Boson I: Cylinder Partition Function (Loop-Channel)

We close this section by determining the cylinder partition function for the free boson. For the free boson, the Laurent modes of the energy-momentum tensor are written using the modes of the current $j(z) = i\,\partial X(z)$. In particular, we have

$$L_0 = \frac{1}{2}\, j_0\, j_0 + \sum_{k\geq 1} j_{-k}\, j_k.$$

Since the current $j(z)$ is a field of conformal dimension one, we find that $j_n|0\rangle = 0$ for $n > -1$ and that states in the Hilbert space have the following form

$$\big|n_1, n_2, n_3, \ldots\big\rangle = j_{-1}^{n_1}\, j_{-2}^{n_2}\, j_{-3}^{n_3} \ldots \big|0\big\rangle \qquad \text{with} \qquad n_i \geq 0 \tag{2.12}$$

and $n_i \in \mathbb{Z}$. The current algebra for the Laurent modes reads

$$\big[j_m, j_n\big] = m\, \delta_{m,-n}.$$

Next, let us compute the action of L_0 on a state (2.12). Clearly, j_0 commutes with all j_{-k} and let is first assume that it annihilates the vacuum. For the other terms we calculate

$$\big[j_{-k} j_k,\ j_{-k}^{n_k}\big] = n_k\, k\, j_{-k}^{n_k}, \tag{2.13}$$

and so we find for the zero Laurent mode of the energy-momentum tensor that

$$L_0\,\big|n_1, n_2, n_3, \ldots\big\rangle = \sum_{k\geq 1} j_{-1}^{n_1}\, j_{-2}^{n_2} \ldots \big(j_{-k} j_k\big)\, j_{-k}^{n_k} \ldots \big|0\big\rangle = \sum_{k\geq 1} k\, n_k\, \big|n_1, n_2, n_3, \ldots\big\rangle.$$

We will utilize this last result in the calculation of the partition function where for simplicity we only focus on the holomorphic part. We compute

$$
\begin{aligned}
&\mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\right) \\
&= q^{-\frac{1}{24}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \ldots \langle n_1, n_2, n_3, \ldots | \sum_{p=0}^{\infty} \frac{1}{p!} (2\pi\tau)^p \, (L_0)^p \, |n_1, n_2, n_3, \ldots\rangle \\
&= q^{-\frac{1}{24}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \ldots \langle n_1, n_2, n_3, \ldots | \sum_{p=0}^{\infty} \frac{1}{p!} (2\pi\tau)^p \, \Big(\sum_{k=1}^{\infty} k\, n_k\Big)^p \, |n_1, n_2, n_3, \ldots\rangle \\
&= q^{-\frac{1}{24}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \ldots \left(q^{1\cdot n_1} \cdot q^{2\cdot n_2} \cdot q^{3\cdot n_3} \cdot \, \ldots\right) \\
&= q^{-\frac{1}{24}} \left(\sum_{n_1=0}^{\infty} q^{1\, n_1}\right) \cdot \left(\sum_{n_2=0}^{\infty} q^{2\, n_2}\right) \cdot \left(\sum_{n_3=0}^{\infty} q^{3\, n_3}\right) \cdot \, \ldots \\
&= q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} q^{k\, n_k} = q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1-q^k}
\end{aligned}
$$

where in the last step we employed the result for the infinite geometric series and the ellipsis indicate that the structure extends to infinity. We then define the Dedekind η-function as

$$
\boxed{\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1-q^n\right)} \tag{2.14}
$$

so that

$$
\mathrm{Tr}_{\mathscr{H}_{\mathscr{B}}}\left(q^{L_0-\frac{c}{24}} \right)\Big|_{\text{without } j_0} = \frac{1}{\eta\,(it)}.
$$

However, recall that we have assumed the action of j_0 on the vacuum to vanish which is in general not applicable. Taking into account the effect of j_0, we now study the three different cases of boundary conditions in turn.

- For the case of Neumann–Neumann boundary conditions, the momentum mode $\pi_0 = \frac{1}{2}\, j_0$ is unconstrained and in principle contributes to the trace. Since it is a continuous variable, the sum is replaced by an integral

$$
\mathrm{Tr}_{\mathscr{H}_{\mathscr{B}}}\left(q^{\frac{1}{2}\, j_0^2}\right) = \sum_{n_0} \langle n_0|\, e^{-\pi t\, j_0^2}\, |n_0\rangle = \sum_{n_0} e^{-\pi t\, n_0^2} \quad \longrightarrow \quad \int_{-\infty}^{\infty} d\pi_0\, e^{-4\pi t\, \pi_0^2},
$$

where we utilised $n_0 = 2\pi_0$. Evaluating this Gaussian integral leads to the following additional factor for the partition function

$$
\frac{1}{2\sqrt{t}}\,. \tag{2.15}
$$

- For the Dirichlet–Dirichlet case, we have seen in equation (2.9) that j_0 is related to the positions of the string endpoints. Therefore, we have a contribution to the partition function of the form

$$q^{\frac{1}{2}j_0^2} = \exp\left(-2\pi t\,\frac{1}{2}\left(\frac{x_0^b - x_0^a}{2\pi}\right)^2\right) = \exp\left(-\frac{t}{4\pi}\left(x_0^b - x_0^a\right)^2\right).$$

- Finally, for the case of mixed Neumann–Dirichlet boundary conditions, the Laurent modes j_n take half-integer values for n. We do not present a detailed calculation for this case. We just mention that there is a twisted sector where the Laurent modes j_n also take half-integer values for n. It is then possible to extract $\mathrm{Tr}_{n\in\mathbb{Z}+\frac{1}{2}}\left(q^{L_0-\frac{c}{24}}\right)$ giving us the partition function in the present case.

In summary, the cylinder partition functions for the example of the free boson read

$$\begin{aligned}
\mathcal{Z}^{\mathcal{C}\,(\mathrm{D,D})}_{\mathrm{bos.}}(t) &= \exp\left(-\frac{t}{4\pi}\left(x_0^b - x_0^a\right)^2\right)\frac{1}{\eta\,(it)},\\
\mathcal{Z}^{\mathcal{C}\,(\mathrm{N,N})}_{\mathrm{bos.}}(t) &= \frac{1}{2\sqrt{t}}\,\frac{1}{\eta\,(it)},\\
\mathcal{Z}^{\mathcal{C}\,(\mathrm{mixed})}_{\mathrm{bos.}}(t) &= \sqrt{\frac{\eta\,(it)}{\vartheta_4(it)}}.
\end{aligned} \tag{2.16}$$

2.2 Boundary States for the Free Boson

In the last section, we have described the boundaries for the free boson CFT implicitly via the boundary conditions for the fields. However, in an abstract CFT usually there is no Lagrangian formulation available and no boundary terms will arise from a variational principle. Therefore, to proceed, we need a more inherent formulation of a boundary.

In the following, we first illustrate the construction of so-called boundary states for the example of the free boson and in the next section, we generalise the structure to Rational Conformal Field Theories with boundaries.

2.2.1 Boundary Conditions

Boundary States

Let us start with the following observation. As it is illustrated in Fig. 2.4, by interchanging τ and σ, we can interpret the cylinder partition function of the Boundary Conformal Field Theory on the left-hand side as a tree-level amplitude of the underlying theory shown on the right-hand side. From a string theory point of view, the

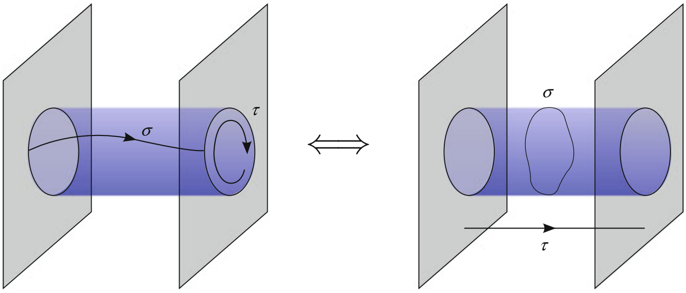

Fig. 2.4 Illustration of world-sheet duality relating the cylinder amplitude in the open and closed sector

tree-level amplitude describes the emission of a closed string at boundary A which propagates to boundary B and is absorbed there. Thus, a boundary can be interpreted as an object, which couples to closed strings. Note that in order to simplify our notation, we call the sector of the BCFT *open* and the sector of the underlying CFT *closed*. The relation above then reads

$$(\sigma, \tau)_{\text{open}} \quad \longleftrightarrow \quad (\tau, \sigma)_{\text{closed}}, \tag{2.17}$$

which in string theory is known as the world-sheet duality between open and closed strings.

The boundary for the closed sector can be described by a coherent state in the Hilbert space $\mathscr{H} \otimes \overline{\mathscr{H}}$ which takes the general form

$$\big|B\big\rangle = \sum_{i,\bar{j} \in \mathscr{H} \otimes \overline{\mathscr{H}}} \alpha_{i\bar{j}} \, \big|i, \bar{j}\big\rangle.$$

Here $i, \bar{j}$ label the states in the holomorphic and anti-holomorphic sector of $\mathscr{H} \otimes \overline{\mathscr{H}}$, and the coefficients $\alpha_{i\bar{j}}$ encode the *strength* of how the closed string mode $|i, \bar{j}\rangle$ couples to the boundary $|B\rangle$. Such a coherent state is called a *boundary state* and provides the CFT description of a D-brane in string theory.

Boundary Conditions

Let us now translate the boundary conditions (2.3) into the picture of boundary states. By using relation (2.17), we readily obtain

$$\boxed{\begin{array}{ll} \partial_\tau X_{\text{closed}}\big|_{\tau=0} \, \big|B_{\text{N}}\big\rangle = 0 & \qquad \text{Neumann condition,} \\ \partial_\sigma X_{\text{closed}}\big|_{\tau=0} \, \big|B_{\text{D}}\big\rangle = 0 & \qquad \text{Dirichlet condition.} \end{array}} \tag{2.18}$$

Next, for the free boson theory we would like to express the boundary conditions (2.18) of a boundary state in terms of the Laurent modes. To do so, we recall (2.5) and set $\tau = 0$ to obtain

$$\begin{aligned} i\cdot\partial_\tau X_{\text{closed}}\big|_{\tau=0} &= \sum_{n\in\mathbb{Z}} \big(j_n\, e^{-in\sigma} + \bar{j}_n\, e^{+in\sigma} \big), \\ \partial_\sigma X_{\text{closed}}\big|_{\tau=0} &= \sum_{n\in\mathbb{Z}} \big(j_n\, e^{-in\sigma} - \bar{j}_n\, e^{+in\sigma} \big). \end{aligned} \tag{2.19}$$

We then relabel $n \to -n$ in the second term of each line and observe again that for generic σ, the summands are linearly independent. Therefore, the boundary conditions (2.18) expressed in terms of the Laurent modes read

$$\boxed{\begin{array}{lr} \big(j_n + \bar{j}_{-n}\big)\, \big|B_{\mathrm{N}}\big\rangle = 0, \quad \big(\, \pi_0\, |B_{\mathrm{N}}\rangle = 0 \,\big) & \text{Neumann condition,} \\ \big(j_n - \bar{j}_{-n}\big)\, \big|B_{\mathrm{D}}\big\rangle = 0 & \text{Dirichlet condition,} \end{array}} \tag{2.20}$$

for each n. Such conditions relating the chiral and anti-chiral modes acting on the boundary state are called *gluing conditions*. Note that for the case of Neumann boundary conditions, in the string theory picture the relation $\pi_0 = 0$ means that there is no momentum transfer through the boundary. On the other hand, for Dirichlet conditions there is no restriction on π_0.

Solutions to the Gluing Conditions

Next, we are going to state the solutions for the gluing conditions for the example of the free boson and verify them thereafter. For now, let us ignore the constraints on j_0. We will come back to this issue later.

The boundary states for Neumann and Dirichlet conditions in terms of the Laurent modes j_n and $\bar{j}_n$ read

$$\boxed{\begin{array}{lr} \big|B_{\mathrm{N}}\big\rangle = \dfrac{1}{\mathcal{N}_{\mathrm{N}}}\, \exp\bigg(- \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\, j_{-k}\, \bar{j}_{-k} \bigg) \big|\, 0\big\rangle & \text{Neumann condition,} \\ \big|B_{\mathrm{D}}\big\rangle = \dfrac{1}{\mathcal{N}_{\mathrm{D}}}\, \exp\bigg(+ \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\, j_{-k}\, \bar{j}_{-k} \bigg) \big|\, 0\big\rangle & \text{Dirichlet condition,} \end{array}} \tag{2.21}$$

where $\mathcal{N}_{\mathrm{N}}$ and $\mathcal{N}_{\mathrm{D}}$ are normalisation constants to be fixed later. One possibility to verify the boundary states is to straightforwardly evaluate the gluing conditions (2.20) for the solutions (2.21) explicitly. However, in order to highlight the underlying structure, we will take a slightly different approach.

Construction of Boundary States

In the following, we focus on a boundary state with Neumann conditions but comment on the Dirichlet case at the end. To start, we rewrite the Neumann boundary state in (2.21) as

$$
\begin{aligned}
\big|B_{\mathrm{N}}\big\rangle &= \frac{1}{\mathcal{N}_{\mathrm{N}}}\, \exp\!\bigg(-\sum_{k=1}^{\infty}\frac{1}{k}\, j_{-k}\, \bar{j}_{-k}\bigg)\big|0\big\rangle \\
&= \frac{1}{\mathcal{N}_{\mathrm{N}}} \prod_{k=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}}\left(\frac{j_{-k}}{\sqrt{k}}\right)^{m}\big|0\big\rangle \otimes \frac{1}{\sqrt{m!}}\left(\frac{-\bar{j}_{-k}}{\sqrt{k}}\right)^{m}\big|\bar{0}\big\rangle \\
&= \frac{1}{\mathcal{N}_{\mathrm{N}}} \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\prod_{k=1}^{\infty} \frac{1}{\sqrt{m_k!}}\left(\frac{j_{-k}}{\sqrt{k}}\right)^{m_k}\big|0\big\rangle \otimes \frac{1}{\sqrt{m_k!}}\left(\frac{-\bar{j}_{-k}}{\sqrt{k}}\right)^{m_k}\big|\bar{0}\big\rangle,
\end{aligned}
\tag{2.22}
$$

where we first have written the sum in the exponential as a product and then we expressed the exponential as an infinite series. Next, we note that the following states form a complete orthonormal basis for all states constructed out of the Laurent modes j_{-k}

$$
\big|\mathbf{m}\big\rangle = \big|m_1, m_2, \ldots\big\rangle = \prod_{k=1}^{\infty}\frac{1}{\sqrt{m_k!}}\left(\frac{j_{-k}}{\sqrt{k}}\right)^{m_k}\big|0\big\rangle. \tag{2.23}
$$

The orthonormal property can be seen by computing

$$
\big\langle \mathbf{n}\mid\mathbf{m}\big\rangle = \prod_{k=1}^{\infty}\frac{1}{\sqrt{n_k!\, m_k!}}\,\frac{1}{\sqrt{k}^{\,n_k+m_k}}\,\big\langle 0\big|\, j_{+k}^{n_k}\, j_{-k}^{m_k}\,\big|0\big\rangle_k = \prod_{k=1}^{\infty}\delta_{n_k,m_k},
$$

where we used that

$$
\big\langle 0\big|\, j_{+k}^{n}\, j_{-k}^{m}\,\big|0\big\rangle = k\, n\, \big\langle 0\big|\, j_{+k}^{n-1}\, j_{-k}^{m-1}\,\big|0\big\rangle = \delta_{m,n}\, k^{n}\, n!.
$$

We now introduce an operator U mapping the chiral Hilbert space to its charge conjugate $U : \mathcal{H} \to \mathcal{H}^{+}$ and similarly for the anti-chiral sector. In particular, the action of U reads

$$
U\, j_k\, U^{-1} = -j_k = -\big(j_{-k}\big)^{\dagger}, \qquad U\, \bar{j}_k\, U^{-1} = -\bar{j}_k = -\big(\bar{j}_{-k}\big)^{\dagger}, \qquad U\, c\, U^{-1} = c^{*},
$$

where c is a constant and * denotes complex conjugation. In the present example, the ground state $|0\rangle$ is non-degenerate and is left invariant by U.[2] Knowing these properties, we can show that U is anti-unitary. For this purpose, we expand a general state as $|a\rangle = \sum_{\mathbf{m}} A_{\mathbf{m}}|\mathbf{m}\rangle$ and compute

$$
\begin{aligned}
U\,\big|a\big\rangle &= \sum_{\mathbf{m}} U\, A_{\mathbf{m}}\, U^{-1}\prod_{k=1}^{\infty}\frac{1}{\sqrt{m_k!}}\left(\frac{U\, j_{-k}\, U^{-1}}{\sqrt{k}}\right)^{m_k} U\,\big|0\big\rangle \\
&= \sum_{\mathbf{m}} A_{\mathbf{m}}^{*}\prod_{k=1}^{\infty}\big(-1\big)^{m_k}\big|\mathbf{m}\big\rangle,
\end{aligned}
\tag{2.24}
$$

where $\mathbf{m}$ denotes the multi-index $\{m_1, m_2, \ldots\}$. By using that $|\mathbf{m}\rangle$ and $|\mathbf{n}\rangle$ form an orthonormal basis, we can now show that U is anti-unitary

[2] For degenerate ground states a non-trivial action on the ground state might need to be defined.

$$\langle\, Ub \mid Ua\,\rangle = \sum_{\mathbf{n},\mathbf{m}} \langle \mathbf{n} \mid B_{\mathbf{n}} \prod_{k=1}^{\infty} (-1)^{n_k+m_k}\, A^*_{\mathbf{m}}\, |\mathbf{m}\rangle = \sum_{\mathbf{m}} A^*_{\mathbf{m}}\, B_{\mathbf{m}} = \langle\, a \mid b\,\rangle.$$

After introducing an orthonormal basis and the anti-unitary operator U, we now express (2.22) in a more general way which will simplify and generalise the following calculations

$$\boxed{|B\rangle = \frac{1}{\mathcal{N}} \sum_{\mathbf{m}} |\mathbf{m}\rangle \otimes |U\, \overline{\mathbf{m}}\rangle}.$$

Verification of the Gluing Conditions

In order to verify the gluing conditions (2.20) for Neumann boundary states, we note that these have to be satisfied also when an arbitrary state $\langle\, \overline{a} \mid \otimes \langle\, b \mid$ is multiplied from the left. We then calculate

$$\begin{aligned}\langle\, \overline{a} \mid \otimes \langle\, b \mid j_n + \overline{j}_{-n}\, |B\rangle &= \frac{1}{\mathcal{N}} \sum_{\mathbf{m}} \langle\, \overline{a} \mid \otimes \langle\, b \mid j_n + \overline{j}_{-n}\, |\mathbf{m}\rangle \otimes |U\, \overline{\mathbf{m}}\rangle \\ &= \frac{1}{\mathcal{N}} \sum_{\mathbf{m}} \langle\, b \mid j_n\, |\mathbf{m}\rangle \langle\, \overline{a}\, |U\, \overline{\mathbf{m}}\rangle + \langle\, b\, |\mathbf{m}\rangle \langle\, \overline{a} \mid \overline{j}_{-n}\, |U\, \overline{\mathbf{m}}\rangle.\end{aligned}$$

Next, due to the identifications on the boundary, the holomorphic and the anti-holomorphic algebra are identical. We can therefore replace matrix elements in the anti-holomorphic sector by those in the holomorphic sector. Using finally the anti-unitarity of U and that $\sum_{\mathbf{m}} |\mathbf{m}\rangle\langle\mathbf{m}| = \mathbb{1}$, we find

$$\begin{aligned}&\langle \overline{a} \mid \otimes \langle b \mid j_n + \overline{j}_{-n}\, |B\rangle \\ &\qquad = \frac{1}{\mathcal{N}} \sum_{\mathbf{m}} \langle\, b \mid j_n\, |\mathbf{m}\rangle \langle\, a\, |U\, \mathbf{m}\rangle + \langle\, b\, |\mathbf{m}\rangle \langle\, a \mid j_{-n}\, |U\, \mathbf{m}\rangle \\ &\qquad = \frac{1}{\mathcal{N}} \sum_{\mathbf{m}} \langle\, b \mid j_n\, |\mathbf{m}\rangle \langle\, \mathbf{m}\, |U^{-1} a\rangle + \langle\, b\, |\mathbf{m}\rangle \langle\, \mathbf{m} \mid (-j_n)\, |U^{-1} a\rangle \\ &\qquad = \frac{1}{\mathcal{N}} \Big(\langle\, b \mid j_n\, |U^{-1} a\rangle - \langle\, b \mid j_n\, |U^{-1} a\rangle \Big) = 0.\end{aligned}$$

Therefore, we have verified that the Neumann boundary state in (2.21) is indeed a solution to the corresponding gluing condition in (2.20).

For the case of Dirichlet boundary conditions, the action of U on the Laurent modes j_n and $\overline{j}_n$ is chosen with a + sign while we still require U to be anti-unitary, i.e. $U\ c\ U^{-1} = c^*$. The calculation is then very similar to the Neumann case presented here. Note furthermore, the construction of boundary states and the verification of the gluing conditions is also applicable for more general CFTs, for instance RCFTs, which we will consider in Sect. 2.3.

Momentum Dependence of Boundary States

In (2.7) we gave the result for the center of mass for an open string. This differs from the closed string case by a factor $\frac{1}{2}$ due to the fact that open strings have by convention length π while closed string have length 2π. In the following, the relation between j_0, $\bar{j}_0$ and π_0 should be clear from the context, but let us summarise that

$$\big(\pi_0\big)_{\text{closed}} = j_0 = \bar{j}_0, \qquad \big(\pi_0\big)_{\text{open}} = \frac{1}{2}\, j_0 = \frac{1}{2}\, \bar{j}_0. \tag{2.25}$$

From a string theory point of view, in addition to the boundary conditions (2.20) there is a further natural constraint on a boundary state with Dirichlet conditions. Namely, the closed string at time $\tau = 0$ is located at the boundary at position x_0^a. We therefore impose

$$X_{\text{closed}}\big(\tau = 0, \sigma\big)\ \big|B_{\text{D}}\big\rangle = x_0^a\ \big|B_{\text{D}}\big\rangle$$

and similarly for $\tau = \pi$. An easy way to realise this constraint is to perform a Fourier transformation from momentum space $|B_{\text{D}}, \pi_0\rangle$ to the position space. Concretely, this reads

$$\big|B_{\text{D}}, x_0^a\big\rangle = \int d\pi_0\ e^{i\pi_0 x_0^a}\ \big|B_{\text{D}}, \pi_0\big\rangle.$$

For the boundary state with Neumann conditions, we have $\pi_0 = 0$ and in position space, there is no definite value for x_0. We thus omit this label.

Conformal Symmetry

In studying the example of the free boson, we have expressed all important quantities in terms of the $U(1)$ current modes j_n and $\bar{j}_n$. However, in more general CFTs such additional symmetries may not be present but the conformal symmetry generated by the energy-momentum tensors always is. In view of generalisations of our present example, let us therefore determine the boundary conditions of the boundary states in terms of the Laurent modes L_n and $\overline{L}_n$.

Mainly guided by the final result, let us compute the following expression by employing that $T(z) = \frac{1}{2}N(jj)(z)$ which implies $L_n = \frac{1}{2}\sum_{k>-1} j_{n-k}j_k + \frac{1}{2}\sum_{k\leq -1} j_k j_{n-k}$

$$
\begin{aligned}
&\left(L_n-\overline{L}_{-n}\right)\,\left|B_{\rm N,D}\right\rangle\\
&=\frac{1}{2}\left(\sum_{k>-1}\left(j_{n-k}j_k-\bar{j}_{-n-k}\bar{j}_k\right)+\sum_{k\leq -1}\left(j_k j_{n-k}-\bar{j}_k\bar{j}_{-n-k}\right)\right)\left|B_{\rm N,D}\right\rangle\\
&=\frac{1}{2}\left(j_n j_0-\bar{j}_{-n}\bar{j}_0+\sum_{k\geq 1}\left(j_{n-k}j_k-\bar{j}_{-n-k}\bar{j}_k+j_{-k}j_{n+k}-\bar{j}_{-k}\bar{j}_{-n+k}\right)\right)\left|B_{\rm N,D}\right\rangle.
\end{aligned}
$$

Note that here we changed the summation index $k \to -k$ in the second sum. Next, we recall (2.20) and $j_0 = \bar{j}_0$ to observe that the terms involving j_0 and $\bar{j}_0$ vanish when applied to $|B_{\rm N,D}\rangle$. The remaining terms can be rewritten as

$$
\begin{aligned}
&\frac{1}{2}\sum_{k\geq 1}\Big(j_{n-k}\left(j_k\pm\bar{j}_{-k}\right)\mp j_{n-k}\bar{j}_{-k}\quad\mp\bar{j}_{-n-k}\left(j_{-k}\pm\bar{j}_k\right)\pm\bar{j}_{-n-k}j_{-k}\\
&+j_{-k}\left(j_{n+k}\pm\bar{j}_{-n-k}\right)\mp j_{-k}\bar{j}_{-n-k}\quad\mp\bar{j}_{-k}\left(j_{n-k}\pm\bar{j}_{-n+k}\right)\pm\bar{j}_{-k}j_{n-k}\Big)\left|B_{\rm N,D}\right\rangle.
\end{aligned}
$$

By again employing the boundary conditions (2.20), we see that half of these terms vanish when acting on the boundary state while the other half cancels among themselves. In summary, we have shown that

$$
\left(L_n-\overline{L}_{-n}\right)\,\left|B_{\rm N,D}\right\rangle=0\ .
$$

2.2.2 *Tree-Level Amplitudes*

Cylinder Diagram in General

We now turn to the cylinder diagram which we compute in the closed sector. Referring again to Fig. 2.4, in string theory we can interpret this diagram as a closed string which is emitted at the boundary A, propagating via the closed sector Hamiltonian $H_{\rm closed} = L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24}$ for a time $\tau = l$ until it reaches the boundary B where it gets absorbed. In analogy to Quantum Mechanics, such an amplitude is given by the overlap

$$
\tilde{\mathscr{Z}}^{\mathscr{C}}(l)=\langle\Theta B|\;e^{-2\pi l\left(L_0+\overline{L}_0-\frac{c+\bar{c}}{24}\right)}\;|B\rangle\ , \tag{2.26}
$$

where the tilde indicates that the computation is performed in the closed sector (or at tree-level) and l is the length of the cylinder connecting the two boundaries.

Let us now explain the notation $\langle\Theta B|$. This bra-vector is understood as the hermitian conjugate of the ket-vector $|\Theta B\rangle$. Furthermore, we have introduced the CPT operator Θ which acts as charge conjugation (C), parity transformation (P) $\sigma \mapsto -\sigma$ and time reversal (T) $\tau \mapsto -\tau$ for the two-dimensional CFT. The reason for considering this operator can roughly be explained by the fact that the orientation of the boundary a closed string is emitted at is opposite to the orientation of the boundary where the closed string gets absorbed. For the momentum dependence of a boundary state $|B, \pi_0\rangle$, this implies in particular that

$$\big\langle\, \pi_0^a \,\big|\, \pi_0^b \,\big\rangle = \delta\big(\pi_0^a + \pi_0^b\big). \tag{2.27}$$

Without a detailed derivation, we finally note that the theory of the free boson is CPT invariant and so the action of Θ on the boundary states (2.21) of the free boson theory (and on ordinary numbers $c \in \mathbb{C}$) reads

$$\Theta\, \big|B, \pi_0\big\rangle = \frac{1}{\mathcal{N}^*}\, \big|B, \pi_0\big\rangle, \qquad \Theta\, c\, \Theta^{-1} = c^*, \tag{2.28}$$

where * denotes complex conjugation.

Free Boson II: Cylinder Diagram (Tree-Channel)

Let us now be more concrete and compute the overlap of two boundary states (2.26) for the example of the free boson. To do so, we note that for the free boson CFT we have $c = \overline{c} = 1$ and that

$$L_0 = \frac{1}{2}\, j_0\, j_0 + \sum_{k \geq 1} j_{-k}\, j_k,$$

and similarly for $\overline{L}_0$. Next, we perform the following calculation in order to evaluate (2.26). In particular, we use $j_{-k} j_k\, j_{-k}^{m_k}|0\rangle = m_k\, k\, j_{-k}^{m_k}|0\rangle$ to find

$$\begin{aligned} q^{\,\sum_{k\geq 1} j_{-k} j_k}\, \big|\mathbf{m}\big\rangle &= \prod_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{\big(-2\pi i\tau\big)^p}{p!}\, \big(j_{-k} j_k\big)^p \prod_{l=1}^{\infty} \frac{1}{\sqrt{m_l!}} \left(\frac{j_{-l}}{\sqrt{l}}\right)^{m_l} |0\rangle \\ &= \prod_{k=1}^{\infty} \sum_{p=0}^{\infty} \frac{\big(-2\pi i\tau\big)^p}{p!}\, \big(m_k\, k\big)^p \prod_{l=1}^{\infty} \frac{1}{\sqrt{m_l!}} \left(\frac{j_{-l}}{\sqrt{l}}\right)^{m_l} |0\rangle \\ &= \prod_{k=1}^{\infty} q^{\,m_k\, k}\, \big|\mathbf{m}\big\rangle. \end{aligned} \tag{2.29}$$

The cylinder diagram for the three possible combinations of boundary conditions is then computed as follows.

- For the case of Neumann–Neumann boundary conditions, we have $j_0|B_{\rm N}\rangle = \bar{j}_0|B_{\rm N}\rangle = 0$ and so the momentum contribution vanishes. For the remaining part, we calculate using (2.29) and (2.24)

$$\begin{aligned}
\widetilde{\mathscr{Z}}^{\mathscr{C}\,(\mathrm{N,N})}_{\mathrm{bos.}}(l) &= \frac{e^{-2\pi l\left(-\frac{2}{24}\right)}}{\mathscr{N}_{\mathrm{N}}^2}\;\sum_{\mathbf{m}}\langle\mathbf{m}|\;e^{-2\pi l\sum_{k\geq 1} j_{-k}j_k}\;|\mathbf{m}\rangle\times\\
&\qquad\qquad\times\left\langle U\overline{\mathbf{m}}\right|\;e^{-2\pi l\sum_{k\geq 1}\bar{j}_{-k}\bar{j}_k}\;\left|U\overline{\mathbf{m}}\right\rangle\\
&= \frac{e^{-2\pi l\left(-\frac{2}{24}\right)}}{\mathscr{N}_{\mathrm{N}}^2}\;\sum_{\mathbf{m}}\prod_{k=1}^{\infty} e^{-2\pi l\, m_k k}\;(-1)^{\sum_{l=1}^{\infty} m_l}\;e^{-2\pi l\, m_k k}\;(-1)^{\sum_{l=1}^{\infty} m_l}\\
&= \frac{e^{\frac{\pi l}{6}}}{\mathscr{N}_{\mathrm{N}}^2}\;\prod_{k=1}^{\infty}\sum_{m_k=0}^{\infty}\left(e^{-4\pi l\, k}\right)^{m_k} = \frac{1}{\mathscr{N}_{\mathrm{N}}^2}\;e^{\frac{\pi l}{6}}\prod_{k=1}^{\infty}\frac{1}{1-e^{-4\pi l\, k}}
\end{aligned}$$

where in the last step we performed a summation of the geometric series. Let us emphasise that due to the action of the CPT operator Θ shown in (2.28), $\mathscr{N}^2$ is just the square of $\mathscr{N}$ and not the absolute value squared. Then, with $q = e^{2\pi i\tau}$ and $\tau = 2il$ we find that the cylinder diagram for Neumann–Neumann boundary conditions is expressed as

$$\widetilde{\mathscr{Z}}^{\mathscr{C}\,(\mathrm{N,N})}_{\mathrm{bos.}}(l) = \frac{1}{\mathscr{N}_{\mathrm{N}}^2}\;\frac{1}{\eta(2il)}. \tag{2.30}$$

- Next, we consider the case of Dirichlet–Dirichlet boundary conditions. Noting that U now acts trivially on the basis states, we see that apart from the momentum contribution the calculation is similar to the case with Neumann–Neumann conditions. However, for the momentum dependence we compute using (2.27) and (2.28)

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\pi_0^a\; d\pi_0^b\; e^{+ix_0^a\pi_0^a}\; e^{+ix_0^b\pi_0^b}\;\left\langle\pi_0^a\right|e^{-2\pi l (j_0)^2}\left|\pi_0^b\right\rangle\\
&=\int_{-\infty}^{\infty} d\pi_0^a\; d\pi_0^b\; e^{+ix_0^a\pi_0^a}\; e^{+ix_0^b\pi_0^b}\; e^{-2\pi l\left(\pi_0^b\right)^2}\;\delta\left(\pi_0^a+\pi_0^b\right)\\
&=\int_{-\infty}^{\infty} d\pi_0^a\; e^{-2\pi l\left(\pi_0^a + i\,\frac{x_0^b - x_0^a}{4\pi l}\right)^2}\; e^{-\frac{\left(x_0^b-x_0^a\right)^2}{8\pi l}} = \frac{1}{\sqrt{2l}}\; e^{-\frac{\left(x_0^b-x_0^a\right)^2}{8\pi l}}
\end{aligned}$$

where we completed a perfect square and performed the Gaussian integration. In order to arrive at the result above, we also employed that in the closed sector $\pi_0 = j_0 = \bar{j}_0$. The cylinder diagram with Dirichlet–Dirichlet boundary conditions therefore reads

$$\widetilde{\mathscr{Z}}^{\mathscr{C}\,(\mathrm{D,D})}_{\mathrm{bos.}}(l) = \frac{1}{\mathscr{N}_{\mathrm{D}}^2}\;\exp\left(-\,\frac{\left(x_0^b - x_0^a\right)^2}{8\pi l}\right)\frac{1}{\sqrt{2l}}\;\frac{1}{\eta(2il)}.$$

- Finally, for mixed Neumann–Dirichlet conditions, the boundary state satisfies $j_0|B_{\mathrm{D}}\rangle = \bar{j}_0|B_{\mathrm{D}}\rangle = \pi_0|B_{\mathrm{D}}\rangle$ which leads us to

$$\int d\pi_0\; e^{i\pi_0 x_0}\langle\pi_0 = 0|\; e^{-2\pi l\, j_0^2}\;|\pi_0\rangle = \int d\pi_0\; e^{i\pi_0 x_0} e^{-2\pi l\,\pi_0^2}\;\delta(\pi_0) = 1.$$

In the anti-holomorphic sector of the Dirichlet boundary state, the action of U on the basis states $|\overline{\mathbf{m}}\rangle$ is trivial and so we obtain a single factor of $(-1)^{\sum_k m_k}$. For the full cylinder diagram, this implies

$$\widetilde{\mathscr{Z}}^{\mathscr{C}\,(\text{mixed})}_{\text{bos.}}(l) = \frac{e^{\frac{\pi l}{6}}}{\mathscr{N}_{\text{N}}\mathscr{N}_{\text{D}}} \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \left(-e^{-4\pi l\, k}\right)^{m_k} = \frac{e^{\frac{\pi l}{6}}}{\mathscr{N}_{\text{N}}\mathscr{N}_{\text{D}}} \prod_{k=1}^{\infty} \frac{1}{1+e^{-4\pi l\, k}}.$$

One defines the $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\tau, z)$-functions as

$$\vartheta\left[\begin{matrix}\alpha\\ \beta\end{matrix}\right](\tau, z) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\alpha)^2}\, e^{2\pi i(n+\alpha)(z+\beta)},$$

which can be shown to also have a representation as an infinite product

$$\begin{aligned}\frac{\vartheta\left[\begin{matrix}\alpha\\ \beta\end{matrix}\right](\tau, z)}{\eta(\tau)} &= e^{2\pi i\alpha(z+\beta)}\, q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1+q^{n+\alpha-\frac{1}{2}}\, e^{2\pi i(z+\beta)}\right)\\ &\quad\times\left(1+q^{n-\alpha-\frac{1}{2}}\, e^{-2\pi i(z+\beta)}\right).\end{aligned}$$

In particular one can write

$$\vartheta_2(\tau) \equiv \vartheta\left[\begin{matrix}1/2\\ 0\end{matrix}\right](\tau, 0) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} \tag{2.31}$$

so that we can express the cylinder diagram for mixed boundary conditions as

$$\widetilde{\mathscr{Z}}^{\mathscr{C}\,(\text{mixed})}_{\text{bos.}}(l) = \frac{\sqrt{2}}{\mathscr{N}_{\text{N}}\mathscr{N}_{\text{D}}}\sqrt{\frac{\eta\,(2il)}{\vartheta_2(2il)}}.$$

Loop-Channel—Tree-Channel Equivalence

Let us come back to Fig. 2.4. As it is illustrated there and motivated at the beginning of this section, we expect the cylinder diagram in the closed and open sector to be related. More specifically, this relation is established by $(\sigma, \tau)_{\text{open}} \leftrightarrow (\tau, \sigma)_{\text{closed}}$ where σ is the world-sheet space coordinate and τ is world-sheet time. However, this mapping does not change the cylinder, in particular, it does not change the modular parameter τ. In the open sector, the cylinder has length $\frac{1}{2}$ and circumference t when measured in units of 2π, while in the closed sector we have length l and circumference 1. Then, the modular parameter in the open and closed sector are

$$\tau_{\text{open}} = \frac{\alpha_2}{\alpha_1} = \frac{it}{1/2} = 2it, \qquad \tau_{\text{closed}} = \frac{\alpha_2}{\alpha_1} = \frac{i}{l}.$$

As we have emphasised, the modular parameters in the open and closed sector have to be equal which leads us to the relation

$$\boxed{t = \frac{1}{2l}}\,.$$

This is the formal expression for the pictorial *loop-channel–tree-channel equivalence* of the cylinder diagram illustrated in Fig. 2.4.

We now verify this relation for the example of the free boson explicitly which will allow us to fix the normalisation constants $\mathcal{N}_{\mathrm{D}}$ and $\mathcal{N}_{\mathrm{N}}$ of the boundary states. Recalling the cylinder partition function (2.16) in the open sector, we compute

$$\mathscr{Z}^{\mathscr{C}(\mathrm{N,N})}_{\mathrm{bos.}}(t) = \frac{1}{2\sqrt{t}}\,\frac{1}{\eta(it)} \xrightarrow{t=\frac{1}{2l}} \sqrt{\frac{l}{2}}\,\frac{1}{\eta\left(-\frac{1}{2il}\right)} = \frac{1}{2\,\eta(2il)} = \frac{\mathcal{N}_{\mathrm{N}}^2}{2}\,\tilde{\mathscr{Z}}^{\mathscr{C}(\mathrm{N,N})}_{\mathrm{bos.}}(l),$$

where we used the modular properties of the Dedekind η function

$$\eta\left(-\tau^{-1}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{2.32}$$

Therefore, requiring the results in the loop- and tree-channel to be related, we can fix

$$\boxed{\mathcal{N}_{\mathrm{N}} = \sqrt{2}}\,. \tag{2.33}$$

Next, for Dirichlet–Dirichlet boundary conditions, we find

$$\begin{aligned}\mathscr{Z}^{\mathscr{C}(\mathrm{D,D})}_{\mathrm{bos.}}(t) &= \exp\left(-\frac{t}{4\pi}\left(x_0^b - x_0^a\right)^2\right)\frac{1}{\eta\,(it)}\\ &\xrightarrow{t=\frac{1}{2l}} \exp\left(-\frac{1}{8\pi l}\left(x_0^b - x_0^a\right)^2\right)\frac{1}{\eta\left(-\frac{1}{2il}\right)} = \mathcal{N}_{\mathrm{D}}^2\,\tilde{\mathscr{Z}}^{\mathscr{C}(\mathrm{D,D})}_{\mathrm{bos.}}(l),\end{aligned}$$

which allows us to fix the normalisation constant as

$$\boxed{\mathcal{N}_{\mathrm{D}} = 1}\,.$$

Finally, the loop-channel–tree-channel equivalence for mixed Neumann–Dirichlet boundary conditions can be verified along similar lines. This discussion shows that indeed the cylinder partition function for the free boson in the open and closed sector are related via a modular transformation, more concretely via a modular S-transformation.

Summary and Remark

Let us now briefly summarise our findings of this section and close with some remarks.

- By performing the so-called world-sheet duality $(\sigma, \tau)_{\text{open}} \leftrightarrow (\tau, \sigma)_{\text{closed}}$, we translated the Neumann and Dirichlet boundary conditions from the open sector to the closed sector. In string theory, the boundary in the closed sector is interpreted as an object which absorbs or emits closed strings.
- Working out the boundary conditions in terms of the Laurent modes of the free boson theory, we obtained the gluing conditions

$$\left(j_n \pm \bar{j}_{-n} \right) \left| B_{\text{N,D}} \right\rangle = 0$$

  which imply that the two $U(1)$ symmetries generated by $j(z)$ and $\bar{j}(\bar{z})$ are broken to a diagonal $U(1)$.
- For the example of the free boson theory, we stated the solution $|B\rangle$ to the gluing conditions and verified them. Along the way, we also outlined the idea for constructing boundary states for more general theories.
- The cylinder amplitude in the closed sector (tree-level) is computed from the overlap of two boundary states

$$\tilde{\mathcal{Z}}^{\mathcal{C}}(l) = \langle \Theta B | \; e^{-2\pi l \left(L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24} \right)} \; |B\rangle \, .$$

  We performed this calculation for the free boson and checked that it is related to the cylinder partition function in the open sector via world-sheet duality. In particular, this transformation is a modular S-transformation.
- Finally, the BCFT also has to preserve the conformal symmetry generated by $T(z)$. The boundary states respect this symmetry in the sense that the following conditions have to be satisfied

$$\left(L_n - \overline{L}_{-n} \right) \left| B_{\text{N,D}} \right\rangle = 0 \, ,$$

  which we checked for the example of the free boson theory.
- Very similarly, one can generalise the concept of boundaries and boundary states to the CFT of a free fermion which is very important for applications in Superstring Theory.

  As we mentioned already, in string theory boundary states are called D-branes to emphasise the space-time point of view of such objects. They are higher dimensional generalisations of strings and membranes, and indeed they play a very important role in understanding the non-perturbative sector of string theory. It was one of the big insights at the end of the last millennium that such higher dimensional objects are naturally contained in string theory (which started as a theory of only one-dimensional objects) and gave rise to various surprising dualities, the most famous surely being the celebrated AdS/CFT correspondence.

2.3 Boundary States for RCFTs

After having studied the Boundary CFT of the free boson in great detail, let us now generalise our findings to theories without a Lagrangian description. In particular, we focus on RCFTs and we will formulate the corresponding Boundary RCFT just in terms of gluing conditions for the theory on the sphere.

Boundary Conditions

We consider Rational Conformal Field Theories with chiral and anti-chiral symmetry algebras $\mathscr{A}$ respectively $\overline{\mathscr{A}}$. For the theory on the sphere the Hilbert space splits into irreducible representations of $\mathscr{A} \otimes \overline{\mathscr{A}}$ as

$$\mathscr{H} = \bigoplus_{i,\bar{j}} M_{i\bar{j}}\ \mathscr{H}_i \otimes \overline{\mathscr{H}}_{\bar{j}}$$

where $M_{i\bar{j}}$ are the same multiplicities of the highest weight representation appearing in the modular invariant torus partition function. Note that for the case of RCFTs we are considering, there is only a finite number of irreducible representations and that the modular invariant torus partition function is given by a combination of chiral and anti-chiral characters as follows

$$\mathscr{Z}(\tau,\overline{\tau}) = \sum_{i,\bar{j}} M_{i\bar{j}}\ \chi_i(\tau)\ \overline{\chi}_{\bar{j}}(\overline{\tau}).$$

Generalising the results from the free boson theory, we state without derivation that a boundary state $|B\rangle$ in the RCFT preserving the symmetry algebra $\mathscr{A} = \overline{\mathscr{A}}$ has to satisfy the following gluing conditions

$$\boxed{\begin{aligned} \left(L_n - \overline{L}_{-n} \right) \big|B\big\rangle = 0 &\quad \text{conformal symmetry,} \\ \left(W_n^i - (-1)^{h^i}\ \overline{W}^i_{-n} \right) \big|B\big\rangle = 0 &\quad \text{extended symmetries,} \end{aligned}} \tag{2.34}$$

where W_n^i is the holomorphic Laurent mode of the extended symmetry generator W^i with conformal weight $h^i = h(W^i)$, and $\overline{W}^i$ denotes the generator in the anti-holomorphic sector. However, the condition for the extended symmetries can be relaxed, so that also Dirichlet boundary conditions similar to the example of a free boson are included

$$\left(W_n^i - (-1)^{h^i}\ \Omega\big(\overline{W}^i_{-n}\big)\right) \big|B\big\rangle = 0,$$

where $\Omega : \mathscr{A} \to \mathscr{A}$ is an automorphism of the chiral algebra $\mathscr{A}$. Such an automorphism Ω is also called a *gluing automorphism* and for our example of the free boson with Dirichlet boundary conditions, it simply is $\Omega : \overline{W}_n \mapsto -\overline{W}_n$.

Ishibashi States

Let us introduce the charge conjugation matrix C which maps highest weight representations i to their charge conjugate i^+. Denoting then the Hilbert space built upon the charge conjugate represention by $\mathscr{H}_i^+$, we can state the important result of Ishibashi:

> For $\overline{\mathscr{A}} = \mathscr{A}$ and $\overline{\mathscr{H}}_i = \mathscr{H}_i^+$, to each highest weight representation ϕ_i of $\mathscr{A}$ one can associate an up to a constant unique state $|\mathscr{B}_i\rangle\rangle$ such that the gluing conditions are satisfied.

Note that since the CFTs we are considering are rational, there is only a finite number of highest weight states and thus only a finite number of such so-called Ishibashi states $|\mathscr{B}_i\rangle\rangle$.

We now construct the Ishibashi states in analogy to the boundary states of the free boson. Denoting by $|\phi_i, \mathbf{m}\rangle$ an orthonormal basis for $\mathscr{H}_i$, the Ishibashi states are written as

$$\big|\mathscr{B}_i\rangle\rangle = \sum_{\mathbf{m}} \big|\phi_i, \mathbf{m}\rangle \otimes U|\overline{\phi}_i, \overline{\mathbf{m}}\rangle, \tag{2.35}$$

where $U : \overline{\mathscr{H}} \to \overline{\mathscr{H}}^+$ is an anti-unitary operator acting on the symmetry generators $\overline{W}^i$ as follows

$$U\, \overline{W}^i_n\, U^{-1} = (-1)^{h^i}\, \left(\overline{W}^i_{-n}\right)^\dagger .$$

The proof that the Ishibashi states are solutions to the gluing conditions (2.34) is completely analogous to the example of the free boson and so we will not present it here.

The Cardy Condition

For later purpose, let us now compute the following overlap of two Ishibashi states

$$\langle\langle \mathscr{B}_j \big|\, e^{-2\pi l\, \left(L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24}\right)}\, \big|\mathscr{B}_i\rangle\rangle. \tag{2.36}$$

Utilising the gluing conditions for the conformal symmetry generator (2.34), we see that we can replace $\overline{L}_0$ by L_0 and $\overline{c}$ by c. Next, because the Hilbert spaces of two different HWRs ϕ_i and ϕ_j are independent of each other, the overlap above is only nonzero for $i = j^+$. Note that here we have written the charge conjugate j^+ of the highest weight ϕ_j because the hermitian conjugation also acts as charge conjugation. We then obtain

$$\begin{aligned}
\langle\langle \mathscr{B}_j \big|\, e^{-2\pi l\, \left(L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24}\right)}\, \big|\mathscr{B}_i\rangle\rangle &= \delta_{ij^+}\, \langle\langle \mathscr{B}_i \big|\, e^{2\pi i\, (2il)\, \left(L_0 - \frac{c}{24}\right)}\, \big|\mathscr{B}_i\rangle\rangle \\
&= \delta_{ij^+}\, \mathrm{Tr}_{\mathscr{H}_i}\Big(q^{L_0 - \frac{c}{24}}\Big) \\
&= \delta_{ij^+}\, \chi_i\big(2il\big)
\end{aligned} \tag{2.37}$$

with χ_i the character defined as

$$\chi_i(\tau) := Tr_{\mathscr{H}_i}\left(q^{L_0-\frac{c}{24}}\right) \tag{2.38}$$

over the Hilbert space $\mathscr{H}_i$ built on the highest weigth state ϕ_i. Performing a modular S-transformation for this overlap, by the same reasoning as for the free boson, we expect to obtain a partition function in the boundary sector. However, because the S-transform of a character $\chi_i(2il)$ in general does not give non-negative integer coefficients in the loop-channel, it is not clear whether to interpret such a quantity as a partition function counting states of a given excitation level.

As it turns out, the Ishibashi states are not the boundary states itself but only building blocks guaranteed to satisfy the gluing conditions. A true boundary state in general can be expressed as a linear combination of Ishibashi states in the following way

$$\big|B_\alpha\big\rangle = \sum_i B^i_\alpha \, \big|\mathscr{B}_i\big\rangle\!\big\rangle. \tag{2.39}$$

The complex coefficients B^i_α in (2.39) are called reflection coefficients and are very constrained by the so-called Cardy condition. This condition essentially ensures the loop-channel–tree-channel equivalence. Indeed, using relation (2.37) and choosing normalisations such that the action of the CPT operator Θ introduced in (2.28) reads

$$\Theta \, \big|B_\alpha\big\rangle = \sum_i \big(B^i_\alpha\big)^* \, \big|\mathscr{B}_{i^+}\big\rangle\!\big\rangle, \tag{2.40}$$

the cylinder amplitude between two boundary states of the form (2.39) can be expressed as follows

$$\begin{aligned}
\widetilde{\mathscr{Z}}_{\alpha\beta}(l) &= \langle\, \Theta B_\alpha |\, e^{-2\pi l\,\left(L_0+\overline{L}_0-\frac{c+\overline{c}}{24}\right)}\, |B_\beta\rangle \\
&= \sum_{i,j} B^j_\alpha\, B^i_\beta \,\big\langle\!\big\langle \mathscr{B}_{j^+}\big|\, e^{-2\pi l\,\left(L_0+\overline{L}_0-\frac{c+\overline{c}}{24}\right)}\, \big|\mathscr{B}_i\big\rangle\!\big\rangle \\
&= \sum_i B^i_\alpha\, B^i_\beta\; \chi_i\big(2il\big).
\end{aligned}$$

Performing a modular S-transformation $l \mapsto \frac{1}{2t}$ on the characters χ_i, this closed sector cylinder diagram is transformed to the following expression in the open sector

$$\widetilde{\mathscr{Z}}_{\alpha\beta}\big(l\big) \to \widetilde{\mathscr{Z}}_{\alpha\beta}\big(\frac{1}{2t}\big) = \sum_{i,j} B^i_\alpha\, B^i_\beta\, S_{ij}\;\chi_j\big(it\big) = \sum_j n^j_{\alpha\beta}\;\chi_j\big(it\big) = \mathscr{Z}_{\alpha\beta}(t),$$

where S_{ij} is the modular S-matrix and where we introduced the new coefficients $n^i_{\alpha\beta}$. Now, the Cardy condition is the requirement that this expression can be interpreted as a partition function in the open sector. That is, for all pairs of boundary states $|B_\alpha\rangle$ and $|B_\beta\rangle$ in a RCFT the following combinations have to be non-negative integers

$$\boxed{n^j_{\alpha\beta} = \sum_i B^i_\alpha \, B^i_\beta \, S_{ij} \quad \in \quad \mathbb{Z}^+_0} \, .$$

Construction of Boundary States

The Cardy condition just illustrated is very reminiscent of the Verlinde formula, where a similar combination of complex numbers leads to non-negative fusion rule coefficients. For the case of a *charge conjugate* modular invariant partition function, that is when the characters $\chi_i(\tau)$ are combined with $\overline{\chi}_{i^+}(\overline{\tau})$ as $\mathscr{Z} = \sum_i \chi_i(\tau) \; \overline{\chi}_{i^+}(\overline{\tau})$, we can construct a generic solution to the Cardy condition by choosing the reflection coefficients in the following way

$$\boxed{B^i_\alpha = \frac{S_{\alpha\, i}}{\sqrt{S_{0\, i}}}} \, .$$

Note, for each highest weight representation ϕ_i in the RCFT, there not only exists an Ishibashi state but also a boundary state, i.e. the index α in $|B_\alpha\rangle$ also runs from one to the number of HWRs. Employing then the Verlinde formula

$$N^k_{ij} = \sum_n \frac{S_{in} S_{jn} S^{kn}}{S_{0n}} \tag{2.41}$$

and denoting the non-negative, integer fusion coefficients by $N^\alpha_{j\beta}$, we find that the Cardy condition for the coefficients $n^j_{\alpha\beta}$ is always satisfied

$$n^j_{\alpha\beta} = \sum_i \frac{S_{\alpha i} \; S_{\beta i} \; S_{ij}}{S_{0i}} = \sum_i \frac{S_{\alpha i} \; S_{\beta i} \; S^*_{ij^+}}{S_{0i}} = N^{j^+}_{\alpha\beta} \quad \in \quad \mathbb{Z}^+_0 \, .$$

Note that here we employed $S^*_{ij} = S_{ij^+}$ which is verified by noting that $S^{-1} = S^*$ as well as that $S^2 = C$ with C the charge conjugation matrix $C_{ij} = \delta_{ij^+}$.

2.4 CFTs on Non-orientable Surfaces

Up to this point, we have studied Conformal Field Theories defined on the Riemann sphere respectively the complex plane, and on the torus. For Boundary CFTs, the corresponding surfaces are the upper half-plane and the cylinder. We note that all these surfaces are orientable, that is an orientation can be chosen globally.

However, in string theory it is necessary to also define CFTs on non-orientable surfaces. One such surface is the so-called crosscap $\mathbb{RP}^2$ which can be viewed as the two-sphere with opposite points identified. Other non-orientable surfaces are the Möbius strip and the Klein bottle, and a summary of all surfaces relevant for the following is shown in Fig. 2.5.

Orientifolds

Before formulating CFTs on non-orientable surfaces, let us briefly explain the string theory origin of such theories. Recalling the action for a free boson (2.1), we observe that this theory has a discrete symmetry denoted as Ω which takes the form

$$\Omega \ : \ X\big(\tau,\sigma\big) \mapsto \widetilde{X}\big(\tau,\sigma\big) = X\big(\tau,-\sigma\big), \tag{2.42}$$

with τ and σ again world-sheet time and space coordinates. To see that the action (2.1) is invariant under Ω, observe that

$$\begin{aligned} \Omega \ \big(\partial_\sigma X\big)(\tau,\sigma)\ \Omega^{-1} &= -\big(\partial_\sigma X\big)(\tau,-\sigma), \\ \Omega \ \big(\partial_\tau X\big)(\tau,\sigma)\ \Omega^{-1} &= +\big(\partial_\tau X\big)(\tau,-\sigma). \end{aligned} \tag{2.43}$$

Next, let us note that from the mapping (2.42), we see that Ω acts as a world-sheet parity operator. In the string theory picture, this means that Ω changes the orientation of a closed string. As with any other symmetry, we can study the quotient of the original theory by the symmetry. Since Ω changes orientation, in analogy to orbifolds, such a quotient is called an *orientifold*.

The Example of the Free Boson in More Detail

Let us further elaborate on the action of the orientifold projection Ω for the free boson. We first note that $-\sigma$ has to be interpreted properly because we normalised the world-sheet space coordinate as $\sigma \in [0,2\pi)$ for the closed sector and as $\sigma \in [0,\pi]$ in the open sector. The correct identification for $-\sigma$ then reads

$$-\sigma_{\text{closed}} \sim 2\pi - \sigma_{\text{closed}}, \qquad -\sigma_{\text{open}} \sim \pi - \sigma_{\text{open}}.$$

Next, we consider the free boson in the closed sector and express $\partial_\sigma X$ in (2.43) in terms of the Laurent modes j_n and $\bar{j}_n$ using (2.5)

$$\begin{aligned} \Omega \ (\partial_\sigma X)\,(\tau,\sigma)\ \Omega^{-1} &= -\,(\partial_\sigma X)\,(\tau,-\sigma) \\ &\sum_{n\in\mathbb{Z}} \Big(\Omega\, j_n\, \Omega^{-1}\, e^{-n(\tau+i\sigma)} - \Omega\, \bar{j}_n\, \Omega^{-1}\, e^{-n(\tau-i\sigma)} \Big) \\ &= \sum_{n\in\mathbb{Z}} \Big(-j_n\, e^{-n(\tau+i(2\pi-\sigma))} + \bar{j}_n\, e^{-n(\tau-i(2\pi-\sigma))} \Big) . \end{aligned} \tag{2.44}$$

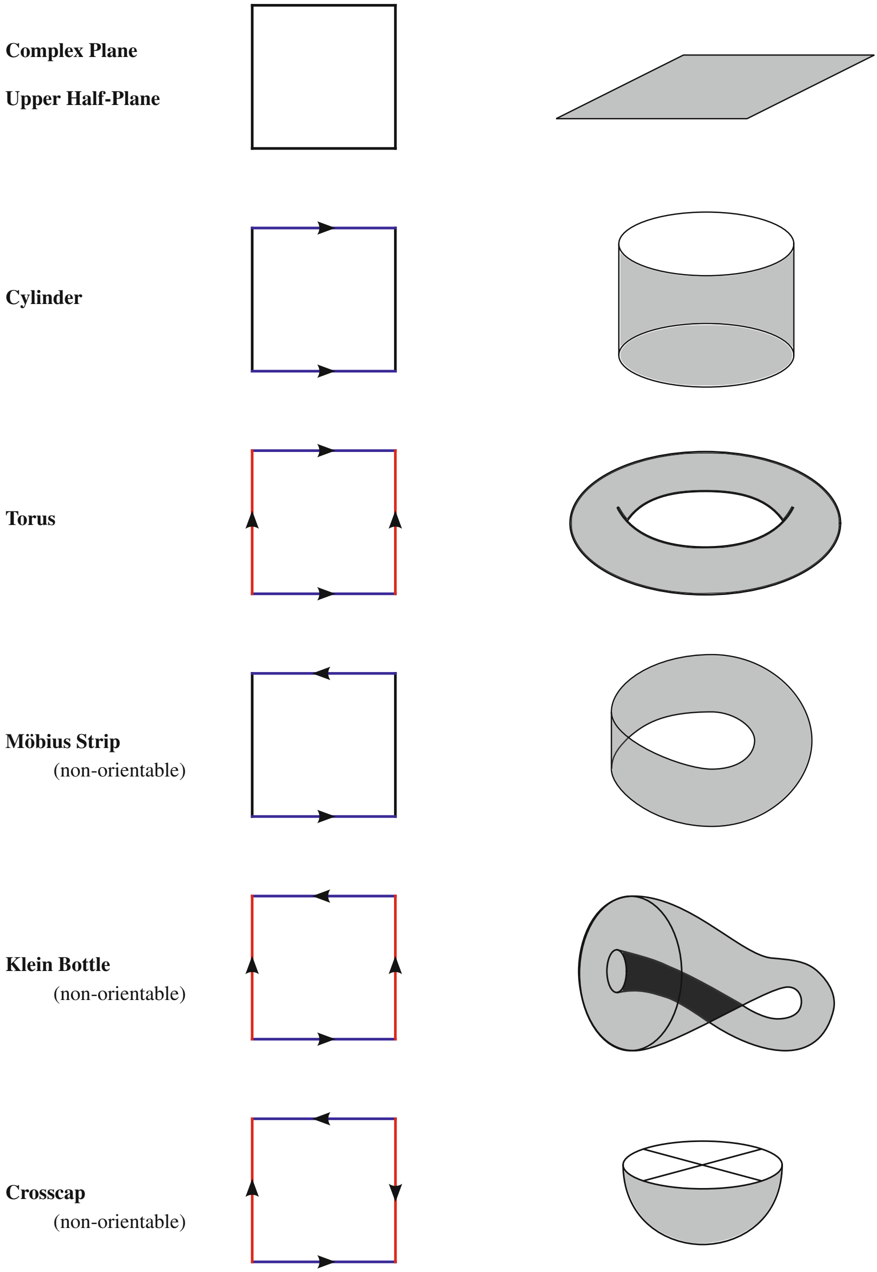

Fig. 2.5 Two-dimensional orientable and non-orientable surfaces. On the *left-hand* side, the fundamental domain can be found and it is indicated how opposite edges are identified leading to the surfaces illustrated on the *right-hand* side. Note that for the identification of opposite edges the orientation given by the arrows is crucial

From this relation we can determine the action of Ω on the modes in the closed sector as follows

$$\boxed{\Omega\, j_n\, \Omega^{-1} = \bar{j}_n, \quad \Omega\, \bar{j}_n\, \Omega^{-1} = j_n.} \tag{2.45}$$

For the open sector, we have to replace 2π on the right-hand side in (2.44) by π which leads to an additional factor of $(-1)^n$. Using then the boundary conditions of an open string (2.6) which relate the Laurent modes as $j_n = \pm\bar{j}_n$, we obtain the action of Ω in the open sector as

$$\boxed{\Omega\, j_n\, \Omega^{-1} = \pm(-1)^n\, j_n} \tag{2.46}$$

where the two signs correspond to Neumann–Neumann respectively Dirichlet–Dirichlet boundary conditions. For the case of mixed boundary conditions, we recall that the Laurent modes have labels $n \in \mathbb{Z} + \frac{1}{2}$ and we note that Ω interchanges the endpoints of an open string as well as the boundary conditions. In particular, we find

$$\Omega\, j_n^{(\mathrm{N,D})}\, \Omega^{-1} = -(-1)^n\, j_n^{(\mathrm{D,N})}, \qquad \Omega\, j_n^{(\mathrm{D,N})}\, \Omega^{-1} = +(-1)^n\, j_n^{(\mathrm{N,D})}. \tag{2.47}$$

Partition Function: Klein Bottle

Let us now consider partition functions for general orientifold theories. We start with the usual form of a modular invariant partition function in a CFT

$$\mathscr{Z}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathscr{H}\times\overline{\mathscr{H}}}\Big(q^{L_0-\frac{c}{24}}\ \overline{q}^{\overline{L}_0-\frac{c}{24}}\Big), \tag{2.48}$$

where we indicated the trace over the combined Hilbert space $\mathscr{H} \times \overline{\mathscr{H}}$ explicitly. Next, we generalise our findings from the example of the free boson and define the action of the world-sheet parity operator Ω on the Hilbert space as follows

$$\Omega \mid i, \bar{j}\ \rangle = \pm \mid \Omega(j), \overline{\Omega(i)}\ \rangle\ , \tag{2.49}$$

where i denotes a state in the holomorphic sector of the theory and $\bar{j}$ stands for the anti-holomorphic sector. The two different signs originate from the two possibilities of Ω acting on the vacuum $|0\rangle$ compatible with the requirement that $\Omega^2 = \mathbb{1}$. The simplest choice for $\Omega(i)$ is $\Omega(i) = i$, but also more general $\mathbb{Z}_2$ involutions are possible, for instance $\Omega(i) = i^+$ where $^+$ denotes charge conjugation.

In order to obtain the partition function we project the entire Hilbert space $\mathscr{H} \times \overline{\mathscr{H}}$ onto those states which are invariant under Ω, i.e. we introduce the projection operator $\frac{1}{2}(1+\Omega)$ into the partition function (2.48). We therefore obtain

$$\begin{aligned}\mathscr{Z}^{\Omega}(\tau, \overline{\tau}) &= \mathrm{Tr}_{\mathscr{H}\times\overline{\mathscr{H}}}\left(\frac{1+\Omega}{2}\ q^{L_0-\frac{c}{24}}\ \overline{q}^{\overline{L}_0-\frac{c}{24}}\right) \\ &= \frac{1}{2}\ \mathscr{Z}(\tau, \overline{\tau}) + \frac{1}{2}\ \mathrm{Tr}_{\mathscr{H}\times\overline{\mathscr{H}}}\Big(\ \Omega\ \ q^{L_0-\frac{c}{24}}\ \overline{q}^{\overline{L}_0-\frac{c}{24}}\Big).\end{aligned}$$

The first term is just one-half of the torus partition function which we already studied. Let us therefore turn to the second term

$$\mathscr{Z}^{\mathscr{K}}(\tau,\overline{\tau}) = \mathrm{Tr}_{\mathscr{H}\times\overline{\mathscr{H}}}\Big(\,\Omega\; q^{L_0-\frac{c}{24}}\;\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\Big). \tag{2.50}$$

The insertion of Ω into the trace has the effect that by looping once around the direction τ of a torus, the closed string comes back to itself up to the action of Ω, that is up to a change of orientation. Geometrically, such a diagram is not a torus but a Klein bottle illustrated in Fig. 2.5. This is also the reason for the superscript $\mathscr{K}$ of the partition function and for its name: the Klein bottle partition function.

We will now specify the action of Ω as $\Omega(i) = i$ and $\Omega\;|0\rangle = +|0\rangle$ in order to make (2.50) more explicit. For this choice we obtain

$$\big\langle\, i,\overline{j}\;\big|\;\Omega\;\big|\;i,\overline{j}\,\big\rangle = \big\langle\, i,\overline{j}\;\big|\;j,\overline{i}\,\big\rangle = \delta_{ij}, \tag{2.51}$$

where we used Eq. (2.49). Therefore, only left-right symmetric states $|i,\overline{i}\rangle$ contribute to the trace in (2.50) and we can simplify the partition function as follows

$$\begin{aligned}
\mathscr{Z}^{\mathscr{K}}(\tau,\overline{\tau}) &= \mathrm{Tr}_{\mathscr{H}\times\overline{\mathscr{H}}}\Big(\,\Omega\; q^{L_0-\frac{c}{24}}\;\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\Big)\\
&= \sum_{i,\overline{j}}\big\langle\, i,\overline{j}\;\big|\;\Omega\; q^{L_0-\frac{c}{24}}\;\Omega^{-1}\;\Omega\;\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\;\Omega^{-1}\;\Omega\;\big|\;i,\overline{j}\,\big\rangle\\
&= \sum_{i}\big\langle\, i,\overline{i}\;\big|\;\Omega\; q^{L_0-\frac{c}{24}}\;\Omega^{-1}\;\Omega\;\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\;\Omega^{-1}\;\big|\;i,\overline{i}\,\big\rangle
\end{aligned}$$

where we employed (2.51). Since only the diagonal subset will contribute to the trace, we see from this expression that effectively L_0 and $\overline{L}_0$ as well as c and $\overline{c}$ can be identified. Observing finally that $q\overline{q} = e^{-4\pi\tau_2}$, we arrive at

$$\mathscr{Z}^{\mathscr{K}}(\tau,\overline{\tau}) = \sum_{i}\big\langle\, i,\overline{i}\;\big|\;\big(q\,\overline{q}\big)^{L_0-\frac{c}{24}}\;\big|\;i,\overline{i}\,\big\rangle = \mathrm{Tr}_{\mathscr{H}_{\mathrm{sym.}}}\Big(\,e^{-4\pi t\left(L_0-\frac{c}{24}\right)}\,\Big), \tag{2.52}$$

with $t = \tau_2$ and $\mathscr{H}_{\mathrm{sym.}}$ denoting the states $\big|\;i,\overline{i}\;\big\rangle$ in the Hilbert space which are combined in a left-right symmetric way.

Free Boson III: Klein Bottle Partition Function (Loop-Channel)

Let us now determine the Klein bottle partition function for the example of the free boson. As it is evident from (2.52), this partition function is the character of the free boson theory with modular parameter $\tau = 2it$. However, for the momentum contribution, we need to perform a calculation similar to the one in the open sector shown on page 57. In particular, from (2.52) we extract the j_0 part, replace the sum by an integral and compute

$$\mathrm{Tr}_{\mathscr{H}_{\mathrm{sym.}}}\Big(\,e^{-4\pi t\,\frac{1}{2}\,j_0^2}\,\Big)\quad\longrightarrow\quad\int_{-\infty}^{+\infty} d\pi_0\; e^{-4\pi t\,\frac{1}{2}\,\pi_0^2} = \frac{1}{\sqrt{2t}}\,,$$

where we observed that in the closed sector $j_0 = \pi_0$. Combining this result with the character of the free boson theory, we obtain the following expression for the full Klein bottle partition function

$$\mathscr{Z}^{\mathscr{K}}_{\text{bos.}}(\tau, \overline{\tau}) = \frac{1}{\sqrt{2t}} \, \frac{1}{\eta\big(2it\big)} \, . \tag{2.53}$$

Partition Function: Möbius Strip

After having studied CFTs on non-orientable surfaces in the closed sector, let us now turn to the open sector. Again, the partition function has to be projected onto states invariant under the orientifold action Ω. Following the same steps as for the closed sector, we find

$$\mathscr{Z}^{\Omega}(t) = \text{Tr}_{\mathscr{H}_B}\left(\frac{1+\Omega}{2} \, e^{-2\pi t\left(L_0 - \frac{c}{24}\right)} \right) = \frac{1}{2}\, \mathscr{Z}^{\mathscr{C}}(t) + \frac{1}{2}\, \text{Tr}_{\mathscr{H}_B}\Big(\Omega \, e^{-2\pi t\left(L_0-\frac{c}{24}\right)} \Big).$$

The first term is the cylinder amplitude, but the second term

$$\mathscr{Z}^{\mathscr{M}}(t) = \text{Tr}_{\mathscr{H}_{\mathscr{B}}}\Big(\Omega \, e^{-2\pi t\left(L_0-\frac{c}{24}\right)} \Big) \tag{2.54}$$

describes an open string whose orientation changes when looping along the t direction. The geometry of such a surface is that of a Möbius strip also shown in Fig. 2.5. The corresponding partition function is called the Möbius strip partition function and hence the superscript $\mathscr{M}$.

Free Boson IV: Möbius Strip Partition Function (Loop-Channel)

We now calculate the Möbius strip partition function for the free boson. The Hilbert space of the the free boson is spanned by states of the form

$$|n_1, n_2, n_3, \ldots\rangle = j_{-1}^{n_1} j_{-2}^{n_2} j_{-3}^{n_3} \ldots |0\rangle, \quad \text{with}\, n_i \geq 0\,, \tag{2.55}$$

where j_n are the modes of the current $j(z)$. Recalling then the mapping (2.46), we see that the action of Ω on a state in the Hilbert space is

$$\Omega \, \big|n_1, n_2, n_3, \ldots\big\rangle = \prod_{k=1}^{\infty} (\pm 1)^{n_k} \, (-1)^{k\, n_k} \, \big|n_1, n_2, n_3, \ldots\big\rangle \, .$$

Taking the action of Ω into account we arrive at

$$\begin{aligned} \text{Tr}_{\mathscr{H}_{\mathscr{B}}}\Big(\Omega \, q^{L_0-\frac{c}{24}} \Big)\Big|_{\text{without } j_0} &= q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} (\pm 1)^{n_k} \, (-1)^{k\, n_k} \, q^{k\, n_k} \\ &= e^{\frac{\pi i}{24}} \, \big(-q\big)^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1 \mp \big(-q\big)^k} \, . \end{aligned} \tag{2.56}$$

We also note that $-q$ with modular parameter τ can be expressed as $+q$ with modular parameter $\tau+\frac{1}{2}$.

For Neumann–Neumann boundary conditions, i.e. for the upper sign in the expression above, we employ the definition of the Dedekind η-function (2.14). However, since the momentum π_0 is unconstrained, we compute

$$\mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}\Big(\,\Omega\, e^{-2\pi t\,\frac{1}{2}\,j_0^2}\,\Big)\quad\longrightarrow\quad\int_{-\infty}^{+\infty} d\pi_0\, e^{-2\pi t\,\frac{1}{2}\,(2\pi_0)^2}=\frac{1}{2\,\sqrt{t}}\,,$$

where we used that j_0 is invariant under Ω as well as that in the open sector $j_0=2\pi_0$. The full Möbius strip partition function in the Neumann–Neumann sector then reads

$$\mathcal{Z}^{\mathcal{M}\,(\mathrm{N,N})}_{\mathrm{bos.}}(t)=e^{\frac{\pi i}{24}}\,\frac{1}{2\sqrt{t}}\,\frac{1}{\eta\big(\frac{1}{2}+it\big)}\,. \tag{2.57}$$

For Dirichlet–Dirichlet conditions, that means the lower sign in (2.56), we find for instance from (2.46) that $j_0=0$ so that there is no additional factor from the momentum integration. Recalling the expression for the ϑ_2-function from equation (2.31) we obtain

$$\mathcal{Z}^{\mathcal{M}\,(\mathrm{D,D})}_{\mathrm{bos.}}(t)=e^{\frac{\pi i}{24}}\,\sqrt{2}\,\sqrt{\frac{\eta\big(\frac{1}{2}+it\big)}{\vartheta_2\big(\frac{1}{2}+it\big)}}\,. \tag{2.58}$$

For mixed boundary conditions, the Möbius strip partition function vanishes as Ω exchanges Neumann–Dirichlet with Dirichlet–Neumann conditions and so there is no contribution to the trace.

Loop-Channel–Tree-Channel Equivalence

For the cylinder partition function, we have seen that the result in the open and closed sector are related via a modular S-transformation. One might therefore suspect that this equivalence between partition functions and overlaps of boundary states can also be found for non-orientable surfaces.

This is indeed the case which we illustrate in Fig. 2.6 for the Klein bottle partition function.

1. The fundamental domain of the Klein bottle shown in Fig. 2.6a is that of a torus up to a change of orientation. However, as opposed to the torus, the modular parameter of the Klein bottle is purely imaginary.
2. In Fig. 2.6b, the fundamental domain is halved and the identification of segments and points is indicated explicitly by arrows and symbols.
3. Next, we shift one half of the fundamental domain as shown in Fig. 2.6c.
4. In Fig. 2.6d, the shifted part has been flipped and the appropriate edges have been identified.

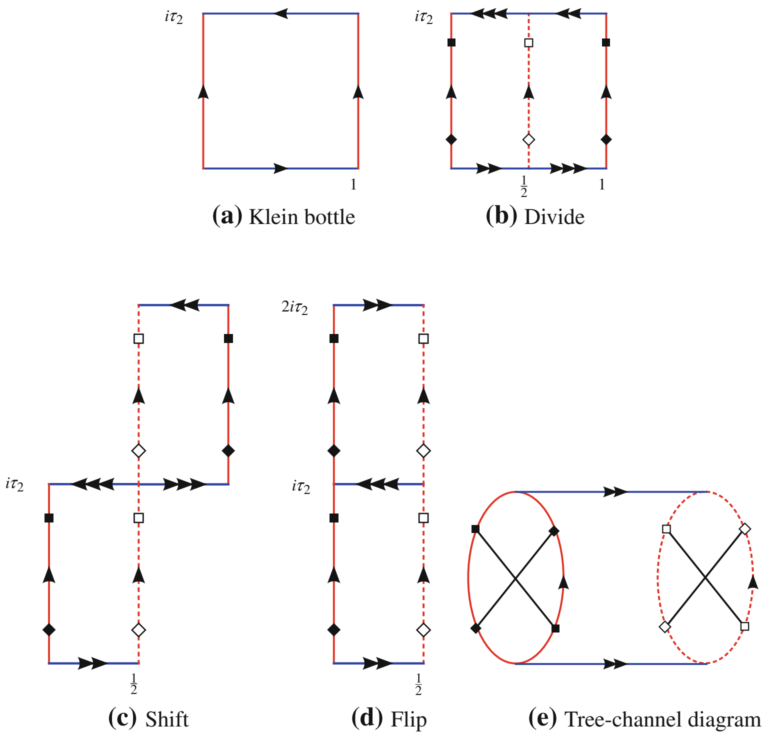

Fig. 2.6a–e Transformation of the fundamental domain of the Klein bottle to a tree-channel diagram between two crosscaps

5. A fundamental domain of this form can be interpreted as a cylinder between two crosscaps as illustrated in 2.6e.

Analogous to the cylinder diagram (2.26), we expect now that the Klein bottle amplitude can be computed as the overlap of two so-called *crosscap* states $|C\rangle$ in the following way

$$\tilde{\mathscr{Z}}^{\mathscr{K}}(l) = \langle \Theta\, C|\, e^{-2\pi l\left(L_0+\overline{L}_0-\frac{c+\overline{c}}{24}\right)}\, |C\rangle\,. \tag{2.59}$$

Considering then again Fig. 2.6d we find the modular parameter in the tree- and loop-channel as

$$\tau_{\text{open}} = \frac{\alpha_2}{\alpha_1} = \frac{2it}{\frac{1}{2}} = 4it, \qquad \tau_{\text{closed}} = \frac{\alpha_2}{\alpha_1} = \frac{i}{l},$$

and because of the tree-channel–loop-channel equivalence, they have to be equal. This implies that the length of the cylinder in Fig. 2.6e and equation (2.59) can be expressed as $l = \frac{1}{4t}$. We will elaborate on these crosscap states in more detail in the next section.

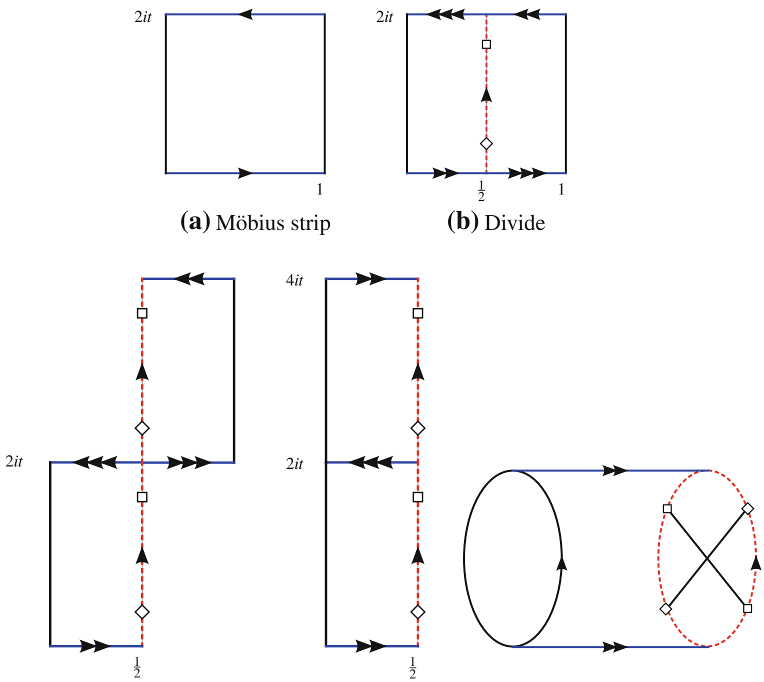

Fig. 2.7a–e Transformation of the fundamental domain of the Möbius strip to a tree-channel diagram between an ordinary boundary and a crosscap

For the Möbius strip amplitude, we can apply the same cuts and shifts as for the Klein bottle amplitude. As it is illustrated in Fig. 2.7, the resulting tree-channel diagram is a cylinder between an ordinary boundary and a crosscap. We thus expect that in the tree-channel, we can calculate the Möbius strip in the following way

$$\tilde{\mathscr{Z}}^{\mathscr{M}}(l) = \langle \Theta \; C |\, e^{-2\pi l \left(L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24} \right)} \, |B\rangle \, . \tag{2.60}$$

Finally, for the modular parameters in the tree- and loop-channel, we obtain

$$\tau_{\text{open}} = \frac{\alpha_2}{\alpha_1} = \frac{4it}{\frac{1}{2}} = 8it, \quad \tau_{\text{closed}} = \frac{\alpha_2}{\alpha_1} = \frac{i}{l},$$

which leads us to $l = \frac{1}{8t}$.

Remarks

- A summary of the various loop-channel and tree-channel expressions together with their modular parameters can be found in Table 2.1.

Table 2.1 Summary of loop-channel–tree-channel relations

	Loop-channel	$\tau = \ldots$	Tree-channel	$l = \ldots$
Torus	$\mathrm{Tr}_{\mathcal{H}\times\overline{\mathcal{H}}}\left(q^{L_0-\frac{c}{24}}\,\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right)$	$\tau=\tau_1+i\tau_2$		
Cylinder	$\mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}\left(q^{L_0-\frac{c}{24}}\right)$	$\tau=it$	$\langle\Theta\, B\|\, e^{-2\pi l\left(L_0+\overline{L}_0-\frac{c+\overline{c}}{24}\right)}\, \|B\rangle$	$l=\frac{1}{2t}$
Klein bottle	$\mathrm{Tr}_{\mathcal{H}_{\text{sym.}}}\left(q^{L_0-\frac{c}{24}}\right)$	$\tau=2it$	$\langle\Theta\, C\|\, e^{-2\pi l\left(L_0+\overline{L}_0-\frac{c+\overline{c}}{24}\right)}\, \|C\rangle$	$l=\frac{1}{4t}$
Möbius strip	$\mathrm{Tr}_{\mathcal{H}_{\mathcal{B}}}\left(\Omega\, q^{L_0-\frac{c}{24}}\right)$	$\tau=it$	$\langle\Theta\, B\|\, e^{-2\pi l\left(L_0+\overline{L}_0-\frac{c+\overline{c}}{24}\right)}\, \|C\rangle$	$l=\frac{1}{8t}$

- Almost all Ω projected CFTs in the closed sector are inconsistent and require the introduction of appropriate boundaries with corresponding boundary states. In string theory, these conditions are known as the tadpole cancellation conditions which we will discuss in the final Sect. 2.7.

2.5 Crosscap States for the Free Boson

Similarly to boundary states which describe the coupling of the closed sector of a CFT to a boundary, for orientifold theories there should exist a coherent state describing the coupling of the closed sector to the crosscap. In particular, analogous to the observation that a world-sheet boundary defines (or is confined to) a space-time D-brane, we say that a world-sheet crosscap defines (or is confined to) a space-time orientifold plane.

In this section, we will discuss crosscap states for the example of the free boson, and in the next section we are going to generalise the appearing structure to RCFTs.

Crosscap Conditions

We start our study of crosscap states by recalling the transformation of the Klein bottle respectively Möbius strip amplitude from the open to the closed sector shown in Figs. 2.6 and 2.7. There, we encountered a new type of boundary, the so-called crosscap, where opposite points are identified. For the construction of the crosscap state, we will employ this geometric intuition, however, later we also compute the tree-channel Klein bottle and Möbius strip amplitudes to check that they are indeed related via a modular transformation to the result in the loop-channel.

As it is illustrated in Fig. 2.8, in an appropriate coordinate system on a crosscap, we observe that points x on a circle are identified with $-x$. Parametrising this circle by $\sigma \in [0, 2\pi)$, we see that the identification $x \sim -x$ corresponds to $\sigma \sim \sigma + \pi$. For a closed string on a crosscap, we thus infer that the field X at (τ, σ) should be identified with the field X at $(\tau, \sigma + \pi)$. More concretely, this reads

$$X\big(\tau,\sigma\big)\,\big|C\big\rangle = X\big(\tau,\sigma+\pi\big)\,\big|C\big\rangle, \tag{2.61}$$

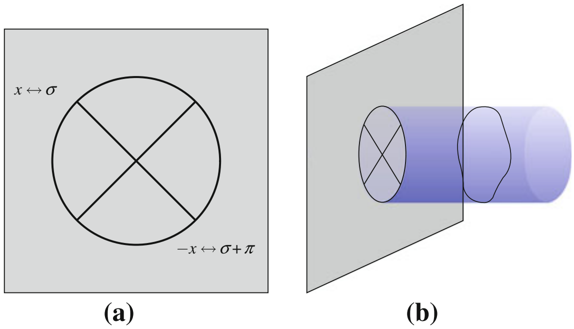

Fig. 2.8 Illustration of how points are identified on a crosscap, and how a closed string couples to a crosscap. **a** Identification of points on a crosscap. **b** Closed string at a cross-cap

and for the derivatives with respect to τ and σ, we impose

$$\begin{aligned} \big(\partial_\sigma X\big)(\tau,\sigma)\,\big|C\big\rangle &= +\big(\partial_\sigma X\big)(\tau,\sigma+\pi)\,\big|C\big\rangle, \\ \big(\partial_\tau X\big)(\tau,\sigma)\,\big|C\big\rangle &= -\big(\partial_\tau X\big)(\tau,\sigma+\pi)\,\big|C\big\rangle. \end{aligned} \tag{2.62}$$

Let us now choose coordinates such that $\tau = 0$ describes the field $X(\tau,\sigma)$ at the crosscap $|C\rangle$. Using then the Laurent mode expansions (2.5) as well as (2.62) with $\tau = 0$, we obtain that

$$\begin{aligned} \big(j_n - \bar{j}_{-n}\big)\,\big|C\big\rangle &= +(-1)^n\big(j_n - \bar{j}_{-n}\big)\,\big|C\big\rangle, \\ \big(j_n + \bar{j}_{-n}\big)\,\big|C\big\rangle &= -(-1)^n\big(j_n + \bar{j}_{-n}\big)\,\big|C\big\rangle, \end{aligned}$$

where, similarly as in the computation for the boundary states, we performed a change in the summation index $n \to -n$. By adding or subtracting these two expressions, we arrive at the gluing conditions for crosscap states

$$\boxed{\big(j_n + (-1)^n\,\bar{j}_{-n}\big)\,\big|C_{\mathrm{O1}}\big\rangle = 0}\,. \tag{2.63}$$

Note that we added the label O1 which stands for *orientifold one-plane*. The reason is that by inserting the expansion (2.8) of $X(\tau,\sigma)$ into (2.61), we see that the center of mass coordinate x_0 of the closed string is unconstrained. In the target space, the location of the crosscap is called an orientifold plane which in the present case fills out one dimension because there is no constraint on x_0. This explains the notation above.

Construction of Crosscap States

Apart from the factor $(-1)^n$, the gluing conditions (2.63) are very similar to those of a boundary state (2.20) with Neumann conditions. The solution to the gluing conditions is therefore also similar to the Neumann boundary state and reads

$$\boxed{\big|C_{\mathrm{O}1}\big\rangle = \frac{\kappa}{\sqrt{2}} \exp\left(- \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\, j_{-k}\, \bar{j}_{-k} \right) \big|\, 0\, \big\rangle} \tag{2.64}$$

where we employed (2.33) and allowed for a relative normalisation factor κ between the boundary state with Neumann conditions $|B_{\mathrm{N}}\rangle$ and the crosscap state $|C_{\mathrm{O}1}\rangle$.

The proof that (2.64) is a solution to the gluing conditions (2.63) is analogous to the one shown on page 62. Note in particular, the crosscap state can be written as

$$\big|C_{\mathrm{O}1}\big\rangle = \frac{\kappa}{\sqrt{2}} \sum_{\mathbf{m}} |\mathbf{m}\rangle \otimes |U\overline{\mathbf{m}}\rangle \tag{2.65}$$

with the anti-unitary operator U acting in the following way

$$U\, \bar{j}_n\, U^{-1} = -(-1)^n\, \big(\bar{j}_{-n}\big)^{\dagger}. \tag{2.66}$$

Remark

Let us make the following remark. In equation (2.42), we have chosen a specific orientifold action Ω for the fields $X(\tau, \sigma)$ which leaves the action (2.1) invariant. However, we can also accompany Ω by another operation, for instance $\mathscr{R} : X(\tau, \sigma) \mapsto -X(\tau, \sigma)$, which also leaves (2.1) invariant. The combined action then reads

$$\Omega\, \mathscr{R} :\; X\big(\tau, \sigma\big) \mapsto \widetilde{X}\big(\tau, \sigma\big) = -X\big(\tau, -\sigma\big).$$

Note that this orientifold action describes a different theory and that there is no direct relation to the results obtained previously.

Performing the same steps as before, we arrive at the following expressions for the combined action $\Omega\mathscr{R}$ on the Laurent modes j_n and $\bar{j}_n$

$$\begin{array}{ll} \text{closed sector} & \Omega\, \mathscr{R}\, j_n\big(\Omega\, \mathscr{R}\big)^{-1} = -\bar{j}_n, \quad \Omega\, \mathscr{R}\, \bar{j}_n\big(\Omega\, \mathscr{R}\big)^{-1} = -j_n, \\ \text{open sector} & \Omega\, \mathscr{R}\, j_n\big(\Omega\, \mathscr{R}\big)^{-1} = \mp(-1)^n\, j_n. \end{array}$$

For the action of $\mathscr{R}$ on the states, we find

$$\mathscr{R}\, \big|\mathbf{m}\big\rangle = \big(-1\big)^{\sum_k m_k}\, \big|\mathbf{m}\big\rangle,$$

which results in additional factors of (-1) in various loop-channel amplitudes. Concerning the construction of crosscap states, also the identification (2.61) receives an factor of (-1) which results in gluing conditions of the form

$$\big(j_n - (-1)^n\, \bar{j}_{-n}\big)\, \big|C_{\mathrm{O}0}\big\rangle = 0,$$

which is similar to the Dirichlet conditions for boundary states. The notation O0 indicates that the orientifold plane does not extend in one dimension but is only a point.

And indeed, using the expansion (2.8) of $X(\tau,\sigma)$ for $X(\tau,\sigma)|C\rangle = -X(\tau,\sigma)|C\rangle$, we see that the center of mass coordinate x_0 is constrained to $x_0 = 0$. Finally, we note that the solution to the gluing conditions in the present case reads

$$\big|C_{\mathrm{O}0}\big\rangle = \kappa\, \exp\left(+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\, j_{-k}\, \bar{j}_{-k} \right) \big|\, 0\, \big\rangle .$$

After this remark about a different possibility for an orientifold projection, let us continue our studies with our original choice (2.42) which leads to O1 crosscap states $|C_{\mathrm{O}1}\rangle$.

Free Boson V: Klein Bottle Amplitude (Tree-Channel)

As we have argued in the previous section, from the overlap of two crosscap states we can compute the Klein bottle amplitude (2.59) in the closed sector, that is in the tree-channel. In order to do so, we recall the crosscap state (2.65) with the action of U given in (2.66). Noting for a basis state (2.23) that

$$U\,\big|\mathbf{m}\big\rangle = \prod_{k=1}^{\infty} (-1)^{m_k}\, (-1)^{m_k k}\, \big|\mathbf{m}\big\rangle, \tag{2.67}$$

and following the same calculation as on page 65 for the overlap of two boundary states in the Neumann–Neumann sector, we obtain

$$\tilde{\mathcal{Z}}^{\mathcal{K}\,(\mathrm{O}1,\mathrm{O}1)}_{\mathrm{bos.}}(l) = \big\langle \Theta\ C_{\mathrm{O}1}\big|\, e^{-2\pi l (L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24})}\, \big|C_{\mathrm{O}1}\big\rangle = \frac{\kappa^2}{2\,\eta(2il)} . \tag{2.68}$$

Note that Θ is again the CPT operator introduced in equation (2.28) which, in particular, acts as complex conjugation on numbers. Finally, recalling from Table 2.1 the relation $l = \frac{1}{4t}$ between the tree-channel and loop-channel modular parameters, we find the loop-channel amplitude to be of the form

$$\tilde{\mathcal{Z}}^{\mathcal{K}\,(\mathrm{O}1,\mathrm{O}1)}_{\mathrm{bos.}}(l) = \frac{\kappa^2}{2\,\eta(2il)} \xrightarrow{\;l=\frac{1}{4t}\;} \frac{\kappa^2}{2\,\eta\left(-\frac{1}{2it}\right)} = \frac{\kappa^2}{2\,\sqrt{2t}}\, \frac{1}{\eta(2it)} ,$$

where we employed the modular property of the Dedekind η-function from equation (2.32). By comparing with the loop-channel result (2.53), we can now fix

$$\boxed{\kappa = \sqrt{2}} .$$

Free Boson VI: Möbius-Strip Amplitude (Tree-Channel)

Eventually, we compute the overlap of a crosscap state and a boundary state giving the tree-level Möbius strip amplitude. Employing equation (2.67) and performing a similar calculation as on page 65, we find for the Möbius strip diagram in the Neumann sector that

$$
\begin{aligned}
\widetilde{\mathscr{Z}}^{\mathscr{M}(\mathrm{O1,N})}_{\mathrm{bos.}}(l) &= \langle \Theta\, C_{\mathrm{O1}} |\, e^{-2\pi l (L_0+\overline{L}_0-\frac{c+\overline{c}}{24})}\, | B_N \rangle \\
&= \frac{1}{\sqrt{2}}\, e^{\frac{\pi l}{6}} \prod_{k=1}^{\infty} \frac{1}{1-\left(-e^{-4\pi l}\right)^k} \\
&= \frac{1}{\sqrt{2}}\, e^{\frac{\pi i}{24}}\, \frac{1}{\eta\left(\frac{1}{2}+2il\right)}
\end{aligned}
\tag{2.69}
$$

where we expressed (-1) as $e^{\pi i}$ and absorbed the additional factor into the definition of the modular parameter. The computation of the Möbius strip amplitude in the Dirichlet sector is very similar to the Neumann sector. We find

$$
\begin{aligned}
\widetilde{\mathscr{Z}}^{\mathscr{M}(\mathrm{O1,D})}_{\mathrm{bos.}}(l) &= \langle \Theta\, C_{\mathrm{O1}} |\, e^{-2\pi l (L_0+\overline{L}_0-\frac{c+\overline{c}}{24})}\, | B_D \rangle \\
&= e^{\frac{\pi l}{6}} \prod_{k=1}^{\infty} \frac{1}{1+\left(-e^{-4\pi l}\right)^k} \\
&= \sqrt{2}\, e^{\frac{\pi i}{24}} \sqrt{\frac{\eta\left(\frac{1}{2}+2il\right)}{\vartheta_2\left(\frac{1}{2}+2il\right)}}
\end{aligned}
$$

where we used again the definition of the ϑ-functions. The momentum integration in this sector is trivial since j_0 acting on the crosscap state vanishes. This is again similar to the computation of the cylinder amplitude for mixed boundary conditions shown on page 67.

Modular transformations

After having computed the tree-channel Möbius strip amplitudes, we would like to transform these results to the loop-channel via the relation $l = \frac{1}{8t}$. However, by comparing with the loop-channel results (2.57) and (2.58), we see that this cannot be achieved by a modular S-transformation. Instead, we have to perform the following combination of T- and S-transformations

$$
\mathscr{P} = TST^2\, S. \tag{2.70}
$$

For the η-function with shifted argument, this transformation reads

$$\begin{aligned}
\eta\left(\frac{1}{2}+2il\right) \quad &\xrightarrow{S} \quad \eta\left(-\frac{1}{\frac{1}{2}+2il}\right)\sqrt{\frac{i}{\frac{1}{2}+2il}} \\
&\xrightarrow{T^2} \quad \eta\left(+\frac{4il}{\frac{1}{2}+2il}\right)\sqrt{\frac{i}{\frac{1}{2}+2il}}\, e^{-\frac{\pi i}{6}} \\
&\xrightarrow{S} \quad \eta\left(-\frac{\frac{1}{2}+2il}{4il}\right)\sqrt{\frac{i}{\frac{1}{2}+2il}}\sqrt{\frac{\frac{1}{2}+2il}{4l}}\, e^{-\frac{\pi i}{6}} \\
&\xrightarrow{T} \quad \eta\left(\frac{1}{2}+\frac{i}{8l}\right)\frac{1}{\sqrt{4l}}\sqrt{i}\, e^{-\frac{\pi i}{6}}\, e^{-\frac{\pi i}{12}} \\
&\qquad = \eta\left(\frac{1}{2}+\frac{i}{8l}\right)\frac{1}{\sqrt{4l}}
\end{aligned}$$

where in the last step we employed that $\sqrt{i} = e^{\frac{\pi i}{4}}$. For the Möbius strip amplitude with Neumann boundary conditions, we then compute the transformation from the tree-channel to the loop-channel as follows

$$\tilde{\mathscr{Z}}^{\mathscr{M}(\mathrm{O1,N})}_{\mathrm{bos.}}(l) = \frac{e^{\frac{\pi i}{24}}}{\sqrt{2}}\,\frac{1}{\eta\left(\frac{1}{2}+2il\right)} \quad \xrightarrow[l=\frac{1}{8t}]{\mathscr{P}} \quad e^{\frac{\pi i}{24}}\,\frac{1}{2\sqrt{t}}\,\frac{1}{\eta\left(\frac{1}{2}+it\right)}.$$

By comparing with the loop-channel result (2.57), we have verified the loop-channel—tree-channel equivalence for the Möbius strip amplitude in the Neumann sector.

In passing, we note that the Möbius strip loop- and tree-channel amplitudes for the Dirichlet sector are also related via a modular $\mathscr{P}$-transformation. In the same manner as above, one can then establish the loop-channel—tree-channel equivalence.

New Characters

In the last paragraph of this section, let us introduce a more general notation for the Möbius strip characters. We define *hatted* characters $\widehat{\chi}(\tau)$ in terms of the usual characters $\chi(\tau)$ as follows

$$\widehat{\chi}(\tau) = e^{-\pi i\,(h-\frac{c}{24})}\,\chi\left(\tau+\frac{1}{2}\right). \tag{2.71}$$

The action of the $\mathscr{P}$-transformation (2.70) for the new characters $\widehat{\chi}(\tau)$ can be deduced as follows. From the mapping of the modular parameter $\tau = 2il$ under the combination of S- and T-transformations

$$2il \quad \xrightarrow{T^{\frac{1}{2}}} \quad 2il+\frac{1}{2} \quad \xrightarrow{TST^2S} \quad \frac{i}{8l}+\frac{1}{2} \quad \xrightarrow{T^{-\frac{1}{2}}} \quad \frac{i}{8l},$$

we can infer the transformation of the hatted characters $\widehat{\chi}(\tau)$ as

$$\widehat{\chi}_i\Big(\frac{i}{8l}\Big) = \sum_j P_{ij}\, \widehat{\chi}_j\big(2il\big) \qquad \text{with} \qquad P = T^{\frac{1}{2}}\, S\, T^2\, S\, T^{\frac{1}{2}},$$

where $T^{\frac{1}{2}}$ is defined as the square root of the entries in the diagonal matrix

$$T_{ij} = \delta_{ij} e^{2\pi i (h_i - \frac{c}{24})}. \tag{2.72}$$

Note that the P-transformation corresponds to the S-transformation of the usual characters, in particular, P realises the loop-channel–tree-channel equivalence.

Finally, using some properties of the S-matrix

$$S^\dagger S = S S^\dagger = \mathbb{1}, \qquad S^T = S \tag{2.73}$$

as well as the relation $S^2 = (ST)^3 = C$ with C the charge conjugation matrix

$$\mathscr{P}^2 = C, \qquad P^2 = C, \qquad PP^\dagger = P^\dagger P = \mathbb{1}, \qquad P^T = P. \tag{2.74}$$

2.6 Crosscap States for RCFTs

Let us now generalise the construction of crosscap states to Conformal Field Theories without a Lagrangian description. In particular, we focus on RCFTs and we mainly state the general structure without explicit derivation.

Construction of Crosscap States

The crosscap gluing conditions for the generators of a symmetry algebra $\mathscr{A} \otimes \overline{\mathscr{A}}$ are in analogy to the conditions (2.63) for the example of the free boson and read

$$\begin{aligned} \big(\, L_n - (-1)^n\, \overline{L}_{-n} \,\big)\, \big|C\big\rangle = 0 \quad & \text{conformal symmetry,} \\ \big(\, W^i_n - (-1)^n\, (-1)^{h^i}\, \overline{W}^i_{-n} \,\big)\, \big|C\big\rangle = 0 \quad & \text{extended symmetries,} \end{aligned} \tag{2.75}$$

with again $h^i = h(W^i)$. For $\mathscr{A} = \overline{\mathscr{A}}$ and $\overline{\mathscr{H}}_i = \mathscr{H}^+_i$, we can define crosscap Ishibashi states $|\mathscr{C}_i\rangle\rangle$ satisfying the crosscap gluing conditions. A crosscap state $|C\rangle$ can then be expressed as a linear combination of the crosscap Ishibashi states in the following way

$$\big|C\big\rangle = \sum_i \varGamma^i \mid \mathscr{C}_i\rangle\rangle\ . \tag{2.76}$$

In fact, the crosscap Ishibashi states and the boundary Ishibashi states are related via

$$|\,\mathscr{C}_i\rangle\rangle = e^{\pi\, i(L_0 - h(\phi_i))}\, |\mathscr{B}_i\rangle\rangle\ . \tag{2.77}$$

Indeed, knowing that the boundary Ishibashi states $|\mathscr{B}_i\rangle\rangle$ satisfy the gluing conditions (2.34), we can show that the (2.77) satisfy the crosscap gluing conditions. To do so, we compute

$$e^{-\pi i L_0}\, L_n\, e^{+\pi i L_0} = (-1)^n\, L_n, \qquad e^{-\pi i L_0}\, W^i_n\, e^{+\pi i L_0} = (-1)^n\, W^i_n,$$

where we used that W^i is a primary field. For the generators of the conformal symmetry, we can then calculate

$$\begin{aligned}
&e^{-\pi\, i(L_0 - h(\phi_i))}\left(L_n - (-1)^n\, \overline{L}_{-n}\right) |\,\mathscr{C}_i\rangle\rangle \\
&= e^{-\pi\, i(L_0 - h(\phi_i))}\left(L_n - (-1)^n\, \overline{L}_{-n}\right) e^{\pi\, i(L_0 - h(\phi_i))}\, |\mathscr{B}_i\rangle\rangle \\
&= (-1)^n \left(L_n - \overline{L}_{-n}\right) |\mathscr{B}_i\rangle\rangle \\
&= 0,
\end{aligned}$$

and the condition for the extended symmetry generators is obtained along the same lines. Therefore, the crosscap Ishibashi states (2.77) satisfy the gluing conditions (2.75).

The Cardy Condition

Similarly to the boundary states, we expect generalisations of the Cardy condition arising from the loop-channel—tree-channel equivalences of the Klein bottle and Möbius strip amplitudes. In order to study this point, we compute the Klein bottle amplitude in the following way

$$\begin{aligned}
\widetilde{\mathscr{Z}}^{\mathscr{K}}(l) &= \big\langle \Theta\, C \big| e^{-2\pi l \left(L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24}\right)} \big|\, C \big\rangle \\
&= \sum_{i,j} \Gamma^i\, \Gamma^j\, \langle\langle \mathscr{B}_{i^+}|\, e^{\pi\, i(L_0 - h(\phi_i))}\, e^{2\pi i (2il)\left(L_0 - \frac{c}{24}\right)}\, e^{\pi\, i\left(L_0 - h(\phi_j)\right)}\, |\mathscr{B}_j\rangle\rangle \\
&= \sum_{i,j} \Gamma^i\, \Gamma^j\, \delta_{ij}\, e^{-2\pi i\left(h(\phi_j) - \frac{c}{24}\right)}\, \langle\langle \mathscr{B}_j|\, e^{2\pi i(2il+1)\left(L_0 - \frac{c}{24}\right)}\, |\mathscr{B}_j\rangle\rangle \\
&= \sum_i \left(\Gamma^i\right)^2 e^{-2\pi i\left(h(\phi_i) - \frac{c}{24}\right)}\, \chi_i(2il+1) \\
&= \sum_i \left(\Gamma^i\right)^2 e^{-2\pi i\left(h(\phi_i) - \frac{c}{24}\right)} \sum_j T_{ij}\, \chi_j(2il) = \sum_i \left(\Gamma^i\right)^2\, \chi_i(2il),
\end{aligned}$$

where Θ is again the CPT operator shown for instance in (2.40), and where we employed Eq. (2.37) as well as the modular T-matrix given in (2.72). In the next step, we perform a modular S-transformation to obtain the result in the loop-channel

$$\tilde{\mathcal{Z}}^{\mathcal{K}}(l) = \sum_i \left(\Gamma^i\right)^2 \chi_i(2il) = \sum_{i,j} \left(\Gamma^i\right)^2 S_{ij}\, \chi_j(2it).$$

Now, the Cardy condition is again the requirement that the expression above can be interpreted as a partition function. Since this partition function includes the action of the orientifold projection Ω, the coefficient in front of the character has to be integer but does not need to be non-negative

$$\sum_i \left(\Gamma^i\right)^2 S_{ij} = \kappa_j \in \mathbb{Z}\,.$$

For the Möbius strip amplitude, we compute along similar lines

$$\begin{aligned}
\tilde{\mathcal{Z}}^{\mathcal{M}}(l) &= \langle \Theta\; C| e^{-2\pi l\left(L_0+\overline{L}_0-\frac{c+\bar{c}}{24}\right)} |B_\alpha\rangle \\
&= \sum_{i,j} \Gamma^i\, B_\alpha^j\;\; \langle\langle \mathcal{B}_{i^+}|\; e^{\pi\, i(L_0-h(\phi_i))}\; e^{2\pi i(2il)\left(L_0-\frac{c}{24}\right)}\; |\mathcal{B}_j\rangle\rangle \\
&= \sum_{i,j} \Gamma^i\, B_\alpha^j\; \delta_{ij}\; e^{-\pi i\left(h(\phi_i)-\frac{c}{24}\right)}\; \langle\langle \mathcal{B}_j|\; e^{2\pi i\left(2il+\frac{1}{2}\right)\left(L_0-\frac{c}{24}\right)}\; |\mathcal{B}_j\rangle\rangle \\
&= \sum_i \Gamma^i\, B_\alpha^i\; e^{-\pi i\left(h(\phi_i)-\frac{c}{24}\right)}\; \chi_i\left(2il+\tfrac{1}{2}\right) \\
&= \sum_i \Gamma^i\, B_\alpha^i\; \widehat{\chi}_i(2il) = \sum_{i,j} \Gamma^i\, B_\alpha^i\; P_{ij}\; \widehat{\chi}_j\left(it\right),
\end{aligned}$$

where we employed the hatted characters (2.71) together with their modular transformation. Interpreting this expression as a loop-channel partition function, we see that the coefficients have to be integer

$$\sum_i \Gamma^i\, B_\alpha^i\, P_{ij} = m_{\alpha j} \quad \in \quad \mathbb{Z}\,.$$

Similar to the Cardy boundary states, for the charge conjugate modular invariant partition function explained on page 73, one can show that these integer conditions are satisfied for the reflection coefficients of the form

$$\Gamma^i = \frac{P_{0i}}{\sqrt{S_{0i}}}, \quad B_\alpha^i = \frac{S_{\alpha\, i}}{\sqrt{S_{0i}}}\;.$$

The Klein bottle and Möbius strip coefficients can then be written as two Verlinde type formulas

$$\kappa_j = \sum_i \frac{P_{0i}P_{0i}S_{ij}}{S_{0i}} = Y_{j0}^0, \qquad m_{\alpha j} = \sum_i \frac{S_{\alpha i}P_{0i}P_{ij}}{S_{0i}} = Y_{\alpha j}^0.$$

From the relations (2.74), we can deduce $P_{ij}^* = P_{ij^+}$ and in particular $P_{0i}^* = P_{0i}$, which allows us to establish the connection to the general coefficients

$$Y_{ij}^k = \sum_l \frac{S_{il}\, P_{jl}\, P_{kl}^*}{S_{0l}} \, .$$

As it turns out, the coefficients Y_{ij}^k are integer, guaranteeing that the loop-channel Klein bottle and Möbius strip amplitudes contain only integer coefficients.

Remark

With the techniques presented in this section, it is possible to construct many orientifolds of Conformal Field Theories. However, one set of essential consistency conditions for the co-existence of crosscap and boundary states is still missing. These are the so-called tadpole cancellation conditions which we are going to discuss in a simple example in the final section of these lecture notes.

2.7 The Orientifold of the Bosonic String

We finally apply the techniques developed in this lecture to orientifold theories with boundaries and crosscaps. In particular, we are going to consider a string theory motivated but still sufficiently simple orientifold model which is the Ω projection of the bosonic string. More interestingly, this theory is actually analogous to the orientifold construction of the Type IIB superstring leading to the so-called Type I superstring. However, this needs a more detailed treatment of free fermions which we have not presented here and which is not necessary to understand the mathematical structure of such theories.

Details on the String Theory Construction

The bosonic string is only consistent in 26 flat space-time dimensions and is thus described by 26 free bosons $X^\mu(\sigma, \tau)$ with $\mu = 0, \ldots, 25$. The quantisation of string theory in this description, the covariant quantisation, is slightly involved. However, by defining

$$X^+ = \frac{1}{\sqrt{2}}\Big(X^0(\sigma,\tau) + X^1(\sigma,\tau)\Big), \quad X^- = \frac{1}{\sqrt{2}}\Big(X^0(\sigma,\tau) - X^1(\sigma,\tau)\Big), \tag{2.78}$$

imposing so-called light-cone gauge and using constraint equations, we are left only left with the momentum p^+ as a degree of freedom. For the computation of the characters, we can therefore simply *ignore* the contribution from $X^0(\sigma, \tau)$ and $X^1(\sigma, \tau)$ so that we are left with the Conformal Field Theory of 24 free bosons $X^I(\tau, \sigma)$ where $I = 2, \ldots, 25$. Since the bosonic string is made out of 24 copies of the free boson CFT, for the computation of the partition functions we can use our previous results. These have been summarized in Table 2.2 for later reference.

Table 2.2 Summary of all loop- and tree-channel amplitudes for the example of the free boson with orientifold projection (2.42)

Loop-channel	Tree-channel
$\mathcal{Z}^{\mathcal{T}}_{\text{bos.}}(\tau,\overline{\tau}) = \frac{1}{\sqrt{\tau_2}}\,\frac{1}{\lvert\eta(\tau)\rvert^2}$	
$\mathcal{Z}^{\mathcal{K}}_{\text{bos.}}(t) = \frac{1}{\sqrt{2t}}\,\frac{1}{\eta\left(2it\right)}$	$\widetilde{\mathcal{Z}}^{\mathcal{K}\,(\text{O1,O1})}_{\text{bos.}}\left(l\right) = \frac{1}{\eta(2il)}$
$\mathcal{Z}^{\mathcal{C}\,(\text{N,N})}_{\text{bos.}}(t) = \frac{1}{2\sqrt{t}}\,\frac{1}{\eta\,(it)}$	$\widetilde{\mathcal{Z}}^{\mathcal{C}\,(\text{N,N})}_{\text{bos.}}(l) = \frac{1}{2\,\eta\left(2il\right)}$
$\mathcal{Z}^{\mathcal{C}\,(\text{D,D})}_{\text{bos.}}(t) = \frac{1}{\eta\,(it)}\,e^{-\frac{t}{4\pi}\left(x_0^b-x_0^a\right)^2}$	$\widetilde{\mathcal{Z}}^{\mathcal{C}\,(\text{D,D})}_{\text{bos.}}(l) = \frac{1}{\sqrt{2l}}\,\frac{1}{\eta\left(2il\right)}\,e^{-\frac{1}{8\pi l}\left(x_0^b-x_0^a\right)^2}$
$\mathcal{Z}^{\mathcal{C}\,(\text{mixed})}_{\text{bos.}}(t) = \sqrt{\frac{\eta\,(it)}{\vartheta_4(it)}}$	$\widetilde{\mathcal{Z}}^{\mathcal{C}\,(\text{mixed})}_{\text{bos.}}(l) = \sqrt{\frac{\eta\,(2il)}{\vartheta_2(2il)}}$
$\mathcal{Z}^{\mathcal{M}\,(\text{N,N})}_{\text{bos.}}(t) = \frac{1}{2\sqrt{t}}\,\frac{1}{\eta\left(\frac{1}{2}+it\right)}\,e^{\frac{\pi i}{24}}$	$\widetilde{\mathcal{Z}}^{\mathcal{M}\,(\text{O1,N})}_{\text{bos.}}\left(l\right) = \frac{1}{\sqrt{2}}\,\frac{1}{\eta\left(\frac{1}{2}+2il\right)}\,e^{\frac{\pi i}{24}}$
$\mathcal{Z}^{\mathcal{M}\,(\text{D,D})}_{\text{bos.}}(t) = \sqrt{2}\,\sqrt{\frac{\eta\left(\frac{1}{2}+it\right)}{\vartheta_2\left(\frac{1}{2}+it\right)}}\,e^{\frac{\pi i}{24}}$	$\widetilde{\mathcal{Z}}^{\mathcal{M}\,(\text{O1,D})}_{\text{bos.}}\left(l\right) = \sqrt{2}\,\sqrt{\frac{\eta\left(\frac{1}{2}+2il\right)}{\vartheta_2\left(\frac{1}{2}+2il\right)}}\,e^{\frac{\pi i}{24}}$

In our previous definition of the open and closed sector partition functions, we employed the notion common to Conformal Field Theory. However, for the relevant quantities in string theory, we have to integrate over the modular parameter of the torus, Klein bottle, cylinder and Möbius strip. After performing the integration over the light-cone momentum p^+, the expressions relevant for the following are

$$\begin{aligned} Z^{\mathcal{T}} &= \int_{\text{Teich}} \frac{d^2\tau}{\tau_2^2}\,\mathcal{Z}^{\mathcal{T}}(\tau,\overline{\tau}), \qquad & Z^{\mathcal{C}} &= \int_0^\infty \frac{dt}{4\,t^2}\,\mathcal{Z}^{\mathcal{C}}(t), \\ Z^{\mathcal{K}} &= \int_0^\infty \frac{dt}{2\,t^2}\,\mathcal{Z}^{\mathcal{K}}(t), & Z^{\mathcal{M}} &= \int_0^\infty \frac{dt}{4\,t^2}\,\mathcal{Z}^{\mathcal{M}}(t). \end{aligned} \tag{2.79}$$

The domain of integration for the torus amplitude $Z^{\mathcal{T}}$ is the so-called Teichmüller space. It is the space of all complex structures τ of a torus $\mathbb{T}^2$ which are not related via the $SL(2,\mathbb{Z})/\mathbb{Z}_2$ symmetry. An illustration can be found in Fig. 2.9 and the precise definition reads

$$\text{Teich} = \left\{\,\tau\in\mathbb{C}\;:\;-\frac{1}{2} < \tau_1 \leq +\frac{1}{2},\;\lvert\tau\rvert \geq 1\right\}. \tag{2.80}$$

Torus Partition Function for the Bosonic String

Let us now become more concrete and determine the torus partition function for the bosonic string in light-cone gauge. Since this theory is a copy of 24 free bosons, we recall from Table 2.2 the form of $\mathcal{Z}^{\mathcal{T}}_{\text{bos.}}$ and combine it into

$$Z^{\mathcal{T}} = \int_{\text{Teich}} \frac{d^2\tau}{\tau_2^2}\left(\mathcal{Z}^{\mathcal{T}}_{\text{bos.}}(\tau,\overline{\tau})\right)^{24} = \int_{\text{Teich}} \frac{d^2\tau}{\tau_2^2}\,\frac{1}{\tau_2^{12}}\,\frac{1}{\lvert\eta^{24}(\tau)\rvert^2}\,. \tag{2.81}$$

In order to become more explicit, let us expand the Dedekind η-function in the following way

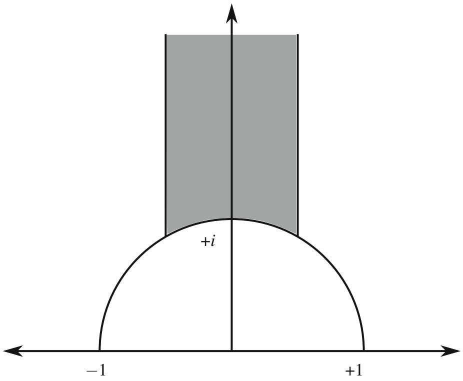

Fig. 2.9 The shaded region in this figure corresponds to the Teichmüller space of the two-torus $\mathbb{T}^2$

$$\frac{1}{\eta^{24}(\tau)} = q^{-1}\Big(1 + 24\, q + 324\, q^2 + \ldots\Big) . \tag{2.82}$$

Using this expansion in (2.81) together with the string theoretical *level-matching condition* which leaves only equal powers of q and $\overline{q}$, we arrive at

$$\begin{aligned} Z^{\mathscr{T}} &= \int_{\text{Teich}} \frac{d^2\tau}{\tau_2^{14}}\, e^{+4\pi\tau_2} \left| \, 1 + 24\, e^{2\pi i\tau} + \ldots \, \right|^2 \\ &\longrightarrow \int_{\text{Teich}} \frac{d^2\tau}{\tau_2^{14}}\, e^{+4\pi\tau_2} \left(1 + (24)^2\, e^{-4\pi\tau_2} + \ldots \right) . \end{aligned} \tag{2.83}$$

Let us now study the divergent behaviour of this integral.

- Although the integrand in (2.83) diverges for $\tau_2 \to 0$ due to the factor of τ_2^{-14}, the whole integral is finite because the domain of integration (2.80) does not include $\tau_2 = 0$. Therefore, this expression is not divergent in the *infrared*, i.e. there is no singularity for small τ_2. Let us emphasize that the finiteness in this parameter region is due to the modular invariance of the torus partition function which restricts the domain of integration to the Teichmüller space.
- Next, we turn to the behaviour of (2.83) for large τ_2. We see that the first term gives rise to a divergence in the region $\tau_2 \to \infty$ which corresponds to a state with negative mass squared, i.e. a tachyon. Thus, the theory of the bosonic string is unstable. In more realistic theories, for instance the superstring, such a tachyon should be absent and we do not expect problems due to divergences in the *ultraviolet*.
- In summary, the torus partition function of the bosonic string is finite in the infrared due to modular invariance. In the ultraviolet, the partition function is divergent due to a tachyon which renders the theory unstable.

Klein Bottle Partition Function for the Bosonic String

As the title of this section suggests, we want to study the orientifold of the bosonic string and so we have to determine the Klein bottle amplitude. Following the same steps as for the torus, we arrive at

$$Z^{\mathscr{K}}(t) = \frac{1}{2}\int_0^\infty \frac{dt}{t^2}\left(\mathscr{Z}^{\mathscr{K}}_{\text{bos.}}(t)\right)^{24} = \frac{1}{2^{13}}\int_0^\infty \frac{dt}{t^{14}}\,\frac{1}{\eta^{24}(2it)}\,.$$

In order to simplify the integrand, we perform a transformation to the tree-channel with modular parameter $t = \frac{1}{4l}$ by employing the modular properties of the Dedekind η-function (2.32)

$$\begin{aligned} Z^{\mathscr{K}}(t) \quad \xrightarrow{t=\frac{1}{4l}} \quad \widetilde{Z}^{\mathscr{K}(\text{O25,O25})}(l) &= \frac{1}{2^{13}}\int_0^\infty \frac{dl}{4\,l^2}\,(4l)^{14}\,\frac{1}{\eta^{24}\left(-\frac{1}{2il}\right)} \\ &= 2\int_0^\infty dl\,\frac{1}{\eta^{24}(2il)}. \end{aligned}$$

The notation O25 deserves some explanation. Since we are studying the bosonic string in a 26-dimensional space-time, the orientifold projection naturally acts also on the light-cone coordinates (2.78). By choosing the orientifold projection (2.42), we have an orientifold plane extending over all 26 dimensions. However, the convention in string theory is such that only the space dimensions are counted which explains the term O25.

Similarly as for the torus partition function, let us now expand the tree-channel Klein bottle amplitude. Using Eq. (2.82) we obtain

$$\widetilde{Z}^{\mathscr{K}(\text{O25,O25})}(l) = 2\int_0^\infty dl\,\left(e^{4\pi\,l} + 24 + 324\,e^{-4\pi\,l} + \ldots\right). \tag{2.84}$$

The first term in (2.84) corresponds again to the tachyon and should be absent in more realistic theories. We therefore ignore this problematic behaviour. However, the second terms corresponds to massless states and gives rise to a divergence since in the present case, the domain of integration includes $t = \frac{1}{4l} = 0$. This term will not be absent in more refined theories and so at this point, the orientifold of the bosonic string is not consistent at a more severe level.

A Stack of D-Branes

As it turns out, the divergence of the Klein bottle diagram can be cancelled by introducing a to be determined number N of D25 branes. The notation D25 means that these D-branes fill out 25 spatial dimensions and it is understood that they always fill the time direction.

If we put a certain number of D-branes on top of each other, we call it a stack of D-branes. However, since there are now multiple branes, we can have new kinds of open strings. In particular, there are strings starting at D-brane i of our stack and ending on D-brane j. We thus include new labels, so-called Chan–Paton labels, to our open string states

$$\big|\mathbf{m},\ i,j\big\rangle = \big|\mathbf{m}\big\rangle \otimes \big|i,j\big\rangle,$$

where $|\mathbf{m}\rangle$ denotes the states for a single string and $i,j = 1,\ldots,N$ label the starting respectively ending points. We furthermore construct the hermitian conjugate $\langle i,j|$ in the usual way such that

$$\big\langle\, i,j \,\big|\, i',j' \,\big\rangle = \delta_{ii'}\delta_{jj'}. \tag{2.85}$$

Next, we define the action of the orientifold projection acting on the Chan–Paton labels. Since Ω changes the orientation of the world-sheet, it clearly interchanges starting and ending points of open strings. But we can also allow for rotations among the D-branes and so a general orientifold action reads

$$\Omega\,\big|\, i,j\,\big\rangle = \sum_{i',j'=1}^{N} \gamma_{jj'}\big|\, j',i'\,\big\rangle\big(\gamma^{-1}\big)_{i'i}\,, \tag{2.86}$$

where γ is a $N\times N$ matrix. Without presenting the detailed argument, we now require that the action of Ω on the Chan–Paton labels squares to the identity. For this we calculate

$$\begin{aligned}
\Omega^2\,\big|\, i,j\,\big\rangle &= \sum_{i'',j''=1}^{N} \gamma_{ii''}\Big[\Omega\,\big|\, i,j\,\big\rangle^{T}\Big]_{i'',j''}\big(\gamma^{-1}\big)_{j''j} \\
&= \sum_{i',j',i'',j''=1}^{N} \gamma_{ii''}\,\big(\gamma^{-1}\big)^{T}_{i''i'}\,\big|\, i',j'\,\big\rangle\,\gamma^{T}_{j'j''}\,\big(\gamma^{-1}\big)_{j''j} \\
&= \sum_{i',j'=1}^{N}\Big[\gamma\big(\gamma^{-1}\big)^{T}\Big]_{ii'}\,\big|\, i',j'\,\big\rangle\,\Big[\gamma^{T}\gamma^{-1}\Big]_{j'j}\,,
\end{aligned}$$

from which we infer the constraint on the matrices γ to be symmetric or anti-symmetric

$$\gamma^{T} = \pm\gamma. \tag{2.87}$$

In string theory, the two different signs correspond to gauge groups $SO(N)$ and $SP(N)$ living on the stack of D-branes.

Let us now come to the final part of this paragraph which is to determine the contribution of the Chan–Paton labels to the partition function. For the Cylinder partition function, we calculate with the help of (2.85)

$$
\begin{aligned}
\mathscr{Z}^{\mathscr{C}}(t) = \mathrm{Tr}_{\mathscr{H}_{\mathscr{B}}}\left(q^{L_0-\frac{c}{24}}\right) &= \sum_n \langle n \mid q^{L_0-\frac{c}{24}} \mid n\rangle \times \sum_{i,j=1}^{N} \langle\, i,j \mid i,j\,\rangle \\
&= \sum_n \langle n \mid q^{L_0-\frac{c}{24}} \mid n\rangle \times N^2 .
\end{aligned}
$$

Therefore, the effect of N D-branes is taken care of by including the factor N^2 for the cylinder partition function. Let us next turn to the Möbius strip partition function. Concentrating only on the Chan–Paton part, we find using (2.85) and (2.86) that

$$
\begin{aligned}
\sum_{i,j=1}^{N} \langle\, i,j \mid \Omega \mid i,j\,\rangle &= \sum_{i,j,i',j'=1}^{N} \langle\, i,j \mid \gamma_{jj'} \mid j',i'\,\rangle (\gamma^{-1})_{i'i} \\
&= \sum_{i,j,i',j'=1}^{N} \delta_{ij'}\, \delta_{ji'}\, \gamma_{jj'}\, \left(\gamma^{-1}\right)_{i'i} \\
&= \mathrm{Tr}\Big(\, \gamma^T\, \gamma^{-1}\,\Big) = \pm N,
\end{aligned}
$$

where in the final step we also employed (2.87). In summary, by including a factor of $\pm N$ in the Möbius strip partition function, we can account for a stack of N D-branes.

Cylinder and Möbius-Strip Partition Function for the Bosonic String

After this discussion about stacks of D-branes, let us now compute the cylinder and Möbius strip partition functions for a stack of N D25-branes. Since the D-branes fill out the 26-dimensional space-time, the open strings always have Neumann–Neumann boundary conditions.

For the cylinder, we recall from Table 2.2 the form of a single cylinder partition function and combine it with the relevant expression from (2.79) to obtain

$$
Z^{\mathscr{C}(\mathrm{N,N})}(t) = \frac{N^2}{4}\int_0^\infty \frac{dt}{t^2}\,\left(\mathscr{Z}^{\mathscr{C}(\mathrm{N,N})}_{\mathrm{bos.}}(t)\right)^{24} = \frac{N^2}{2^{26}}\int_0^\infty \frac{dt}{t^{14}}\,\frac{1}{\eta^{24}(it)}\,,
$$

where we included the factor N^2 as explained above. In order to extract the divergences, we perform a transformation from the loop- to the tree-channel via $t=\frac{1}{2l}$ to find

$$
\begin{aligned}
Z^{\mathscr{C}(\mathrm{N,N})}(t) \quad \xrightarrow{t=\frac{1}{2l}} \quad \widetilde{Z}^{\mathscr{C}(\mathrm{N,N})}(l) &= \frac{N^2}{2^{26}}\int_0^\infty \frac{dl}{2\,l^2}\,(2l)^{14}\,\frac{1}{\eta^{24}\left(-\frac{1}{2il}\right)} \\
&= \frac{N^2}{2^{25}}\int_0^\infty dl\,\frac{1}{\eta^{24}(2il)}.
\end{aligned}
$$

With the help of (2.82), we can again expand this expression. The first terms read as follows

$$\widetilde{Z}^{\mathscr{C}(\mathrm{N,N})}(l) = \frac{N^2}{2^{25}} \int_0^\infty dl\, \Big(e^{4\pi\, l} + 24 + 324\, e^{-4\pi\, l} + \ldots \Big).$$

Next, we turn to the Möbius strip contribution. Along similar lines as above, we recall from Table 2.2 the expression for the partition function of a single free boson and combine 24 copies of it into the Möbius partition function

$$Z^{\mathscr{M}(\mathrm{N,N})}(t) = \pm \frac{N}{4} \int_0^\infty \frac{dt}{t^2} \left(\mathscr{Z}^{\mathscr{M}(\mathrm{N,N})}_{\mathrm{bos.}}(t) \right)^{24} = \pm \frac{N}{2^{26}} \int_0^\infty \frac{dt}{t^{14}}\, \frac{e^{\pi i}}{\eta^{24}(\frac{1}{2} + it)}\,.$$

In order to extract the divergences more easily, we transform this expression into the tree-channel via the relation $t = \frac{1}{8l}$ and the modular $\mathscr{P}$ transformation (2.70)

$$\begin{aligned} Z^{\mathscr{M}(\mathrm{N,N})}(t) \quad \xrightarrow{t=\frac{1}{8l}} \quad \widetilde{Z}^{\mathscr{M}(\mathrm{N,N})}(l) &= \pm \frac{N}{2^{26}} \int_0^\infty \frac{dl}{8\, l^2}\, (8l)^{14}\, \frac{e^{\pi i}}{\eta^{24}\left(\frac{1}{2} + \frac{i}{8l}\right)} \\ &= \pm \frac{N}{2^{11}} \int_0^\infty dl\, \frac{e^{\pi i}}{\eta^{24}(\frac{1}{2} + 2il)}\,. \end{aligned}$$

Expanding this expression with the help of (2.82), we find

$$\widetilde{Z}^{\mathscr{M}(\mathrm{N,N})}(l) = \pm \frac{N}{2^{11}} \int_0^\infty dl\, \Big(e^{4\pi\, l} - 24 + 324\, e^{-4\pi\, l} - \ldots \Big).$$

Tadpole Cancellation Condition

After having determined the divergent contributions of the one-loop amplitudes, we can now combine them into the full expression. Leaving out the torus amplitude, we find

$$\begin{aligned} &\frac{1}{2} \Big(\widetilde{Z}^{\mathscr{K}(\mathrm{O25,O25})}(l) + \widetilde{Z}^{\mathscr{C}(\mathrm{N,N})}(l) + \widetilde{Z}^{\mathscr{M}(\mathrm{N,N})}(l) \Big) \\ &\quad = 2^{-26} \int_0^\infty dl\, \Big(e^{4\pi l} \left(2^{26} \pm 2 \cdot 2^{13} N + N^2 \right) \\ &\qquad\qquad + 24 \left(2^{26} \mp 2 \cdot 2^{13} N + N^2 \right) \\ &\qquad\qquad + 324\, e^{-4\pi l} \left(2^{26} \pm 2 \cdot 2^{13} N + N^2 \right) \quad + \quad \ldots \Big). \end{aligned} \tag{2.88}$$

The first terms with prefactor $e^{4\pi l}$ stem again from the tachyon which in a more realistic theory, e.g. Superstring Theory, should be absent. We will therefore ignore this divergence. The next line with prefactor 24 corresponds to massless states which will not be absent in more refined theories. However, we can simplify this expression by noting that

$$\left(2^{26} \mp 2 \cdot 2^{13} N + N^2 \right) = \left(2^{13} \mp N \right)^2.$$

We thus see that by taking $N = 2^{13} = 8192$ D25-branes and choosing the minus sign corresponding to $SO(N)$ gauge groups, the divergence is cancelled. In summary, we have found that

> For the orientifold of the bosonic string with $N = 8192$D25-branes and gauge group SO (8192), the divergence due to massless states is cancelled. This is the famous tadpole cancellation condition for the bosonic string.

Finally, it is easy to see that the proceeding terms in (2.88) with prefactors $e^{-4\pi l}$ and powers thereof do not give rise to divergences in the integral.

Remarks

- Here we have discussed a very simple example for a CFT with boundaries. The next step is to generalise these methods for the superstring, in which case we have to define boundary and crosscap states for the CFT of the free fermion. The orientifold of the Type IIB superstring defines the so-called Type I string living in ten-dimensions and carrying gauge group $SO(32)$ instead of $SO(8192)$.
- Many examples of such orientifold models have been discussed for compactified dimensions. These include orientifolds on toroidal orbifolds and also orientifolds of Gepner models. For this purpose, one first has to find classes of boundary and crosscap states for the $\mathcal{N} = 2$ unitary models and then for Gepner models, in which the simple current construction is utilised in an essential way. Finally, one has to derive and solve the tadpole cancellation conditions. All this is a feasible exercise but beyond the scope of these lecture notes.

Acknowledgements I want to thank Erik Plauschinn for collaboration on the book [1], where these lectures notes are essentially taken from. Moreover, I am grateful to Marco Baumgartl for helping with the manuscript.

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http://www.springer.com/978-3-642-25946-3

Strings and Fundamental Physics
Baumgartl, M.; Brunner, I.; Haack, M. (Eds.)
2012, XIII, 291 p. 24 illus., Softcover
ISBN: 978-3-642-25946-3