Chapter 2
Spherical Harmonics

This chapter presents a theory of spherical harmonics from the viewpoint of invariant linear function spaces on the sphere. It is shown that the system of spherical harmonics is the only system of invariant function spaces that is both complete and closed, and cannot be reduced further. In this chapter, the dimension $d \geq 2$. Spherical harmonics are introduced in Sect. 2.1 as the restriction to the unit sphere of harmonic homogeneous polynomials. Two very important properties of the spherical harmonics are the addition theorem and the Funk–Hecke formula, and these are discussed in Sects. 2.2 and 2.5, respectively. A projection operator into spherical harmonic function subspaces is introduced in Sect. 2.3; this operator is useful in proving various properties of the spherical harmonics. Since several polynomial spaces are used, it is convenient to include a discussion on relations of these spaces and this is done in Sect. 2.4. Legendre polynomials play an essential role in the study of the spherical harmonics. Representation formulas for Legendre polynomials are given in Sect. 2.6, whereas numerous properties of the polynomials are discussed in Sect. 2.7. Completeness of the spherical harmonics in $C(\mathbb{S}^{d-1})$ and $L^2(\mathbb{S}^{d-1})$ is the topic of Sect. 2.8, and this refers to the property that linear combinations of the spherical harmonics are dense in $C(\mathbb{S}^{d-1})$ and in $L^2(\mathbb{S}^{d-1})$. As an extension of the Legendre polynomials, the Gegenbauer polynomials are introduced in Sect. 2.9. The last two sections of the chapter, Sects. 2.10 and 2.11, are devoted to a discussion of the associated Legendre functions and their role in generating orthonormal bases for spherical harmonic function spaces.

2.1 Spherical Harmonics Through Primitive Spaces

We start with more notation. We use $\mathbb{O}^d$ for the set of all real orthogonal matrices of order d. Recall that $A \in \mathbb{R}^{d\times d}$ is orthogonal if $A^T A = I$, or alternatively, $AA^T = I$, $I = I_d$ being the identity matrix of order d.

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The product of two orthogonal matrices is again orthogonal. In algebra terminology, $\mathbb{O}^d$ is a group; but in this book, we will avoid using this term. It is easy to see that $\det(A) = \pm 1$ for any $A \in \mathbb{O}^d$. The subset of those matrices in $\mathbb{O}^d$ with the determinant equal to 1 is denoted as $\mathbb{SO}^d$. For any non-zero vector $\boldsymbol{\eta} \in \mathbb{R}^d$,

$$\mathbb{O}^d(\boldsymbol{\eta}) := \left\{ A \in \mathbb{O}^d : A\boldsymbol{\eta} = \boldsymbol{\eta} \right\}$$

is the subset of orthogonal matrices that leave the one-dimensional subspace $\operatorname{span}\{\boldsymbol{\eta}\} := \{\alpha\,\boldsymbol{\eta} : \alpha \in \mathbb{R}\}$ unchanged.

For a function $f : \mathbb{R}^d \to \mathbb{C}$ and a matrix $A \in \mathbb{R}^{d\times d}$, we define $f_A : \mathbb{R}^d \to \mathbb{C}$ by the formula

$$f_A(\boldsymbol{x}) = f(A\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

We will use this definition mainly for $A \in \mathbb{O}^d$ and for study of symmetry properties of functions.

Proposition 2.1. *If $f_A = f$ for any $A \in \mathbb{O}^d$, then $f(\boldsymbol{x})$ depends on $\boldsymbol{x}$ through $|\boldsymbol{x}|$, so that f is constant on a sphere of an arbitrary radius.*

Proof. For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ with $|\boldsymbol{x}| = |\boldsymbol{y}|$, we can find a matrix $A \in \mathbb{O}^d$ such that $A\boldsymbol{x} = \boldsymbol{y}$. Thus, $f(\boldsymbol{x}) = f_A(\boldsymbol{x}) = f(\boldsymbol{y})$ and the proof is completed. $\square$

Consider the subset $\mathbb{O}^d(\boldsymbol{e}_d)$. It is easy to show that any $A \in \mathbb{O}^d(\boldsymbol{e}_d)$ is of the form

$$A = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad A_1 \in \mathbb{O}^{d-1}. \tag{2.1}$$

Similar to Proposition 2.1, if $f_A = f$ for any $A \in \mathbb{O}^d(\boldsymbol{e}_d)$, then $f(\boldsymbol{x})$ depends on $\boldsymbol{x}$ through $|\boldsymbol{x}_{(d-1)}|$ and x_d.

We will introduce spherical harmonic spaces of different orders as primitive subspaces of $C(\mathbb{S}^{d-1})$. Consider a general subspace $\mathbb{V}$ of functions defined in $\mathbb{R}^d$ or over a subset of $\mathbb{R}^d$.

Definition 2.2. $\mathbb{V}$ is said to be *invariant* if $f \in \mathbb{V}$ and $A \in \mathbb{O}^d$ imply $f_A \in \mathbb{V}$.

Assume $\mathbb{V}$ is an invariant subspace of an inner product function space with the inner product $(\cdot,\cdot)$. Then $\mathbb{V}$ is said to be *reducible* if $\mathbb{V} = \mathbb{V}_1 + \mathbb{V}_2$ with $\mathbb{V}_1 \neq \emptyset$, $\mathbb{V}_2 \neq \emptyset$, both invariant, and $\mathbb{V}_1 \perp \mathbb{V}_2$. $\mathbb{V}$ is *irreducible* if it is not reducible. $\mathbb{V}$ is said to be *primitive* if it is both invariant and irreducible.

We note that $\mathbb{V}_1 \perp \mathbb{V}_2$ refers to the property that $(f,g) = 0\ \forall\, f \in \mathbb{V}_1$, $\forall\, g \in \mathbb{V}_2$.

Definition 2.3. Given $f : \mathbb{R}^d \to \mathbb{C}$, define $\operatorname{span}\left\{ f_A : A \in \mathbb{O}^d \right\}$, the space of functions constructed through f and $\mathbb{O}^d$, to be the space of all the convergent combinations of the form $\sum_{j\ge 1} c_j f_{A_j}$ with $A_j \in \mathbb{O}^d$ and $c_j \in \mathbb{C}$.

For the above definition, it is easy to see span $\left\{f_A : A \in \mathbb{O}^d\right\}$ is a function subspace. Moreover, if $\mathbb{V}$ is a finite dimensional primitive space, then

$$\mathbb{V} = \operatorname{span}\left\{f_A : A \in \mathbb{O}^d\right\} \quad \forall\, 0 \neq f \in \mathbb{V}.$$

2.1.1 Spaces of Homogeneous Polynomials

We start with $\mathbb{H}_n^d$, the space of all homogeneous polynomials of degree n in d dimensions. The space $\mathbb{H}_n^d$ consists of all the functions of the form

$$\sum_{|\boldsymbol{\alpha}|=n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \ a_{\boldsymbol{\alpha}} \in \mathbb{C}.$$

As some concrete examples,

$$\begin{aligned}
\mathbb{H}_2^2 &= \left\{a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 : a_j \in \mathbb{C}\right\},\\
\mathbb{H}_2^3 &= \left\{a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_1 x_3 + a_4 x_2^2 + a_5 x_2 x_3 + a_6 x_3^2 : a_j \in \mathbb{C}\right\},\\
\mathbb{H}_3^2 &= \left\{a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_2^3 : a_j \in \mathbb{C}\right\}.
\end{aligned}$$

It is easy to see that $\mathbb{H}_n^d$ is a finite dimensional invariant space. To determine the dimension $\dim \mathbb{H}_n^d$, we need to count the number of monomials of degree n: $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with $\alpha_i \geq 0$ and $\alpha_1 + \cdots + \alpha_d = n$. We consider a set of $n + d - 1$ numbers: 1, 2, …, $n + d - 1$. Let us remove from the set $d - 1$ numbers, say $\beta_1 < \cdots < \beta_{d-1}$. Denote $\beta_0 = 0$ and $\beta_d = n + d$. Then define

$$\alpha_i = \beta_i - \beta_{i-1} - 1, \quad 1 \leq i \leq d,$$

i.e., define α_i to be the number of integers between β_{i-1} and β_i, exclusive. Note that $\sum_{i=1}^n \alpha_i = d$. This establishes a one-to-one correspondence between the set of non-negative integers $\alpha_1, \ldots, \alpha_d$ with a sum n and the set of $d - 1$ distinct positive integers $\beta_1 < \cdots < \beta_{d-1}$ between 1 and $n + d - 1$. Since the number of ways of selecting $d - 1$ different numbers from a set of $n + d - 1$ numbers is

$$\binom{n+d-1}{d-1},$$

we have

$$\dim \mathbb{H}_n^d = \binom{n+d-1}{d-1} = \binom{n+d-1}{n}. \tag{2.2}$$

In particular, for $d = 2$ and 3, we have

$$\dim \mathbb{H}_n^2 = n + 1, \qquad \dim \mathbb{H}_n^3 = \frac{1}{2}\,(n+1)\,(n+2). \tag{2.3}$$

We give in passing a compact formula for the generating function of the sequence $\{\dim \mathbb{H}_n^d\}_{n\geq 0}$,

$$\sum_{n=0}^{\infty} \left(\dim \mathbb{H}_n^d\right) z^n.$$

Recall the Taylor expansion (e.g., deduced from [9, (1.1.7)])

$$(1+x)^s = \sum_{n=0}^{\infty} \binom{s}{n} x^n, \quad |x| < 1, \quad \binom{s}{n} := \frac{s\,(s-1)\cdots(s-n+1)}{n!}.$$

Replacing x by $(-x)$ and choosing $s = -d$, we obtain

$$(1-x)^{-d} = \sum_{n=0}^{\infty} \binom{n+d-1}{n} x^n, \quad |x| < 1. \tag{2.4}$$

Thus,

$$\sum_{n=0}^{\infty} \left(\dim \mathbb{H}_n^d\right) z^n = \frac{1}{(1-z)^d}, \quad |z| < 1. \tag{2.5}$$

For $n \geq 2$,

$$|\cdot|^2 \mathbb{H}_{n-2}^d := \left\{ |\boldsymbol{x}|^2 H_{n-2}(\boldsymbol{x}) : H_{n-2} \in \mathbb{H}_{n-2}^d \right\}$$

is a proper invariant subspace of $\mathbb{H}_n^d$. Hence $\mathbb{H}_n^d|_{\mathbb{S}^{d-1}}$, the restriction of $\mathbb{H}_n^d$ to $\mathbb{S}^{d-1}$, is reducible. Let us identify the subspace of $\mathbb{H}_n^d$ that does not contain the factor $|\boldsymbol{x}|^2$.

Any $H_n \in \mathbb{H}_n^d$ can be written in the form

$$H_n(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=n} a_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}},\ a_{\boldsymbol{\alpha}} \in \mathbb{C}.$$

For this polynomial H_n, define

$$H_n(\boldsymbol{\nabla}) = \sum_{|\boldsymbol{\alpha}|=n} a_{\boldsymbol{\alpha}} \boldsymbol{\nabla}^{\boldsymbol{\alpha}}.$$

Given any two polynomials in $\mathbb{H}_n^d$,

$$H_{n,1}(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=n} a_{\boldsymbol{\alpha},1} \boldsymbol{x}^{\boldsymbol{\alpha}}, \qquad H_{n,2}(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=n} a_{\boldsymbol{\alpha},2} \boldsymbol{x}^{\boldsymbol{\alpha}},$$

it is straightforward to show

$$H_{n,1}(\boldsymbol{\nabla})\overline{H_{n,2}(\boldsymbol{x})} = \sum_{|\boldsymbol{\alpha}|=n} \boldsymbol{\alpha}! a_{\boldsymbol{\alpha},1}\overline{a_{\boldsymbol{\alpha},2}} = \overline{H_{n,2}(\boldsymbol{\nabla})\overline{H_{n,1}(\boldsymbol{x})}}.$$

Thus,

$$(H_{n,1}, H_{n,2})_{\mathbb{H}_n^d} := H_{n,1}(\boldsymbol{\nabla})\overline{H_{n,2}(\boldsymbol{x})} \tag{2.6}$$

defines an inner product in the subspace $\mathbb{H}_n^d$.

Recall that a function f is *harmonic* if $\Delta f(\boldsymbol{x}) = 0$. Being harmonic is an invariant property for functions.

Lemma 2.4. *If $\Delta f = 0$, then $\Delta f_A = 0\ \forall\, A \in \mathbb{O}^d$.*

Proof. Denote $\boldsymbol{y} = A\boldsymbol{x}$. Then $\boldsymbol{\nabla}_{\boldsymbol{x}} = A\boldsymbol{\nabla}_{\boldsymbol{y}}$. Since $A \in \mathbb{O}^d$, we have

$$\Delta_{\boldsymbol{x}} = \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}_{\boldsymbol{y}} = \Delta_{\boldsymbol{y}}.$$

So the stated property holds. □

We now introduce an important subspace of $\mathbb{H}_n^d$.

Definition 2.5. The space of the homogeneous harmonics of degree n in d dimensions, $\mathbb{Y}_n(\mathbb{R}^d)$, consists of all homogeneous polynomials of degree n in $\mathbb{R}^d$ that are also harmonic.

We comment that non-trivial functions in $\mathbb{Y}_n(\mathbb{R}^d)$ do not contain the factor $|\boldsymbol{x}|^2$. This is shown as follows. Suppose $Y_n(\boldsymbol{x}) = |\boldsymbol{x}|^2 Y_{n-2}(\boldsymbol{x})$ is harmonic, where $Y_{n-2}(\boldsymbol{x})$ is a homogeneous polynomial of degree $(n-2)$. Then

$$(Y_n, Y_n)_{\mathbb{H}_{n,d}} = Y_{n-2}(\boldsymbol{\nabla})\overline{\Delta Y_n(\boldsymbol{x})} = 0.$$

Hence, $Y_n(\boldsymbol{x}) \equiv 0$.

Example 2.6. Obviously, $\mathbb{Y}_n(\mathbb{R}^d) = \mathbb{H}_n^d$ if $n = 0$ or 1.

For $d = 1$, $\mathbb{Y}_n(\mathbb{R}) = \emptyset$ for $n \geq 2$.

For $d = 2$, $\mathbb{Y}_2(\mathbb{R}^2)$ consists of all polynomials of the form $a\left(x_1^2 - x_2^2\right) + b\, x_1 x_2$, $a, b \in \mathbb{C}$. Polynomials of the form $(x_1 + i\, x_2)^n$ belong to $\mathbb{Y}_n(\mathbb{R}^2)$.

For $d = 3$, any polynomial of the form $(x_3 + i\, x_1 \cos\theta + i\, x_2 \sin\theta)^n$, $\theta \in \mathbb{R}$ being fixed, belongs to $\mathbb{Y}_n(\mathbb{R}^3)$. □

Let us determine the dimension $N_{n,d} := \dim \mathbb{Y}_n(\mathbb{R}^d)$. The number $N_{n,d}$ will appear at various places in this text. Any polynomial $H_n \in \mathbb{H}_n^d$ can be written in the form

$$H_n(x_1, \cdots, x_d) = \sum_{j=0}^{n} (x_d)^j h_{n-j}(x_1, \cdots, x_{d-1}), \quad h_{n-j} \in \mathbb{H}_{n-j}^{d-1}. \tag{2.7}$$

Apply the Laplacian operator to this polynomial,

$$\Delta_{(d)} H_n(\boldsymbol{x}_{(d)}) = \sum_{j=0}^{n-2} (x_d)^j \left[\Delta_{(d-1)} h_{n-j}(\boldsymbol{x}_{(d-1)}) \right.$$
$$\left. + (j+2)\,(j+1)\, h_{n-j-2}(\boldsymbol{x}_{(d-1)})\right].$$

Thus, if $H_n \in \mathbb{Y}_n(\mathbb{R}^d)$ so that $\Delta_{(d)} H_n(\boldsymbol{x}_{(d)}) \equiv 0$, then

$$h_{n-j-2} = -\frac{1}{(j+2)\,(j+1)}\,\Delta_{(d-1)} h_{n-j}, \quad 0 \le j \le n-2. \tag{2.8}$$

Consequently, a homogeneous harmonic $H_n \in \mathbb{Y}_n(\mathbb{R}^d)$ is uniquely determined by $h_n \in \mathbb{H}_n^{d-1}$ and $h_{n-1} \in \mathbb{H}_{n-1}^{d-1}$ in the expansion (2.7). From this, we get the following relation on the polynomial space dimensions:

$$N_{n,d} = \dim \mathbb{H}_n^{d-1} + \dim \mathbb{H}_{n-1}^{d-1}. \tag{2.9}$$

Using the formula (2.2) for $\dim \mathbb{H}_n^{d-1}$ and $\dim \mathbb{H}_{n-1}^{d-1}$, we have, for $d \ge 2$,

$$N_{n,d} = \frac{(2n+d-2)\,(n+d-3)!}{n!(d-2)!}, \quad n \in \mathbb{N}. \tag{2.10}$$

In particular, with $n \in \mathbb{N}$, for $d = 2$, $N_{n,2} = 2$, and for $d = 3$, $N_{n,3} = 2n + 1$. It can be verified directly that $N_{0,d} = 1$ for any $d \ge 1$, and

$$N_{0,1} = N_{1,1} = 1, \quad N_{n,1} = 0 \;\forall\, n \ge 2. \tag{2.11}$$

Note the asymptotic behavior

$$N_{n,d} = \mathcal{O}(n^{d-2}) \quad \text{for } n \text{ sufficiently large.} \tag{2.12}$$

For the generating function of the sequence $\{N_{n,d}\}_n$, we apply the relation (2.9) for $n \ge 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} N_{n,d} z^n &= 1 + \sum_{n=1}^{\infty} N_{n,d} z^n \\
&= 1 + \sum_{n=1}^{\infty} \left(\dim \mathbb{H}_n^{d-1}\right) z^n + \sum_{n=1}^{\infty} \left(\dim \mathbb{H}_{n-1}^{d-1}\right) z^n \\
&= \sum_{n=0}^{\infty} \left(\dim \mathbb{H}_n^{d-1}\right) z^n + z \sum_{n=0}^{\infty} \left(\dim \mathbb{H}_n^{d-1}\right) z^n \\
&= (1+z) \sum_{n=0}^{\infty} \left(\dim \mathbb{H}_n^{d-1}\right) z^n.
\end{aligned}$$

Thus, using the formula (2.5), we get a compact formula for the generating function of the sequence $\{N_{n,d}\}_n$:

$$\sum_{n=0}^{\infty} N_{n,d} z^n = \frac{1+z}{(1-z)^{d-1}}, \quad |z| < 1. \tag{2.13}$$

We can use (2.13) to derive a recursion formula for $N_{n,d}$ with respect to the dimension parameter d. Write

$$\frac{1+z}{(1-z)^{d-1}} = \frac{1+z}{(1-z)^{d-2}} \cdot \frac{1}{1-z} = \left(\sum_{m=0}^{\infty} N_{m,d-1} z^m\right)\left(\sum_{k=0}^{\infty} z^k\right).$$

We have

$$\frac{1+z}{(1-z)^{d-1}} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} N_{m,d-1}\right) z^n.$$

Comparing this formula with (2.13), we obtain

$$N_{n,d} = \sum_{m=0}^{n} N_{m,d-1}. \tag{2.14}$$

2.1.2 Legendre Harmonic and Legendre Polynomial

We now introduce a special homogeneous harmonic, the Legendre harmonic of degree n in d dimensions, $L_{n,d} : \mathbb{R}^d \to \mathbb{R}$, by the following three conditions:

$$L_{n,d} \in \mathbb{Y}_n(\mathbb{R}^d), \tag{2.15}$$

$$L_{n,d}(A\boldsymbol{x}) = L_{n,d}(\boldsymbol{x}) \quad \forall\, A \in \mathbb{O}^d(\boldsymbol{e}_d),\ \forall\, \boldsymbol{x} \in \mathbb{R}^d, \tag{2.16}$$

$$L_{n,d}(\boldsymbol{e}_d) = 1. \tag{2.17}$$

The condition (2.16) expresses the isotropical symmetry of $L_{n,d}$ with respect to the x_d-axis, whereas the condition (2.17) is a normalizing condition. Write $L_{n,d}$ in the form (2.7) and $A \in \mathbb{O}^d(\boldsymbol{e}_d)$ in the form (2.1). Then the condition (2.16) implies

$$h_{n-j}(A_1 \boldsymbol{x}_{(d-1)}) = h_{n-j}(\boldsymbol{x}_{(d-1)}) \quad \forall\, A_1 \in \mathbb{O}^{d-1},\ \boldsymbol{x}_{(d-1)} \in \mathbb{R}^{d-1},\ 0 \le j \le n.$$

From Proposition 2.1, h_{n-j} depends on $\boldsymbol{x}_{(d-1)}$ through $|\boldsymbol{x}_{(d-1)}|$. Since h_{n-j} is a homogeneous polynomial, this is possible only if $(n-j)$ is even and we have

$$h_{n-j}(\boldsymbol{x}_{(d-1)}) = \begin{cases} c_k |\boldsymbol{x}_{(d-1)}|^{2k} & \text{if } n-j=2k, \\ 0 & \text{if } n-j=2k+1, \end{cases} \qquad c_k \in \mathbb{R}.$$

Hence,

$$L_{n,d}(\boldsymbol{x}) = \sum_{k=0}^{[n/2]} c_k |\boldsymbol{x}_{(d-1)}|^{2k} (x_d)^{n-2k},$$

where $[n/2]$ denotes the integer part of $n/2$. To determine the coefficients $\{c_k\}_{k=0}^{[n/2]}$, we apply the relation (2.8) to obtain

$$c_k = -\frac{(n-2k+2)\,(n-2k+1)}{2k\,(2k+d-3)}\, c_{k-1}, \quad 1 \le k \le [n/2].$$

The normalization condition (2.17) implies $c_0 = 1$. Then

$$c_k = (-1)^k \frac{n!\,\Gamma(\frac{d-1}{2})}{4^k k!\,(n-2k)!\,\Gamma(k+\frac{d-1}{2})}, \quad 0 \le k \le [n/2].$$

Therefore, we have derived the following formula for the Legendre harmonic

$$L_{n,d}(\boldsymbol{x}) = n!\,\Gamma\Big(\frac{d-1}{2}\Big) \sum_{k=0}^{[n/2]} (-1)^k \frac{|\boldsymbol{x}_{(d-1)}|^{2k} (x_d)^{n-2k}}{4^k k!\,(n-2k)!\,\Gamma(k+\frac{d-1}{2})}. \tag{2.18}$$

Using the polar coordinates

$$\boldsymbol{x}_{(d)} = r\boldsymbol{\xi}_{(d)}, \quad \boldsymbol{\xi}_{(d)} = t\,\boldsymbol{e}_d + (1-t^2)^{1/2}\boldsymbol{\xi}_{(d-1)},$$

we define the Legendre polynomial of degree n in d dimensions, $P_{n,d}(t) := L_{n,d}(\boldsymbol{\xi}_{(d)})$, as the restriction of the Legendre harmonic on the unit sphere. Then from the formula (2.18), we have

$$P_{n,d}(t) = n!\,\Gamma\Big(\frac{d-1}{2}\Big) \sum_{k=0}^{[n/2]} (-1)^k \frac{(1-t^2)^k t^{n-2k}}{4^k k!\,(n-2k)!\,\Gamma(k+\frac{d-1}{2})}. \tag{2.19}$$

Corresponding to (2.17), we have

$$P_{n,d}(1) = 1. \tag{2.20}$$

This property can be deduced straightforward from the formula (2.19). Note the relation

$$L_{n,d}(\boldsymbol{x}) = L_{n,d}(r\boldsymbol{\xi}_{(d)}) = r^n P_{n,d}(t). \tag{2.21}$$

The polynomial $P_{n,3}(t)$ is the standard Legendre polynomial of degree n. Following [85], we also call $P_{n,d}(t)$ of (2.19) Legendre polynomial.

Detailed discussion of the Legendre polynomials $P_{n,d}(t)$ is given in Sects. 2.6 and 2.7.

2.1.3 Spherical Harmonics

We are now ready to introduce spherical harmonics.

Definition 2.7. $\mathbb{Y}_n^d := \mathbb{Y}_n(\mathbb{R}^d)|_{\mathbb{S}^{d-1}}$ is called the spherical harmonic space of order n in d dimensions. Any function in $\mathbb{Y}_n^d$ is called a spherical harmonic of order n in d dimensions.

By the definition, we see that any spherical harmonic $Y_n \in \mathbb{Y}_n^d$ is related to a homogeneous harmonic $H_n \in \mathbb{Y}_n(\mathbb{R}^d)$ as follows:

$$H_n(r\boldsymbol{\xi}) = r^n Y_n(\boldsymbol{\xi}).$$

Thus the dimension of $\mathbb{Y}_n^d$ is the same as that of $\mathbb{Y}_n(\mathbb{R}^d)$:

$$\dim \mathbb{Y}_n^d = N_{n,d}$$

and $N_{n,d}$ is given by (2.10).

Take the case of $d = 2$ as an example. The complex-valued function $(x_1 + i\,x_2)^n$ is a homogeneous harmonic of degree n, and so are the real part and the imaginary part of the function. In polar coordinates (r, θ), $\boldsymbol{\xi} = (\cos\theta, \sin\theta)^T$ and the restriction of the function $(x_1 + i\,x_2)^n$ on the unit circle is

$$(\cos\theta + i\,\sin\theta)^n = e^{in\theta} = \cos(n\theta) + i\,\sin(n\theta).$$

Thus,

$$y_{n,1}(\boldsymbol{\xi}) = \cos(n\theta), \quad y_{n,2}(\boldsymbol{\xi}) = \sin(n\theta) \tag{2.22}$$

are elements of the space $\mathbb{Y}_n^2$.

Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ be fixed. A function $f : \mathbb{S}^{d-1} \to \mathbb{C}$ is said to be invariant with respect to $\mathbb{O}^d(\boldsymbol{\xi})$ if

$$f(A\boldsymbol{\eta}) = f(\boldsymbol{\eta}) \quad \forall\, A \in \mathbb{O}^d(\boldsymbol{\xi}),\, \forall\, \boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

We have the following result, which will be useful later on several occasions.

Theorem 2.8. *Let $Y_n \in \mathbb{Y}_n^d$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. Then Y_n is invariant with respect to $\mathbb{O}^d(\boldsymbol{\xi})$ if and only if*

$$Y_n(\boldsymbol{\eta}) = Y_n(\boldsymbol{\xi})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) \quad \forall\, \boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{2.23}$$

Proof. ($\Longrightarrow$) Since $\boldsymbol{\xi}$ is a unit vector, we can find an $A_1 \in \mathbb{O}^d$ such that $\boldsymbol{\xi} = A_1 \boldsymbol{e}_d$. Consider the function

$$\tilde{Y}_n(\boldsymbol{\eta}) := Y_n(A_1\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

Then $\tilde{Y}_n$ is invariant with respect to $\mathbb{O}^d(\boldsymbol{e}_d)$. From the definition of the Legendre harmonic $L_{n,d}(\boldsymbol{x})$, we know that the homogeneous harmonic $r^n\tilde{Y}_n(\boldsymbol{\eta})$ is a multiple of $L_{n,d}(r^n\boldsymbol{\eta})$,

$$r^n\tilde{Y}_n(\boldsymbol{\eta}) = c_1 L_{n,d}(r^n\boldsymbol{\eta}), \quad r \ge 0,\ \boldsymbol{\eta} \in \mathbb{S}^{d-1}$$

with some constant c_1. Thus,

$$\tilde{Y}_n(\boldsymbol{\eta}) = c_1 L_{n,d}(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

Choosing $\boldsymbol{\eta} = \boldsymbol{e}_d$, we find

$$c_1 = \tilde{Y}_n(\boldsymbol{e}_d).$$

Hence,

$$\tilde{Y}_n(\boldsymbol{\eta}) = \tilde{Y}_n(\boldsymbol{e}_d)\, L_{n,d}(\boldsymbol{\eta}) = \tilde{Y}_n(\boldsymbol{e}_d)\, P_{n,d}(\boldsymbol{\eta}\cdot\boldsymbol{e}_d), \quad \boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

Then,

$$\begin{aligned} Y_n(\boldsymbol{\eta}) &= \tilde{Y}_n(A_1^T\boldsymbol{\eta}) \\ &= Y_n(A_1\boldsymbol{e}_d)\, P_{n,d}(A_1^T\boldsymbol{\eta}\cdot\boldsymbol{e}_d) \\ &= Y_n(A_1\boldsymbol{e}_d)\, P_{n,d}(\boldsymbol{\eta}\cdot A_1\boldsymbol{e}_d) \\ &= Y_n(\boldsymbol{\xi})\, P_{n,d}(\boldsymbol{\eta}\cdot\boldsymbol{\xi}), \end{aligned}$$

i.e., the formula (2.23) holds.

($\Longleftarrow$) The function $Y_n(\boldsymbol{\eta})$ satisfying (2.23) is obviously invariant with respect to $\mathbb{O}^d(\boldsymbol{\xi})$. □

Consequently, the subspaces of isotropically invariant functions from $\mathbb{Y}_n^d$ are one-dimensional.

2.2 Addition Theorem and Its Consequences

One important property regarding the spherical harmonics is the addition theorem.

Theorem 2.9 (Addition Theorem). *Let* $\{Y_{n,j} : 1 \le j \le N_{n,d}\}$ *be an orthonormal basis of* $\mathbb{Y}_n^d$, *i.e.,*

$$\int_{\mathbb{S}^{d-1}} Y_{n,j}(\boldsymbol{\eta})\overline{Y_{n,k}(\boldsymbol{\eta})}\, dS^{d-1}(\boldsymbol{\eta}) = \delta_{jk}, \quad 1 \le j,k \le N_{n,d}.$$

Then

$$\sum_{j=1}^{N_{n,d}} Y_{n,j}(\boldsymbol{\xi})\overline{Y_{n,j}(\boldsymbol{\eta})} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) \quad \forall\, \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{2.24}$$

Proof. For any $A \in \mathbb{O}^d$ and $1 \le k \le N_{n,d}$, $Y_{n,k}(A\boldsymbol{\xi}) \in \mathbb{Y}_n^d$ and we can write

$$Y_{n,k}(A\boldsymbol{\xi}) = \sum_{j=1}^{N_{n,d}} c_{kj} Y_{n,j}(\boldsymbol{\xi}), \quad c_{kj} \in \mathbb{C}. \tag{2.25}$$

From

$$\int_{\mathbb{S}^{d-1}} Y_{n,j}(A\boldsymbol{\xi})\overline{Y_{n,k}(A\boldsymbol{\xi})}\, dS^{d-1}(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} Y_{n,j}(\boldsymbol{\eta})\overline{Y_{n,k}(\boldsymbol{\eta})}\, dS^{d-1}(\boldsymbol{\eta}) = \delta_{jk},$$

we have

$$\delta_{jk} = \sum_{l,m=1}^{N_{n,d}} c_{jl}\overline{c_{km}}\,(Y_{n,l}, Y_{n,m}) = \sum_{l=1}^{N_{n,d}} c_{jl}\overline{c_{kl}}.$$

In matrix form, $CC^H = I$. Here C^H is the conjugate transpose of C. Thus, the matrix $C := (c_{jl})$ is unitary and so $C^H C = I$, i.e.,

$$\sum_{j=1}^{N_{n,d}} \overline{c_{jl}}c_{jk} = \delta_{lk}, \quad 1 \le l,k \le N_{n,d}. \tag{2.26}$$

Now consider the sum

$$Y(\boldsymbol{\xi},\boldsymbol{\eta}) := \sum_{j=1}^{N_{n,d}} Y_{n,j}(\boldsymbol{\xi})\overline{Y_{n,j}(\boldsymbol{\eta})}, \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

For any $A \in \mathbb{O}^d$, use the expansion (2.25),

$$Y(A\boldsymbol{\xi}, A\boldsymbol{\eta}) = \sum_{j=1}^{N_{n,d}} Y_{n,j}(A\boldsymbol{\xi})\overline{Y_{n,j}(A\boldsymbol{\eta})} = \sum_{j,k,l=1}^{N_{n,d}} c_{jk}\overline{c_{jl}}Y_{n,k}(\boldsymbol{\xi})\overline{Y_{n,l}(\boldsymbol{\eta})},$$

and then use the property (2.26),

$$Y(A\boldsymbol{\xi}, A\boldsymbol{\eta}) = \sum_{k=1}^{N_{n,d}} Y_{n,k}(\boldsymbol{\xi})\overline{Y_{n,k}(\boldsymbol{\eta})} = Y(\boldsymbol{\xi}, \boldsymbol{\eta}).$$

So for fixed $\boldsymbol{\xi}$, $Y(\boldsymbol{\xi}, \cdot) \in \mathbb{Y}_n^d$ and is invariant with respect to $\mathbb{O}^d(\boldsymbol{\xi})$. By Theorem 2.8,

$$Y(\boldsymbol{\xi}, \boldsymbol{\eta}) = Y(\boldsymbol{\xi}, \boldsymbol{\xi})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}).$$

Similarly, we have the equality

$$Y(\boldsymbol{\xi}, \boldsymbol{\eta}) = Y(\boldsymbol{\eta}, \boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}).$$

Thus, $Y(\boldsymbol{\xi}, \boldsymbol{\xi}) = Y(\boldsymbol{\eta}, \boldsymbol{\eta})$ and is a constant on $\mathbb{S}^{d-1}$. To determine this constant, we integrate the equality

$$Y(\boldsymbol{\xi}, \boldsymbol{\xi}) = \sum_{j=1}^{N_{n,d}} |Y_{n,j}(\boldsymbol{\xi})|^2$$

over $\mathbb{S}^{d-1}$ to obtain

$$Y(\boldsymbol{\xi}, \boldsymbol{\xi})\, |\mathbb{S}^{d-1}| = \sum_{j=1}^{N_{n,d}} \int_{\mathbb{S}^{d-1}} |Y_j(\boldsymbol{\xi})|^2 dS^{d-1} = N_{n,d}.$$

Therefore,

$$Y(\boldsymbol{\xi}, \boldsymbol{\xi}) = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}$$

and the equality (2.24) holds. □

The equality (2.24) is, for $d = 3$,

$$\sum_{j=1}^{2n+1} Y_{n,j}(\boldsymbol{\xi})\overline{Y_{n,j}(\boldsymbol{\eta})} = \frac{2n+1}{4\,\pi}\, P_{n,3}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) \quad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^2, \tag{2.27}$$

and for $d = 2$,

$$\sum_{j=1}^{2} Y_{n,j}(\boldsymbol{\xi})\overline{Y_{n,j}(\boldsymbol{\eta})} = \frac{1}{\pi}\, P_{n,2}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) \quad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^1. \tag{2.28}$$

For the case $d = 2$, we write $\boldsymbol{\xi} = (\cos\theta, \sin\theta)^T$ and $\boldsymbol{\eta} = (\cos\psi, \sin\psi)^T$. Then, $\boldsymbol{\xi}\cdot\boldsymbol{\eta} = \cos(\theta - \psi)$. As an orthonormal basis for $\mathbb{Y}_n^2$, take (cf. (2.22))

$$Y_{n,1}(\boldsymbol{\xi}) = \frac{1}{\sqrt{\pi}} \cos(n\theta), \quad Y_{n,2}(\boldsymbol{\xi}) = \frac{1}{\sqrt{\pi}} \sin(n\theta).$$

By (2.28),

$$P_{n,2}(\cos(\theta-\psi)) = \cos(n\theta)\,\cos(n\psi) + \sin(n\theta)\,\sin(n\psi) = \cos(n\,(\theta-\psi))\,.$$

Thus,

$$P_{n,2}(t) = \cos(n\,\arccos t)\,,\quad |t|\le 1, \tag{2.29}$$

i.e., $P_{n,2}$ is the ordinary Chebyshev polynomial of degree n.

We note that for $d=2$,

$$\sum_{k=0}^{n} \frac{1}{\pi}\,P_{k,2}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = \frac{1}{2\,\pi}\,\frac{\sin((n+1/2)\,\phi)}{\sin(\phi/2)},\quad \cos\phi := \boldsymbol{\xi}\cdot\boldsymbol{\eta},$$

is the Dirichlet kernel, whereas for $d=3$,

$$\sum_{k=0}^{n}\sum_{j=1}^{2k+1} Y_{k,j}(\boldsymbol{\xi})\overline{Y_{k,j}(\boldsymbol{\eta})} = \frac{n+1}{4\,\pi}\,P_n^{(1,0)}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\quad \forall\,\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{S}^2. \tag{2.30}$$

Here $P_n^{(1,0)}(t)$ is the Jacobi polynomial of degree n on $[-1,1]$, based on the weight function $w(t)=1-t$; and as a normalization, $P_n^{(1,0)}(1)=n+1$. This identity is noted in [50]. See Sect. 4.3.1 for an introduction of the Jacobi polynomials.

We now discuss several applications of the addition theorem.

The addition theorem can be used to find a compact expression of the reproducing kernel of $\mathbb{Y}_n^d$. Any $Y_n\in\mathbb{Y}_n^d$ can be written in the form

$$Y_n(\boldsymbol{\xi}) = \sum_{j=1}^{N_{n,d}} (Y_n, Y_{n,j})_{\mathbb{S}^{d-1}} Y_{n,j}(\boldsymbol{\xi}). \tag{2.31}$$

Applying (2.24),

$$\begin{aligned} Y_n(\boldsymbol{\xi}) &= \int_{\mathbb{S}^{d-1}} Y_n(\boldsymbol{\eta}) \sum_{j=1}^{N_{n,d}} Y_{n,j}(\boldsymbol{\xi})\overline{Y_{n,j}(\boldsymbol{\eta})}\,dS^{d-1}(\boldsymbol{\eta}) \\ &= \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,Y_n(\boldsymbol{\eta})\,dS^{d-1}(\boldsymbol{\eta}). \end{aligned}$$

Hence,

$$K_{n,d}(\boldsymbol{\xi},\boldsymbol{\eta}) := \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) \tag{2.32}$$

is the reproducing kernel of $\mathbb{Y}_n^d$, i.e.,

$$Y_n(\boldsymbol{\xi}) = (Y_n, K_{n,d}(\boldsymbol{\xi},\cdot))_{\mathbb{S}^{d-1}}\quad \forall\,Y_n\in\mathbb{Y}_n^d,\ \boldsymbol{\xi}\in\mathbb{S}^{d-1}. \tag{2.33}$$

Define

$$\mathbb{Y}^d_{0:m} := \bigoplus_{n=0}^{m} \mathbb{Y}^d_n$$

to be the space of all the spherical harmonics of order less than or equal to m. Then by (2.33),

$$K_{0:m,d}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \frac{1}{|\mathbb{S}^{d-1}|} \sum_{n=0}^{m} N_{n,d} P_{n,d}(\boldsymbol{\xi} \cdot \boldsymbol{\eta}) \tag{2.34}$$

is the reproducing kernel of $\mathbb{Y}^d_{0:m}$ in the sense that

$$Y(\boldsymbol{\xi}) = (Y, K_{0:m,d}(\boldsymbol{\xi}, \cdot))_{\mathbb{S}^{d-1}} \quad \forall\, Y \in \mathbb{Y}^d_{0:m},\ \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

We now derive some bounds for any spherical harmonic and for the Legendre polynomial, see (2.38) and (2.39) below, respectively.

Since $P_{n,d}(1) = 1$, we get from (2.24) that

$$\sum_{j=1}^{N_{n,d}} |Y_{n,j}(\boldsymbol{\xi})|^2 = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \quad \forall\, \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.35}$$

This provides an upper bound for the maximum value of any member of an orthonormal basis in $\mathbb{Y}^d_n$:

$$\max\left\{ |Y_{n,j}(\boldsymbol{\xi})| : \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ 1 \le j \le N_{n,d} \right\} \le \left(\frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \right)^{1/2}. \tag{2.36}$$

Consider an arbitrary $Y_n \in \mathbb{Y}^d_n$. From (2.31), we find

$$\int_{\mathbb{S}^{d-1}} |Y_n(\boldsymbol{\xi})|^2 dS^{d-1}(\boldsymbol{\xi}) = \sum_{j=1}^{N_{n,d}} |(Y_n, Y_{n,j})_{\mathbb{S}^{d-1}}|^2. \tag{2.37}$$

By (2.31) again,

$$|Y_n(\boldsymbol{\xi})|^2 \le \sum_{j=1}^{N_{n,d}} |Y_{n,j}(\boldsymbol{\xi})|^2 \sum_{j=1}^{N_{n,d}} |(Y_n, Y_{n,j})_{\mathbb{S}^{d-1}}|^2.$$

Then using (2.35) and (2.37),

$$|Y_n(\boldsymbol{\xi})|^2 \le \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \|Y_n\|^2_{L^2(\mathbb{S}^{d-1})}.$$

Thus we have the inequality

$$\|Y_n\|_\infty \le \left(\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\right)^{1/2} \|Y_n\|_{L^2(\mathbb{S}^{d-1})} \quad \forall\, Y_n \in \mathbb{Y}_n^d, \tag{2.38}$$

which extends the bound (2.36).

By (2.24) and (2.35), we have

$$\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\,|P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})| \le \left[\sum_{j=1}^{N_{n,d}} |Y_{n,j}(\boldsymbol{\xi})|^2\right]^{1/2} \left[\sum_{j=1}^{N_{n,d}} |Y_{n,j}(\boldsymbol{\eta})|^2\right]^{1/2} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}.$$

Therefore,

$$|P_{n,d}(t)| \le 1 = P_{n,d}(1) \quad \forall\, n \in \mathbb{N},\ d \ge 2,\ t \in [-1,1]. \tag{2.39}$$

We have an integral formula

$$\int_{\mathbb{S}^{d-1}} |P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})|^2 dS^{d-1}(\boldsymbol{\eta}) = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}}. \tag{2.40}$$

This formula is proved as follows. First we use (2.24) to get

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} |P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})|^2 dS^{d-1}(\boldsymbol{\eta}) \\
&\quad = \left(\frac{|\mathbb{S}^{d-1}|}{N_{n,d}}\right)^2 \int_{\mathbb{S}^{d-1}} \left|\sum_{j=1}^{N_{n,d}} Y_{n,j}(\boldsymbol{\xi})\overline{Y_{n,j}(\boldsymbol{\eta})}\right|^2 dS^{d-1}(\boldsymbol{\eta}) \\
&\quad = \left(\frac{|\mathbb{S}^{d-1}|}{N_{n,d}}\right)^2 \sum_{j=1}^{N_{n,d}} |Y_{n,j}(\boldsymbol{\xi})|^2.
\end{aligned}$$

Then we apply the identity (2.35).

As one more application of the addition theorem, we have the following result.

Theorem 2.10. *For any $n \in \mathbb{N}_0$ and any $d \in \mathbb{N}$, the spherical harmonic space $\mathbb{Y}_n^d$ is irreducible.*

Proof. We argue by contradiction. Suppose $\mathbb{Y}_n^d$ is reducible so that it is possible to write $\mathbb{Y}_n^d = \mathbb{V}_1 + \mathbb{V}_2$ with $\mathbb{V}_1 \ne \emptyset$, $\mathbb{V}_2 \ne \emptyset$, and $\mathbb{V}_1 \perp \mathbb{V}_2$. Choose an orthonormal basis of $\mathbb{Y}_n^d$ in such a way that the first N_1 functions span $\mathbb{V}_1$ and the remaining $N_2 = N_{n,d} - N_1$ functions span $\mathbb{V}_2$. For both $\mathbb{V}_1$ and $\mathbb{V}_2$, we can apply the addition theorem with the corresponding Legendre functions $P_{n,d,1}$ and $P_{n,d,2}$. Since $\mathbb{V}_1 \perp \mathbb{V}_2$,

$$\int_{\mathbb{S}^{d-1}} P_{n,d,1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) P_{n,d,2}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = 0 \quad \forall\, \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.41}$$

For an arbitrary but fixed $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$, consider the function $\boldsymbol{\eta} \mapsto P_{n,d,1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})$. For any $A \in \mathbb{O}^d(\boldsymbol{\xi})$, we have $A^T A = I$ and $A\boldsymbol{\xi} = \boldsymbol{\xi}$, implying $A^T\boldsymbol{\xi} = \boldsymbol{\xi}$. Then

$$P_{n,d,1}(\boldsymbol{\xi}\cdot A\boldsymbol{\eta}) = P_{n,d,1}(A^T\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = P_{n,d,1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}),$$

i.e., the function $\boldsymbol{\eta} \mapsto P_{n,d,1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})$ is invariant with respect to $\mathbb{O}^d(\boldsymbol{\xi})$. By Theorem 2.8,

$$P_{n,d,1}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = P_{n,d,1}(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}).$$

Similarly,

$$P_{n,d,2}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}).$$

But then the integral in (2.41) equals $|\mathbb{S}^{d-1}|/N_{n,d}$ by (2.40) and we reach a contradiction. $\square$

2.3 A Projection Operator

Consider the problem of finding the best approximation in $\mathbb{Y}_n^d$ of a function $f \in L^2(\mathbb{S}^{d-1})$:

$$\inf\left\{\|f - Y_n\|_{L^2(\mathbb{S}^{d-1})} : Y_n \in \mathbb{Y}_n^d\right\}. \tag{2.42}$$

In terms of an orthonormal basis $\{Y_{n,j} : 1 \le j \le N_{n,d}\}$ of $\mathbb{Y}_n^d$, the solution of the problem (2.42) is

$$(\mathcal{P}_{n,d} f)(\boldsymbol{\xi}) = \sum_{j=1}^{N_{n,d}} (f, Y_{n,j})_{\mathbb{S}^{d-1}} Y_{n,j}(\boldsymbol{\xi}). \tag{2.43}$$

This is the projection of any f into $\mathbb{Y}_n^d$ and it is defined for $f \in L^1(\mathbb{S}^{d-1})$. The disadvantage of using this formula is the requirement of explicit knowledge of an orthonormal basis. We can circumvent this weakness by applying (2.24) to rewrite the right side of (2.43).

Definition 2.11. The projection of $f \in L^1(\mathbb{S}^{d-1})$ into $\mathbb{Y}_n^d$ is

$$(\mathcal{P}_{n,d} f)(\boldsymbol{\xi}) := \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.44}$$

The operator $\mathcal{P}_{n,d}$ is obviously linear. Let us derive some bounds for the operator $\mathcal{P}_{n,d}$. First, we obtain from (2.39) that

$$|(\mathcal{P}_{n,d}f)(\boldsymbol{\xi})| \le \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\|f\|_{L^1(\mathbb{S}^{d-1})}, \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

Then, for all $f \in L^1(\mathbb{S}^{d-1})$,

$$\|\mathcal{P}_{n,d}f\|_{C(\mathbb{S}^{d-1})} \le \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\|f\|_{L^1(\mathbb{S}^{d-1})}, \tag{2.45}$$

$$\|\mathcal{P}_{n,d}f\|_{L^1(\mathbb{S}^{d-1})} \le N_{n,d}\|f\|_{L^1(\mathbb{S}^{d-1})}. \tag{2.46}$$

Next, assume $f \in L^2(\mathbb{S}^{d-1})$. For any $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$,

$$\begin{aligned}|(\mathcal{P}_{n,d}f)(\boldsymbol{\xi})|^2 \le \left(\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\right)^2 &\int_{\mathbb{S}^{d-1}} |P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})|^2 dS^{d-1}(\boldsymbol{\eta}) \\ &\cdot \int_{\mathbb{S}^{d-1}} |f(\boldsymbol{\eta})|^2 dS^{d-1}(\boldsymbol{\eta}).\end{aligned}$$

Use (2.40),

$$|(\mathcal{P}_{n,d}f)(\boldsymbol{\xi})|^2 \le \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\|f\|^2_{L^2(\mathbb{S}^{d-1})}.$$

Hence, for all $f \in L^2(\mathbb{S}^{d-1})$,

$$\|\mathcal{P}_{n,d}f\|_{L^2(\mathbb{S}^{d-1})} \le N_{n,d}^{1/2}\|f\|_{L^2(\mathbb{S}^{d-1})}, \tag{2.47}$$

$$\|\mathcal{P}_{n,d}f\|_{C(\mathbb{S}^{d-1})} \le \left(\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\right)^{1/2}\|f\|_{L^2(\mathbb{S}^{d-1})}. \tag{2.48}$$

We remark that (2.47) can be improved to

$$\|\mathcal{P}_{n,d}f\|_{L^2(\mathbb{S}^{d-1})} \le \|f\|_{L^2(\mathbb{S}^{d-1})};$$

see (2.134) later. Furthermore, if $f \in C(\mathbb{S}^{d-1})$, a similar argument leads to

$$\|\mathcal{P}_{n,d}f\|_{C(\mathbb{S}^{d-1})} \le N_{n,d}^{1/2}\|f\|_{C(\mathbb{S}^{d-1})}. \tag{2.49}$$

Proposition 2.12. *The projection operator $\mathcal{P}_{n,d}$ and orthogonal transformations commute:*

$$\mathcal{P}_{n,d}f_A = (\mathcal{P}_{n,d}f)_A \quad \forall\, A \in \mathbb{O}^d.$$

Proof. We start with the left side of the equality,

$$
\begin{aligned}
(\mathcal{P}_{n,d} f_A)(\boldsymbol{\xi}) &= \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, f(A\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) \\
&= \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} P_{n,d}(A\boldsymbol{\xi}\cdot\boldsymbol{\zeta})\, f(\boldsymbol{\zeta})\, dS^{d-1}(\boldsymbol{\zeta}),
\end{aligned}
$$

which is $(\mathcal{P}_{n,d} f)_A(\boldsymbol{\xi})$ by definition. □

A useful consequence of Proposition 2.12 is the following result.

Corollary 2.13. *If $\mathbb{V}$ is an invariant space, then $\mathcal{P}_{n,d}\mathbb{V} := \{\mathcal{P}_{n,d} f : f \in \mathbb{V}\}$ is an invariant subspace of $\mathbb{Y}_n^d$.*

Since $\mathbb{Y}_n^d$ is irreducible, by Theorem 2.10, Corollary 2.13 implies that if $\mathbb{V}$ is an invariant space, then either $\mathbb{V}$ is orthogonal to $\mathbb{Y}_n^d$ or $\mathcal{P}_{n,d}\mathbb{V} = \mathbb{Y}_n^d$. Moreover, we have the next result.

Theorem 2.14. *If $\mathbb{V}$ is a primitive subspace of $C(\mathbb{S}^{d-1})$, then either $\mathbb{V} \perp \mathbb{Y}_n^d$ or $\mathcal{P}_{n,d}$ is a bijection from $\mathbb{V}$ to $\mathbb{Y}_n^d$. In the latter case, $\mathbb{V} = \mathbb{Y}_n^d$.*

Proof. We only need to prove that if $\mathcal{P}_{n,d} : \mathbb{V} \to \mathbb{Y}_n^d$ is a bijection, then $\mathbb{V} = \mathbb{Y}_n^d$. The two spaces are finite dimensional and have the same dimension $N_{n,d} = \dim(\mathbb{Y}_n^d)$. Let $\{V_j : 1 \le j \le N_{n,d}\}$ be an orthonormal basis of $\mathbb{V}$. Since $\mathbb{V}$ is primitive, for any $A \in \mathbb{O}^d$, we can write

$$
V_j(A\boldsymbol{\xi}) = \sum_{k=1}^{N_{n,d}} c_{jk} V_k(\boldsymbol{\xi}), \quad c_{jk} \in \mathbb{C},
$$

and the matrix (c_{jk}) is unitary as in the proof of Theorem 2.9. Consider the function

$$
V(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{j=1}^{N_{n,d}} V_j(\boldsymbol{\xi}) \overline{V_j(\boldsymbol{\eta})}.
$$

Then again as in the proof of Theorem 2.9, we have

$$
V(A\boldsymbol{\xi}, A\boldsymbol{\eta}) = V(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad \forall\, A \in \mathbb{O}^d.
$$

Given $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1}$, we can find an $A \in \mathbb{O}^d$ such that

$$
A\boldsymbol{\xi} = \boldsymbol{e}_d, \quad A\boldsymbol{\eta} = t\, \boldsymbol{e}_d + (1-t^2)^{1/2} \boldsymbol{e}_{d-1} \text{ with } t = \boldsymbol{\xi}\cdot\boldsymbol{\eta}.
$$

Then

$$
V(\boldsymbol{\xi}, \boldsymbol{\eta}) = V(\boldsymbol{e}_d, t\, \boldsymbol{e}_d + (1-t^2)^{1/2} \boldsymbol{e}_{d-1})
$$

is a function of $t = \boldsymbol{\xi}\cdot\boldsymbol{\eta}$. Denote this function by $P_d(t)$. For fixed $\boldsymbol{\xi}$, the mapping $\boldsymbol{\eta} \mapsto \overline{P_d(\boldsymbol{\xi}\cdot\boldsymbol{\eta})}$ is a function in $\mathbb{V}$, whereas for fixed $\boldsymbol{\zeta}$, the mapping $\boldsymbol{\eta} \mapsto P_{n,d}(\boldsymbol{\zeta}\cdot\boldsymbol{\eta})$ is a function in $\mathbb{Y}_n^d$. Consider the function

$$\phi(\boldsymbol{\xi},\boldsymbol{\zeta}) = \int_{\mathbb{S}^{d-1}} \overline{P_d(\boldsymbol{\xi}\cdot\boldsymbol{\eta})} P_{n,d}(\boldsymbol{\zeta}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}).$$

We have the property

$$\phi(A\boldsymbol{\xi}, A\boldsymbol{\zeta}) = \phi(\boldsymbol{\xi},\boldsymbol{\zeta}) \quad \forall\, A \in \mathbb{O}^d.$$

So $\phi(\boldsymbol{\xi},\boldsymbol{\zeta})$ depends on $\boldsymbol{\xi}\cdot\boldsymbol{\zeta}$ only. This function belongs to both $\mathbb{V}$ and $\mathbb{Y}_n^d$. Thus, either $\mathbb{V} = \mathbb{Y}_n^d$ or $\phi \equiv 0$. In the latter case, we have

$$\sum_{j,k=1}^{N_{n,d}} \overline{V_j(\boldsymbol{\xi})} Y_{n,k}(\boldsymbol{\zeta})\, (V_j, Y_{n,k})_{L^2(\mathbb{S}^{d-1})} = 0 \quad \forall\, \boldsymbol{\xi},\boldsymbol{\zeta} \in \mathbb{S}^{d-1},$$

where $\{Y_{n,k} : 1 \le k \le N_{n,d}\}$ is an orthonormal basis of $\mathbb{Y}_n^d$. Since each of the sets $\{V_j : 1 \le j \le N_{n,d}\}$ and $\{Y_{n,j} : 1 \le j \le N_{n,d}\}$ consists of linearly independent elements, we obtain from the above identity that

$$(V_j, Y_{n,k})_{L^2(\mathbb{S}^{d-1})} = 0, \quad 1 \le j,k \le N_{n,d}.$$

This implies $\mathbb{V} \perp \mathbb{Y}_n^d$. $\square$

We let $\mathbb{V} = \mathbb{Y}_m^d$, $m \ne n$, in Theorem 2.14 to obtain the following result concerning orthogonality of spherical harmonics of different order.

Corollary 2.15. *For $m \ne n$, $\mathbb{Y}_m^d \perp \mathbb{Y}_n^d$.*

This result can be proved directly as follows. Let $Y_m \in \mathbb{Y}_m^d$ and $Y_n \in \mathbb{Y}_n^d$ be the restrictions on $\mathbb{S}^{d-1}$ of $H_m \in \mathbb{Y}_m(\mathbb{R}^d)$ and $H_n \in \mathbb{Y}_n(\mathbb{R}^d)$, respectively. Since $\Delta H_m(\boldsymbol{x}) = \Delta H_n(\boldsymbol{x}) = 0$, we have

$$\int_{\|\boldsymbol{x}\|<1} (H_m \Delta H_n - H_n \Delta H_m)\, d\boldsymbol{x} = 0.$$

Apply Green's formula,

$$\int_{\mathbb{S}^{d-1}} \left(H_m \frac{\partial H_n}{\partial r} - H_n \frac{\partial H_m}{\partial r} \right) dS^{d-1} = 0. \tag{2.50}$$

Since H_m is a homogeneous polynomial of degree m,

$$\left.\frac{\partial H_m(\boldsymbol{x})}{\partial r}\right|_{\boldsymbol{x}=\boldsymbol{\xi}} = m\, Y_m(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

Similarly,

$$\left.\frac{\partial H_n(\boldsymbol{x})}{\partial r}\right|_{\boldsymbol{x}=\boldsymbol{\xi}} = n\,Y_n(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

Thus, from (2.50),

$$\int_{\mathbb{S}^{d-1}} (n-m)\,Y_m(\boldsymbol{\xi})\,Y_n(\boldsymbol{\xi})\,dS^{d-1}(\boldsymbol{\xi}) = 0.$$

Hence, since $m \neq n$,

$$\int_{\mathbb{S}^{d-1}} Y_m(\boldsymbol{\xi})\,Y_n(\boldsymbol{\xi})\,dS^{d-1}(\boldsymbol{\xi}) = 0.$$

2.4 Relations Among Polynomial Spaces

We have introduced several polynomial spaces in the previous sections. Here we discuss some relations among these polynomial spaces.

Proposition 2.16. *The Laplacian operator Δ is surjective from $\mathbb{H}_n^d$ to $\mathbb{H}_{n-2}^d$ for $n \geq 2$.*

Proof. Obviously, the operator Δ maps $\mathbb{H}_n^d$ to $\mathbb{H}_{n-2}^d$. By (2.2) and (2.10), we have

$$\begin{aligned}
\dim \mathbb{H}_n^d - \dim \mathbb{Y}_n(\mathbb{R}^d) &= \frac{(n+d-1)!}{n!\,(d-1)!} - \frac{(2n+d-2)\,(n+d-3)!}{n!\,(d-2)!} \\
&= \frac{(n-2+d-1)!}{(n-2)!\,(d-1)!} \\
&= \dim \mathbb{H}_{n-2}^d.
\end{aligned}$$

Therefore, $\Delta : \mathbb{H}_n^d \to \mathbb{H}_{n-2}^d$ is surjective. $\square$

It is possible to give another proof of Proposition 2.16 using the inner product (2.6). Suppose $\Delta : \mathbb{H}_n^d \to \mathbb{H}_{n-2}^d$ is not surjective. Then there exists a non-zero function $H_{n-2} \in \mathbb{H}_{n-2}^d$ such that

$$(\Delta H_n, H_{n-2})_{\mathbb{H}_{n-2}^d} = 0 \quad \forall\, H_n \in \mathbb{H}_n^d.$$

Take $H_n(\boldsymbol{x}) = |\boldsymbol{x}|^2 H_{n-2}(\boldsymbol{x})$ to get

$$\begin{aligned}
(H_n, H_n)_{\mathbb{H}_n^d} &= H_n(\boldsymbol{\nabla})\overline{H_n(\boldsymbol{x})} = H_{n-2}(\boldsymbol{\nabla})\overline{\Delta H_n(\boldsymbol{x})} \\
&= (H_{n-2}, \Delta H_n)_{\mathbb{H}_{n-2}^d} = 0.
\end{aligned}$$

Hence, $H_n(\boldsymbol{x}) = 0$ and then $H_{n-2}(\boldsymbol{x}) = 0$. This contradicts the assumption that $H_{n-2} \neq 0$.

Lemma 2.17. *For $n \geq 2$, $\mathbb{H}_n^d = \mathbb{Y}_n(\mathbb{R}^d) \oplus |\cdot|^2\mathbb{H}_{n-2}^d$, with respect to the inner product (2.6).*

Proof. It is shown in the proof of Proposition 2.16 that

$$\dim \mathbb{H}_n^d = \dim \mathbb{Y}_n(\mathbb{R}^d) + \dim \mathbb{H}_{n-2}^d.$$

Thus, it remains to show $\mathbb{Y}_n(\mathbb{R}^d) \perp |\cdot|^2\mathbb{H}_{n-2}^d$. For any $Y_n \in \mathbb{Y}_n(\mathbb{R}^d)$ and any $H_{n-2} \in \mathbb{H}_{n-2}^d$, there holds

$$\big(Y_n, |\cdot|^2 H_{n-2}\big)_{\mathbb{H}_n^d} = (\Delta Y_n, H_{n-2})_{\mathbb{H}_{n-2}^d} = 0.$$

Therefore, the statement is valid. □

The orthogonal decomposition stated in Lemma 2.17 can be applied repeatedly, leading to the next result.

Theorem 2.18. *With respect to the inner product (2.6), we have*

$$\mathbb{H}_n^d = \mathbb{Y}_n(\mathbb{R}^d) \oplus |\cdot|^2\mathbb{Y}_{n-2}(\mathbb{R}^d) \oplus \cdots \oplus |\cdot|^{2\,[n/2]}\mathbb{Y}_{n-2\,[n/2]}(\mathbb{R}^d). \tag{2.51}$$

Proof. For any $H_n \in \mathbb{H}_n^d$, by Lemma 2.17, we have

$$H_n(\boldsymbol{x}) = Y_n(\boldsymbol{x}) + |\boldsymbol{x}|^2 H_{n-2}(\boldsymbol{x})$$

with uniquely determined $Y_n \in \mathbb{Y}_n(\mathbb{R}^d)$ and $H_{n-2} \in \mathbb{H}_{n-2}^d$. Applying Lemma 2.17 to $H_{n-2} \in \mathbb{H}_{n-2}^d$, we can uniquely determine a pair of functions $Y_{n-2} \in \mathbb{Y}_{n-2}(\mathbb{R}^d)$ and $H_{n-4} \in \mathbb{H}_{n-4}^d$ such that

$$H_{n-2}(\boldsymbol{x}) = Y_{n-2}(\boldsymbol{x}) + |\boldsymbol{x}|^2 H_{n-4}(\boldsymbol{x}).$$

Hence,

$$H_n(\boldsymbol{x}) = Y_n(\boldsymbol{x}) + |\boldsymbol{x}|^2 Y_{n-2}(\boldsymbol{x}) + |\boldsymbol{x}|^4 H_{n-4}(\boldsymbol{x}).$$

Continue this process to obtain the unique decomposition

$$H_n(\boldsymbol{x}) = Y_n(\boldsymbol{x}) + |\boldsymbol{x}|^2 Y_{n-2}(\boldsymbol{x}) + \cdots + |\boldsymbol{x}|^{2\,[n/2]} Y_{n-2\,[n/2]}(\boldsymbol{x}), \tag{2.52}$$

where $Y_{n-2j} \in \mathbb{Y}_{n-2j}(\mathbb{R}^d)$. Note that the terms on the right side of (2.52) are mutually orthogonal with respect to the inner product (2.6). □

As consequences of Theorem 2.18, we have the following two results.

Corollary 2.19.

$$\left(\sum_{j=0}^{n} \mathbb{H}_j^d\right)\Bigg|_{\mathbb{S}^{d-1}} = \sum_{j=0}^{n} \mathbb{Y}_j^d.$$

So the restriction of any polynomial on $\mathbb{S}^{d-1}$ *is a sum of some spherical harmonics and the restriction of the space of the polynomials of* d *variables on* $\mathbb{S}^{d-1}$ *is* $\sum_{j=0}^{\infty} \mathbb{Y}_j^d$.

Corollary 2.20. *A polynomial* $H_n \in \mathbb{H}_n^d$ *is harmonic if and only if*

$$\int_{\mathbb{S}^{d-1}} H_n(\boldsymbol{\xi})\, \overline{H_{n-2}(\boldsymbol{\xi})}\, dS^{d-1}(\boldsymbol{\xi}) = 0 \quad \forall\, H_{n-2} \in \mathbb{H}_{n-2}^d. \tag{2.53}$$

Proof. ($\Longleftarrow$) Use (2.52) to obtain

$$H_n(\boldsymbol{\xi}) = Y_n(\boldsymbol{\xi}) + Y_{n-2}(\boldsymbol{\xi}) + \cdots + Y_{n-2\,[n/2]}(\boldsymbol{\xi}).$$

Then by (2.53) and the orthogonality of spherical harmonics of different order (Corollary 2.15), we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{d-1}} H_n(\boldsymbol{\xi})\, \overline{Y_{n-2j}(\boldsymbol{\xi})}\, dS^{d-1}(\boldsymbol{\xi}) \\ &= \int_{\mathbb{S}^{d-1}} |Y_{n-2j}(\boldsymbol{\xi})|^2 dS^{d-1}(\boldsymbol{\xi}), \quad 1 \le j \le [n/2]. \end{aligned}$$

So $Y_{n-2j} \equiv 0$ for $1 \le j \le [n/2]$ and $H_n(\boldsymbol{x}) = Y_n(\boldsymbol{x})$ is harmonic.

($\Longrightarrow$) Assume $H_n \in \mathbb{Y}_n(\mathbb{S}^d)$ is harmonic. Recalling (2.52), we write an arbitrary $H_{n-2} \in \mathbb{H}_{n-2}^d$ as

$$H_{n-2}(\boldsymbol{x}) = Y_{n-2}(\boldsymbol{x}) + |\boldsymbol{x}|^2 Y_{n-4}(\boldsymbol{x}) + \cdots + |\boldsymbol{x}|^{2\,[(n-2)/2]} Y_{n-2-2\,[(n-2)/2]}(\boldsymbol{x}).$$

Then,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} H_n(\boldsymbol{\xi})\, \overline{H_{n-2}(\boldsymbol{\xi})}\, dS^{d-1}(\boldsymbol{\xi}) &= \sum_{j=0}^{[(n-2)/2]} \int_{\mathbb{S}^{d-1}} H_n(\boldsymbol{\xi})\, \overline{Y_{n-2-2j}(\boldsymbol{\xi})}\, dS^{d-1}(\boldsymbol{\xi}) \\ &= 0, \end{aligned}$$

again using the fact that spherical harmonics of different order are orthogonal. □

Now we discuss the question of how to determine the harmonic polynomials $Y_n,\ Y_{n-2},\ \ldots,\ Y_{n-2\,[n/2]}$ in the decomposition (2.52) for an arbitrary homogeneous polynomial H_n of degree n. Since $H_n(\boldsymbol{x})$ is homogeneous of degree n,

$$H_n(\lambda \boldsymbol{x}) = \lambda^n H_n(\boldsymbol{x}) \quad \forall\, \lambda \in \mathbb{R},\, \boldsymbol{x} \in \mathbb{R}^d.$$

We differentiate this equality with respect to λ and then set $\lambda = 1$ to obtain

$$\sum_{i=1}^{d} x_i \frac{\partial H_n(\boldsymbol{x})}{\partial x_i} = n\, H_n(\boldsymbol{x}), \quad H_n \in \mathbb{H}_n^d. \tag{2.54}$$

Consider the function $r^m H_n(\boldsymbol{x})$ with $r = |\boldsymbol{x}|$ and $m \in \mathbb{N}_0$. Note that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \quad 1 \le i \le d.$$

We take derivatives of the function $r^m H_n(\boldsymbol{x})$ to obtain

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(r^m H_n(\boldsymbol{x})\right) &= m\, r^{m-2} x_i H_n(\boldsymbol{x}) + r^m \frac{\partial H_n(\boldsymbol{x})}{\partial x_i},\\
\frac{\partial^2}{\partial x_i^2}\left(r^m H_n(\boldsymbol{x})\right) &= \left[m\,(m-2)\, r^{m-4} x_i^2 + m\, r^{m-2}\right] H_n(\boldsymbol{x})\\
&\qquad + 2\, m\, r^{m-2} x_i \frac{\partial H_n(\boldsymbol{x})}{\partial x_i} + r^m \frac{\partial^2 H_n(\boldsymbol{x})}{\partial x_i^2},
\end{aligned}$$

and hence, using (2.54),

$$\begin{aligned}
\Delta\left(r^m H_n(\boldsymbol{x})\right) &= m\,(d + 2n + m - 2)\, r^{m-2} H_n(\boldsymbol{x}) + r^m \Delta H_n(\boldsymbol{x})\\
&\qquad \forall\, H_n \in \mathbb{H}_n^d.
\end{aligned} \tag{2.55}$$

In particular, if $H_n(\boldsymbol{x}) = Y_n(\boldsymbol{x})$ is harmonic, then

$$\Delta\left(r^m Y_n(\boldsymbol{x})\right) = m\,(d + 2n + m - 2)\, r^{m-2} Y_n(\boldsymbol{x}) \quad \forall\, Y_n \in \mathbb{Y}_n(\mathbb{R}^d). \tag{2.56}$$

For $H_n \in \mathbb{H}_n^d$, we write (2.52) in a compact form

$$H_n(\boldsymbol{x}) = \sum_{j=0}^{[n/2]} |\boldsymbol{x}|^{2j} Y_{n-2j}(\boldsymbol{x}). \tag{2.57}$$

Apply the Laplacian operator Δ to both sides of (2.57) and use the formula (2.56),

$$\Delta H_n(\boldsymbol{x}) = \sum_{j=1}^{[n/2]} 2j\,(d + 2n - 2j - 2)\, |\boldsymbol{x}|^{2(j-1)} Y_{n-2j}(\boldsymbol{x}).$$

In general, for $k \ge 1$ an integer, we have

$$\Delta^k H_n(\boldsymbol{x}) = \sum_{j=k}^{[n/2]} 2j \cdot 2(j-1) \cdots 2(j-(k-1))\ (d+2n-2j-2) \\ \cdot (d+2n-2j-4) \cdots (d+2n-2j-2k)\, |\boldsymbol{x}|^{2(j-k)} Y_{n-2j}(\boldsymbol{x}).$$

Using the notation of double factorial,

$$\Delta^k H_n(\boldsymbol{x}) = \sum_{j=k}^{[n/2]} \frac{(2j)!!\,(d+2n-2j-2)!!}{(2j-2k)!!\,(d+2n-2j-2k-2)!!} |\boldsymbol{x}|^{2(j-k)} Y_{n-2j}(\boldsymbol{x}). \tag{2.58}$$

By taking $k = [n/2],\ [n/2]-1,\ \ldots,\ 1,\ 0$ in (2.58), we can obtain in turn $Y_{n-2\,[n/2]},\ \ldots,\ Y_n(\boldsymbol{x})$. In particular, for n even,

$$\Delta^{n/2} H_n(\boldsymbol{x}) = \frac{n!!\,(d+n-2)!!}{(d-2)!!} Y_0(\boldsymbol{x}).$$

Hence,

$$Y_0(\boldsymbol{x}) = \frac{(d-2)!!}{n!!\,(d+n-2)!!} \Delta^{n/2} H_n(\boldsymbol{x}). \tag{2.59}$$

Example 2.21. Write

$$x_i^2 = Y_2(\boldsymbol{x}) + |\boldsymbol{x}|^2 Y_0(\boldsymbol{x}).$$

We first apply (2.59) to get

$$Y_0(\boldsymbol{x}) = \frac{1}{d}.$$

We then use (2.58) with $n = 2$ and $k = 0$ to obtain

$$Y_2(\boldsymbol{x}) = x_i^2 - \frac{1}{d}\,|\boldsymbol{x}|^2.$$

Hence, we have the decomposition

$$x_i^2 = \left(x_i^2 - \frac{1}{d}\,|\boldsymbol{x}|^2\right) + |\boldsymbol{x}|^2 \frac{1}{d}, \quad 1 \le i \le d.$$

The same technique can be applied for higher degree homogeneous polynomials. $\square$

2.5 The Funk–Hecke Formula

The Funk–Hecke formula is useful in simplifying calculations of certain integrals over $\mathbb{S}^{d-1}$, cf. Sect. 3.7 for some examples. Introduce a weighted L^1 space

$$L^1_{(d-3)/2}(-1,1) := \left\{ f \text{ measurable on } (-1,1) : \|f\|_{L^1_{(d-3)/2}(-1,1)} < \infty \right\} \tag{2.60}$$

with the norm

$$\|f\|_{L^1_{(d-3)/2}(-1,1)} := \int_{-1}^{1} |f(t)| \, (1-t^2)^{(d-3)/2} dt.$$

Note that for $d \geq 2$, $C[-1,1] \subset L^1_{(d-3)/2}(-1,1)$. In the rest of the section, we assume $d \geq 2$.

Recall the projection operator $\mathcal{P}_{n,d}$ defined in (2.44). Given $f \in L^1_{(d-3)/2}(-1,1)$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$, define $f_{\boldsymbol{\xi}}(\boldsymbol{\eta}) = f(\boldsymbol{\xi}\cdot\boldsymbol{\eta})$ for $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$. Then $(\mathcal{P}_{n,d} f_{\boldsymbol{\xi}})_A = \mathcal{P}_{n,d} f_{\boldsymbol{\xi}}$ for any $A \in \mathbb{O}^d(\boldsymbol{\xi})$. Since $\mathcal{P}_{n,d} f_{\boldsymbol{\xi}} \in \mathbb{Y}^d_n$, by Theorem 2.8, it is a multiple of $P_{n,d}(\boldsymbol{\xi}\cdot)$:

$$(\mathcal{P}_{n,d} f_{\boldsymbol{\xi}})(\boldsymbol{\eta}) = \lambda_n \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}).$$

This is rewritten as, following the definition (2.44),

$$\lambda_n P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = \int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\zeta}\cdot\boldsymbol{\eta}) \, f(\boldsymbol{\xi}\cdot\boldsymbol{\zeta}) \, dS^{d-1}(\boldsymbol{\zeta}). \tag{2.61}$$

We determine the constant λ_n by setting $\boldsymbol{\eta} = \boldsymbol{\xi}$ in (2.61):

$$\lambda_n = \int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\zeta}) \, f(\boldsymbol{\xi}\cdot\boldsymbol{\zeta}) \, dS^{d-1}(\boldsymbol{\zeta}).$$

The integral does not depend on $\boldsymbol{\xi}$ and we may take $\boldsymbol{\xi} = \boldsymbol{e}_d$. Then using (1.16),

$$\lambda_n = |\mathbb{S}^{d-2}| \int_{-1}^{1} P_{n,d}(t) \, f(t) \, (1-t^2)^{\frac{d-3}{2}} \, dt. \tag{2.62}$$

Let $Y_n \in \mathbb{Y}^d_n$ be arbitrary yet fixed. Multiply (2.61) by Y_n and integrate over $\mathbb{S}^{d-1}$ with respect to $\boldsymbol{\eta}$:

$$\begin{aligned} &\lambda_n \int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) Y_n(\boldsymbol{\eta}) \, dS^{d-1}(\boldsymbol{\eta}) \\ &\quad = \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi}\cdot\boldsymbol{\zeta}) \left(\int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\zeta}\cdot\boldsymbol{\eta}) Y_n(\boldsymbol{\eta}) \, dS^{d-1}(\boldsymbol{\eta}) \right) dS^{d-1}(\boldsymbol{\zeta}). \end{aligned} \tag{2.63}$$

Applying the addition theorem, Theorem 2.9, we see that

$$\int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\eta}\cdot\boldsymbol{\zeta}) Y_n(\boldsymbol{\eta}) \, dS^{d-1}(\boldsymbol{\eta}) = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}} \sum_{j=1}^{N_{n,d}} (Y_n, Y_{n,j})_{\mathbb{S}^{d-1}} \, Y_{n,j}(\boldsymbol{\zeta}),$$

i.e.,

$$\int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\eta}\cdot\boldsymbol{\zeta}) Y_n(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}}\, Y_n(\boldsymbol{\zeta}). \tag{2.64}$$

Hence, from (2.63),

$$\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, Y_n(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = \lambda_n Y_n(\boldsymbol{\xi}). \tag{2.65}$$

We summarize the result in the form of a theorem.

Theorem 2.22 (Funk–Hecke Formula). *Let $f \in L^1_{(d-3)/2}(-1,1)$, $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $Y_n \in \mathbb{Y}^d_n$. Then the Funk–Hecke formula* (2.65) *holds with the constant λ_n given by* (2.62).

From (2.65), we can deduce the following statement using the formula (2.24). Assume $f \in L^1_{(d-3)/2}(-1,1)$. Then

$$\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi}\cdot\boldsymbol{\zeta})\, P_{n,d}(\boldsymbol{\eta}\cdot\boldsymbol{\zeta})\, dS^{d-1}(\boldsymbol{\zeta}) = \lambda_n P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) \quad \forall\, \boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{S}^{d-1},\ n \in \mathbb{N}_0, \tag{2.66}$$

where λ_n is given by the formula (2.62).

Letting $f = P_{n,d}$ in (2.65) and comparing it with (2.64), we deduce the formula

$$\int_{-1}^{1} [P_{n,d}(t)]^2 \left(1-t^2\right)^{\frac{d-3}{2}} dt = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}|\mathbb{S}^{d-2}|}, \tag{2.67}$$

which is equivalent to (2.40).

2.6 Legendre Polynomials: Representation Formulas

Further studies of spherical harmonics require a deeper knowledge of the Legendre polynomials. In this section, we present compact formulas for the Legendre polynomial $P_{n,d}$ defined in (2.19): one differential formula (Rodrigues representation formula) and some integral representation formulas. These formulas are used in proving properties of the Legendre polynomials in Sect. 2.7.

2.6.1 Rodrigues Representation Formula

By Corollary 2.15,

$$\int_{\mathbb{S}^{d-1}} P_{m,d}(\boldsymbol{\xi}\cdot\boldsymbol{\zeta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\zeta})\, dS^{d-1}(\boldsymbol{\xi}) = 0 \quad \text{for } m \neq n.$$

By the formula (1.17), the left side integral equals

$$\int_{\mathbb{S}^{d-2}} \left(\int_{-1}^{1} P_{m,d}(t)\, P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt \right) dS^{d-2}$$
$$= |\mathbb{S}^{d-2}| \int_{-1}^{1} P_{m,d}(t)\, P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt.$$

So

$$\int_{-1}^{1} P_{m,d}(t)\, P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt = 0 \quad \text{for } m \neq n. \tag{2.68}$$

Consequently, denoting P_m a polynomial of degree less than or equal to m, we have the orthogonality

$$\int_{-1}^{1} P_m(t)\, P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt = 0, \quad m < n. \tag{2.69}$$

The Legendre polynomials are determined by the orthogonality relation (2.68) and the normalization condition $P_{n,d}(1) = 1$.

Theorem 2.23 (Rodrigues representation formula).

$$P_{n,d}(t) = (-1)^n R_{n,d} (1-t^2)^{\frac{3-d}{2}} \left(\frac{d}{dt}\right)^n (1-t^2)^{n+\frac{d-3}{2}} \quad \text{for } d \geq 2, \tag{2.70}$$

where the Rodrigues constant

$$R_{n,d} = \frac{\Gamma(\frac{d-1}{2})}{2^n \Gamma(n + \frac{d-1}{2})}. \tag{2.71}$$

Proof. The function

$$p_n(t) = (1-t^2)^{\frac{3-d}{2}} \left(\frac{d}{dt}\right)^n (1-t^2)^{n+\frac{d-3}{2}}$$

is easily seen to be a polynomial of degree n. Let us show that these polynomials are orthogonal with respect to the weight $\left(1-t^2\right)^{\frac{d-3}{2}}$. For $n > m$,

$$\int_{-1}^{1} p_n(t)\, p_m(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt = \int_{-1}^{1} p_m(t) \left(\frac{d}{dt}\right)^n \left(1-t^2\right)^{n+\frac{d-3}{2}} dt.$$

Performing integration by parts n times shows that the integral is zero.

The value $p_n(1)$ is calculated as follows:

$$
\begin{aligned}
p_n(1) &= (1-t^2)^{\frac{3-d}{2}} \left(\frac{d}{dt}\right)^n \left[(1+t)^{n+\frac{d-3}{2}}(1-t)^{n+\frac{d-3}{2}}\right]\Big|_{t=1} \\
&= (1-t^2)^{\frac{3-d}{2}}(1+t)^{n+\frac{d-3}{2}} \left(\frac{d}{dt}\right)^n (1-t)^{n+\frac{d-3}{2}}\Big|_{t=1} \\
&= (-1)^n \left(\frac{d-1}{2}\right)_n (1+t)^n\Big|_{t=1} \\
&= (-1)^n \frac{2^n\Gamma(n+\frac{d-1}{2})}{\Gamma(\frac{d-1}{2})},
\end{aligned}
$$

where the formula (1.12) for Pochhammer's symbol $((d-1)/2)_n$ is used. Hence,

$$P_{n,d}(t) = (-1)^n R_{n,d}\, p_n(t),$$

which is the stated formula. □

In the case $d = 3$, we recover the Rodrigues representation formula for the standard Legendre polynomials:

$$P_{n,3}(t) = \frac{1}{2^n n!}\left(\frac{d}{dt}\right)^n (t^2-1)^n, \quad n \in \mathbb{N}_0.$$

In the case $d = 2$, we use the relation

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\,\Gamma\left(\frac{1}{2}\right),$$

derived from a repeated application of (1.6), and obtain

$$P_{n,2}(t) = (-1)^n \frac{2^n n!}{(2n)!}(1-t^2)^{\frac{1}{2}}\left(\frac{d}{dt}\right)^n (1-t^2)^{n-\frac{1}{2}}, \quad n \in \mathbb{N}_0.$$

This formula is not convenient to use. A more familiar form is given by the Chebyshev polynomial:

$$P_{n,2}(t) = \cos(n \arccos t), \quad t \in [-1,1].$$

This result is verified by showing $\cos(n \arccos t)$ is a polynomial of degree n, has a value 1 at $t = 1$, and these polynomials satisfy the orthogonality condition (2.68) with $d = 2$. See also the derivation leading to (2.29).

In the case $d = 4$, we can similarly verify the formula

$$P_{n,4}(t) = \frac{1}{n+1}\,U_n(t), \quad t \in [-1,1],$$

where

$$U_n(t) = \frac{1}{n+1}\, P'_{n+1,2}(t)$$

is the nth degree Chebyshev polynomial of the second kind. For $-1 < t < 1$, we have the formula

$$U_n(t) = \frac{\sin((n+1)\arccos t)}{\sin(\arccos t)}.$$

We note that the Legendre polynomial $P_{n,d}(t)$ is proportional to the Jacobi polynomial $P_n^{(\alpha,\alpha)}(t)$ with $\alpha = (d-3)/2$. The Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ are introduced in Sect. 4.3.1.

2.6.2 Integral Representation Formulas

In addition to the Rodrigues representation formula (2.70), there are integral representation formulas for the Legendre polynomials which are useful in showing certain properties of the Legendre polynomials.

Let $d \geq 3$. For a fixed $\boldsymbol{\eta} \in \mathbb{S}^{d-2}$, the function $\boldsymbol{x} \mapsto (x_d + i\, \boldsymbol{x}_{(d-1)} \cdot \boldsymbol{\eta})^n$ is a homogeneous harmonic polynomial of degree n. Consider its average with respect to $\boldsymbol{\eta} \in \mathbb{S}^{d-2}$,

$$L_n(\boldsymbol{x}) = \frac{1}{|\mathbb{S}^{d-2}|} \int_{\mathbb{S}^{d-2}} \left(x_d + i\, \boldsymbol{x}_{(d-1)} \cdot \boldsymbol{\eta}\right)^n dS^{d-2}(\boldsymbol{\eta}).$$

This function is a homogeneous harmonic of degree n. For $A \in \mathbb{O}^d(\boldsymbol{e}_d)$, we recall (2.1) and write

$$A\boldsymbol{x} = \begin{pmatrix} A_1 \boldsymbol{x}_{(d-1)} \\ x_d \end{pmatrix}, \quad A_1 \in \mathbb{O}^{d-1}.$$

Note that here we view $\boldsymbol{x}_{(d-1)}$ as a vector in $\mathbb{S}^{d-2}$. Then

$$L_n(A\boldsymbol{x}) = \frac{1}{|\mathbb{S}^{d-2}|} \int_{\mathbb{S}^{d-2}} \left(x_d + i\, \boldsymbol{x}_{(d-1)} \cdot A_1^T \boldsymbol{\eta}\right)^n dS^{d-2}(\boldsymbol{\eta}).$$

With a change of variable $\boldsymbol{\zeta} = A_1^T \boldsymbol{\eta}$, we have

$$L_n(A\boldsymbol{x}) = \frac{1}{|\mathbb{S}^{d-2}|} \int_{\mathbb{S}^{d-2}} \left(x_d + i\, \boldsymbol{x}_{(d-1)} \cdot \boldsymbol{\zeta}\right)^n dS^{d-2}(\boldsymbol{\zeta}),$$

which coincides with $L_n(\boldsymbol{x})$. Moreover, $L_n(\boldsymbol{e}_d) = 1$. Thus, $L_n(\boldsymbol{x})$ is the Legendre harmonic of degree n in dimension d. By the relation (2.21), we see that

$$P_{n,d}(t) = \frac{1}{|\mathbb{S}^{d-2}|} \int_{\mathbb{S}^{d-2}} \left[t + i\,(1-t^2)^{1/2} \boldsymbol{\xi}_{(d-1)} \cdot \boldsymbol{\eta} \right]^n dS^{d-2}(\boldsymbol{\eta}), \quad t \in [-1,1].$$

In this formula, $\boldsymbol{\xi}_{(d-1)} \in \mathbb{S}^{d-2}$ is arbitrary. In particular, choosing $\boldsymbol{\xi}_{(d-1)} = (0, \cdots, 0, 1)^T$ in $\mathbb{S}^{d-2}$ and applying (1.17), we obtain the first integral representation formula for the Legendre polynomials.

Theorem 2.24. *For $n \in \mathbb{N}_0$ and $d \geq 3$,*

$$P_{n,d}(t) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \int_{-1}^{1} \left[t + i\,(1-t^2)^{1/2} s \right]^n (1-s^2)^{\frac{d-4}{2}} ds, \quad t \in [-1,1]. \tag{2.72}$$

An easy consequence of the representation formula (2.72) is that $P_{n,d}(t)$ has the same parity as the integer n, i.e.,

$$P_{n,d}(-t) = (-1)^n P_{n,d}(t), \quad -1 \leq t \leq 1. \tag{2.73}$$

There is another useful integral representation formula that can be derived from (2.72). Recall definitions of hyper-trigonometric functions:

$$\sinh x := \frac{e^x - e^{-x}}{2}, \quad \cosh x := \frac{e^x + e^{-x}}{2},$$
$$\tanh x := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and differentiation formulas

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x, \quad (\tanh x)' = \frac{1}{\cosh^2 x}.$$

Use the change of variable

$$s = \tanh u, \quad u \in \mathbb{R}. \tag{2.74}$$

We have $s \to 1-$ as $u \to \infty$, $s \to -1+$ as $u \to -\infty$, and

$$ds = \frac{1}{\cosh^2 u}\, du, \quad 1 - s^2 = \frac{1}{\cosh^2 u}. \tag{2.75}$$

Since $P_{n,d}(-t) = (-1)^n P_{n,d}(t)$ by (2.73), it is sufficient to consider the case $t \in (0,1]$ for the second integral representation formula. Write

$$t + i\,(1-t^2)^{1/2} = e^{i\theta}$$

for a uniquely determined $\theta \in [0, \pi/2)$. Then $t = \cos\theta$ and

$$t + i\,(1-t^2)^{1/2} s = \cos\theta + i \tanh u \sin\theta.$$

The hyper-trigonometric functions are defined for complex variables and it can be verified that

$$\cos\theta + i\tanh u \sin\theta = \frac{\cosh(u+i\theta)}{\cosh u}.$$

Thus,

$$\int_{-1}^{1}\left[t+i\,(1-t^2)^{1/2}s\right]^n(1-s^2)^{\frac{d-4}{2}}\,ds = \int_{-\infty}^{\infty}\frac{\cosh^n(u+i\theta)}{\cosh^{n+d-2}u}\,du.$$

The integrand is a meromorphic function of u with poles at $u = i\pi\,(k+1/2)$, $k\in\mathbb{Z}$. We then apply the Cauchy integral theorem in complex analysis [2] to obtain

$$\int_{-\infty}^{\infty}\frac{\cosh^n(u+i\theta)}{\cosh^{n+d-2}u}\,du = \int_{-\infty}^{\infty}\frac{\cosh^n u}{\cosh^{n+d-2}(u-i\theta)}\,du.$$

Return back to the variable s, using the relation

$$\cosh(u-i\theta) = \cosh u\left[t-i(1-t^2)^{1/2}s\right]$$

together with (2.74) and (2.75),

$$P_{n,d}(t) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\int_{-1}^{1}\frac{(1-s^2)^{\frac{d-4}{2}}}{\left[t-i\,(1-t^2)^{1/2}s\right]^{n+d-2}}ds.$$

Note that changing s to $-s$ for the integrand leads to another integral representation formula for $P_{n,d}(t)$. In summary, the following result holds.

Theorem 2.25. *For $n\in\mathbb{N}_0$ and $d\geq 3$,*

$$P_{n,d}(t) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\int_{-1}^{1}\frac{(1-s^2)^{\frac{d-4}{2}}}{\left[t\pm i\,(1-t^2)^{1/2}s\right]^{n+d-2}}ds,\quad t\in(0,1]. \tag{2.76}$$

2.7 Legendre Polynomials: Properties

In this section, we explore properties of the Legendre polynomials by using the compact presentation formulas given in Sect. 2.6.

2.7.1 Integrals, Orthogonality

The following result is useful in computing integrals involving the Legendre polynomials.

Proposition 2.26. *If $f \in C^n([-1,1])$, then*

$$\int_{-1}^{1} f(t)\, P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt = R_{n,d} \int_{-1}^{1} f^{(n)}(t) \left(1-t^2\right)^{n+\frac{d-3}{2}} dt, \quad (2.77)$$

where the constant $R_{n,d}$ is given in (2.71).

Proof. By the Rodrigues representation formula (2.70), the left side of (2.77) is

$$(-1)^n R_{n,d} \int_{-1}^{1} f(t) \left(\frac{d}{dt}\right)^n \left(1-t^2\right)^{n+\frac{d-3}{2}} dt.$$

Performing integration by parts n times on this integral leads to (2.77). □

Recall the formula (2.40) or (2.67),

$$\int_{-1}^{1} \left[P_{n,d}(t)\right]^2 \left(1-t^2\right)^{\frac{d-3}{2}} dt = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}\,|\mathbb{S}^{d-2}|}. \quad (2.78)$$

Combining (2.68) and (2.78), we have the orthogonality relation

$$\int_{-1}^{1} P_{m,d}(t)\, P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}\,|\mathbb{S}^{d-2}|}\, \delta_{mn}. \quad (2.79)$$

Using (1.18), we can rewrite (2.78) as

$$\int_{-1}^{1} \left[P_{n,d}(t)\right]^2 \left(1-t^2\right)^{\frac{d-3}{2}} dt = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{N_{n,d}\,\Gamma(\frac{d}{2})}.$$

In particular, for $d=3$, $N_{n,3} = 2n+1$ and

$$\int_{-1}^{1} \left[P_{n,3}(t)\right]^2 dt = \frac{2}{2n+1}.$$

For $d=2$, $N_{n,2} = 2$ and

$$\int_{-1}^{1} \left[P_{n,2}(t)\right]^2 (1-t^2)^{-\frac{1}{2}} dt = \frac{\pi}{2}.$$

We can verify this result easily by a direct calculation using the formula

$$P_{n,2}(t) = \cos(n \arccos t).$$

2.7.2 Differential Equation and Distribution of Roots

First we derive a differential equation satisfied by the Legendre polynomial $P_{n,d}(t)$. Introduce a second-order differential operator L_d defined by

$$L_d g(t) := \left(1-t^2\right)^{\frac{3-d}{2}} \frac{d}{dt}\left[\left(1-t^2\right)^{\frac{d-1}{2}} \frac{d}{dt} g(t)\right], \quad g \in C^2[-1,1].$$

Also introduce a weighted inner product

$$(f,g)_d := \int_{-1}^{1} f(t)\, g(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt.$$

Then through integration by parts, we have

$$(L_d f, g)_d = (f, L_d g)_d \quad \forall\, f, g \in C^2[-1,1]. \tag{2.80}$$

Thus, the operator L_d is self-adjoint with respect to the weighted inner product $(\cdot,\cdot)_d$.

Consider the function $L_d P_{n,d}(t)$. Since

$$L_d g(t) = (1-t^2)\, g''(t) - (d-1)\, t\, g'(t),$$

we see that if $p_n(t)$ is a polynomial of degree n, then so is $L_d p_n(t)$. Let $0 \le m \le n-1$. By the weighted orthogonality relation (2.69), we have

$$(P_{n,d}, L_d P_{m,d})_d = 0.$$

Then by (2.80),

$$(P_{m,d}, L_d P_{n,d})_d = 0, \quad 0 \le m \le n-1.$$

Thus, the polynomial $L_d P_{n,d}(t)$ must be a multiple of $P_{n,d}(t)$. Writing

$$P_{n,d}(t) = a_{n,d}^0 t^n + \text{l.d.t.} \tag{2.81}$$

Here l.d.t. stands for the lower degree terms. We have

$$L_d P_{n,d}(t) = -n\,(n+d-2)\,a^0_{n,d} t^n + \text{l.d.t.}$$

Hence,

$$L_d P_{n,d}(t) + n\,(n+d-2)\,P_{n,d}(t) = 0.$$

So $P_{n,d}$ is an eigenfunction for the differential operator $-L_d$ corresponding to the eigenvalue $n\,(n+d-2)$. In other words, the Legendre polynomial $P_{n,d}(t)$ satisfies the differential equation

$$\left(1-t^2\right)^{\frac{3-d}{2}} \frac{d}{dt}\left[\left(1-t^2\right)^{\frac{d-1}{2}} \frac{d}{dt} P_{n,d}(t)\right] + n\,(n+d-2)\,P_{n,d}(t) = 0, \quad (2.82)$$

which can also be written as

$$(1-t^2)\,P''_{n,d}(t) - (d-1)\,t\,P'_{n,d}(t) + n\,(n+d-2)\,P_{n,d}(t) = 0. \quad (2.83)$$

Next, we present a result regarding distributions of the roots of the Legendre polynomials. This result plays an important role in the theory of Gaussian quadratures. From the differential equation (2.83), we deduce that $P_{n,d}(t)$ and $P'_{n,d}(t)$ cannot both vanish at any point in $(-1,1)$; in other words, $P_{n,d}(t)$ has no multiple roots in $(-1,1)$. Assume $P_{n,d}(t)$ has k distinct roots $t_1, \cdots, t_k$ in the interval $(-1,1)$, and $k<n$. Then

$$p_k(t) = (t-t_1)\cdots(t-t_k)$$

is a polynomial of degree k, $p_k(1) > 0$, and $P_{n,d}(t) = q_{n-k}(t)\,p_k(t)$ with a polynomial q_{n-k} of degree $n-k$. Since the polynomial $q_{n-k}(t)$ does not change sign in $(-1,1)$ and is positive at 1, it is positive in $(-1,1)$. So

$$\int_{-1}^{1} P_{n,d}(t)\,p_k(t)\left(1-t^2\right)^{\frac{d-3}{2}} dt = \int_{-1}^{1} q_{n-k}(t)\,p_k(t)^2\left(1-t^2\right)^{\frac{d-3}{2}} dt > 0.$$

However, since $k<n$, the integral on the left side is zero and this leads to contradiction. We summarize the result in the form of a proposition.

Proposition 2.27. *The Legendre polynomial $P_{n,d}(t)$ has exactly n distinct roots in $(-1,1)$.*

For n even, $P_{n,d}(t)$ is an even function so that its roots can be written as $\pm t_1, \ldots, \pm t_{n/2}$ with $0 < t_1 < \cdots < t_{n/2} < 1$. For n odd, $P_{n,d}(t)$ is an odd function so that its roots can be written as $0, \pm t_1, \ldots, \pm t_{(n-1)/2}$ with $0 < t_1 < \cdots < t_{(n-1)/2} < 1$.

In the particular case $d=2$, it is easy to find the n roots of the equation

$$P_{n,2}(t) = \cos(n \arccos t) = 0$$

to be

$$t_j = \cos\frac{(2j+1)\,\pi}{2\,n}, \quad 0 \le j \le n-1.$$

For $n = 2k$ even, noting that $t_{2k-1-j} = -t_j$, we can list the roots as

$$\pm t_0,\ \pm t_1,\ \cdots,\ \pm t_{k-1} \text{ with } t_j = \cos\frac{(2j+1)\,\pi}{4k},\ 0 \le j \le k-1.$$

For $n = 2k+1$ odd, noting that $t_k = 0$ and $t_{2k-j} = -t_j$, we can list the roots as

$$0,\ \pm t_0,\ \pm t_1,\ \cdots,\ \pm t_{k-1} \text{ where } t_j = \cos\frac{(2j+1)\,\pi}{2(2k+1)},\ 0 \le j \le k-1.$$

2.7.3 Recursion Formulas

Recursion formulas are useful in computing values of the Legendre polynomials, especially those of a higher degree.

Let us first determine the leading coefficient $a_{n,d}^0$ of $P_{n,d}(t)$ (see (2.81)). We start with the equality

$$\int_{-1}^{1} [P_{n,d}(t)]^2 \left(1-t^2\right)^{\frac{d-3}{2}} dt = a_{n,d}^0 \int_{-1}^{1} t^n P_{n,d}(t) \left(1-t^2\right)^{\frac{d-3}{2}} dt, \qquad (2.84)$$

obtained by an application of the orthogonality property (2.69). By (2.78), the left side of (2.84) equals

$$\frac{|\mathbb{S}^{d-1}|}{N_{n,d}\,|\mathbb{S}^{d-2}|}.$$

Applying Proposition 2.26, we see that the right side of (2.84) equals

$$a_{n,d}^0 R_{n,d}\, n! \int_{-1}^{1} \left(1-t^2\right)^{n+\frac{d-3}{2}} dt.$$

To compute the integral, we let $s = t^2$:

$$\begin{aligned}\int_{-1}^{1} \left(1-t^2\right)^{n+\frac{d-3}{2}} dt &= \int_0^1 s^{\frac{1}{2}-1}(1-s)^{n+\frac{d-1}{2}-1} ds\\ &= \frac{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{d-1}{2})}{\Gamma(n+\frac{d}{2})}.\end{aligned}$$

Hence,

$$\frac{|\mathbb{S}^{d-1}|}{N_{n,d}\,|\mathbb{S}^{d-2}|} = a^0_{n,d} R_{n,d}\, n! \frac{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{d-1}{2})}{\Gamma(n+\frac{d}{2})}.$$

Therefore, the leading coefficient of the Legendre polynomial $P_{n,d}(t)$ is

$$a^0_{n,d} = \frac{2^{n-1}\Gamma(d-1)\,\Gamma(n+\frac{d-2}{2})}{\Gamma(\frac{d}{2})\Gamma(n+d-2)}. \tag{2.85}$$

As an application of the formula (2.85), we note that

$$\frac{a^0_{n,d}}{a^0_{n-1,d}} = \frac{2n+d-4}{n+d-3}.$$

So

$$(n+d-3)\,P_{n,d}(t) - (2n+d-4)\,t\,P_{n-1,d}(t)$$

is a polynomial of degree $\le n-1$ and is orthogonal to $P_{k,d}(t)$ with respect to the weighted inner product $(\cdot,\cdot)_d$ for $0 \le k \le n-3$. Thus, when this polynomial is expressed as a linear combination of $P_{j,d}(t)$, $0 \le j \le n-1$, only the two terms involving $P_{n-2,d}(t)$ and $P_{n-1,d}(t)$ remain. In other words, for two suitable constants c_1 and c_2,

$$(n+d-3)\,P_{n,d}(t) - (2n+d-4)\,t\,P_{n-1,d}(t) = c_1 P_{n-1,d}(t) + c_2 P_{n-2,d}(t).$$

The constants c_1 and c_2 can be found from the above equality at $t = \pm 1$, since $P_{k,d}(1) = 1$ and $P_{k,d}(-1) = (-1)^k$ (cf. (2.73)):

$$\begin{aligned} c_1 + c_2 &= 1-n, \\ c_1 - c_2 &= n-1. \end{aligned}$$

The solution of this system is $c_1 = 0$, $c_2 = 1-n$. Thus, the Legendre polynomials satisfy the recursion relation

$$P_{n,d}(t) = \frac{2n+d-4}{n+d-3}\,t\,P_{n-1,d}(t) - \frac{n-1}{n+d-3} P_{n-2,d}(t), \quad n \ge 2,\ d \ge 2. \tag{2.86}$$

The initial conditions for the recursion formula (2.86) are

$$P_{0,d}(t) = 1, \quad P_{1,d}(t) = t. \tag{2.87}$$

It is convenient to use the recursion formula (2.86) to derive expressions of the Legendre polynomials. The following are some examples. Note that in any dimension d, the first two Legendre polynomials are the same, given by (2.87).

For $d = 2$,

$$\begin{aligned}
P_{2,2}(t) &= 2\,t^2 - 1, \\
P_{3,2}(t) &= 4\,t^3 - 3\,t, \\
P_{4,2}(t) &= 8\,t^4 - 8\,t^2 + 1, \\
P_{5,2}(t) &= 16\,t^5 - 20\,t^3 + 5\,t.
\end{aligned}$$

For $d = 3$,

$$\begin{aligned}
P_{2,3}(t) &= \frac{1}{2}\left(3\,t^2 - 1\right), \\
P_{3,3}(t) &= \frac{1}{2}\left(5\,t^3 - 3\,t\right), \\
P_{4,3}(t) &= \frac{1}{8}\left(35\,t^4 - 30\,t^2 + 3\right), \\
P_{5,3}(t) &= \frac{1}{8}\left(63\,t^5 - 70\,t^3 + 15\,t\right).
\end{aligned}$$

For $d = 4$,

$$\begin{aligned}
P_{2,4}(t) &= \frac{1}{3}\left(4\,t^2 - 1\right), \\
P_{3,4}(t) &= 2\,t^3 - t, \\
P_{4,4}(t) &= \frac{1}{5}\left(16\,t^4 - 12\,t^2 + 1\right), \\
P_{5,4}(t) &= \frac{1}{3}\left(16\,t^5 - 16\,t^3 + 3\,t\right).
\end{aligned}$$

For $d = 5$,

$$\begin{aligned}
P_{2,5}(t) &= \frac{1}{4}\left(5\,t^2 - 1\right), \\
P_{3,5}(t) &= \frac{1}{4}\left(7\,t^3 - 3\,t\right), \\
P_{4,5}(t) &= \frac{1}{8}\left(21\,t^4 - 14\,t^2 + 1\right), \\
P_{5,5}(t) &= \frac{1}{8}\left(33\,t^5 - 30\,t^3 + 5\,t\right).
\end{aligned}$$

Graphs of these Legendre polynomials are found in Figs. 2.1–2.4.

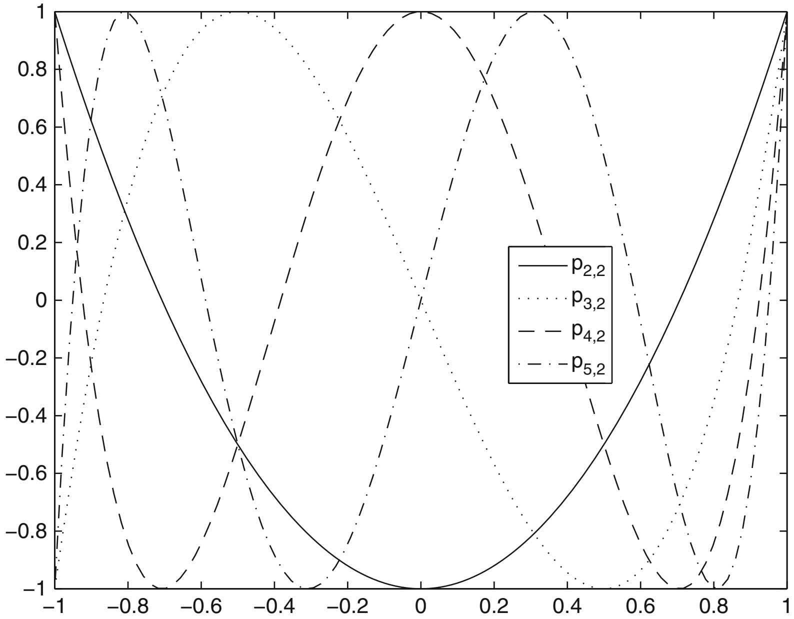

Fig. 2.1 Legendre polynomials for dimension 2

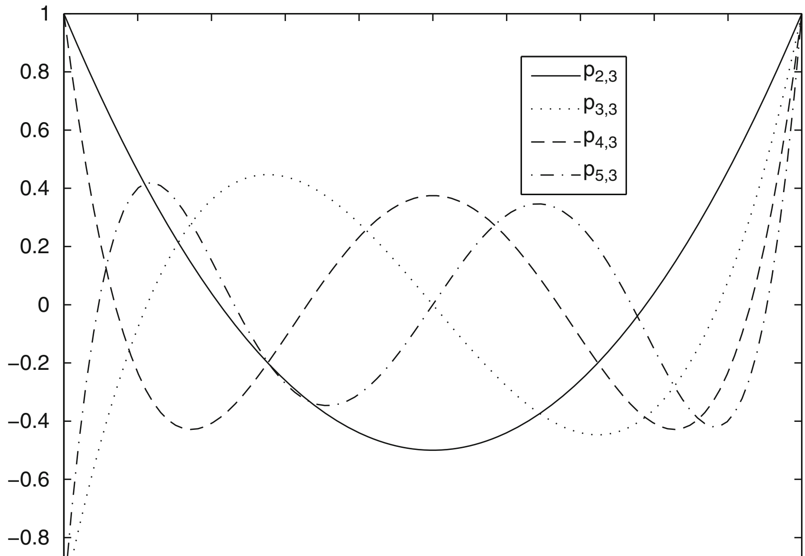

Fig. 2.2 Legendre polynomials for dimension 3

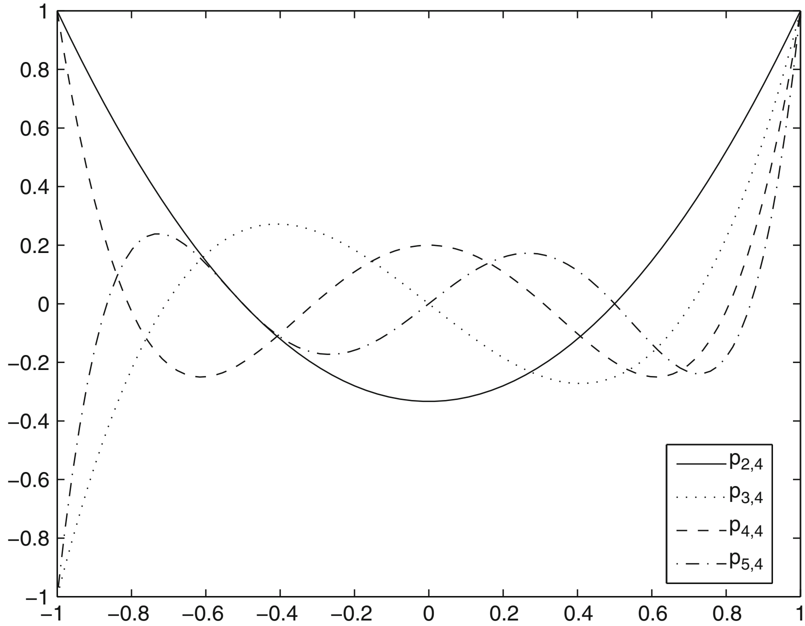

Fig. 2.3 Legendre polynomials for dimension 4

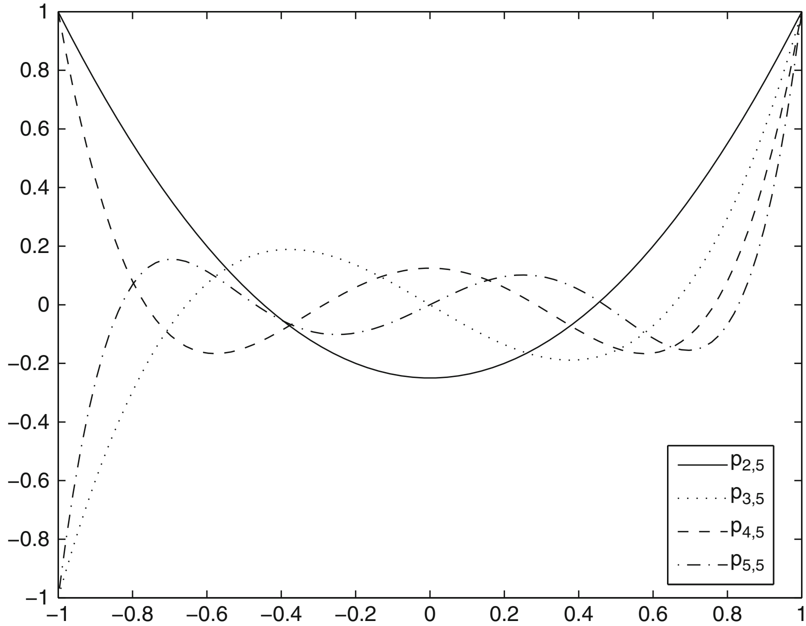

Fig. 2.4 Legendre polynomials for dimension 5

As another application of the formula (2.85), we derive a formula for derivatives of the Legendre polynomials in terms of the polynomials themselves. Note that

$$\frac{a_{n,d}^0}{a_{n-1,d+2}^0} = \frac{n+d-2}{d-1}.$$

So

$$(d-1)\, P_{n,d}'(t) - n\,(n+d-2)\, P_{n-1,d+2}(t) \tag{2.88}$$

is a polynomial of degree $\le n-2$. For $k \le n-2$,

$$\begin{aligned}
&\int_{-1}^{1} P_{n,d}'(t)\, P_{k,d+2}(t) \left(1-t^2\right)^{\frac{d-1}{2}} dt \\
&\quad = -\int_{-1}^{1} P_{n,d}(t)\, \frac{d}{dt}\left[P_{k,d+2}(t) \left(1-t^2\right)^{\frac{d-1}{2}}\right] dt \\
&\quad = -\int_{-1}^{1} P_{n,d}(t) \left[(1-t^2)\, P_{k,d+2}'(t) - (d-1)\, t\, P_{k,d+2}(t)\right] \left(1-t^2\right)^{\frac{d-3}{2}} dt.
\end{aligned}$$

Since

$$(1-t^2)\, P_{k,d+2}'(t) - (d-1)\, t\, P_{k,d+2}(t)$$

is a polynomial of degree $\le n-1$,

$$\int_{-1}^{1} P_{n,d}'(t)\, P_{k,d+2}(t) \left(1-t^2\right)^{\frac{d-1}{2}} dt = 0, \quad 0 \le k \le n-2.$$

Thus, the polynomial (2.88) is of degree $\le n-2$ and is orthogonal to all the polynomials of degree $\le n-2$ with respect to the weighted inner product $(\cdot,\cdot)_{d+2}$. Then the polynomial (2.88) must be zero. Summarizing, we have shown the following relation

$$P_{n,d}'(t) = \frac{n\,(n+d-2)}{d-1}\, P_{n-1,d+2}(t), \quad n \ge 1,\ d \ge 2. \tag{2.89}$$

Applying (2.89) recursively, we see that

$$P_{n,d}^{(j)}(t) = c_{n,d,j} P_{n-j,d+2j}(t)$$

where the constant $c_{n,d,j}$ is

$$\frac{n\,(n-1)\cdots(n-(j-1))\cdot(n+d-2)\,(n+d-1)\cdots(n+j+d-3)}{(d-1)\,(d+1)\cdots(d+2j-3)}.$$

The denominator of the above fraction can be rewritten as

$$2^j \left(\frac{d-1}{2}\right)_j = \frac{2^j \Gamma(j+\frac{d-1}{2})}{\Gamma(\frac{d-1}{2})},$$

where (1.12) is applied. Thus,

$$P_{n,d}^{(j)}(t) = \frac{n!\,(n+j+d-3)!\,\Gamma(\frac{d-1}{2})}{2^j (n-j)!\,(n+d-3)!\,\Gamma(j+\frac{d-1}{2})}\, P_{n-j,d+2j}(t), \quad n \ge j,\ d \ge 2. \tag{2.90}$$

Note that for $n < j$, $P_{n,d}^{(j)}(t) = 0$.

The formula (2.90) provides one way to compute the Legendre polynomials in higher dimensions $d \ge 4$ through differentiating the Legendre polynomials for $d = 3$ and $d = 2$. This is done as follows. First, rewrite (2.90) as

$$P_{n,d}(t) = \frac{2^j n!\,(n+d-j-3)!\,\Gamma(\frac{d-1}{2})}{(n+j)!\,(n+d-3)!\,\Gamma(\frac{d-1}{2}-j)}\, P_{n+j,d-2j}^{(j)}(t). \tag{2.91}$$

For $d = 2k$ even, take $j = k-1$. Then from (2.91),

$$P_{n,2k}(t) = \frac{2^{k-1} n!\,(n+k-2)!\,\Gamma(k-\frac{1}{2})}{(n+k-1)!\,(n+2k-3)!\,\Gamma(\frac{1}{2})}\, P_{n+k-1,2}^{(k-1)}(t).$$

Applying (1.10), we have

$$P_{n,2k}(t) = \frac{(2k-2)!\,n!}{2^{k-1}(n+k-1)\,(k-1)!\,(n+2k-3)!}\, P_{n+k-1,2}^{(k-1)}(t).$$

For $d = 2k+1$ odd, take $j = k-1$. Then from (2.91),

$$\begin{aligned} P_{n,2k+1}(t) &= \frac{2^{k-1} n!\,(k-1)!\,(n+k-1)!}{(n+k-1)!\,(n+2k-2)!}\, P_{n+k-1,3}^{(k-1)}(t) \\ &= \frac{2^{k-1} n!\,(k-1)!}{(n+2k-2)!}\, P_{n+k-1,3}^{(k-1)}(t). \end{aligned}$$

Let us derive some recursion formulas for the computation of the derivative $P_{n,d}'(t)$. First, we differentiate (2.76) to obtain

$$(1-t^2)\, P_{n,d}'(t) = -(n+d-2)\left[P_{n+1,d}(t) - t\, P_{n,d}(t)\right]. \tag{2.92}$$

Since (2.76) is valid for $d \ge 3$ and $t \in (0,1]$, the relation (2.92) is proved for $d \ge 3$ and $t \in (0,1)$. For $d = 2$, $P_{n,2}(t) = \cos(n\theta)$ with $\theta = \arccos\theta$, and it is easy to verify that both sides of (2.92) are equal to $n \sin\theta \sin(n\theta)$. Then the relation (2.92) is valid for $d \ge 2$ and $t \in (0,1)$. Since $P_{n,d}(-t) = (-1)^n P_{n,d}(t)$, we know that (2.92) holds for $t \in (-1, 0)$ as well. Finally, since both sides of (2.92) are polynomials, we conclude that the relation remains true for $t = \pm 1$ and 0, i.e.,

$$(1-t^2)\, P'_{n,d}(t) = -(n+d-2)\left[P_{n+1,d}(t) - t\, P_{n,d}(t)\right], \qquad n \in \mathbb{N}_0,\ d \ge 2,\ t \in [-1,1]. \tag{2.93}$$

Then, from (2.86), we have

$$t\, P_{n,d}(t) = \frac{1}{2n+d-2}\left[(n+d-2)\, P_{n+1,d}(t) + n\, P_{n-1,d}(t)\right].$$

Using this equality in (2.93) we obtain another relation

$$(1-t^2)\, P'_{n,d}(t) = \frac{n\,(n+d-2)}{2n+d-2}\left[P_{n-1,d}(t) - P_{n+1,d}(t)\right], \qquad n \in \mathbb{N},\ d \ge 2,\ t \in [-1,1]. \tag{2.94}$$

Finally, we differentiate the integral representation formula (2.72),

$$P'_{n,d}(t) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\int_{-1}^{1} n\left[t + i\,(1-t^2)^{1/2} s\right]^{n-1} \cdot \left[1 - i\,t\,(1-t^2)^{-1/2} s\right](1-s^2)^{\frac{d-4}{2}}\, ds.$$

Then we find out

$$(1-t^2)\, P'_{n,d}(t) = n\left[P_{n-1,d}(t) - t\, P_{n,d}(t)\right].$$

This equality is proved for $d \ge 3$. For $d = 2$, $P_{n,2}(t) = \cos(n \arccos t)$ and one can verify directly the equality. So we have the relation

$$(1-t^2)\, P'_{n,d}(t) = n\left[P_{n-1,d}(t) - t\, P_{n,d}(t)\right], \quad n \ge 1,\ d \ge 2,\ t \in [-1,1]. \tag{2.95}$$

2.7.4 Generating Function

Consider the following generating function of the Legendre polynomials

$$\phi(r) = \sum_{n=0}^{\infty} \binom{n+d-3}{d-3} P_{n,d}(t)\, r^n, \quad |t| \le 1,\ |r| < 1. \tag{2.96}$$

Let us first derive a compact formula for $\phi(r)$.

Since $|P_{n,d}(t)| \le 1$ for any n, d and t, it is easy to verify that the series converges absolutely for any r with $|r| < 1$. We differentiate (2.96) with respect to r to find

$$\phi'(r) = \sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^{n-1} \tag{2.97}$$

$$= \sum_{n=0}^{\infty} (n+1) \binom{n+d-2}{d-3} P_{n+1,d}(t)\, r^{n}. \tag{2.98}$$

Using (2.97) and (2.98), we can write

$$\begin{aligned}\left(1+r^2-2\,r\,t\right)\phi'(r) = {} & \sum_{n=0}^{\infty} (n+1) \binom{n+d-2}{d-3} P_{n+1,d}(t)\, r^{n} \\ & + \sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^{n+1} \\ & - 2\,t \sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^{n}. \end{aligned} \tag{2.99}$$

In the first sum of (2.99), for $n \ge 1$, use the following relation from (2.86):

$$P_{n+1,d}(t) = \frac{2n+d-2}{n+d-2}\, t\, P_{n,d}(t) - \frac{n}{n+d-2} P_{n-1,d}(t).$$

Then after some straightforward algebraic manipulations, we obtain from (2.99) that

$$\left(1+r^2-2\,r\,t\right)\phi'(r) = (d-2)\,(t-r)\,\phi(r). \tag{2.100}$$

The unique solution of the differential equation (2.100) with the initial condition

$$\phi(0) = P_{0,d}(0) = 1$$

is

$$\phi(r) = \left(1+r^2-2\,r\,t\right)^{-\frac{d-2}{2}}.$$

Therefore, we have the following compact formula for the generating function of the Legendre polynomials:

$$\sum_{n=0}^{\infty} \binom{n+d-3}{d-3} P_{n,d}(t)\, r^{n} = \left(1+r^2-2\,r\,t\right)^{-\frac{d-2}{2}}, \quad |t| \le 1,\ |r| < 1. \tag{2.101}$$

In particular, we have, for $P_n(t) := P_{n,3}(t)$,

$$\sum_{n=0}^{\infty} r^n P_n(t) = \frac{1}{(1+r^2-2rt)^{1/2}}, \quad |t| \le 1,\ |r| < 1. \tag{2.102}$$

The Legendre polynomials $P_{n,3}(t)$ were originally introduced as coefficients of the expansion (2.102).

For $d \geq 3$, we differentiate (2.101) with respect to r for $|r| < 1$:

$$\sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^{n-1} = \frac{(d-2)\,(t-r)}{(1+r^2-2\,r\,t)^{\frac{d}{2}}}. \tag{2.103}$$

Note that

$$\frac{1}{(1+r^2-2\,r\,t)^{\frac{d-2}{2}}} + \frac{2\,r\,(t-r)}{(1+r^2-2\,r\,t)^{\frac{d}{2}}} = \frac{1-r^2}{(1+r^2-2\,r\,t)^{\frac{d}{2}}}.$$

Multiply both sides by $(d-2)$ and apply (2.101) and (2.103). Then we obtain

$$\sum_{n=0}^{\infty} (2\,n+d-2) \binom{n+d-3}{d-3} P_{n,d}(t)\, r^n = \frac{(d-2)\,(1-r^2)}{(1+r^2-2\,r\,t)^{\frac{d}{2}}}.$$

This identity can be rewritten as

$$\sum_{n=0}^{\infty} N_{n,d} r^n P_{n,d}(t) = \frac{1-r^2}{(1+r^2-2rt)^{\frac{d}{2}}}$$

and has been proved for $d \geq 3$. It can be verified that the identity holds also for $d = 2$. Therefore, we have the next result.

Proposition 2.28. (Poisson identity) *For $d \geq 2$,*

$$\sum_{n=0}^{\infty} N_{n,d} r^n P_{n,d}(t) = \frac{1-r^2}{(1+r^2-2rt)^{\frac{d}{2}}}, \quad |r| < 1,\ t \in [-1,1]. \tag{2.104}$$

Consider the special case $d = 2$. Then $P_{n,2}(t) = \cos(n \arccos t)$. With $t = \cos\theta$, the Poisson identity (2.104) is

$$1 + 2 \sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta} \quad |r| < 1,\ 0 \leq \theta \leq \pi. \tag{2.105}$$

With $d = 3$, the Poisson identity (2.104) is

$$\sum_{n=0}^{\infty} (2n+1)\, r^n P_{n,3}(t) = \frac{1-r^2}{(1+r^2-2rt)^{\frac{3}{2}}}, \quad |r| < 1,\ t \in [-1,1].$$

This Poisson identity provides the expansion of the Henyey–Greenstein phase function (1.2) with respect to the Legendre polynomials.

We now use (2.101) to derive a few more recursive relations involving the first order derivative of the Legendre polynomials. Differentiate (2.101) with respect to t,

$$\sum_{n=0}^{\infty} \binom{n+d-3}{d-3} P'_{n,d}(t)\, r^n = (d-2)\, r \left(1+r^2-2\,r\,t\right)^{-\frac{d}{2}}.$$

Differentiate (2.101) with respect to r,

$$\sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^{n-1} = (d-2)\,(t-r) \left(1+r^2-2\,r\,t\right)^{-\frac{d}{2}}.$$

Combining these two equalities we have

$$(t-r) \sum_{n=1}^{\infty} \binom{n+d-3}{d-3} P'_{n,d}(t)\, r^n = \sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^n,$$

i.e.,

$$t \sum_{n=1}^{\infty} \binom{n+d-3}{d-3} P'_{n,d}(t)\, r^n - \sum_{n=2}^{\infty} \binom{n+d-4}{d-3} P'_{n-1,d}(t)\, r^n$$
$$= \sum_{n=1}^{\infty} n \binom{n+d-3}{d-3} P_{n,d}(t)\, r^n.$$

Thus, for $n \ge 2$,

$$t \binom{n+d-3}{d-3} P'_{n,d}(t) - \binom{n+d-4}{d-3} P'_{n-1,d}(t) = n \binom{n+d-3}{d-3} P_{n,d}(t),$$

which can be simplified to

$$(n+d-3)\, t\, P'_{n,d}(t) - n\, P'_{n-1,d}(t) = n\,(n+d-3)\, P_{n,d}(t). \tag{2.106}$$

Differentiate (2.86) with respect to t,

$$(n+d-2)\, P'_{n+1,d}(t) = (2\,n+d-2) \left[t\, P'_{n,d}(t) + P_{n,d}(t)\right] - n\, P'_{n-1,d}(t). \tag{2.107}$$

Add (2.106) and (2.107) to obtain

$$(n+d-2)\, P'_{n+1,d}(t) - (n+1)\, t\, P'_{n,d}(t) = \left[n^2 + (d-1)\, n + d - 2\right] P_{n,d}(t). \tag{2.108}$$

We can use either (2.106) or (2.108) to express a Legendre polynomial in terms of derivatives of Legendre polynomials:

$$P_{n,d}(t) = \frac{1}{n}\, t\, P'_{n,d}(t) - \frac{1}{n+d-3}\, P'_{n-1,d}(t), \tag{2.109}$$

and

$$\begin{aligned} P_{n,d}(t) &= \frac{n+d-2}{n^2+(d-1)\,n+d-2}\, P'_{n+1,d}(t) \\ &\quad - \frac{n+1}{n^2+(d-1)\,n+d-2}\, t\, P'_{n,d}(t). \end{aligned} \tag{2.110}$$

Replace n by $(n-1)$ in (2.108),

$$(n+d-3)\, P'_{n,d}(t) - n\, t\, P'_{n-1,d}(t) = n\,(n+d-3)\, P_{n-1,d}(t).$$

Then subtract from this relation the identity obtained from (2.106) multiplied by t,

$$(1-t^2)\, P'_{n,d}(t) = n\left[P_{n-1,d}(t) - t\, P_{n,d}(t)\right].$$

This is the formula (2.95).

From (2.86),

$$P_{n-1,d}(t) = \frac{2n+d-2}{n}\, t\, P_{n,d}(t) - \frac{n+d-2}{n} P_{n+1,d}(t).$$

We can use this relation in (2.95) to recover (2.93).

2.7.5 Values and Bounds

First, we recall the parity property (2.73),

$$P_{n,d}(-t) = (-1)^n P_{n,d}(t), \quad -1 \le t \le 1. \tag{2.111}$$

We know from (2.20) that

$$P_{n,d}(1) = 1.$$

Using the property (2.111), we further have

$$P_{n,d}(-1) = (-1)^n. \tag{2.112}$$

This result also follows from the value

$$p_n(-1) = \frac{2^n \Gamma(n + \frac{d-1}{2})}{\Gamma(\frac{d-1}{2})},$$

computed with a similar technique used in evaluating $p_n(1)$ in the proof of Theorem 2.23.

We use (2.72) to compute $P_{n,d}(0)$ for $d \geq 3$.

$$P_{n,d}(0) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \int_{-1}^{1} i^n s^n (1-s^2)^{\frac{d-4}{2}} ds.$$

For n odd, $n = 2k+1$, $k \in \mathbb{N}_0$, obviously,

$$P_{2k+1,d}(0) = 0. \tag{2.113}$$

For n even, $n = 2k$, $k \in \mathbb{N}_0$,

$$P_{2k,d}(0) = (-1)^k \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \, 2 \int_0^1 s^{2k} (1-s^2)^{\frac{d-4}{2}} ds.$$

Use the change of variable $t = s^2$,

$$P_{2k,d}(0) = (-1)^k \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \int_0^1 t^{k-1/2} (1-t)^{\frac{d-4}{2}} dt.$$

Therefore,

$$P_{2k,d}(0) = (-1)^k \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \frac{\Gamma(\frac{d-2}{2})\, \Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{d-1}{2})}. \tag{2.114}$$

As an example,

$$P_{2k,3}(0) = (-1)^k \frac{(2k-1)!!}{2^k k!}.$$

Alternatively, we may use the generating function formula (2.101) to compute the values. For example, take $x = -1$ in (2.101):

$$\sum_{n=0}^{\infty} \binom{n+d-3}{d-3} P_{n,d}(-1)\, r^n = (1+r)^{-(d-2)}.$$

Apply (2.4) to expand the right side to obtain

$$\sum_{n=0}^{\infty} \binom{n+d-3}{d-3} P_{n,d}(-1)\, r^n = \sum_{n=0}^{\infty} \binom{n+d-3}{d-3} (-r)^n, \quad |r| < 1.$$

Hence,

$$P_{n,d}(-1) = (-1)^n.$$

We may also apply (2.90) to find derivative values at particular points. For instance, since

$$P_{n-j,d+2j}(1) = 1,$$
$$P_{n-j,d+2j}(-1) = (-1)^{n-j},$$

we have for $n \ge j$ and $d \ge 2$,

$$P_{n,d}^{(j)}(1) = \frac{n!(n+j+d-3)!\Gamma(\frac{d-1}{2})}{2^j(n-j)!(n+d-3)!\Gamma(j+\frac{d-1}{2})},$$
$$P_{n,d}^{(j)}(-1) = \frac{(-1)^n n!(n+j+d-3)!\Gamma(\frac{d-1}{2})}{2^j(n-j)!(n+d-3)!\Gamma(j+\frac{d-1}{2})}.$$

In particular, for $d = 3$,

$$P_{n,3}^{(j)}(1) = \frac{(n+j)!}{2^j j!(n-j)!},$$

from which,

$$P_{n,3}'(1) = \frac{1}{2}\,n\,(n+1)\,, \qquad P_{n,3}''(1) = \frac{1}{8}\,(n-1)\,n\,(n+1)\,(n+2)\,.$$

Next we provide some bounds for the Legendre polynomials and their derivatives. We use (2.72) to bound $P_{n,d}(t)$. For $s, t \in [-1, 1]$,

$$\left|t + i\,(1-t^2)^{1/2}s\right| = \left[t^2 + (1-t^2)\,s^2\right]^{1/2} \le (t^2+1-t^2)^{1/2} = 1. \quad (2.115)$$

So for $d \ge 3$,

$$|P_{n,d}(t)| \le \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\int_{-1}^{1}(1-s^2)^{\frac{d-4}{2}}ds = 1, \quad t \in [-1,1].$$

This bound is valid also for $d = 2$. Thus,

$$|P_{n,d}(t)| \le 1, \quad n \in \mathbb{N}_0,\ d \ge 2,\ t \in [-1,1]. \quad (2.116)$$

Instead of (2.115), we can use the bound

$$\left|t + i\,(1-t^2)^{1/2}s\right| = \left[1 - (1-t^2)\,(1-s^2)\right]^{1/2} \le e^{-(1-t^2)\,(1-s^2)/2}$$

for $s, t \in [-1, 1]$. Then,

$$|P_{n,d}(t)| \le \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \int_{-1}^{1} e^{-n\,(1-t^2)\,(1-s^2)/2}(1-s^2)^{\frac{d-4}{2}}\,ds$$
$$= 2\frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \int_{0}^{1} e^{-n\,(1-t^2)\,(1-s^2)/2}(1-s^2)^{\frac{d-4}{2}}\,ds.$$

Let $t \in (-1, 1)$. Use the change of variable $s = 1 - u$ and the relation $u \le 1 - s^2 \le 2u$ for $s \in [0, 1]$,

$$|P_{n,d}(t)| < 2^{\frac{d-2}{2}} \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \int_{0}^{\infty} e^{-n\,(1-t^2)\,u/2} u^{\frac{d-4}{2}}\,du.$$

For the integral, we apply the formula (1.4),

$$\int_{0}^{\infty} e^{-n\,(1-t^2)\,u/2} u^{\frac{d-4}{2}}\,du = \left[\frac{2}{n\,(1-t^2)}\right]^{\frac{d-2}{2}} \Gamma\!\left(\frac{d-2}{2}\right).$$

Then,

$$|P_{n,d}(t)| < 2^{d-2}\frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \frac{\Gamma(\frac{d-2}{2})}{[n\,(1-t^2)]^{\frac{d-2}{2}}}$$
$$= \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}} \left[\frac{4}{n\,(1-t^2)}\right]^{\frac{d-2}{2}}.$$

This inequality is valid also for $d = 2$. Therefore,

$$|P_{n,d}(t)| < \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}} \left[\frac{4}{n\,(1-t^2)}\right]^{\frac{d-2}{2}}, \quad n \in \mathbb{N}_0,\ d \ge 2,\ t \in (-1, 1). \tag{2.117}$$

From (2.90), we have bounds for derivatives of $P_{n,d}(t)$ of any order:

$$\left|P_{n,d}^{(j)}(t)\right| \le P_{n,d}^{(j)}(1) = \frac{n!(n+j+d-3)!\Gamma(\frac{d-1}{2})}{2^j(n-j)!(n+d-3)!\Gamma(j+\frac{d-1}{2})}.$$

In particular,

$$\max_{t\in[-1,1]} \left|P_{n,d}^{(j)}(t)\right| = \mathcal{O}(n^{2j}). \tag{2.118}$$

As an application of (2.118), we observe that for any $t, s \in [-1, 1]$,

$$P_{n,d}(t) - P_{n,d}(s) = P'_{n,d}(\tau)\,(t-s)$$

for some τ between t and s. Applying (2.118) with $j = 1$, we have

$$|P_{n,d}(t) - P_{n,d}(s)| \le c\,n^2 |t - s| \quad \forall\, t, s \in [-1, 1]. \tag{2.119}$$

Hence,

$$|P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\zeta}) - P_{n,d}(\boldsymbol{\eta}\cdot\boldsymbol{\zeta})| \le c\,n^2 |\boldsymbol{\xi} - \boldsymbol{\eta}| \quad \forall\, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{S}^{d-1}. \tag{2.120}$$

2.8 Completeness

In this section, we show in a constructive way that the spherical harmonics are complete in $C(\mathbb{S}^{d-1})$ and in $L^2(\mathbb{S}^{d-1})$, i.e., linear combinations of the spherical harmonics are dense in $C(\mathbb{S}^{d-1})$ and in $L^2(\mathbb{S}^{d-1})$.

2.8.1 Completeness in $C(\mathbb{S}^{d-1})$

Let $f \in C(\mathbb{S}^{d-1})$. Formally,

$$f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \delta(1 - \boldsymbol{\xi}\cdot\boldsymbol{\eta})\, f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}$$

using a Dirac delta function $\delta(t)$ whose value is 0 at $t \neq 0$, $+\infty$ at $t = 0$, and which satisfies formally

$$\int_{\mathbb{S}^{d-1}} \delta(1 - \boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = 1 \quad \forall\, \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

The idea to demonstrate the completeness of the spherical harmonics in $C(\mathbb{S}^{d-1})$ is to construct a sequence of kernel functions $\{k_n(t)\}$ such that $k_n(\boldsymbol{\xi}\cdot\boldsymbol{\eta})$ approaches $\delta(1 - \boldsymbol{\xi}\cdot\boldsymbol{\eta})$ and is such that for each $n \in \mathbb{N}$, the function $\int_{\mathbb{S}^{d-1}} k_n(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta})$ is a linear combination of spherical harmonics of order less than or equal to n. One possibility is to choose $k_n(t)$ proportional to $(1 + t)^n / 2^n$. Thus, we let

$$k_n(t) = E_{n,d} \left(\frac{1 + t}{2} \right)^n,$$

where $E_{n,d}$ is a scaling constant so that

$$\int_{\mathbb{S}^{d-1}} k_n(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = 1 \quad \forall\, \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.121}$$

To satisfy the condition (2.121), we have

$$E_{n,d} = \frac{(n+d-2)!}{(4\pi)^{\frac{d-1}{2}}\Gamma(n+\frac{d-1}{2})}. \tag{2.122}$$

This formula is derived as follows. First,

$$\int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n dS^{d-1}(\boldsymbol{\eta}) = |\mathbb{S}^{d-2}| \int_{-1}^{1} \left(\frac{1+t}{2}\right)^n (1-t^2)^{\frac{d-3}{2}}\, dt.$$

Use the change of variable $s = (1+t)/2$,

$$\int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n dS^{d-1}(\boldsymbol{\eta}) = 2^{d-2}|\mathbb{S}^{d-2}| \int_{0}^{1} s^{n+\frac{d-3}{2}}(1-s)^{\frac{d-3}{2}}\, ds.$$

By (1.19),

$$|\mathbb{S}^{d-2}| = \frac{2\,\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}.$$

Moreover,

$$\int_0^1 s^{n+\frac{d-3}{2}}(1-s)^{\frac{d-3}{2}}\, ds = B\left(n+\frac{d-1}{2}, \frac{d-1}{2}\right) = \frac{\Gamma(n+\frac{d-1}{2})\,\Gamma(\frac{d-1}{2})}{\Gamma(n+d-1)}.$$

Thus

$$\int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n dS^{d-1}(\boldsymbol{\eta}) = (4\pi)^{\frac{d-1}{2}} \frac{\Gamma(n+\frac{d-1}{2})}{\Gamma(n+d-1)}.$$

Hence, (2.122) holds.

Now we introduce an operator $\Pi_{n,d}$ by the following formula

$$(\Pi_{n,d} f)(\boldsymbol{\xi}) := E_{n,d} \int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}), \quad f \in C(\mathbb{S}^{d-1}). \tag{2.123}$$

Let us express $(\Pi_{n,d} f)(\boldsymbol{\xi})$ as a linear combination of spherical harmonics of order less than or equal to n. For this purpose, we write

$$E_{n,d}\left(\frac{1+t}{2}\right)^n = \sum_{k=0}^{n} \mu_{n,k,d} \frac{N_{k,d}}{|\mathbb{S}^{d-1}|} P_{k,d}(t). \tag{2.124}$$

To determine the coefficients $\{\mu_{n,k,d}\}_{k=0}^n$, multiply both sides by the function $P_{l,d}(t)\,(1-t^2)^{\frac{d-3}{2}}$, $0 \le l \le n$, integrate from $t=-1$ to $t=1$ and use the orthogonality relation (2.79) to obtain

$$\mu_{n,l,d} = |\mathbb{S}^{d-2}| E_{n,d} \int_{-1}^{1} \left(\frac{1+t}{2}\right)^n P_{l,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\,dt.$$

Applying Proposition 2.26, we have

$$\begin{aligned}\mu_{n,l,d} &= |\mathbb{S}^{d-2}| E_{n,d} R_{l,d} \int_{-1}^{1} \left(\frac{d}{dt}\right)^l \left(\frac{1+t}{2}\right)^n (1-t^2)^{l+\frac{d-3}{2}}\,dt \\ &= |\mathbb{S}^{d-2}| E_{n,d} R_{l,d} \frac{n!}{2^n (n-l)!} \int_{-1}^{1} (1+t)^{n-l}\,(1-t^2)^{l+\frac{d-3}{2}}\,dt.\end{aligned}$$

To compute the integral, we let $t = 2\,s - 1$. Then

$$\begin{aligned}\int_{-1}^{1} (1+t)^{n-l}\,(1-t^2)^{l+\frac{d-3}{2}}\,dt &= 2^{n+l+d-2} \int_{0}^{1} s^{n+\frac{d-3}{2}}\,(1-s)^{l+\frac{d-3}{2}}\,ds \\ &= 2^{n+l+d-2} \frac{\Gamma(n+\frac{d-1}{2})\,\Gamma(l+\frac{d-1}{2})}{\Gamma(n+l+d-1)}.\end{aligned}$$

Hence, using the formulas (1.19), (2.71), and (2.122), we have

$$\mu_{n,l,d} = \frac{n!(n+d-2)!}{(n-l)!(n+l+d-2)!}.$$

It is easy to see that $\mu_{n,l,d} < \mu_{n+1,l,d}$ and $\mu_{n,l,d} \to 1$ as $n \to \infty$. From the expansion (2.124), we get, by making use of the projection operator $\mathcal{P}_{n,d}$ defined in Definition 2.11,

$$(\Pi_{n,d} f)(\boldsymbol{\xi}) = \sum_{k=0}^{n} \mu_{n,k,d} (\mathcal{P}_{k,d} f)(\boldsymbol{\xi}). \tag{2.125}$$

In other words, $\Pi_{n,d} f$ is a linear combination of spherical harmonics of order less than or equal to n.

To prove the completeness, we note the following property.

Lemma 2.29. *If $t \in [-1, 1)$, then*

$$\lim_{n\to\infty} E_{n,d} \left(\frac{1+t}{2}\right)^n = 0.$$

Proof. By Stirling's formula (1.11),

$$\Gamma(x) \sim \sqrt{2\pi}\, x^{x-1/2} e^{-x} \text{ for } x \to \infty.$$

Then,

$$E_{n,d} \sim \frac{n^{\frac{d}{2}}}{(4\pi)^{\frac{d-1}{2}}}$$

and the statement holds. $\square$

Now we state and prove a completeness result.

Theorem 2.30.

$$\lim_{n\to\infty} \|\Pi_{n,d} f - f\|_{C(\mathbb{S}^{d-1})} = 0 \quad \forall\, f \in C(\mathbb{S}^{d-1}). \tag{2.126}$$

Proof. Use the modulus of continuity

$$\omega(f;\delta) = \sup\{|f(\boldsymbol{\xi}) - f(\boldsymbol{\eta})| : \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1},\ |\boldsymbol{\xi} - \boldsymbol{\eta}| \le \delta\}, \quad \delta > 0,$$

and recall that since $f \in C(\mathbb{S}^{d-1})$,

$$\omega(f;\delta) \to 0 \quad \text{as } \delta \to 0.$$

Denote

$$M := \sup\{|f(\boldsymbol{\xi}) - f(\boldsymbol{\eta})| : \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1}\} < \infty.$$

Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ be arbitrary but fixed. Using (2.121), we have

$$\begin{aligned}(\Pi_{n,d} f)(\boldsymbol{\xi}) - f(\boldsymbol{\xi}) &= E_{n,d} \int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n [f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})]\, dS^{d-1}(\boldsymbol{\eta}) \\ &\equiv \mathrm{I}_1(\boldsymbol{\xi}) + \mathrm{I}_2(\boldsymbol{\xi}),\end{aligned}$$

where

$$\mathrm{I}_1(\boldsymbol{\xi}) = E_{n,d} \int_{\{\boldsymbol{\eta}\in\mathbb{S}^{d-1} : |\boldsymbol{\xi}-\boldsymbol{\eta}|\le\delta\}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n [f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})]\, dS^{d-1}(\boldsymbol{\eta}),$$

$$\mathrm{I}_2(\boldsymbol{\xi}) = E_{n,d} \int_{\{\boldsymbol{\eta}\in\mathbb{S}^{d-1} : |\boldsymbol{\xi}-\boldsymbol{\eta}|>\delta\}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n [f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})]\, dS^{d-1}(\boldsymbol{\eta}).$$

We bound each term as follows:

$$|\mathrm{I}_1(\boldsymbol{\xi})| \le \omega(f;\delta)\, E_{n,d} \int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n dS^{d-1}(\boldsymbol{\eta}) = \omega(f;\delta),$$

$$|\mathrm{I}_2(\boldsymbol{\xi})| \le M\, E_{n,d} |\mathbb{S}^{d-1}| \left(1 - \frac{\delta^2}{2}\right)^n.$$

In bounding $\mathrm{I}_2(\boldsymbol{\xi})$, we used the relation

$$|\boldsymbol{\xi}-\boldsymbol{\eta}|>\delta \quad\Longrightarrow\quad \boldsymbol{\xi}\cdot\boldsymbol{\eta}<1-\frac{\delta^2}{2}$$

for $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{S}^{d-1}$. Thus, for any $\delta\in(0,1)$, applying Lemma 2.29, we have

$$\limsup_{n\to\infty}\|\Pi_{n,d}f-f\|_{C(\mathbb{S}^{d-1})}\le\omega(f;\delta).$$

Note that $\omega(f;\delta)\to 0$ as $\delta\to 0$. So the stated result holds. □

Using the formula (2.125), we can restate Theorem 2.30 as follows.

Theorem 2.31. *For any $f\in C(\mathbb{S}^{d-1})$,*

$$f(\boldsymbol{\xi})=\lim_{n\to\infty}\sum_{k=0}^{n}\mu_{n,k,d}(\mathcal{P}_{k,d}f)(\boldsymbol{\xi})\ \text{uniformly in}\ \boldsymbol{\xi}\in\mathbb{S}^{d-1}.$$

If $\mathcal{P}_{k,d}f=0$ for all $n\in\mathbb{N}_0$, then $f=0$.

Theorem 2.31 combined with Theorem 2.14 implies that $\{\mathbb{Y}_n^d:n\in\mathbb{N}_0\}$ is the only system of primitive spaces in $C(\mathbb{S}^{d-1})$ since any primitive space not identical with one of $\mathbb{Y}_n^d$, $n\in\mathbb{N}_0$, is orthogonal to all and is therefore trivial.

2.8.2 Completeness in $C(\mathbb{S}^{d-1})$ via the Poisson Identity

We now use the Poisson identity (2.104) to give another constructive proof of the completeness of the spherical harmonics. First we introduce a lemma.

Lemma 2.32. *The function*

$$G_d(r,t):=\frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\,\frac{1-r^2}{(1+r^2-2rt)^{\frac{d}{2}}},\quad |r|<1,\ t\in[-1,1] \tag{2.127}$$

is positive and has the properties:

$$\int_{-1}^{1}G_d(r,t)\,(1-t^2)^{\frac{d-3}{2}}\,dt=1, \tag{2.128}$$

$$\lim_{r\to 1-}G_d(r,t)=0\ \text{uniformly for}\ t\in[-1,t_0]\quad \text{with any fixed}\ t_0\in(-1,1). \tag{2.129}$$

Proof. For (2.128),

$$\int_{-1}^{1} G_d(r,t)\,(1-t^2)^{\frac{d-3}{2}}\,dt = \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\int_{-1}^{1}\sum_{n=0}^{\infty} N_{n,d} r^n P_{n,d}(t)(1-t^2)^{\frac{d-3}{2}}\,dt$$
$$= \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\int_{-1}^{1}(1-t^2)^{\frac{d-3}{2}}\,dt$$
$$= 1.$$

For (2.129), note the bound

$$\frac{1-r^2}{(1+r^2-2rt)^{\frac{d}{2}}} = \frac{1-r^2}{[(1-r)^2+2r(1-t)]^{\frac{d}{2}}} \le \frac{1-r^2}{[2r(1-t_0)]^{\frac{d}{2}}}$$

which is valid for $t \in [-1, t_0]$. □

Define an operator $\mathcal{G}_d(r)$ by

$$(\mathcal{G}_d(r)f)(\boldsymbol{\xi}) = \frac{1}{|\mathbb{S}^{d-2}|}\int_{\mathbb{S}^{d-1}} G_d(r,\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}).$$

Note that for $|r|<1$,

$$(\mathcal{G}_d(r)f)(\boldsymbol{\xi}) = \frac{1}{|\mathbb{S}^{d-2}|}\sum_{n=0}^{\infty} N_{n,d} r^n \int_{\mathbb{S}^{d-1}} P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}),$$

i.e.,

$$(\mathcal{G}_d(r)f)(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} r^n (\mathcal{P}_{n,d} f)(\boldsymbol{\xi}). \tag{2.130}$$

Thus, $\mathcal{G}_d(r)f$ is the limit of a sequence of finite linear combinations of the spherical harmonics.

Theorem 2.33 (Completeness).

$$\lim_{r\to 1-} \|\mathcal{G}_d(r)f - f\|_{C(\mathbb{S}^{d-1})} = 0 \quad \forall\, f \in C(\mathbb{S}^{d-1}). \tag{2.131}$$

Proof. The proof is similar to that of Theorem 2.30. Using (2.128),

$$(\mathcal{G}_d(r)f)(\boldsymbol{\xi}) - f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} G_d(r,\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,[f(\boldsymbol{\eta}) - f(\boldsymbol{\xi})]\, dS^{d-1}(\boldsymbol{\eta})$$
$$\equiv \mathrm{I}_1(\boldsymbol{\xi}) + \mathrm{I}_2(\boldsymbol{\xi}),$$

where

$$\mathrm{I}_1(\boldsymbol{\xi}) = \int_{|\boldsymbol{\xi}-\boldsymbol{\eta}|\geq\delta} G_d(r,\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,[f(\boldsymbol{\eta})-f(\boldsymbol{\xi})]\,dS^{d-1}(\boldsymbol{\eta}),$$

$$\mathrm{I}_2(\boldsymbol{\xi}) = \int_{|\boldsymbol{\xi}-\boldsymbol{\eta}|<\delta} G_d(r,\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,[f(\boldsymbol{\eta})-f(\boldsymbol{\xi})]\,dS^{d-1}(\boldsymbol{\eta}).$$

For any $\delta > 0$, by (2.129),

$$|\mathrm{I}_1(\boldsymbol{\xi})| \to 0 \text{ uniformly as } r \to 1- .$$

Also,

$$|\mathrm{I}_2(\boldsymbol{\xi})| \leq \omega(f;\delta).$$

So

$$\limsup_{r\to 1-} \|\mathcal{G}_d(r)f - f\|_{C(\mathbb{S}^{d-1})} \leq \omega(f;\delta)$$

and (2.131) follows. □

2.8.3 Convergence of Fourier–Laplace Series

We now consider convergence in average and uniform convergence of the Fourier–Laplace series. For a given function f, the series

$$\sum_{k=0}^{\infty} \mathcal{P}_{k,d} f$$

is called the Fourier–Laplace series of the function f. Recall Definition 2.11 for the projection $\mathcal{P}_{k,d} f$.

First, we present a result for convergence in average.

Theorem 2.34. *We have the convergence in average of the Fourier–Laplace series:*

$$\lim_{n\to\infty} \Big\| f - \sum_{k=0}^{n} \mathcal{P}_{k,d} f \Big\|_{L^2(\mathbb{S}^{d-1})} = 0 \quad \forall\, f \in L^2(\mathbb{S}^{d-1}). \tag{2.132}$$

Proof. Note that the operator $\mathcal{P}_{k,d}$ is self-adjoint:

$$(f, \mathcal{P}_{k,d} g) = (\mathcal{P}_{k,d} f, g) \quad \forall\, f, g \in L^2(\mathbb{S}^{d-1}).$$

Also, $(\mathcal{P}_{k,d})^2 = \mathcal{P}_{k,d}$. Therefore,

$$(f, \mathcal{P}_{k,d} f) = (f, (\mathcal{P}_{k,d})^2 f) = (\mathcal{P}_{k,d} f, \mathcal{P}_{k,d} f) = \|\mathcal{P}_{k,d} f\|^2_{L^2(\mathbb{S}^{d-1})}$$

and

$$(\mathcal{P}_{k,d}f, \mathcal{P}_{n,d}f) = \delta_{kn}\|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})}.$$

Apply the above two equalities to obtain

$$\Big\|f - \sum_{k=0}^{n} \mathcal{P}_{k,d}f\Big\|^2_{L^2(\mathbb{S}^{d-1})} = \|f\|^2_{L^2(\mathbb{S}^{d-1})} - \sum_{k=0}^{n} \|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})}. \tag{2.133}$$

Hence,

$$\sum_{k=0}^{n} \|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})} \le \|f\|^2_{L^2(\mathbb{S}^{d-1})} \quad \forall\, n \in \mathbb{N}_0.$$

Then,

$$\sum_{k=0}^{\infty} \|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})} \le \|f\|^2_{L^2(\mathbb{S}^{d-1})} \quad \forall\, f \in L^2(\mathbb{S}^{d-1}). \tag{2.134}$$

First we assume $f \in C(\mathbb{S}^{d-1})$. From the formula (2.130),

$$\|\mathcal{G}_d(r)f\|^2_{L^2(\mathbb{S}^{d-1})} = \sum_{k=0}^{\infty} r^{2k}\|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})}. \tag{2.135}$$

By Theorem 2.33, $\mathcal{G}_d(r)f$ converges uniformly to f on $\mathbb{S}^{d-1}$ as $r \to 1-$. Take the limit $r \to 1-$ in (2.135) to obtain

$$\|f\|^2_{L^2(\mathbb{S}^{d-1})} = \sum_{k=0}^{\infty} \|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})}. \tag{2.136}$$

Then by (2.133) we obtain (2.132) for $f \in C(\mathbb{S}^{d-1})$.

Extension of the result from a $C(\mathbb{S}^{d-1})$ function to an $L^2(\mathbb{S}^{d-1})$ function is achieved by using the density of $C(\mathbb{S}^{d-1})$ in $L^2(\mathbb{S}^{d-1})$, by noticing that since spherical harmonics of different order are orthogonal,

$$\Big\|\sum_{k=0}^{n} \mathcal{P}_{k,d}f\Big\|^2_{L^2(\mathbb{S}^{d-1})} = \sum_{k=0}^{n} \|\mathcal{P}_{k,d}f\|^2_{L^2(\mathbb{S}^{d-1})}$$

and by applying the bound (2.134). □

Then we turn to a study of uniform convergence of the Fourier–Laplace series.

Define $\mathcal{S}_n : C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ to be the linear operator given by the partial sum of the spherical harmonic expansion

$$\mathcal{S}_n f(\boldsymbol{\xi}) := \sum_{k=0}^{n} (\mathcal{P}_{k,d} f)(\boldsymbol{\xi}), \quad f \in C(\mathbb{S}^{d-1}). \tag{2.137}$$

Denote by $\|\mathcal{S}_n\|$ the norm of the operator. To answer the question when do the partial sums $\{\mathcal{S}_n f\}$ converge uniformly to f, an important tool is the following result, due to Lebesgue.

Theorem 2.35. *For $f \in C(\mathbb{S}^{d-1})$,*

$$\|f - \mathcal{S}_n f\|_{C(\mathbb{S}^{d-1})} \leq (1 + \|\mathcal{S}_n\|)\, E_{n,\infty}(f), \tag{2.138}$$

where

$$E_{n,\infty}(f) := \inf\left\{\|f - p_n\|_{C(\mathbb{S}^{d-1})} : p_n \in \mathbb{Y}^d_{0:n}\right\} \tag{2.139}$$

and

$$\mathbb{Y}^d_{0:n} := \bigoplus_{j=0}^{n} \mathbb{Y}^d_j.$$

Proof. Note that

$$\mathcal{S}_n p_n = p_n \quad \forall\, p_n \in \mathbb{Y}^d_{0:n}.$$

Thus,

$$f - \mathcal{S}_n f = (f - p_n) - \mathcal{S}_n (f - p_n) \quad \forall\, p_n \in \mathbb{Y}^d_{0:n}.$$

Apply the $C(\mathbb{S}^{d-1})$-norm,

$$\|f - \mathcal{S}_n f\|_{C(\mathbb{S}^{d-1})} \leq (1 + \|\mathcal{S}_n\|)\, \|f - p_n\|_{C(\mathbb{S}^{d-1})}.$$

Then take the infimum with respect to p_n over the subspace $\mathbb{Y}^d_{0:n}$ to get (2.138). $\square$

The operator norm $\|\mathcal{S}_n\|$ is called the "Lebesgue constant". In [94], it is shown that

$$\|\mathcal{S}_n\| = \mathcal{O}\Big(n^{(d-2)/2}\Big), \quad d \geq 3.$$

Based on this bound, the next result regarding the uniform convergence of the Fourier–Laplace series can be proved.

Theorem 2.36. *Let $d \geq 3$ and $f \in C^{k,\alpha}\big(\mathbb{S}^{d-1}\big)$ for some $k \geq 0$ and $\alpha \in (0,1]$. Assume $k+\alpha > d/2-1$. Then $\mathcal{S}_n f$ converges uniformly to f over $\mathbb{S}^{d-1}$.*

The spaces $C^k\big(\mathbb{S}^{d-1}\big)$ and $C^{k,\alpha}\big(\mathbb{S}^{d-1}\big)$ can be defined in a variety of ways, some of which are discussed in Sects. 4.2.1 and 4.2.2. We say $f \in C^{k,\alpha}\big(\mathbb{S}^{d-1}\big)$ if all of its k^{th}-order derivatives are Hölder continuous with exponent $\alpha \in (0,1]$.

This theorem is proven in [94], based on results from [93] and [54]. Results from these papers are discussed in greater detail in Sect. 4.2 for the special case of $\mathbb{S}^2$.

In the case $d = 2$, the Fourier–Laplace series reduces to the ordinary Fourier series. The Lebesgue constant is [123, Chap. 2, p. 67]

$$\|\mathcal{S}_n\| = \frac{4}{\pi^2}\ln n + \mathcal{O}(1).$$

The following uniform convergence result on the Fourier series holds (see, e.g., [13, Sect. 3.7]).

Theorem 2.37. *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function, with 2π being an integer multiple of its period. If $f \in C^{k,\alpha}(\mathbb{R})$ with $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$, then for the n^{th} order partial sum $\mathcal{S}_n f$ of the Fourier series of the function f,*

$$\|f - \mathcal{S}_n f\|_{C[0,2\pi]} \le c\,\frac{\ln(n+2)}{n^{k+\alpha}}.$$

In particular, this implies the uniform convergence of the Fourier series of the function f.

2.8.4 Completeness in $L^2(\mathbb{S}^{d-1})$

Theorem 2.34 implies the completeness of spherical harmonics in $L^2(\mathbb{S}^{d-1})$, i.e., the subspace of linear combinations of spherical harmonics is dense in $L^2(\mathbb{S}^{d-1})$.

An alternative way to show the completeness of spherical harmonics in $L^2(\mathbb{S}^{d-1})$ is through using the operator $\Pi_{n,d}$ defined in (2.123). First, we show the operator $\Pi_{n,d}$ is bounded as a mapping from $L^2(\mathbb{S}^{d-1})$ to $L^2(\mathbb{S}^{d-1})$:

$$\|\Pi_{n,d} f\|_{L^2(\mathbb{S}^{d-1})} \le \|f\|_{L^2(\mathbb{S}^{d-1})} \quad \forall\, f \in L^2(\mathbb{S}^{d-1}). \tag{2.140}$$

This is proved as follows:

$$\begin{aligned}
\|\Pi_{n,d} f\|^2_{L^2(\mathbb{S}^{d-1})} &= \int_{\mathbb{S}^{d-1}} E^2_{n,d}\left[\int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n f(\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta})\right]^2 dS^{d-1}(\boldsymbol{\xi}) \\
&\le \int_{\mathbb{S}^{d-1}} E^2_{n,d}\Bigg[\int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n dS^{d-1}(\boldsymbol{\eta}) \\
&\qquad \int_{\mathbb{S}^{d-1}} \left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n |f(\boldsymbol{\eta})|^2 dS^{d-1}(\boldsymbol{\eta})\Bigg]\, dS^{d-1}(\boldsymbol{\xi}).
\end{aligned}$$

Apply (2.121),

$$\begin{aligned}\|\Pi_{n,d}f\|^2_{L^2(\mathbb{S}^{d-1})} &\le \int_{\mathbb{S}^{d-1}} E_{n,d}\left[\int_{\mathbb{S}^{d-1}}\left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n |f(\boldsymbol{\eta})|^2 dS^{d-1}(\boldsymbol{\eta})\right] dS^{d-1}(\boldsymbol{\xi})\\ &= \int_{\mathbb{S}^{d-1}} |f(\boldsymbol{\eta})|^2 \left[E_{n,d}\int_{\mathbb{S}^{d-1}}\left(\frac{1+\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2}\right)^n dS^{d-1}(\boldsymbol{\xi})\right] dS^{d-1}(\boldsymbol{\eta}).\end{aligned}$$

Apply (2.121) again to obtain

$$\|\Pi_{n,d}f\|^2_{L^2(\mathbb{S}^{d-1})} \le \|f\|^2_{L^2(\mathbb{S}^{d-1})},$$

i.e., (2.140) holds.

Let $f \in L^2(\mathbb{S}^{d-1})$. For any $\varepsilon > 0$, by the density of $C(\mathbb{S}^{d-1})$ in $L^2(\mathbb{S}^{d-1})$, we can find a function $f_\varepsilon \in C(\mathbb{S}^{d-1})$ such that

$$\|f - f_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} < \frac{\varepsilon}{3}.$$

Choose n sufficiently large so that, following Theorem 2.30,

$$\|\Pi_{n,d}f_\varepsilon - f_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} < \frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned}\|\Pi_{n,d}f - f\|_{L^2(\mathbb{S}^{d-1})} &\le \|\Pi_{n,d}(f - f_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})} + \|\Pi_{n,d}f_\varepsilon - f_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &\quad + \|f - f_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &\le 2\,\|f - f_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} + \|\Pi_{n,d}f_\varepsilon - f_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &< \varepsilon.\end{aligned}$$

Thus, the spherical harmonics are dense in $L^2(\mathbb{S}^{d-1})$.

Since spherical harmonics of different orders are orthogonal, we can also deduce the next result.

Theorem 2.38. *We have the orthogonal decomposition*

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{n=0}^{\infty} \mathbb{Y}_n^d.$$

Thus, any function $f \in L^2(\mathbb{S}^{d-1})$ can be uniquely represented as

$$f(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} f_n(\boldsymbol{\xi}) \text{ in } L^2(\mathbb{S}^{d-1}),\ f_n \in \mathbb{Y}_n^d,\ n \ge 0. \tag{2.141}$$

We call $f_n \in \mathbb{Y}_n^d$ the n-spherical harmonic component of f and have the following formula

$$f_n(\boldsymbol{\xi}) = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}), \quad n \geq 0. \tag{2.142}$$

This formula is derived from (2.141) as follows. Replace $\boldsymbol{\xi}$ by $\boldsymbol{\eta}$ in (2.141), multiply both sides by $P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})$ and integrate with respect to $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$ to obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) &= \int_{\mathbb{S}^{d-1}} \sum_{j=0}^{\infty} f_j(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) \\ &= \sum_{j=0}^{\infty} \int_{\mathbb{S}^{d-1}} f_j(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}).\end{aligned}$$

By the orthogonality of spherical harmonics of different orders,

$$\int_{\mathbb{S}^{d-1}} f_j(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = 0 \quad \forall\, j \neq n.$$

Moreover, by (2.33),

$$\int_{\mathbb{S}^{d-1}} f_n(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}}\, f_n(\boldsymbol{\xi}).$$

Hence,

$$\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\, dS^{d-1}(\boldsymbol{\eta}) = \frac{|\mathbb{S}^{d-1}|}{N_{n,d}}\, f_n(\boldsymbol{\xi})$$

and the formula (2.142) is proved. Notice that $f_n(\boldsymbol{\xi}) = (\mathcal{P}_{n,d} f)(\boldsymbol{\xi})$ with the projection operator $\mathcal{P}_{n,d}$ defined in (2.44).

As a consequence of (2.141), we have the Parseval equality on $L^2(\mathbb{S}^{d-1})$:

$$\|f\|^2_{L^2(\mathbb{S}^{d-1})} = \sum_{n=0}^{\infty} \|f_n\|^2_{L^2(\mathbb{S}^{d-1})} \quad \forall\, f \in L^2(\mathbb{S}^{d-1}), \tag{2.143}$$

where f_n is given by (2.142). This equality extends (2.136) from $C(\mathbb{S}^{d-1})$ functions to $L^2(\mathbb{S}^{d-1})$ functions.

2.9 The Gegenbauer Polynomials

The Gegenbauer polynomials are useful in generalizing the expansion (2.102). Recall the integral representation formula (2.72) for the Legendre polynomials.

Definition 2.39. For $\nu > 0$, $n \in \mathbb{N}_0$,

$$C_{n,\nu}(t) := \binom{n+2\nu-1}{n} \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \int_{-1}^{1} \left[t + i\,(1-t^2)^{1/2} s\right]^n (1-s^2)^{\nu-1} ds \tag{2.144}$$

is called the Gegenbauer polynomial of degree n with index ν.

Note that for an arbitrary number a, the binomial coefficient

$$\binom{a}{n} := \frac{a\,(a-1)\cdots(a-(n-1))}{n!}, \quad n \in \mathbb{N}.$$

Why $C_{n,\nu}(t)$ is a polynomial of degree n? First,

$$\left[t + i\,(1-t^2)^{1/2} s\right]^n = \sum_{j=0}^{n} \binom{n}{j} t^{n-j} (1-t^2)^{j/2} (is)^j.$$

For $j = 2k+1$ odd, the integral of the corresponding term is

$$\int_{-1}^{1} s^{2k+1} (1-s^2)^{\nu-1} ds = 0.$$

So $C_{n,\nu}(t)$ is real valued and

$$C_{n,\nu}(t) = \binom{n+2\nu-1}{n} \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \sum_{k=0}^{[n/2]} \binom{n}{2k} t^{n-2k} (-1)^k (1-t^2)^k \cdot \int_{-1}^{1} s^{2k} (1-s^2)^{\nu-1} ds$$

is a polynomial of degree $\leq n$. The coefficient of t^n in $C_{n,\nu}(t)$ is

$$\binom{n+2\nu-1}{n} \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \sum_{k=0}^{[n/2]} \binom{n}{2k} \int_{-1}^{1} s^{2k} (1-s^2)^{\nu-1} ds > 0.$$

Hence, $C_{n,\nu}(t)$ is a polynomial of degree n.

Observe that, recalling the formula (2.72),

$$C_{n,\frac{d-2}{2}}(t) = \binom{n+d-3}{n} P_{n,d}(t), \quad d \geq 3. \tag{2.145}$$

Proposition 2.40. (Gegenbauer identity)

$$\sum_{n=0}^{\infty} r^n C_{n,\nu}(t) = \frac{1}{(1+r^2-2rt)^{\nu}}, \quad |r| < 1,\ t \in [-1,1]. \tag{2.146}$$

Proof. First we calculate $C_{n,\nu}(1)$:

$$C_{n,\nu}(1) = \binom{n+2\nu-1}{n} \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \int_{-1}^{1} (1-s^2)^{\nu-1} ds.$$

Let $t = s^2$. Then,

$$C_{n,\nu}(1) = \binom{n+2\nu-1}{n} \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \int_{0}^{1} t^{-\frac{1}{2}} (1-t)^{\nu-1} dt.$$

Since

$$\int_0^1 t^{-\frac{1}{2}} (1-t)^{\nu-1} dt = \frac{\Gamma(\frac{1}{2})\,\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2})},$$

we have

$$C_{n,\nu}(1) = \binom{n+2\nu-1}{n}. \tag{2.147}$$

From the power series (2.4),

$$\sum_{n=0}^{\infty} \binom{n+2\nu-1}{n} z^n = \frac{1}{(1-z)^{2\nu}}, \quad |z|<1.$$

For $|r|<1$ and $|t|\le 1$,

$$\sum_{n=0}^{\infty} r^n C_{n,\nu}(t) = \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \int_{-1}^{1} \frac{(1-s^2)^{\nu-1}}{[1-rt-i\,r\,(1-t^2)^{1/2}s]^{2\nu}} ds. \tag{2.148}$$

Write

$$1-rt-i\,r\,(1-t^2)^{1/2} = (1+r^2-2rt)^{1/2} e^{-i\alpha}$$

for some $\alpha \in \left[0, \frac{\pi}{2}\right)$. Use the substitution (2.74), recall the relations (2.75), and note that

$$\begin{aligned} 1-rt-i\,r\,(1-t^2)^{1/2}s &= (1+r^2-2rt)^{1/2}\,(\cos\alpha - i\tanh u \sin\alpha) \\ &= (1+r^2-2rt)^{1/2} \frac{\cosh(u-i\alpha)}{\cosh u}. \end{aligned}$$

So from (2.148), we have

$$\sum_{n=0}^{\infty} r^n C_{n,\nu}(t) = \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)} \frac{1}{(1+r^2-2rt)^{\nu}} \int_{-\infty}^{\infty} \frac{1}{\cosh^{2\nu}(u-i\alpha)}\, du.$$

Since the poles of the function $(\cosh u)^{-2\nu}$ are $u = i\pi\,(k+1/2)$, $k \in \mathbb{Z}$, and since $0 \le \alpha < \pi/2$, we can apply the Cauchy integral theorem in complex analysis to get

$$\int_{-\infty}^{\infty} \frac{1}{\cosh^{2\nu}(u - i\alpha)}\, du = \int_{-\infty}^{\infty} \frac{1}{\cosh^{2\nu} u}\, du,$$

which is a constant. Thus, for some constant c,

$$\sum_{n=0}^{\infty} r^n C_{n,\nu}(t) = \frac{c}{(1 + r^2 - 2rt)^{\nu}}.$$

Let $t = 1$ and use the value (2.147):

$$\frac{c}{(1-r)^{2\nu}} = \sum_{n=0}^{\infty} \binom{n+2\nu-1}{n} r^n = \frac{1}{(1-r)^{2\nu}}.$$

So the constant $c = 1$. □

Obviously, (2.102) is a special case of (2.146) by taking $\nu = 1/2$.

2.10 The Associated Legendre Functions

We have seen that the Legendre polynomials play an important role in the study of spherical harmonics. In an increasing order of complexity, we next introduce associated Legendre functions which are useful in constructing spherical harmonics from those in a lower dimension.

2.10.1 Definition and Representation Formulas

Recall the first integral representation formula (2.72) for the Legendre polynomials. We then introduce the following definition.

Definition 2.41. For $d \ge 3$ and $n, j \in \mathbb{N}_0$,

$$P_{n,d,j}(t) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} i^{-j} \int_{-1}^{1} \left[t + i\,(1-t^2)^{1/2} s\right]^n P_{j,d-1}(s)\,(1-s^2)^{\frac{d-4}{2}}\, ds,$$
$$t \in [-1, 1]. \tag{2.149}$$

is called the associated Legendre function of degree n with order j in dimension d.

When $j = 0$, $P_{n,d,0}(t) = P_{n,d}(t)$ is the Legendre polynomial of degree n in d-dimensions. The factor i^{-j} is included in (2.149) to make $P_{n,d,j}(t)$ real-valued. To see this, note that

$$\left[t + i\,(1-t^2)^{1/2} s\right]^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} (1-t^2)^{k/2} i^k s^k.$$

Thus,

$$P_{n,d,j}(t) = \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \sum_{k=0}^{n} \binom{n}{k} t^{n-k} (1-t^2)^{k/2} i^{k-j} \int_{-1}^{1} s^k P_{j,d-1}(s)\,(1-s^2)^{\frac{d-4}{2}}\,ds.$$

By the parity property (2.111) for the Legendre polynomials, when $|k-j|$ is odd, $s^k P_{j,d-1}(s)$ is an odd function and then

$$\int_{-1}^{1} s^k P_{j,d-1}(s)\,(1-s^2)^{\frac{d-4}{2}}\,ds = 0.$$

Consequently, $P_{n,d,j}(t)$ is real-valued.

The associated Legendre functions can be used to generate orthonormal systems of spherical harmonics on $\mathbb{S}^{d-1}$; see Sect. 2.11.

Applying Proposition 2.26, we have

$$\begin{aligned} P_{n,d,j}(t) = {} & R_{j,d-1} \frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|} \frac{n!}{(n-j)!} (1-t^2)^{\frac{j}{2}} \\ & \cdot \int_{-1}^{1} \left[t + i\,(1-t^2)^{1/2} s\right]^{n-j} (1-s^2)^{j+\frac{d-4}{2}}\,ds, \end{aligned}$$

where by (2.71),

$$R_{j,d-1} = \frac{\Gamma(\frac{d-2}{2})}{2^j \Gamma(j + \frac{d-2}{2})}.$$

Since

$$\int_{-1}^{1} \left[t + i\,(1-t^2)^{1/2} s\right]^{n-j} (1-s^2)^{j+\frac{d-4}{2}}\,ds = \frac{\pi^{\frac{1}{2}}\Gamma(j+\frac{d-2}{2})}{\Gamma(j+\frac{d-1}{2})} P_{n-j,d+2j}(t)$$

by an application of the integral representation formula (2.72), we have thus shown the following result.

Proposition 2.42. *For $d \geq 3$ and $0 \leq j \leq n$,*

$$P_{n,d,j}(t) = \frac{n!\,\Gamma(\frac{d-1}{2})}{2^j (n-j)!\,\Gamma(j+\frac{d-1}{2})} (1-t^2)^{\frac{j}{2}} P_{n-j,d+2j}(t), \quad t \in [-1,1].$$

In some references, the associated Legendre functions are also called the associated Legendre polynomials. From Proposition 2.42, it is evident that the associated Legendre function $P_{n,d,j}(t)$ is a polynomial in t if and only if j is even.

Combining Theorem 2.23 and Proposition 2.42, we obtain the formula

$$P_{n,d,j}(t) = \frac{(-1)^{n-j} n!\Gamma(\frac{d-1}{2})}{2^n (n-j)!\Gamma(n+\frac{d-1}{2})} (1-t^2)^{\frac{3-d-j}{2}} \left(\frac{d}{dt}\right)^{n-j} (1-t^2)^{n+\frac{d-3}{2}}$$

for $d \geq 3$, $0 \leq j \leq n$ and $t \in [-1,1]$. For the particular case $d = 3$, with $0 \leq j \leq n$ and $t \in [-1,1]$,

$$P_{n,3,j}(t) = \frac{(-1)^{n-j}}{2^n (n-j)!} (1-t^2)^{-\frac{j}{2}} \left(\frac{d}{dt}\right)^{n-j} (1-t^2)^n. \tag{2.150}$$

Furthermore, by the formula (2.90), we obtain the next result.

Proposition 2.43. *For $d \geq 3$ and $0 \leq j \leq n$,*

$$P_{n,d,j}(t) = \frac{(n+d-3)!}{(n+j+d-3)!} (1-t^2)^{\frac{j}{2}} P_{n,d}^{(j)}(t), \quad t \in [-1,1].$$

Thus, the associated Legendre functions can be computed through differentiating the Legendre polynomials.

Combining Theorem 2.23 and Proposition 2.43, we obtain the formula

$$\begin{aligned} P_{n,d,j}(t) = {} & \frac{(-1)^n (n+d-3)!\Gamma(\frac{d-1}{2})}{2^n (n+j+d-3)!\Gamma(n+\frac{d-1}{2})} (1-t^2)^{\frac{j}{2}} \\ & \cdot \left(\frac{d}{dt}\right)^j \left[(1-t^2)^{\frac{3-d}{2}} \left(\frac{d}{dt}\right)^n (1-t^2)^{n+\frac{d-3}{2}}\right] \end{aligned}$$

for $d \geq 3$, $0 \leq j \leq n$ and $t \in [-1,1]$. For $d = 3$, with $0 \leq j \leq n$ and $t \in [-1,1]$,

$$P_{n,3,j}(t) = \frac{(-1)^n}{2^n (n+j)!} (1-t^2)^{\frac{j}{2}} \left(\frac{d}{dt}\right)^{n+j} (1-t^2)^n. \tag{2.151}$$

From (2.150) and (2.151), we obtain an identity

$$(1-t^2)^j \left(\frac{d}{dt}\right)^{n+j} (1-t^2)^n = (-1)^j \frac{(n+j)!}{(n-j)!} \left(\frac{d}{dt}\right)^{n-j} (1-t^2)^n, \quad 0 \leq j \leq n.$$

For $d = 2$, we use the formulas given in Proposition 2.42 or Proposition 2.43 to define $P_{n,2,j}(t)$ for $0 \leq j \leq n$.

2.10.2 Properties

First we present an addition theorem for the associated Legendre functions. The function $\left[t+i\,(1-t^2)^{1/2}s\right]^n$ is a polynomial of degree n in the variable s. Consider the expansion

$$\left[t+i\,(1-t^2)^{1/2}s\right]^n=\sum_{j=0}^{n}c_j(t)\,P_{j,d-1}(s)$$

and let us determine $c_j(t)$. By Definition 2.41, for $0\le k\le n$,

$$P_{n,d,k}(t)=\frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}i^{-k}\sum_{j=0}^{n}c_j(t)\int_{-1}^{1}P_{k,d-1}(s)\,P_{j,d-1}(s)\,(1-s^2)^{\frac{d-4}{2}}ds.$$

Using (2.79), we have

$$P_{n,d,k}(t)=\frac{1}{i^k N_{k,d-1}}\,c_k(t).$$

So

$$c_k(t)=i^k N_{k,d-1}P_{n,d,k}(t)$$

and then we can write the expansion as

$$\left[t+i\,(1-t^2)^{1/2}s\right]^n=\sum_{j=0}^{n}i^jN_{j,d-1}P_{n,d,j}(t)\,P_{j,d-1}(s). \tag{2.152}$$

Temporarily assume $m\ge n\ge 0$. We use the identity (2.152) to obtain

$$\begin{aligned}
P_{m+n,d}(t)&=\frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\int_{-1}^{1}\left[t+i\,(1-t^2)^{1/2}s\right]^{m+n}(1-s^2)^{\frac{d-4}{2}}ds\\
&=\frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\int_{-1}^{1}\left[t+i\,(1-t^2)^{1/2}s\right]^{m}\sum_{j=0}^{n}i^jN_{j,d-1}P_{n,d,j}(t)\\
&\qquad\cdot P_{j,d-1}(s)\,(1-s^2)^{\frac{d-4}{2}}ds\\
&=\frac{|\mathbb{S}^{d-3}|}{|\mathbb{S}^{d-2}|}\sum_{j=0}^{n}i^jN_{j,d-1}P_{n,d,j}(t)\int_{-1}^{1}\left[t+i\,(1-t^2)^{1/2}s\right]^{m}\\
&\qquad\cdot P_{j,d-1}(s)\,(1-s^2)^{\frac{d-4}{2}}ds\\
&=\sum_{j=0}^{n}(-1)^jN_{j,d-1}P_{m,d,j}(t)P_{n,d,j}(t),
\end{aligned}$$

recalling the defining relation (2.149). Thus,

$$P_{m+n,d}(t) = \sum_{j=0}^{\min\{m,n\}} (-1)^j N_{j,d-1} P_{m,d,j}(t) P_{n,d,j}(t), \quad m,n \in \mathbb{N}_0. \tag{2.153}$$

This is an addition theorem for the associated Legendre functions.

For the case $d = 2$, $P_{n,2}(t) = \cos(n \arccos t)$. With the new variable $\theta = \arccos t$, we have $P_{n,2}(\cos\theta) = \cos(n\theta)$. Also, in this case, $N_{n,1}$ is given by (2.11). By Proposition 2.43,

$$P_{n,2,j}(t) = \frac{(n-1)!}{(n+j-1)!} (1-t^2)^{\frac{j}{2}} \left(\frac{d}{dt}\right)^j \cos(n \arccos t). \tag{2.154}$$

In particular, with $j = 1$, we obtain from (2.154) that

$$P_{n,2,1}(t) = \sin(n \arccos t).$$

The addition theorem formula (2.153) with $d = 2$

$$P_{m+n,2}(t) = \sum_{j=0}^{\min\{m,n\}} (-1)^j N_{j,1} P_{m,2,j}(t) P_{n,2,j}(t)$$

takes the following familiar form, with $\theta = \arccos t$,

$$\cos((m+n)\theta) = \cos(m\theta)\cos(n\theta) - \sin(m\theta)\sin(n\theta), \quad m,n \in \mathbb{N}_0.$$

Next, we derive a differential equation for $P_{n,d,j}(t)$. Differentiate (2.83) j times,

$$\left(\frac{d}{dt}\right)^j \left[(1-t^2)\, P_{n,d}''(t) - (d-1)\, t\, P_{n,d}'(t) + n\,(n+d-2)\, P_{n,d}(t)\right] = 0.$$

Since

$$\begin{aligned}\left(\frac{d}{dt}\right)^j \left[(1-t^2)\, P_{n,d}''(t)\right] &= (1-t^2)\, P_{n,d}^{(j+2)}(t) - 2\,j\,t\, P_{n,d}^{(j+1)}(t) \\ &\quad - j\,(j-1)\, P_{n,d}^{(j)}(t), \\ \left(\frac{d}{dt}\right)^j \left[t\, P_{n,d}'(t)\right] &= t\, P_{n,d}^{(j+1)}(t) + j\, P_{n,d}^{(j)}(t),\end{aligned}$$

we get

$$
\begin{aligned}
&(1-t^2)\, P_{n,d}^{(j+2)}(t) - (2j+d-1)\, t\, P_{n,d}^{(j+1)}(t) \\
&\qquad + \left[n\,(n+d-2) - j\,(j+d-2)\right] P_{n,d}^{(j)}(t) = 0.
\end{aligned} \tag{2.155}
$$

By Proposition 2.43,

$$
P_{n,d}^{(j)}(t) = c_0 (1-t^2)^{-\frac{j}{2}} P_{n,d,j}(t), \qquad c_0 = \frac{(n+j+d-3)!}{(n+d-3)!}.
$$

Then,

$$
\begin{aligned}
P_{n,d}^{(j+1)}(t) &= c_0 (1-t^2)^{-\frac{j}{2}} \left[P'_{n,d,j}(t) + j\, t\, (1-t^2)^{-1} P_{n,d,j}(t)\right], \\
P_{n,d}^{(j+2)}(t) &= c_0 (1-t^2)^{-\frac{j}{2}-1} \big[(1-t^2)\, P''_{n,d,j}(t) + 2\, j\, t\, P'_{n,d,j}(t) \\
&\qquad + j\left((j+2)(1-t^2)^{-1} - (j+1)\right) P_{n,d,j}(t)\big].
\end{aligned}
$$

Substitute these expressions in (2.155) and rearrange the terms to get the differential equation

$$
\begin{aligned}
&(1-t^2)\, P''_{n,d,j}(t) - (d-1)\, t\, P'_{n,d,j}(t) \\
&\qquad + \left[n\,(n+d-2) - \frac{j\,(j+d-3)}{1-t^2}\right] P_{n,d,j}(t) = 0.
\end{aligned} \tag{2.156}
$$

Taking $j = 0$ in (2.156), we recover the differential equation (2.83) for the Legendre polynomials $P_{n,d}(t) = P_{n,d,0}(t)$.

We now use the differential equation (2.156) to prove the following orthogonality property.

Proposition 2.44.

$$
\int_{-1}^{1} P_{m,d,j}(t)\, P_{n,d,j}(t)\, (1-t^2)^{\frac{d-3}{2}}\, dt = 0, \quad m \neq n. \tag{2.157}
$$

Proof. We rewrite (2.156) in the form

$$
\begin{aligned}
&(1-t^2)^{-\frac{d-3}{2}} \frac{d}{dt}\left[(1-t^2)^{\frac{d-1}{2}} \frac{d}{dt} P_{n,d,j}(t)\right] \\
&\qquad + \left[n\,(n+d-2) - \frac{j\,(j+d-3)}{1-t^2}\right] P_{n,d,j}(t) = 0.
\end{aligned}
$$

From this equation, we deduce that

$$P_{m,d,j}(t)\,\frac{d}{dt}\left[(1-t^2)^{\frac{d-1}{2}}\frac{d}{dt}P_{n,d,j}(t)\right]-P_{n,d,j}(t)\,\frac{d}{dt}\left[(1-t^2)^{\frac{d-1}{2}}\frac{d}{dt}P_{m,d,j}(t)\right]+(m-n)\,(m+n+d-2)\,P_{m,d,j}(t)\,P_{n,d,j}(t)\,(1-t^2)^{\frac{d-3}{2}}=0.$$

Integrate this equation for $t\in[-1,1]$ to get

$$(m-n)\,(m+n+d-2)\int_{-1}^{1}P_{m,d,j}(t)\,P_{n,d,j}(t)\,(1-t^2)^{\frac{d-3}{2}}\,dt=0.$$

Thus, (2.157) holds. □

Various recursion formulas for the associated Legendre functions exist; see [49, Sect. 3.12] in the case $d=3$. The recursion formulas are useful for pointwise evaluation of the functions.

2.10.3 Normalized Associated Legendre Functions

In explicit calculations involving the associated Legendre functions, usually it is more convenient to use the normalized ones. From the formula given in Proposition 2.42,

$$\int_{-1}^{1}\left[P_{n,d,j}(t)\right]^2(1-t^2)^{\frac{d-3}{2}}\,dt=\left[\frac{n!\,\Gamma(\frac{d-1}{2})}{2^j\,(n-j)!\,\Gamma(j+\frac{d-1}{2})}\right]^2\int_{-1}^{1}\left[P_{n-j,d+2j}(t)\right]^2(1-t^2)^{j+\frac{d-3}{2}}\,dt.$$

Use (2.79) for the integral,

$$\int_{-1}^{1}\left[P_{n-j,d+2j}(t)\right]^2(1-t^2)^{j+\frac{d-3}{2}}\,dt=\frac{|\mathbb{S}^{d+2j-1}|}{N_{n-j,d+2j}|\mathbb{S}^{d+2j-2}|}.$$

Then

$$\int_{-1}^{1}\left[P_{n,d,j}(t)\right]^2(1-t^2)^{\frac{d-3}{2}}\,dt=\frac{2^{d-2}(n!)^2\,\Gamma(\frac{d-1}{2})^2}{(2n+d-2)\,(n-j)!(n+d+j-3)!}.$$

Thus, we define normalized associated Legendre functions

$$\tilde{P}_{n,d,j}(t) = \frac{[(2n+d-2)\,(n-j)!\,(n+d+j-3)!]^{\frac{1}{2}}}{2^{\frac{d-2}{2}}\,n!\,\Gamma(\frac{d-1}{2})}\,P_{n,d,j}(t),$$
$$t \in [-1,1]. \tag{2.158}$$

We can also write, with the help of Proposition 2.43,

$$\tilde{P}_{n,d,j}(t) = \frac{(n+d-3)!}{n!\,\Gamma(\frac{d-1}{2})}\left[\frac{(2n+d-2)\,(n-j)!}{2^{d-2}(n+d+j-3)!}\right]^{\frac{1}{2}}(1-t^2)^{\frac{j}{2}}P_{n,d}^{(j)}(t),$$
$$t \in [-1,1]. \tag{2.159}$$

These functions are normalized:

$$\int_{-1}^{1}\left[\tilde{P}_{n,d,j}(t)\right]^2(1-t^2)^{\frac{d-3}{2}}dt = 1.$$

Moreover, note that $\tilde{P}_{n,d,j}(t)$ is proportional to $P_{n,d,j}(t)$. Hence, these functions are orthonormal:

$$\int_{-1}^{1}\tilde{P}_{n,d,j}(t)\,\tilde{P}_{m,d,j}(t)\,(1-t^2)^{\frac{d-3}{2}}dt = \delta_{nm}. \tag{2.160}$$

In the case $d=3$,

$$\tilde{P}_{n,3,j}(t) = \left[\frac{(n+\frac{1}{2})\,(n-j)!}{(n+j)!}\right]^{\frac{1}{2}}(1-t^2)^{\frac{j}{2}}P_{n,3}^{(j)}(t). \tag{2.161}$$

In the case $j=0$, $\tilde{P}_{n,d,0}(t)$ is proportional to the Legendre polynomial $P_{n,d}(t)$,

$$\tilde{P}_{n,d,0}(t) = \frac{1}{\Gamma(\frac{d-1}{2})}\left[\frac{(2n+d-2)\,(n+d-3)!}{2^{d-2}n!}\right]^{\frac{1}{2}}P_{n,d}(t).$$

2.11 Generating Orthonormalized Bases for Spherical Harmonic Spaces

We now discuss a procedure to generate an orthonormal basis in $\mathbb{Y}_n^d$ from orthonormal bases in $(d-1)$ dimensions, by making use of the associated Legendre functions introduced in Sect. 2.10.

Let $d \ge 3$. Consider a vector $\boldsymbol{\zeta} = \boldsymbol{\zeta}_{(d)} \in \mathbb{C}^d$ of the form $\boldsymbol{\zeta}_{(d)} = \boldsymbol{e}_d + i\left(\boldsymbol{\eta}^T, 0\right)^T$ with $\boldsymbol{\eta} \in \mathbb{S}^{d-2}$. A simple calculation shows $\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = 0$ and hence $\Delta_{\boldsymbol{x}}(\boldsymbol{\zeta}\cdot\boldsymbol{x})^n = 0$. So the function $\boldsymbol{x} \mapsto (\boldsymbol{\zeta}\cdot\boldsymbol{x})^n = (x_d + i\boldsymbol{x}_{(d-1)}\cdot\boldsymbol{\eta})^n$ is homogeneous and harmonic. Then

$$f(\boldsymbol{x}) := \frac{i^{-j}}{|\mathbb{S}^{d-2}|}\int_{\mathbb{S}^{d-2}} (x_d + i\boldsymbol{x}_{(d-1)}\cdot\boldsymbol{\eta})^n Y_{j,d-1}(\boldsymbol{\eta})\, dS^{d-2}(\boldsymbol{\eta})$$

is a homogeneous harmonic polynomial of degree n, i.e., it is an element of $\mathbb{Y}_n(\mathbb{R}^d)$. Use the polar coordinates (1.15),

$$\boldsymbol{x} = |\boldsymbol{x}|\,\boldsymbol{\xi}, \quad \boldsymbol{\xi} = t\,\boldsymbol{e}_d + \sqrt{1-t^2}\,\boldsymbol{\xi}_{(d-1)}, \quad |t| \le 1,\ \boldsymbol{\xi}_{(d-1)} \in \mathbb{S}^{d-1},$$

noting that $\boldsymbol{\xi}_{(d-1)}$ denotes a d-dimensional vector $(\xi_1, \cdots, \xi_{d-1}, 0)^T$. The restriction of the function $f(\boldsymbol{x})$ to $\mathbb{S}^{d-1}$ is

$$f(\boldsymbol{\xi}) = \frac{i^{-j}}{|\mathbb{S}^{d-2}|}\int_{\mathbb{S}^{d-2}} (t + i\,(1-t^2)^{\frac{1}{2}}\boldsymbol{\xi}_{(d-1)}\cdot\boldsymbol{\eta})^n Y_{j,d-1}(\boldsymbol{\eta})\, dS^{d-2}(\boldsymbol{\eta}).$$

Applying the Funk–Hecke formula (Theorem 2.22), we have

$$\int_{\mathbb{S}^{d-2}} (t + i\,(1-t^2)^{\frac{1}{2}}\boldsymbol{\xi}_{(d-1)}\cdot\boldsymbol{\eta})^n Y_{j,d-1}(\boldsymbol{\eta})\, dS^{d-2}(\boldsymbol{\eta}) = \lambda\, Y_{j,d-1}(\boldsymbol{\xi}_{(d-1)}),$$

where

$$\lambda = |\mathbb{S}^{d-3}| \int_{-1}^{1} P_{j,d-1}(s)\left(t + i\,(1-t^2)^{\frac{1}{2}} s\right)^j (1-t^2)^{\frac{d-4}{2}}\, dt.$$

Thus,

$$f(\boldsymbol{\xi}) = P_{n,d,j}(t)\, Y_{j,d-1}(\boldsymbol{\xi}_{(d-1)})$$

is a spherical harmonic of order n in dimension d. So we have shown the following result.

Proposition 2.45. *If $Y_{j,d-1} \in \mathbb{Y}_j^{d-1}$, then $P_{n,d,j}(t)Y_{j,d-1}(\boldsymbol{\xi}_{(d-1)}) \in \mathbb{Y}_n^d$ in polar coordinates* (1.15)*.*

This result allows us to construct a basis for $\mathbb{Y}_n^d$ in d dimensions in terms of bases in $\mathbb{Y}_0^{d-1}, \ldots, \mathbb{Y}_n^{d-1}$ in $(d-1)$ dimensions. In the following we use the normalized associated functions $\tilde{P}_{n,d,j}$ since most formulas will then have a simpler form.

Definition 2.46. For $d \ge 3$ and $m \le n$, define an operator

$$\tilde{\mathcal{P}}_{n,m} : \mathbb{Y}_m^{d-1} \to \mathbb{Y}_n^d$$

by the formula

$$(\tilde{\mathcal{P}}_{n,m}Y_{m,d-1})(\boldsymbol{\xi}) = \tilde{P}_{n,d,m}(t)Y_{m,d-1}(\boldsymbol{\xi}_{(d-1)}), \quad Y_{m,d-1} \in \mathbb{Y}_m^{d-1}.$$

Then define $\mathbb{Y}_{n,m}^d := \tilde{\mathcal{P}}_{n,m}(\mathbb{Y}_m^{d-1})$, called the associated space of order m in $\mathbb{Y}_n^d$.

The spherical harmonic space $\mathbb{Y}_n^d$ can be decomposed as an orthogonal sum of the associated spaces $\mathbb{Y}_{n,m}^d$, $0 \le m \le n$.

Theorem 2.47. *For $d \ge 3$ and $n \ge 0$,*

$$\mathbb{Y}_n^d = \mathbb{Y}_{n,0}^d \oplus \cdots \oplus \mathbb{Y}_{n,n}^d. \tag{2.162}$$

Proof. First we show that the subspaces on the right side of (2.162) are pairwise orthogonal. Let $0 \le k, m \le n$ with $k \ne m$. For any $Y_{k,d-1} \in \mathbb{Y}_k^{d-1}$ and any $Y_{m,d-1} \in \mathbb{Y}_m^{d-1}$,

$$\begin{aligned}
&(\tilde{\mathcal{P}}_{n,k}Y_{k,d-1}, \tilde{\mathcal{P}}_{n,m}Y_{m,d-1})_{L^2(\mathbb{S}^{d-1})} \\
&\quad = (Y_{k,d-1}, Y_{m,d-1})_{L^2(\mathbb{S}^{d-2})} \int_{-1}^{1} \tilde{P}_{n,d,k}(t)\, \tilde{P}_{n,d,m}(t)\, (1-t^2)^{\frac{d-3}{2}} dt \\
&\quad = 0
\end{aligned}$$

using the orthogonality (2.160). Thus, $\mathbb{Y}_{n,k}^d \perp \mathbb{Y}_{n,m}^d$ for $k \ne m$.

For each m, $0 \le m \le n$, $\mathbb{Y}_{n,m}^d$ is a subspace of $\mathbb{Y}_n^d$ and so

$$\mathbb{Y}_n^d \supset \mathbb{Y}_{n,0}^d \oplus \cdots \oplus \mathbb{Y}_{n,n}^d. \tag{2.163}$$

Since the mapping $\tilde{\mathcal{P}}_{n,m} : \mathbb{Y}_m^{d-1} \to \mathbb{Y}_{n,m}^d$ is a bijection,

$$\dim \mathbb{Y}_{n,m}^d = \dim \mathbb{Y}_m^{d-1} = N_{m,d-1}.$$

Hence, recalling the identity (2.14),

$$\sum_{m=0}^{n} \dim \mathbb{Y}_{n,m}^d = \sum_{m=0}^{n} N_{m,d-1} = N_{n,d} = \dim \mathbb{Y}_n^d.$$

In other words, the two sides of equality (2.162) are finite-dimensional spaces of equal dimension. Then the equality (2.162) holds in view of the relation (2.163). $\square$

From Theorem 2.47 and its proof, we see that if

$$\{Y_{m,d-1,j} : 1 \le j \le N_{m,d-1}\}$$

is an orthonormal basis for $\mathbb{Y}_m^{d-1}$, $0 \le m \le n$, then

$$\left\{\tilde{P}_{n,d,m}(t) Y_{m,d-1,j}(\boldsymbol{\xi}_{(d-1)}) : 1 \le j \le N_{m,d-1},\ 0 \le m \le n\right\} \tag{2.164}$$

is an orthonormal basis for $\mathbb{Y}_n^d$.

Example 2.48. An orthonormal basis for $\mathbb{Y}_n^2$ is presented in Sect. 2.2. Let us apply the above result and use the orthonormal basis for $\mathbb{Y}_n^2$ to construct an orthonormal basis for $\mathbb{Y}_n^3$. We use the relation

$$\boldsymbol{\xi}_{(3)} = t\,\boldsymbol{e}_3 + \sqrt{1-t^2}\begin{pmatrix}\boldsymbol{\xi}_{(2)}\\ 0\end{pmatrix},$$

where $t = \cos\theta$ for $0 \le \theta \le \pi$, $\boldsymbol{\xi}_{(2)} = (\cos\phi, \sin\phi)^T$ for $0 \le \phi \le 2\pi$. In the notation of the above discussion,

$$\left\{Y_{m,2,1}(\boldsymbol{\xi}_{(2)}) = \frac{1}{\sqrt{\pi}}\cos(m\phi),\ Y_{m,2,2}(\boldsymbol{\xi}_{(2)}) = \frac{1}{\sqrt{\pi}}\sin(m\phi)\right\}$$

is an orthonormal basis for $\mathbb{Y}_m^2$. Recall the formula (2.161),

$$\tilde{P}_{n,3,m}(t) = \left[\frac{(n+\frac{1}{2})\,(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} (1-t^2)^{\frac{m}{2}} P_{n,3}^{(m)}(t).$$

Here $P_{n,3}^{(m)}(t)$ denotes the m^{th} derivative of the function $P_{n,3}(t)$. Then, an orthonormal basis for $\mathbb{Y}_n^3$ is given by the functions

$$\left[\frac{(2\,n+1)\,(n-m)!}{2\,\pi\,(n+m)!}\right]^{\frac{1}{2}} (\sin\theta)^m P_{n,3}^{(m)}(\cos\theta)\,\cos(m\phi), \quad 0 \le m \le n,$$

$$\left[\frac{(2\,n+1)\,(n-m)!}{2\,\pi\,(n+m)!}\right]^{\frac{1}{2}} (\sin\theta)^m P_{n,3}^{(m)}(\cos\theta)\,\sin(m\phi), \quad 1 \le m \le n.$$

The basis is usually also written as

$$(-1)^{(m+|m|)/2}\left[\frac{(2\,n+1)\,(n-|m|)!}{4\,\pi\,(n+|m|)!}\right]^{\frac{1}{2}} (\sin\theta)^m P_{n,3}^{(m)}(\cos\theta)\,e^{im\phi},$$
$$-\,n \le m \le n.$$

This latter form is more convenient to use in some calculations. □

We now use the orthonormal system (2.164) to express the addition theorem. Set

$$\boldsymbol{\xi}_{(d)} = t\,\boldsymbol{e}_d + (1-t^2)^{\frac{1}{2}}\boldsymbol{\xi}_{(d-1)}, \quad -1 \le t \le 1,$$
$$\boldsymbol{\eta}_{(d)} = s\,\boldsymbol{e}_d + (1-s^2)^{\frac{1}{2}}\boldsymbol{\eta}_{(d-1)}, \quad -1 \le s \le 1.$$

Then the identity (2.24)

$$\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\,P_{n,d}(\boldsymbol{\xi}\cdot\boldsymbol{\eta}) = \sum_{k=1}^{N_{n,d}} Y_{n,k}(\boldsymbol{\xi})\,\overline{Y_{n,k}(\boldsymbol{\eta})}$$

is rewritten as

$$\begin{aligned}
&\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\,P_{n,d}(s\,t + (1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}\boldsymbol{\xi}_{(d-1)}\cdot\boldsymbol{\eta}_{(d-1)}) \\
&\quad = \sum_{m=0}^{n} \tilde{P}_{n,d,m}(s)\tilde{P}_{n,d,m}(t) \sum_{k=1}^{N_{n,d-1}} Y_{m,k}(\boldsymbol{\xi}_{(d-1)})\,\overline{Y_{m,k}(\boldsymbol{\eta}_{(d-1)})} \\
&\quad = \sum_{m=0}^{n} \frac{N_{m,d-1}}{|\mathbb{S}^{d-2}|}\,\tilde{P}_{n,d,m}(s)\tilde{P}_{n,d,m}(t)P_{m,d-1}(\boldsymbol{\xi}_{(d-1)}\cdot\boldsymbol{\eta}_{(d-1)}),
\end{aligned}$$

where in the last step, the identity (2.24) is applied again. Denote $u = \boldsymbol{\xi}_{(d-1)}\cdot\boldsymbol{\eta}_{(d-1)}$. Then for $d \ge 3$ and $s, t, u \in [-1, 1]$,

$$\begin{aligned}
&\sum_{m=0}^{n} N_{m,d-1}\tilde{P}_{n,d,m}(s)\tilde{P}_{n,d,m}(t)P_{m,d-1}(u) \\
&\qquad = \frac{N_{n,d}|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\,P_{n,d}(s\,t + (1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}u).
\end{aligned} \tag{2.165}$$

Another identity can be derived from (2.165) as follows. Multiply both sides of (2.165) by $P_{k,d-1}(u)\,(1-u^2)^{\frac{d-4}{2}}$, $0 \le k \le n$, integrate with respect to u from -1 to 1, and use the orthogonality relation (2.79) for the Legendre polynomials,

$$\begin{aligned}
&\frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\int_{-1}^{1} P_{n,d}(s\,t + (1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}u)P_{k,d-1}(u)\,(1-u^2)^{\frac{d-4}{2}}\,du \\
&\quad = \frac{1}{|\mathbb{S}^{d-3}|}\tilde{P}_{n,d,k}(s)\tilde{P}_{n,d,k}(t),
\end{aligned}$$

i.e.,

$$\int_{-1}^{1} P_{n,d}(s\,t + u\,(1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}})P_{k,d-1}(u)\,(1-u^2)^{\frac{d-4}{2}}\,du$$
$$= \frac{2\pi}{(d-2)\,N_{n,d}}\tilde{P}_{n,d,k}(s)\tilde{P}_{n,d,k}(t). \tag{2.166}$$

In particular, taking $k = 0$ in (2.166) and noting that

$$\tilde{P}_{n,d,0}(t) = \left(\frac{N_{n,d}|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\right)^{\frac{1}{2}} P_{n,d}(t),$$

we arrive at an identity for the Legendre polynomials,

$$\int_{-1}^{1} P_{n,d}(s\,t + (1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}u)(1-u^2)^{\frac{d-4}{2}}\,du = \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}P_{n,d}(s)P_{n,d}(t) \tag{2.167}$$

for $d \geq 3$.

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