Applications of Boundary Harnack Inequalities for p Harmonic Functions and Related Topics

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1 Outline of the Course

This course will be concerned with applications of recent work—techniques concerning the boundary behavior of positive p harmonic functions vanishing on a portion of the boundary of Lipschitz, chord arc, and Reifenberg flat domains. An optimistic outline follows:

1. Fundamental properties of p harmonic functions and elliptic measure.
2. The dimension of p harmonic measure.
3. Boundary Harnack inequalities and the Martin boundary problem in Reifenberg flat and Lipschitz domains.
4. Uniqueness and regularity in free boundary—inverse type problems.

The lectures concerning 2 will be drawn from [6, 8, 48, 63]. Lectures involving 3 will be based on [49, 50, 52, 62]. Lectures on 4 will be concerned with [55–60] and [49–54].

1.1 Ode to the p Laplacian

I used to be in love with the Laplacian so worked hard to please her with beautiful theorems. However she often scorned me for the likes of Albert Baernstein, Björn Dahlberg, Carlos Kenig, and Thomas Wolff. Gradually I became interested in her sister the p Laplacian, $1 < p < \infty, p \neq 2$. I did not find her as pretty as the Laplacian and she was often difficult to handle

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because of her nonlinearity. However over many years I took a shine to her and eventually developed an understanding of her disposition. Today she is my girl and the Laplacian pales in comparison to her.

1.2 My Introduction to p Harmonic Functions

I was trained in function theory—subharmonic functions by my advisor, Maurice Heins and postdoctoral advisor Matts Essén. My first paper on elliptic PDE and p harmonic functions (see [47]) was entitled 'Capacitary Functions in Convex Rings.' The catalyst for this paper was a problem in [31] which read as follows:

'If D is a convex domain in space of 3 or more dimensions can we assert any inequalities for the Green's function $g(P,Q)$ which generalize the results for two dimensions, that follow from the classical inequalities for schlicht functions. Gabriel [29] has shown that the level surfaces of $g(P,Q) = \lambda$ are convex but his proof is long. It would be interesting to find a simpler proof.'

I tried to find a simpler proof of Gabriel's result but failed so eventually read his paper. In contrast to the author of the problem, I found Gabriel's proof easy to follow and quite ingenious. Thus instead of finding a different proof I found a different PDE, the p Laplacian, to use Gabriel's technique on. Moreover in writing the above paper I was forced to learn some classical PDE techniques (Möser iteration, Schauder techniques) in order to deal with this degenerate nonlinear divergence form elliptic PDE.

2 Basic Estimates for the p Laplacian

We shall be working in Euclidean n space $\mathbf{R}^n$. Points will be denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $\bar{E}, \partial E$, be the usual closure, boundary, of the set E and $d(y,E)$ = the distance from y to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and $|x| = \langle x, x\rangle^{1/2}$ is the Euclidean norm of x. $B(x,r) = \{y \in \mathbf{R}^n : |x-y| < r\}$ and dx denotes Lebesgue n measure on $\mathbf{R}^n$. Let e_i be the i unit coordinate vector. If $O \subset \mathbf{R}^n$ is open and $1 \leq q \leq \infty$, let $W^{1,q}(O)$, denote the usual Sobolev space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are qth power integrable on O with Sobolev norm, $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$. Next let $C_0^\infty(O)$ be infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$.

Given G a bounded domain (i.e., a connected open set) and $1 < p < \infty$, we say that u is p harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1}$$

whenever $\theta \in W_0^{1,p}(G)$. Observe that if u is smooth and $\nabla u \neq 0$ in G, then

$$\nabla \cdot (|\nabla u|^{p-2} \, \nabla u) \equiv 0 \text{ in } G \tag{2}$$

so u is a classical solution in G to the p Laplace partial differential equation. Equation (1) arises in the study of the following Dirichlet problem: Given $g \in W^{1,p}(\mathbf{R}^n)$ let $\mathcal{F} = \{h : h - g \in W_0^{1,p}(G)\}$. Find

$$\inf_{h \in \mathcal{F}} \int_G |\nabla h|^p \, dx. \tag{3}$$

it is well known that the infimum in (3) occurs for a unique function u with $u - g \in W_0^{1,p}(G)$. Moreover u satisfies (1) as follows from the fact that u is a minimum and the usual calculus of variations type argument.

v is said to be subpharmonic (superpharmonic) in G if $v \in W^{1,p}(G)$ and whenever $\theta \geq 0 \in W_0^{1,p}(G)$,

$$\int |\nabla v|^{p-2} \langle \nabla v, \nabla \theta \rangle \, dx \leq 0 \, (\geq 0) \tag{4}$$

Lemma 1.1. *(Boundary Maximum Principle) If v is subpharmonic in G, while w is superpharmonic in G with* $\min\{v-w, 0\} \in W_0^{1,p}(G)$, *then $v - w \leq 0$ a.e in G.*

Lemma 1.2. *(Interior Estimates for u) Given $p, 1 < p < \infty$, let u be a positive p harmonic function in $B(w, 2r)$. Then*

(i) Caccioppoli Inequality:

$$r^{p-n} \int\limits_{B(w,r/2)} |\nabla u|^p \, dx \, \leq \, c \, (\max_{B(w,r)} u)^p,$$

(ii) Harnack's Inequality:

$$\max_{B(w,r)} \, u \leq c \, \min_{B(w,r)} u.$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $x, y \in B(w, r/2)$ then

(iii) Hölder Continuity:

$$|u(x) - u(y)| \leq c \left(\tfrac{|x-y|}{r} \right)^{\alpha} \max_{B(w,r)} u.$$

Lemma 1.3. *(Interior Estimates for ∇u) Let $1 < p < \infty$ and suppose u is p harmonic in $B(w, 2r)$. Then u has a representative in $W^{1,p}(B(w, 2r))$ with Hölder continuous partial derivatives in $B(w, 2r)$. In particular there exists $\sigma \in (0, 1]$, depending only on p, n, such that if $x, y \in B(w, r/2)$, then for some $c = c(p, n)$,*

$$
\begin{aligned}
c^{-1} |\nabla u(x) - \nabla u(y)| &\leq (|x - y|/r)^{\sigma} \max_{B(w,r)} |\nabla u| \\
&\leq c\, r^{-1} (|x - y|/r)^{\sigma} \max_{B(w,2r)} |u|.
\end{aligned}
$$

Also if $\nabla u(x) \neq 0$, then u is C^{∞} near x.

For a proof of Lemmas 1.1 and 1.2, see [69]. Numerous proofs have been given of Hölder continuity of ∇u in Lemma 1.3. Perhaps the first was due to Ural'tseva for $p > 2$ while DiBenedetto, myself, and Tolksdorff all gave proofs independently and nearly at the same time (1983, 1984) for $1 < p < \infty$. A proof which even applies to the parabolic p Laplacian and systems can be found in [19].

If $p > 2$ it is known that u (as above) need not be C^2 locally. For $1 < p < 2$ this question is still open in $\mathbf{R}^n, n \geq 3$. In two dimensions, Iwaniec and Manfredi [36] showed that solutions are C^k where $k = k(p) \geq 2$ when $1 < p < 2$ and $k \to \infty$ as $p \to 1$.

2.1 p Harmonic Functions in NTA Domains

Definition A. *A domain Ω is called non tangentially accessible (NTA), if there exist $M \geq 2$ and $0 < r_0 \leq \infty$ such that the following are fulfilled,*

(i) Corkscrew condition: For any $w \in \partial\Omega, 0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying $M^{-1} r < |a_r(w) - w| < r, d(a_r(w), \partial\Omega) > M^{-1} r$,
(ii) $\mathbf{R}^n \setminus \bar{\Omega}$ satisfies the corkscrew condition,
(iii) Uniform condition: If $w \in \partial\Omega$, and $w_1, w_2 \in B(w, r_0) \cap \Omega$, then there is a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and

(a) $H^1(\gamma) \leq M |w_1 - w_2|$,
(b) $\min\{H^1(\gamma([0, t])),\, H^1(\gamma([t, 1]))\} \leq M\, d(\gamma(t), \partial\Omega)$.

In Definition A, H^1 denotes length or Hausdorff one measure. Often in our applications Ω will at least be an NTA domain with constants M, r_0 while p is fixed, $1 < p < \infty$. Also, $c \geq 1$ will be a positive constant which may only depend on p, n, M unless otherwise stated. Let $w \in \partial\Omega, 0 < r < r_0$, and suppose $u > 0$ is p harmonic in $\Omega \cap B(w, 2r)$ with $u = 0$ on $\partial\Omega \cap B(w, 2r)$ in the usual Sobolev sense. We extend u to $B(w, 2r)$ by putting $u = 0$ on $B(w, 2r) \setminus \Omega$. Under this scenario we state

Lemma 1.4. *Let u, p, w, Ω be as above. Then $u \in W^{1,p}(B(w,2r))$ and*

$$r^{p-n} \int_{B(w,r/2)} |\nabla u|^p \, dx \le c \, (\max_{B(w,r)} u)^p.$$

Moreover there exists $\beta = \beta(p,n,M) \in (0,1)$ such that u has a Hölder continuous representative in $B(w,2r)$ with

$$|u(x) - u(y)| \le c \, (|x-y|/r)^\beta \max_{B(w,r)} u$$

whenever $x, y \in B(w,r)$.

Lemma 1.5. *Let $u, p, w, \Omega, r,$ be as in Lemma 1.4. There exists c such that if $\hat{r} = r/c$, then*

$$\max_{B(w,\hat{r})} u \le c u(a_{\hat{r}}(w)).$$

2.1.1 Outline of Proofs

The first display in Lemma 1.4 is a standard subsolution type inequality (use u times a cutoff as a test function in (1)). As for the last display in Lemma 1.4 if $p > n$, this display is a consequence of Morrey's Theorem and the first display. If $1 < p \le n$, then from the interior estimates in Lemma 1.2, we deduce that it suffices to consider only the case when $y \in \partial\Omega \cap B(w,r)$. One then shows for some $\theta = \theta(p,n,M), 0 < \theta < 1$, that

$$\max_{B(z,\rho/2)} u \le \theta \max_{B(z,\rho)} u \tag{5}$$

whenever $0 < \rho < r/4$ and $z \in \partial\Omega \cap B(w,r)$. Equation (5) can then be iterated to get Hölder continuity in Lemma 1.4 for x, y as above. To prove (5) one uses the fact that $(\mathbf{R}^n \setminus \Omega) \cap B(z,\rho/2)$ and $B(z,\rho/2)$ have comparable p capacities, as well as estimates for subsolutions to elliptic partial differential equations of p Laplacian type. These estimates are due to [68] for the p Laplacian (see also [30]). Lemma 1.5 for harmonic functions is often called Carleson's lemma although apparently it could be due to Domar. This lemma for uniformly elliptic PDE in divergence form is usually attributed to [14]. All proofs use only Harnack's inequality and Hólder continuity near the boundary. Thus Lemma 1.5 is also valid for solutions to many PDE's including the p Laplacian. $\square$

In our study of p harmonic measure we shall outline a proof of a similar inequality when the geometry is considerably more complicated. That is when $\Omega \subset \mathbf{R}^2$ is only a bounded simply connected domain.

2.2 The p Laplacian and Elliptic PDE

Let u be a solution to the p Laplace equation in (2) and suppose ∇u is nonzero as well as sufficiently smooth in a neighborhood of $x \in \Omega$. Let $\eta \in \mathbf{R}^n$ with $|\eta| = 1$ and put $\zeta = \langle \nabla u, \eta \rangle$. Then differentiating $\nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = 0$ with respect to η one gets that ζ is a strong solution at x to

$$L\zeta \; = \; \nabla \cdot [(p-2)|\nabla u|^{p-4} \langle \nabla u, \nabla \zeta \rangle \, \nabla u + |\nabla u|^{p-2} \nabla \zeta] = 0. \tag{6}$$

Clearly, (7)

$$Lu \; = \; (p-1) \, \nabla \cdot \left[\, |\nabla u|^{p-2} \, \nabla u \, \right] = 0 \; \text{ at } x. \tag{7}$$

Equation (6) can be rewritten in the form

$$L\zeta \; = \; \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[\, b_{ij}(x) \zeta_{x_j}(x) \, \right] = 0, \tag{8}$$

$$\text{where } b_{ij}(x) = |\nabla u|^{p-4} [(p-2) u_{x_i} u_{x_j} + \delta_{ij} |\nabla u|^2](x), \tag{9}$$

for $1 \leq i, j \leq n$, and δ_{ij} is the Kronecker δ. In many of our applications it is of fundamental importance that u, derivatives of u, both satisfy the same divergence form PDE in (6), (7). For example, in several of our papers we integrate by parts functions of $u, \nabla u$ and the bad terms drop out because both functions satisfy the same PDE. Thus we study (8), (9). We note that if $\xi \in \mathbf{R}^n$, then

$$\begin{aligned} \min\{p-1, 1\} |\xi|^2 \, |\nabla u(x)|^{p-2} \; &\leq \; \sum_{i,k=1}^{n} b_{ik} \, \xi_i \xi_k \\ &\leq \; \max\{1, p-1\} |\nabla u(x)|^{p-2} \, |\xi|^2 \, . \end{aligned} \tag{10}$$

Observe from (10) that L can be degenerate elliptic if $\nabla u = 0$. Thus in many of our papers we also prove the fundamental inequality:

$$c^{-1} u(x)/d(x, \partial\Omega) \leq |\nabla u(x)| \leq c u(x)/d(x, \partial\Omega), \tag{11}$$

for some constant c and x near $\partial\Omega$. Note that (10), (11), and Harnack's inequality for u imply that $(b_{ik}(x))$ are locally uniformly elliptic in Ω.

Behaviour near the boundary, such as boundary Harnack inequalities, are more involved. The easiest case for our methods to work is when $\partial\Omega$ is sufficiently flat in the sense of Reifenberg (to be defined later). In this case we will be able to show that $|\nabla u|^{p-2}$ is an A_2 weight (also to be defined). Thus we list some theorems on degenerate elliptic equations whose degeneracy is given in terms of an A_2 weight.

2.3 Degenerate Elliptic Equations

Let $w \in \mathbf{R}^n$, $0 < r$ and let λ be a real valued Lebesgue measurable function defined almost everywhere on $B(w, 2r)$. λ is said to belong to the class $A_2(B(w,r))$ if there exists a constant γ such that

$$\tilde{r}^{-2n} \int\limits_{B(\tilde{w},\tilde{r})} \lambda \, dx \cdot \int\limits_{B(\tilde{w},\tilde{r})} \lambda^{-1} dx \leq \gamma$$

whenever $\tilde{w} \in B(w,r)$ and $0 < \tilde{r} \leq r$. If $\lambda(x)$ belongs to the class $A_2(B(w,r))$ then λ is referred to as an $A_2(B(w,r))$-weight. The smallest γ such that the above display holds is referred to as the constant of the weight.

Once again let $\Omega \subset \mathbf{R}^n$ be a NTA domain with NTA-constants M, r_0. Let $w \in \partial\Omega$, $0 < r < r_0$, and consider the operator

$$\hat{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\hat{b}_{ij}(x) \frac{\partial}{\partial x_j}\right) \tag{12}$$

in $\Omega \cap B(w, 2r)$. We assume that the coefficients $\{\hat{b}_{ij}(x)\}$ are bounded, Lebesgue measurable functions defined almost everywhere on $B(w, 2r)$. Moreover, $\hat{b}_{ij} = \hat{b}_{ji}$ for all $i, j \in \{1, .., n\}$, and

$$c^{-1}\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} \hat{b}_{ij}(x)\xi_i\xi_j \leq c|\xi|^2\lambda(x) \tag{13}$$

for almost every $x \in B(w,r)$, where $\lambda \in A_2(B(w,r))$. If $O \subset B(w, 2r)$ is open let $\tilde{W}^{1,2}(O)$ be the weighted Sobolev space of equivalence classes of functions v with distributional gradient ∇v and norm

$$\|v\tilde{\|}_{1,2}^2 = \int\limits_O v^2 \lambda dx + \int\limits_O |\nabla v|^2 \lambda dx < \infty.$$

Let $\tilde{W}_0^{1,2}(O)$ be the closure of $C_0^\infty(O)$ in the norm $\tilde{W}^{1,2}(O)$. We say that v is a weak solution to $\hat{L}v = 0$ in O provided $v \in \tilde{W}^{1,2}(O)$ and

$$\int\limits_O \sum_{i,j} \hat{b}_{ij} v_{x_i} \phi_{x_j} dx = 0 \tag{14}$$

whenever $\phi \in C_0^\infty(O)$.

The following three lemmas, Lemmas 1.6–1.8, are tailored to our situation and based on the results in [21–23].

Lemma 1.6. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant γ. Suppose that v is a positive weak solution to $\hat{L}v = 0$ in $\Omega \cap B(w,2r)$. Then there exists a constant c, $1 \le c < \infty$, depending only on n, M and γ, such that if $\tilde{w} \in \Omega$ and $B(\tilde{w}, 2\tilde{r}) \subset \Omega \cap B(w,r)$, then*

(i)

$$c^{-1}\tilde{r}^2 \int\limits_{B(\tilde{w},\tilde{r})} |\nabla v|^2 \lambda dx \le c \int\limits_{B(\tilde{w},2\tilde{r})} |v|^2 \lambda dx,$$

(ii)

$$\max_{B(\tilde{w},\tilde{r})} v \le c \min_{B(\tilde{w},\tilde{r})} v.$$

Furthermore, there exists $\alpha = \alpha(n, M, \gamma) \in (0,1)$ such that if $x, y \in B(\tilde{w}, \tilde{r})$ then

(iii)

$$|v(x) - v(y)| \le c \left(\frac{|x-y|}{\tilde{r}} \right)^{\alpha} \max_{B(\tilde{w},2\tilde{r})} v.$$

Lemma 1.7. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant γ. Suppose that v is a positive weak solution to $\hat{L}v = 0$ in $\Omega \cap B(w,2r)$ and that $v = 0$ on $\partial\Omega \cap B(w,2r)$ in the weighted Sobolev sense. Extend v to $B(w,2r)$ by setting $v \equiv 0$ in $B(w,2r) \setminus \bar{\Omega}$. Then $v \in \tilde{W}^{1,2}(B(w,2r))$ and there exists $\tilde{c} = \tilde{c}(n, M, \gamma)$, $1 \le \tilde{c} < \infty$, such that the following holds with $\tilde{r} = r/\tilde{c}$.*

(i)

$$r^2 \int\limits_{\Omega \cap B(w,r/2)} |\nabla v|^2 \lambda dx \le \tilde{c} \int\limits_{\Omega \cap B(w,r)} |v|^2 \lambda dx,$$

(ii)

$$\max_{\Omega \cap B(w,\tilde{r})} v \le \tilde{c} v(a_{\tilde{r}}(w)).$$

Furthermore, there exists $\alpha = \alpha(n, M, \gamma) \in (0,1)$ such that if $x, y \in \Omega \cap B(w, \tilde{r})$, then

(iii)

$$|v(x) - v(y)| \le c \left(\frac{|x-y|}{r} \right)^{\alpha} \max_{\Omega \cap B(w,2\tilde{r})} v.$$

Lemma 1.8. *Let $\Omega \subset \mathbf{R}^n$ be a NTA-domain with constant M, $w \in \partial\Omega$, $0 < r < r_0$, and let λ be an $A_2(B(w,r))$-weight with constant γ. Suppose that v_1 and v_2 are two positive weak solutions to $\hat{L}v = 0$ in $\Omega \cap B(w,2r)$ and $v_1 = 0 = v_2$ on $\partial\Omega \cap B(w,2r)$ in the weighted Sobolev sense. Then there exist*

$c = c(n, M, \gamma)$, $1 \le c < \infty$, *and* $\alpha = \alpha(n, M, \gamma) \in (0,1)$ *such that if* $\tilde{r} = r/c$, *and* $y_1, y_2 \in \Omega \cap B(w, r/c)$, *then*

$$\left| \frac{v_1(y_1)}{v_2(y_1)} - \frac{v_1(y_2)}{v_2(y_2)} \right| \le c \frac{v_1(y_1)}{v_2(y_1)} \left(\frac{|y_1 - y_2|}{r} \right)^{\alpha}.$$

Note: The last display implies v_1/v_2 is Hölder continuous, as well as bounded above and below by its value at any one point in $\Omega \cap B(w, r/c)$. We refer to the last display as a boundary Harnack inequality.

3 p Harmonic Measure

If $\gamma > 0$ is a positive function on $(0, r_0)$ with $\lim_{r \to 0} \gamma(r) = 0$ define H^γ Hausdorff measure on $\mathbf{R}^n$ as follows: For fixed $0 < \delta < r_0$ and $E \subseteq \mathbf{R}^2$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, ...$ Set

$$\phi_\delta^\gamma(E) = \inf_{L(\delta)} \sum \gamma(r_i).$$

Then

$$H^\gamma(E) = \lim_{\delta \to 0} \phi_\delta^\gamma(E).$$

In case $\gamma(r) = r^k$ we write H^k for H^γ.

Next let $\Omega \subset \mathbf{R}^n, n \ge 2$, be a bounded domain, p fixed, $1 < p < \infty$, and N an open neighborhood of $\partial\Omega$. Let v be p harmonic in $\Omega \cap N$ and suppose that v is positive on $\Omega \cap N$ with boundary value zero on $\partial\Omega$, in the $W^{1,p}$ Sobolev sense. Extend v to a function in $W^{1,p}(N)$ by setting $v \equiv 0$ on $N \setminus \Omega$.

Then there exists (see [33]) a unique positive Borel measure ν on $\mathbf{R}^n$ with support $\subset \partial\Omega$, for which

$$\int |\nabla v|^{p-2} \langle \nabla v, \nabla \phi \rangle \, dx = - \int \phi \, d\nu \tag{15}$$

whenever $\phi \in C_0^\infty(N)$. In fact if $\partial\Omega, |\nabla v|$, are smooth

$$d\nu = |\nabla v|^{p-1} dH^{n-1} \text{ on } \partial\Omega.$$

Existence of ν follows if one can show for $\phi \ge 0$ as above,

$$\int |\nabla v|^{p-2} \langle \nabla v, \nabla \phi \rangle \, dx \le 0. \tag{16}$$

Indeed, assuming (15) one can define a positive operator on the space of continuous functions and using basic Caccioppoli inequalities—the Riesz representation theorem, get the existence of ν. If v has continuous boundary value zero one can get (16) as follows. Let $\theta = [(\eta + \max[v-\epsilon, 0])^{\epsilon} - \eta^{\epsilon}\,]\ \phi$. Then one can show that θ may be used as a test function in (1). Doing this we deduce

$$\int_{\{v \geq \epsilon\} \cap N} [(\eta + \max[v-\epsilon, 0])^{\epsilon} - \eta^{\epsilon}]$$

$$\times\ |\nabla v|^{p-2} \langle \nabla v,\, \nabla \phi \rangle dx \,\leq\, 0.$$

Using dominated convergence, letting η and then $\epsilon \to 0$, we get (16).

If $p = 2$ and v is the Green's function with pole at $x_0 \in \Omega$, then $\nu = \omega(\cdot, x_0)$ is harmonic measure with respect to $x_0 \in \Omega$. Green's functions can be defined for the p Laplacian when $1 < p < \infty$, but are not very useful due to the nonlinearity of the p Laplacian when $p \neq 2$. Instead we often study the measure, μ, associated with a p capacitary function, say u, in $\Omega \setminus \bar{B}(x_0, r)$, where $B(x_0, 4r) \subset \Omega$. That is, u is p harmonic in $\Omega \setminus \bar{B}(x_0, r)$ with continuous boundary values, $u \equiv 1$ on $\partial B(x_0, r)$ and $u \equiv 0$ on $\partial\Omega$.

Remark 1. μ is different from the so called p harmonic measure introduced by Martio, which in fact is not a measure (see [65]).

Define the Hausdorff dimension of μ by

$$\begin{aligned} \text{H-dim}\ \mu\ = \inf\{k :\ & \text{there exists } E \text{ Borel} \subset \partial\Omega \\ & \text{with } H^k(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}. \end{aligned}$$

Remark 2. We discuss for a fixed $p, 1 < p < \infty$, what is known about H-dim μ when μ corresponds to a positive p harmonic function u in $\Omega \cap N$ with boundary value 0 in the $W^{1,p}$ Sobolev sense. It turns out that H-dim μ is independent of u as above. Thus we often refer to H-dim μ as the dimension of p harmonic measure in Ω. For $p = 2, n = 2$, and harmonic measure, Carleson [15] used ideas from ergodic theory and boundary Harnack inequalities for harmonic functions to deduce H-dim $\omega = 1$ when $\Omega \subset \mathbf{R}^2$ is a 'snowflake' type domain and H-dim $\omega \leq 1$ when $\Omega \subset \mathbf{R}^2$ is the complement of a self similar Cantor set. He was also the first to recognize the importance of

$$\int_{\partial\Omega_n} |\nabla g_n| \log |\nabla g_n|\, dH^1$$

(g_n is Green's function for Ω_n with pole at zero and (Ω_n) is an increasing sequence of domains whose union is Ω). Wolff [72] used Carleson's ideas and brilliant ideas of his own to study the dimension of harmonic measure, ω, with respect to a point in domains bounded by 'Wolff snowflakes' $\subset \mathbf{R}^3$. He constructed snowflakes for which H-dim $\omega > 2$ and snowflakes for which H-dim $\omega < 2$.

In [61] we constructed Wolff Snowflakes, for which the harmonic measures on both sides of the snowflake were of H-dim $< n-1$ and also a snowflake for which the harmonic measures on both sides were of H-dim $> n-1$. Soon after we finished this paper, Björn Bennewitz became my Ph.D. student. We began studying the dimension of the measure μ as in (15) for fixed $p, 1 < p < \infty, p \neq 2$. We tried to imitate the Carleson–Wolff construction in order to produce examples of snowflakes where we could estimate H-dim μ, when $\Omega \subset \mathbf{R}^2$ is a snowflake. To indicate the difficulties involved we note that Wolff showed Carleson's integral over $\partial\Omega_n$ can be estimated at the nth step in the construction of certain snowflakes $\subset \mathbf{R}^3$. His calculations make key use of a boundary Harnack inequality for positive harmonic functions vanishing on a portion of the boundary of a NTA domain. Thus we proved $u_1/u_2 \leq c$ on $B(z, r/2) \cap \Omega$ whenever $z \in \partial\Omega$ and $0 < r \leq r_0$. Here $u_1, u_2 > 0$ are p harmonic in $B(z, r) \cap \Omega$ and vanish continuously on $B(z, r) \cap \partial\Omega$. Using our boundary Harnack inequality, we were able to deduce that μ had a certain weak mixing property and consequently, arguing as in Carleson–Wolff, we obtained an ergodic measure $\approx \mu$ on $\partial\Omega$. Applying the ergodic theorem of Birkhoff and entropy theorem of Shannon–McMillan–Breiman it followed that

$$\lim_{r \to 0} \frac{\log \mu[B(x, r)]}{\log r} = \text{H-dim}\, \mu \ \text{ for } \mu \text{ almost every } x \in \partial\Omega. \tag{17}$$

Wolff uses Hölder continuity of the ratio and other arguments in order to make effective use of (17) in his estimates of H-dim μ. We first tried to avoid many of these estimates by a finesse type argument. However, later this argument fell through because of a calculus mistake. Finally we decided that instead of Wolff's argument we should use the divergence theorem and try to find a partial differential equation for which u is a solution and $\log |\nabla u|$ is a subsolution (supersolution) when $p > 2\,(1 < p < 2)$. We succeeded, in fact the PDE is given in (8), (9):

$$L\zeta(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (b_{ij} \zeta_{x_j})(x)$$

$$b_{ij}(x) = |\nabla u|^{p-4}[(p-2) u_{x_i} u_{x_j} + \delta_{ij} |\nabla u|^2](x), 1 \leq i, j \leq n.$$

Moreover for domains $\subset \mathbf{R}^2$ whose boundary is a quasi circle. we were able to prove the fundamental inequality in (11):

$$c^{-1} u(x)/d(x, \partial\Omega) \leq |\nabla u(x)| \leq c u(x)/d(x, \partial\Omega),$$

for some constant c and x near $\partial\Omega$. Thus interior estimates for uniformly elliptic non divergence form PDE could be applied to solutions of L. Armed with this knowledge we eventually proved in [6]:

Theorem 2.1. *Fix $p, 1 < p < \infty$, and let $u > 0$ be p harmonic in $\Omega \cap N \subset \mathbf{R}^2$ with $u = 0$ continuously on $\partial\Omega$. If $\partial\Omega$ is a snowflake and $1 < p < 2$, then H-dim $\mu > 1$ while if $2 < p < \infty$, then H-dim $\mu < 1$.*

Theorem 2.2. *Let p, u, μ be as in Theorem 2.1. If $\partial\Omega \subset \mathbf{R}^2$ is a self similar Cantor set and $2 < p < \infty$, then H-dim $\mu < 1$.*

Theorem 2.3. *Let p, u, μ be as in Theorem 2.2. If $\partial\Omega \subset \mathbf{R}^2$ is a k quasicircle, then H-dim $\mu \leq 1$ for $2 < p < \infty$, while H-dim $\mu \geq 1$ for $1 < p < 2$.*

To outline the proof of Theorem 2.1 we note that the boundary Harnack inequality mentioned earlier implies that all measures associated with functions in Theorems 2.1–2.3, have the same Hausdorff dimension. Also, since the p Laplacian is translation, dilation, and rotation invariant we may assume that u is the p capacitary function for $\Omega \setminus \bar{B}(0,1)$ where $d(0, \partial\Omega) = 4$.

That is u is p harmonic in $\Omega \setminus \bar{B}(0,1)$ with continuous boundary values, $u \equiv 1$ on $\partial B(0,1)$ and $u \equiv 0$ on $\partial\Omega$. Let μ be the measure associated with u as in (15). Let $\Omega_n \subset \Omega$ be a sequence of approximating domains constructed in the usual way and for large n let u_n be the p capacitary function for $\Omega_n \setminus \bar{B}(0,1)$. Then one first proves:

Lemma 2.4. *For fixed $p, 1 < p < \infty$,*

$$\eta = \lim_{n \to \infty} n^{-1} \int_{\partial\Omega_n} |\nabla u_n|^{p-1} \log |\nabla u_n| \, dH^1 x$$

exists. If $\eta > 0$ then H-dim $\mu < 1$ while if $\eta < 0$, then H-dim $\mu > 1$.

To prove Lemma 2.4 we followed Carleson–Wolff but our argument was necessarily more complicated, due to the non-linearity of the p Laplacian. □

To prove Theorem 2.1 let u_n, Ω_n be as in Lemma 2.4. We note that one can show $\nabla u_n \neq 0$ in $\Omega_n \setminus \bar{B}(0,1)$ and if $v = \log |\nabla u_n|, p \neq 2, 1 < p < \infty$, that

$$\frac{Lv}{p-2} \geq \min(1, p-1) \sum_{i,j=1}^{2} |\nabla u|^{p-4} (u_{x_i x_j})^2. \tag{18}$$

where L is as in (8), (9).

Next we apply the divergence theorem for large n in $O_n = \Omega_n \setminus \bar{B}(0,1)$ to the vector field whose ith component, $i = 1, 2$, is

$$u_n \sum_{k=1}^{2} b_{ik} \, v_{x_k} - v \sum_{k=1}^{2} b_{ik} \, (u_n)_{x_k}.$$

We get

$$\int_{O_n} u_n \, Lv \, dx = \int_{\partial\Omega_n} \sum_{i,k=1}^{2} b_{ik} \xi_i [\, u_n \, v_{x_k} - v \, (u_n)_{x_k} \,] \, dH^1 x + O(1) \tag{19}$$

where ξ denotes the outer unit normal to $\partial\Omega_n$. Using the fact that $\xi = -\nabla u_n / |\nabla u_n|$ and the definition of (b_{ik}), we find that

$$\int_{\partial\Omega_n} \sum_{i,k=1}^{2} b_{ik}\xi_i \left[u_n\, v_{x_k} - v\,(u_n)_{x_k}\right] dH^1x$$
$$= (p-1)\int_{\partial\Omega_n} |\nabla u_n|^{p-1}\log|\nabla u_n|\, dH^1x + O(1). \tag{20}$$

From (18)–(20), and Lemma 2.4 we conclude that in order to prove Theorem 2.1 it suffices to show

$$\liminf_{n\to\infty}\left(n^{-1}\int_{O_n} u_n\,|\nabla u_n|^{p-4}\sum_{i,j=1}(u_n)^2_{x_i x_j}\,dx\,dx\right) > 0. \tag{21}$$

To prove (21) we showed the existence of $\lambda \in (0,1)$ such that if $x \in \Omega_n \setminus B(0,2)$ and $d(x,\partial\Omega_n) \geq 3^{-n}$, then

$$c\int_{O_n\cap B(x,\lambda d(x,\partial\Omega_n))} u_n|\nabla u_n|^{p-4}\,(u_n)^2_{y_i y_j}\,dy \;\geq\; \mu_n(B(x,2d(x,\partial\Omega_n))) \tag{22}$$

where c depends on p and the k quasi-conformality of Ω. Covering $\{3^{-m-1} \leq d(x,\partial\Omega_n) \leq 3^{-m}\}$ by balls and summing over $1 \leq m \leq n-1$ we obtain first (21) and then Theorem 2.1. □

To prove Theorem 2.3 let $w(x) = \max(v-c,0)$ when $1 < p < 2$ and $w(x) = \max(-v-c,0)$ when $p > 2$. Here c is chosen so large that $|v| \leq c$ on $B(0,2)$. Following Makarov [66] we prove:

Lemma 2.5. *Let m be a nonnegative integer. There exists $c_+ = c_+(k,p) \geq 1$ such that for $0 < t < 1$,*

$$\int_{\{x:u(x)=t\}} |\nabla u|^{p-1}\, w^{2m}\, dH^1x \;\leq\; c_+^{m+1}\, m!\,[\log(2/t)]^m\,.$$

To outline the proof of Lemma 2.5 let $\Omega(t) = \Omega \setminus \{x : u(x) \leq t\}$, whenever $0 < t < 1$. Using the fact that $Lw \leq 0$ for $1 < p < \infty$ when $w > 0$, one computes,

$$L(w^{2m})(x) \;\leq\; 2m(2m-1)p\,|\nabla u|^{p-2}(x)\, w^{2m-2}(x)\,|\nabla w|^2(x). \tag{23}$$

Next we use (23) and apply the divergence theorem in $\Omega(t)$ to the vector field whose ith component for $i = 1,2$ is

$$(u-t)\sum_{j=1}^{2} b_{ij}(w^{2m})_{x_j} - w^{2m}\sum_{j=1}^{2} b_{ij}u_{x_j}\,.$$

We get

$$
\begin{aligned}
&(p-1)\int_{\{x:u(x)=t\}} |\nabla u|^{p-1}\, w^{2m}\, dH^1 x \\
&\quad \le\; 2m(2m-1)p\int_{\Omega(t)} u\,|\nabla u|^{p-2}\, w^{2m-2}\,|\nabla w|^2\, dx.
\end{aligned}
\tag{24}
$$

Using interior estimates for solutions to the p Laplace equation from Section 2, the coarea formula and once again and our fundamental inequality, $|\nabla u(\cdot)| \ge u(\cdot)/d(\cdot, \partial\Omega)$, we deduce from (24) that

$$
I_m(t) = \int_{\{x:u(x)=t\}} |\nabla u|^{p-1}\, w^{2m}\, dH^1 x \;\le\; 2m(2m-1)c\int_t^1 I_{m-1}(\tau)\,\tau^{-1}\, d\tau
$$

where $c = c(p,k)$. Lemma 2.5 then follows from an inductive type argument, using $I_0 \equiv$ constant on (0,1). □

Dividing the display in Lemma 2.5 by $(2c_+)^m\, m!\, \log^m(2/t)$ and summing we see for $0 < t < 1$ that

$$
\int_{\{x:u(x)=t\}} |\nabla u|^{p-1}\, \exp\left[\frac{w^2}{2c_+\log(2/t)}\right] dH^1 x \;\le\; 2\,c_+. \tag{25}
$$

Using (25) and weak type estimates it follows that if

$$
\begin{aligned}
&\lambda(t) = \sqrt{4\,c_+\,\log(2/t)}\,\sqrt{\log(-\log t)} \text{ for } 0 < t < e^{-2}, \\
&F(t) = \{x : u(x) = t \text{ and } w(x) \;\ge\; \lambda(t)\}
\end{aligned}
\tag{26}
$$

then

$$
\int_{F(t)} |\nabla u|^{p-1}\, dH^1 x \le \frac{2c^+}{\log^2(1/t)} \tag{27}
$$

One can use (27) to show that if Hausdorff measure (denoted H^γ) is defined with respect to

$$
\gamma(r) = \begin{cases} r\, e^{a\lambda(r)} & \text{when } 1 < p < 2 \\ r\, e^{-a\lambda(r)} & \text{when } p > 2. \end{cases} \tag{28}
$$

and a is sufficiently large then

$$
\mu \text{ is absolutely continuous with respect to } H^\gamma \text{ when } 1 < p < 2 \tag{29}
$$

$$
\mu \text{ is concentrated on a set of } \sigma \text{ finite } H^\gamma \text{ measure when } p > 2. \tag{30}
$$

Clearly (29), (30) imply Theorem 2.3. □

3.1 p Harmonic Measure in Simply Connected Domains

Recently in [63] we have managed to prove the following theorem.

Theorem 2.6. *Fix $p, 1 < p < \infty$, and let $u > 0$ be p harmonic in $\Omega \cap N$, where Ω is simply connected, $\partial\Omega$ is compact, and N is a neighborhood of $\partial\Omega$. Suppose u has continuous boundary value 0 on $\partial\Omega$ and let μ be the measure associated with u as in (1). If λ, γ, are as in (26), (28), then (29), (30) are valid for $a = a(p)$ sufficiently large. Hence Theorem 2.3 remains valid in simply connected domains.*

We note that Makarov in [66] proved for harmonic measure (i.e., $p = 2$) the stronger theorem:

Theorem 2.7. *Let ω be harmonic measure with respect to a point in the simply connected domain Ω. Then*

(a) ω is concentrated on a set of σ finite H^1 measure
(b) ω is absolutely continuous with respect to $H^{\hat{\gamma}}$ measure defined relative to $\hat{\gamma}(r) = r\,\exp[A\sqrt{\log 1/r\,\log\log\log 1/r}]$ for A sufficiently large.

The best known value of A in the definition of $\hat{\gamma}$ appears to be $A = 6\sqrt{\frac{\sqrt{24}-3}{5}}$ given in [32].

3.2 Preliminary Reductions for Theorem 2.6

To outline the proof of Theorem 2.6 we first claim, as in the proof of Theorem 2.5, that all measures associated with functions satisfying the hypotheses of Theorem 2.6, will have the same Hausdorff dimension. Indeed let $u_1, u_2 > 0$ be p harmonic functions in $\Omega \cap N$ and let μ_1, μ_2, be the corresponding measures as in (15). Observe from the maximum principle for p harmonic functions and continuity of u_1, u_2, that there is a neighborhood $N_1 \subset N$ of $\partial\Omega$ with

$$M^{-1}u_1 \leq u_2 \leq M u_1 \text{ in } N_1 \cap \Omega. \tag{31}$$

One can also show there exists $r_0 > 0, c = c(p) < \infty$, such that whenever $w \in \partial\Omega, 0 < r \leq r_0$, and $i = 1, 2$,

$$c^{-1}\, r^{p-2}\, \mu_i[B(w, r/2)] \leq \max_{B(w,r)} u_i^{p-1} \leq c\, r^{p-2}\, \mu_i[B(w, 2r)]. \tag{32}$$

Using (31), (32), and a covering argument it follows that μ_1, μ_2 are mutually absolutely continuous. Mutual absolute continuity is easily seen to imply H-dim μ_1 = H-dim μ_2. Thus we may assume, as in the proof of Theorem 2.3, that u is the p capacitary function for $D = \Omega \setminus \bar{B}(0,1)$ and $d(0, \partial\Omega) = 4$. The major obstacle to proving Theorem 2.6 in our earlier paper was that we could not prove the fundamental inequality in (11). That is, in our new paper, we prove

Theorem 2.8. *If u is the p capacitary function for D, then there exists $c = c(p) \geq 1$, such that*

$$c|\nabla u|(z) \geq \frac{u(z)}{d(z, \partial\Omega)} \text{ whenever } z \in D.$$

Given Theorem 2.8 we can copy the argument leading to (27) in the proof of Theorem 2.3. However one has to work harder in order to deduce (29), (30) from (27) as previously we used the doubling property of μ in (32) and this property is not available in the simply connected case. Still we omit the additional measure theoretic argument and shall regard the proof of Theorem 2.6 as complete once we sketch the proof of Theorem 2.8.

3.3 Proof of Theorem 2.8

To prove Theorem 2.8 we assume, as we may, that $\partial\Omega$ is a Jordan curve, since otherwise we can approximate Ω in the Hausdorff distance sense by Jordan domains and use the fact that the constant in Theorem 2.8 depends only on p to eventually get this theorem for Ω. We continue under this assumption and shall use complex notation. Let $z = x + iy$, where $i = \sqrt{-1}$ and for $a, b \in D$, let $\rho(a, b)$ denote the hyperbolic distance from $a, b \in D$ to $\partial\Omega$.

Fact A. *u is real analytic in $D, \nabla u \neq 0$ in D, and $u_z = (1/2)(u_x - iu_y)$, is $k = k(p)$ quasi-regular in D. Consequently, $\log |\nabla u|$ is a weak solution to a divergence form PDE for which a Harnack inequality holds. That is, if $h \geq 0$ is a weak solution to this PDE in $B(\zeta, r) \subset D$, then $\max\limits_{B(\zeta,r/2)} h \leq \tilde{c} \min\limits_{B(\zeta,r/2)} h$, where $\tilde{c} = \tilde{c}(p)$.*

From Fact A and Lemma 1.3 one can show that

$$|\nabla u(z)| \leq cu(z)/d(z, \partial\Omega) \text{ in } D \tag{33}$$

and that Theorem 2.8 is valid in $B(0,2) \setminus \bar{B}(0,1)$. Next we use Fact A and (33) to show that Theorem 2.8 for $x \in D \setminus B(0,2)$ is a consequence of the following lemma.

Lemma 2.9. *There is a constant $c = c(p) \geq 1$ such that for any point $z_1 \in D \setminus B(0,2)$, there exists $z^\star \in D \setminus B(0,2)$ with $u(z^\star) = u(z_1)/2$ and $\rho(z_1, z^\star) \leq c$.*

Assuming Lemma 2.9 one gets Theorem 2.8 from the following argument. Let Γ be the hyperbolic geodesic connecting z_1 to z^* and suppose that $\Gamma \subset D$. From properties of ρ one sees for some $c = c(p)$ that

$$H^1(\Gamma) \leq cd(z_1, \partial\Omega) \text{ and } d(\Gamma, \partial\Omega) \geq c^{-1} d(z_1, \partial\Omega). \tag{34}$$

Thus

$$\begin{aligned} \frac{1}{2} u(z_1) \leq u(z_1) - u(z^\star) &\leq \int_\Gamma |\nabla u(z)||dz| \\ &\leq cH^1(\Gamma) \max_\Gamma |\nabla u| \leq cd(z_1, \partial\Omega) \max_\Gamma |\nabla u|. \end{aligned}$$

So for some $\zeta \in \Gamma$ and $c^* = c^*(p) \geq 1$,

$$c^\star |\nabla u(\zeta)| \ \geq \ \frac{u(z_1)}{d(z_1, \partial\Omega)}. \tag{35}$$

Also from (35), we deduce the existence of Whitney balls $\{B(w_j, r_j\}$, with $w_j \in \Gamma$, $r_j \approx d(z_1, \partial\Omega)$, connecting ζ to z_1 and

$$|\nabla u(z)| \leq cu(z_1)/d(z_1, \partial\Omega) \text{ when } z \in \bigcup_j B(w_j, r_j)\,. \tag{36}$$

From (35), (36), we see that if $c = c(p)$ is large enough and

$$h(z) = \log\left(\frac{c\, u(z_1)}{d(z_1, \partial\Omega)\, |\nabla u(z)|}\right) \text{ for } z \in \bigcup_j B(w_i, r_i)$$

then $h > 0$ in $\cup_i B(w_i, r_i)$ and $h(\zeta) \leq c$. From Fact A we see that Harnack's inequality can be applied to h in successive balls of the form $B(w_i, r_i/2)$. Doing this we obtain $h(z_1) \leq c'$ where $c' = c'(p)$. Clearly, this inequality implies Theorem 2.8.

We note that if $\partial\Omega$ is a quasicircle one can choose z^* to be a point on the line segment connecting z_1 to $w \in \partial\Omega$ where $|w - z_1| = d(z_1, \partial\Omega)$ The proof uses Hölder continuity of u near $\partial\Omega$ and the fact that for some $c = c(p,k),\ cu(z_1) \geq \max\limits_{B(z_1, 2d(z_1, \partial\Omega))} u$. (see Lemmas 1.4 and 1.5). This inequality need not hold in a Jordan domain and so we have to give a more complicated argument to get Lemma 2.9. To this end, we construct a Jordan arc $\sigma : (-1,1) \to D$ with $\sigma(0) = z_1$, $\sigma(\pm 1) = \lim\limits_{t \to \pm 1} \sigma(t) \in \partial\Omega$, and $\sigma(1) \neq \sigma(-1)$. Moreover, for some $c = c(p)$,

$$\begin{aligned} &(\alpha)\ H^1(\sigma) \leq cd(z_1, \partial\Omega) \\ &(\beta)\ u \leq cu(z_1) \text{ on } \sigma. \end{aligned} \tag{37}$$

Let Ω_1 be the component of $\Omega \setminus \sigma$ not containing $B(0,1)$. Then we also require that there is a point w_0 on $\partial\Omega \cap \partial\Omega_1$ with

$$|w_o - z_1| \leq cd(z_1, \partial\Omega) \text{ and } d(w_0, \sigma) \geq c^{-1} d(z_1, \partial\Omega). \tag{38}$$

Finally we shall show the existence of a Lipschitz curve $\tau : (0,1) \to \Omega_1$ with $\tau(0) = z_1$, $\tau(1) = w_0$, satisfying the cigar condition:

$$\min\{H^1(\tau[0,t]), H^1(\tau[t,1])\} \leq \hat{c} d(\tau(t), \partial\Omega), \tag{39}$$

for $0 < t < 1$ and some absolute constant $\hat{c}$.

To get Lemma 2.9 from (37)–(39) let $u_1 = u$ in Ω_1 and $u_1 \equiv 0$ outside of Ω_1. From PDE estimates, (37) (β), and (38) one finds $\theta > 0, c < \infty$ such that

$$\max_{B(w_0,t)} u_1 \leq cu(z_1) \left(\frac{t}{d(z_1, \partial\Omega)} \right)^\theta \text{ for } 0 < t < d(w_0, \sigma). \tag{40}$$

From (40), (39) we conclude the existence of z^* with $\rho(z_1, z^*) \leq c$ and $u(z^*) = 1/2$, which is Lemma 2.9. To construct σ, τ let f be the Riemann mapping function from the upper half plane, $\mathbb{H}$, onto Ω with $f(i) = 0$ and $f(a) = z_1$, where $a = is$ for some $s, 0 < s < 1$. Note that f has a continuous extension to $\bar{\mathbb{H}}$, since $\partial\Omega$ is a Jordan curve. Let $I(b) = [\text{ Re } b - \text{ Im } b, \text{ Re } b + \text{ Im } b]$ whenever $b \in \mathbb{H}$. We need the following lemmas.

Lemma 2.10. *There is a set $E(b) \subset I(b)$ such that for $x \in E(b)$*

$$\int_0^{Im\ b} |f'(x+iy)| dy \leq c^\star d(f(b), \partial\Omega)$$

for some absolute constant $c^\star$, and also

$$H^1(E(b)) \geq (1 - 10^{-100}) H^1(I(b)).$$

Lemma 2.11. *Given $0 < \delta < 10^{-1000}$, there is an absolute constant $\hat{c}$ such that if $\delta_\star = e^{-\hat{c}/\delta}$ then, whenever $x \in E(b)$ there is an interval $J = J(x)$ centered at x with*

$$2\delta_\star\ Im\ b \leq H^1(J) \leq c\delta^{1/2}\ Im\ b \leq \frac{Im\ b}{10000}$$

(for some absolute constant c) and a subset $F = F(x) \subset J$ with $H^1(F) \geq (1 - 10^{-100}) H^1(J)$ so that

$$\int_0^{\delta_\star \text{ Im } b} |f'(t+iy)| dy \leq \delta d(f(b), \partial\Omega) \quad \text{for every } t \in F.$$

Lemma 2.12. *Let* $\hat{F} = \bigcup_{x \in E(b)} F(x)$. *If* $L \subset I(b)$ *is an interval with* $H^1(L) \geq \dfrac{Im\ b}{100}$, *then*

$$H^1(E(b) \cap \hat{F} \cap L) \geq \frac{Im\ b}{1000}\,.$$

Moreover, if $\{\tau_1, \tau_2, \dots, \tau_m\}$ *is a subset of* $I(b)$, *then there exists* τ_{m+1} *in* $E(b) \cap \hat{F} \cap L$ *with*

$$|f(\tau_{m+1}) - f(\tau_j)| \geq \frac{d(f(b), \partial\Omega)}{10^{10}\ m^2} \text{ whenever } 1 \leq j \leq m.$$

To construct σ, τ from Lemma 2.12 we put $b = a = is$, and deduce for given $\delta, 0 < \delta < 10^{-1000}$, the existence of $x_1, x_2, x_3 \in E(a)$ with $-s < x_1 < -s/2, -\frac{1}{8}s < x_3 < \frac{1}{8}s$, and $\frac{1}{2}s < x_2 < s$. Moreover,

$$\int_0^{\delta_* s} |f'(x_j + iy|\, dy \leq \ \delta d(z_1, \partial\Omega) \text{ for } 1 \leq j \leq 3, \tag{41}$$

$$\min\{|f(x_1) - f(x_3)|, |f(x_2) - f(x_3)|\} \geq 10^{-11} d(z_1, \partial\Omega) \tag{42}$$

Let $\tilde{Q}(a)$ be the rectangle whose boundary in $\mathbb{H}, \xi$, consists of the horizontal line segment from $x_1 + is$ to $x_2 + is$, and the vertical line segments from x_j to $x_j + is$, for $j = 1, 2$. Put $\sigma = f(\xi)$ and note from (41) that (37) (α) is valid. To construct τ we put $t_0 = 0, s_0 = s, a_0 = t_0 + is_0$. Let $s_1 = \delta_* s_0, t_1 = x_3$, and $a_1 = t_1 + is_1$. By induction, suppose $a_m = s_m + it_m$ has been defined for $1 \leq m \leq k-1$. We then choose $t_k \in E(a_{k-1})$ so that the last display in Lemma 2.11 holds with $t = t_k$. Set $s_k = \delta_* s_{k-1}$ and $a_k = s_k + it_k$.

Let λ_k be the curve consisting of the horizontal segment from a_{k-1} to $t_k + is_{k-1}$ and the vertical line segment from a_{k-1} to a_k. Put $\lambda = \bigcup \lambda_k$ and $\tau = f(\lambda)$. From our construction we deduce that τ satisfies the cigar condition in (39) for $\delta > 0$ small. Also $x_0 = \lim_{t \to 1} \lambda(t)$ exists, $|x_0| < 1/4$, and (38) holds for $w_0 = f(x_0)$, thanks to (42) and our construction.

3.4 The Final Proof

It remains to prove $u \leq cu(z_1)$ on σ which is (37) (β). The proof is by contradiction. Suppose $u > Au(z_1)$ on σ. We shall obtain a contradiction if $A = A(p)$ is suitably large. Our argument is based on the recurrence type scheme mentioned after Lemma 1.5 (often attributed to Carleson–Domar in the complex world and Caffarelli et al. in the PDE world). Given the rectangle $\tilde{Q}(a)$ we let $b_{j,1} = x_j + i\delta_* \operatorname{Im} a, j = 1,2$, and note that $b_{j,1}, j = 1,2$, are points on the vertical sides of $\tilde{Q}(a)$. These points will spawn two new boxes $\tilde{Q}(b_{j,1})$, $j = 1,2$, which in turn will each spawn two more new boxes, and so on. Without loss of generality, we focus on $\tilde{Q}(b_{1,1})$.

This box is constructed in the same way as $\tilde{Q}(a)$ and we also construct, using Lemma 2.12 once again, a polygonal path $\lambda_{1,1}$ from $b_{1,1}$ to some point $x_{0,1} \in I(b_{1,1})$. λ_{11} is defined relative to $b_{1,1}$ in the same way that λ was defined relative to a. Also in view of Lemma 2.12 we can require that $\lambda_{1,1} \subset \{ \operatorname{Re} z < \operatorname{Re} b_{1,1}\}$. $\lambda_{2,1}$ with endpoints, $b_{2,1}, x_{0,2}$ is constructed similarly to lie in $\{ \operatorname{Re} z > \operatorname{Re} b_{2,1}\}$. Next let Λ be a Harnack constant such that

$$\max\{u(f(b_{1,1})), u(f(b_{2,1}))\} \leq \Lambda u(z_1). \tag{43}$$

From Harnack's inequality for u and Lemma 2.12 with δ fixed, it is clear that Λ in (43) can be chosen to depend only on p, so can also be used in further iterations.

Let $U = u \circ f$. By the maximum principle, since $A > \Lambda$ and $\lambda_{1,1}$, $\lambda_{2,1}$ lie outside of $\tilde{Q}(a)$, there will be a point $z \in \lambda_{1,1} \cup \lambda_{2,1}$ such that $U(z) > AU(a) = Au(z_1)$. Suppose $z \in \lambda_{1,1}$. The larger the constant A, the closer z will be to $\mathbf{R}$. More precisely, if $A > \Lambda^k$ then $\operatorname{Im} z \leq \delta_\star^k \operatorname{Im} a$, as follows from Harnack's inequality for u, and the construction of $\lambda_{1,1}$. In fact we can show that

$$|f(z) - f(x_{0,1})| \leq C\delta^{k-1} d(f(b_{1,1}), \partial\Omega).$$

The argument now is similar to the argument showing the existence of z^* given σ, τ. Let $\xi_{1,1}$ be the boundary of $\tilde{Q}(b_{1,1})$ which is in $\mathbb{H}$ and let $\sigma_{1,1} = f(\xi_{1,1})$. Set $w_{0,1} = f(x_{0,1})$. Then

$$B(w_{0,1}, d(w_{0,1}, \sigma_{1,1})) \subset f(\tilde{Q}(b_{1,1})).$$

and since $d(w_{0,1}, \sigma_{1,1}) \approx d(f(b_{1,1}), \partial\Omega)$ it follows from Hölder continuity of u that

$$U(z) \leq C\delta^{\theta k} \max_{\tilde{Q}(b_{1,1})} U.$$

Choose k, depending only on p, to be the least positive integer such that

$$C\delta^{\theta k} < \Lambda^{-1}.$$

This choice of k determines A (say $A = 2\Lambda^k$) which therefore also depends only on p. With this choice of A we have

$$\max_{\xi_{1,1}} U > \Lambda U(z) > \Lambda A U(a). \tag{44}$$

Since $U(b_{1,1}) \leq \Lambda U(a)$ we see from (44) that we can now repeat the above argument with $\tilde{Q}(b_{1,1})$ playing the role of $\tilde{Q}(a)$. That is, we find $b_{1,2}$ on the vertical sides of $\tilde{Q}(b_{1,1})$ with $\text{Im } b_{1,2} = \delta_\star^2 \text{ Im } a$ and a box $\tilde{Q}(b_{1,2})$ with boundary $\xi_{1,2}$ such that

$$\max_{\xi_{1,2}} U > \Lambda^2 A U(a) \geq A U(b_{1,2}).$$

Continuing by induction we get a contradiction because $U = 0$ continuously on $\partial\Omega$. The proof of (37) (β), Theorem 2.8, and Theorem 2.6 is now complete.

□

3.5 p Harmonic Measure in Space

In [48] we proved

Theorem 2.13. *Let p, u, μ, be as in Theorem 2.3. There exists $k_0(p) > 0$ such that if $\partial\Omega$ is a k quasicircle, $0 < k < k_0(p)$, then*

(a) μ is concentrated on a set of σ finite H^1 measure when p ¿ 2.
(b) There exists $A = A(p), 0 < A(p) < \infty$, such that if $1 < p < 2$, then μ is absolutely continuous with respect to $H^{\hat{\lambda}}$ where $\hat{\lambda}(r) = r \exp[A\sqrt{\log 1/r \, \log\log\log 1/r}]$.

In [8] we prove an analogue of Theorem 2.13 when $p \geq n$. To be more specific we need a definition.

Definition B. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0) NTA domain and $0 < r \leq r_0$. Then Ω and $\partial\Omega$ are said to be (δ, r_0), Reifenberg flat provided that whenever $w \in \partial\Omega$, there exists a hyperplane, $P = P(w, r)$, containing w such that*

(a) $\Psi(\partial\Omega \cap B(w, r), P \cap B(w, r)) \leq \delta r$
(b) $\{x \in \Omega \cap B(w, r) : d(x, \partial\Omega) \geq 2\delta r\} \subset$ one component of $\mathbf{R}^n \setminus P$.

In Definition B, $\Psi(E, F)$ denotes the Hausdorff distance between the sets E and F defined by

$$\Psi(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$$

Theorem 2.14. *Let $\Omega \subset \mathbf{R}^n, n \geq 3$, be a (δ, r_0) Reifenberg flat domain, $w \in \partial\Omega$, and p fixed, $n \leq p < \infty$. Let $u > 0$ be p harmonic in Ω with $u = 0$ continuously on $\partial\Omega$. Let μ be the measure associated with u as in (15). There exists, $\hat{\delta} = \hat{\delta}(p, n) > 0$, such that if $0 < \delta \leq \hat{\delta}$, then μ is concentrated on a set of σ finite H^{n-1} measure.* To outline the proof of Theorem 2.14, we shall need the following result from [50]:

Theorem 2.15. *Let Ω be (δ, r_0) Reifenberg flat, $1 < p < \infty$, and $u > 0$, a p harmonic function in Ω with $u \equiv 0$ on $\partial\Omega$. Then there exists, $\delta_0 > 0, c_1 \geq 1$,* *depending only on* p, n, *such that if* $0 < \delta \leq \delta_0$ *and* $x \in \Omega$, *then* $u \in C^\infty(\Omega)$ *and*

(a) $c_1^{-1}|\nabla u(x)| \leq u(x)/d(x, \partial\Omega) \leq c_1|\nabla u(x)|, x \in \Omega,$
(b) $|\nabla u|^{p-2}$ *extends to an* A_2 *weight on* $\mathbf{R}^n$ *with constant* $\leq c_1$.

An outline of the proof of Theorem 2.15 will be given in the next lecture. From Theorem 2.15 we see that $(b_{ik}(x))$ in (8), (9) are locally uniformly elliptic in Ω with ellipticity constants given in terms of an A_2 weight on $\mathbf{R}^n$. Thus Lemmas 1.6–1.8 can be used. To prove Theorem 2.14 we need a key lemma:

Lemma 2.16. *Let* u, Ω, *be as in Theorem 2.14 and* $p \geq n$. *Then* $Lv \geq 0$ *where* L *is as in (8), (9), and* $v = \log|\nabla u|$.

Using Lemma 2.16 and Theorem 2.15 we can essentially repeat the proof in [48] which in turn was based on the a proof in [66]. The main difficulty involves showing that if

$$\Theta = \{y \in \partial\Omega : v(x) \to -\infty \text{ as } x \to y \text{ nontangentially}\}$$

then $\mu(\Theta) = 0$. To accomplish this we use some results on elliptic PDE whose degeneracy is given in terms of an A_2 weight. □

3.6 Open Problems for p Harmonic Measure

Note. In problems (1)–(8) the surrounding space is $\mathbf{R}^2$.

1. Can Theorem 2.6 for simply connected domains be generalized to:
   (a) μ is concentrated on a set of σ finite H^1 measure whenever $p > 2$.
   (b) If $a = a(p) > 1$ is large enough and $1 < p < 2$, then μ is absolutely continuous with respect to $H^{\hat{\gamma}}$ measure where $\hat{\gamma}$ is defined in Theorem 2.7
2. Is H-dim μ concentrated on a set of σ finite H^1 measure when $p > 2$ and Ω is any planar domain. For harmonic measure this result is in [39, 71].
3. What is the exact value of H-dim μ for a given p when $\partial\Omega$ is the Van Koch snowflake and $p \neq 2$?
4. For a given p, what is the supremum $(p < 2)$ or infimum $(p > 2)$ of H-dim μ taken over the class of quasi-circles and/or simply connected domains?.
5. Is H-dim μ continuous and/or decreasing as a function of p when $\partial\Omega$ is the Van Koch snowflake?

   Regarding this question, the proof of Theorem 2.1 gives that H-dim $\mu = 1 + O(|p-2|)$ as $p \to 2$ for a snowflake domain.
6. Are the p harmonic measures defined on each side of a snowflake mutually singular? The answer is yes when $p = 2$ as shown in [9].

7. Is it always true for $1 < p < \infty$ that H-dim $\mu <$ Hausdorff dimension of $\partial\Omega$ when $\partial\Omega$ is a snowflake or a self similar Cantor set? The answer is yes when $p = 2$ for the snowflake as shown in [40]. The answer is also yes for self similar Cantor sets when $p = 2$. This question and continuity questions for H-dim ω on certain four cornered Cantor sets are answered by Batakis in [3–5].
8. We noted in Remark 2 that H-dim μ was independent of the choice of u vanishing on $\partial\Omega$. However in more general scenarios we do not know whether H-dim μ is independent of u. For example, suppose $x_0 \in \partial\Omega$ and $u > 0$ is p harmonic in $\Omega \cap B(x_0, r)$ with $u = 0$ on $\partial\Omega \cap B(x_0, r)$ in the $W^{1,p}$ sense. If $\partial\Omega \cap B(x_0, r)$ has positive p capacity, then there exists a measure μ satisfying (15) with $\phi \in C_0^\infty(N)$ replaced by $\phi \in C_0^\infty(B(x_0, r))$. Is H-dim $\mu|_{B(x_0,r/2)}$ independent of u? If Ω is simply connected and $p = 2$, then I believe the answer to this question is yes. In general this problem appears to be linked with boundary Harnack inequalities.
9. Is it true for $p \geq n$ that H-dim $\mu \leq 1$ whenever $\Omega \subset \mathbf{R}^n$? If not is there a more general class of domains than Reifenberg flat domains (see Theorem 2.14) for which this inequality holds? Compare with Problem 2.
10. What can be said about the dimension of Wolff snowflakes? Regarding this question it appears that we can perturb off the $p = 2$ case (see [61]) in order to construct Wolff snowflakes for $0 < |p - 2| < \epsilon, \epsilon > 0$ small, for which the H-dim of the corresponding p harmonic measures on both sides of the snowflake are $< n - 1$ and also examples for which the H-dim of these measures are $> n - 1$.
11. What can be said for the dimension of p harmonic measure, $p > 3 - \log 4/\log 3$, or even just harmonic measure in $\Omega = \mathbf{R}^3 \setminus J$ where J is the Van Koch snowflake?
12. The existence of a measure μ, corresponding to a weak solution u with vanishing boundary values, as in (2), exists for a large class of divergence form partial differential equations. What can be said about analogues of Theorems 2.1–2.3 or Theorems 2.6, 2.8, 2.14, 2.15 for the measures corresponding to these solutions? What can be said about analogues of problems (1)–(11)?

4 Boundary Harnack Inequalities and the Martin Boundary Problem for p Harmonic Functions

Recall from Section 3 the definition of nontangentially accessible and Reifenberg flat domains. We note that a Lipschitz domain (i.e., a domain which is locally the graph of a Lipschitz function) is NTA. Also a Reifenberg flat domain need not have a rectifiable boundary or tangent planes in the geometric measure sense anywhere (e.g., the Van Koch snowflake, in two dimensions). In [50] we prove

Theorem 3.1. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain. Suppose that u, v are positive p-harmonic functions in $\Omega \cap B(w, 4r)$, that u, v are continuous in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0 = v$ on $\partial\Omega \cap B(w, 4r)$. There exists $\tilde{\delta}, \sigma > 0$ and $c_1 \geq 1$, all depending only on p, n, such that*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c_1 \left(\frac{|y_1 - y_2|}{r} \right)^{\sigma}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_1)$. Here $w \in \partial\Omega, 0 < r < r_0,$ and $0 < \delta < \tilde{\delta}$.

Observe that the last display in Theorem 3.1 is equivalent to:

$$\left| \frac{u(y_1)}{v(y_1)} - \frac{u(y_2)}{v(y_2)} \right| \leq c_1 \frac{u(y_1)}{v(y_1)} \left(\frac{|y_1 - y_2|}{r} \right)^{\sigma}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_1)$ so Theorem 3.1 is a boundary Harnack inequality for positive p harmonic functions vanishing on a portion of a sufficiently flat Reifenberg domain.

4.1 History of Theorem 3.1

The term boundary Harnack inequality for harmonic functions was first introduced by Kemper in [41]. He attempted to show the ratio of two positive harmonic functions vanishing on a portion of a Lipschitz domain was bounded. Unfortunately Kemper's proof was not correct, as Brelot later pointed out. This inequality for harmonic functions in Lipschitz domains was later proved independently and at about the same time in [2, 17, 73]. Jerison and Kenig in [37] proved Theorem 3.1 for NTA domains. Moreover boundary Harnack inequalities for solutions to linear divergence form uniformly elliptic PDE are proved in [14] while these inequalities for degenerate linear divergence form elliptic PDE whose degeneracy is specified in terms of an A_2 weight were proved by [21–23], as mentioned earlier. Theorem 3.1 will be proved in the following steps:

Step 1: We prove Theorem 3.1 for

$$Q = \{x : |x_i| < 1, 1 \leq i \leq n-1, 0 < x_n < 2\}.$$

Step 2: (The 'fundamental inequality' for $|\nabla u|$) In this step, for u as in Theorem 3.1, we show there exist $\hat{c} = \hat{c}(p, n)$ and $\bar{\lambda} = \bar{\lambda}(p, n)$, such that if $0 < \delta \leq \delta_1$, then

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \partial\Omega)} \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \partial\Omega)} \tag{45}$$

whenever $y \in \Omega \cap B(w, r/\hat{c})$.

Step 3: In this step we show that $|\nabla u|^{p-2}$ extends to an A_2-weight locally with constants depending only on p, n (provided δ is small enough).

Step 4: (Deformation of p harmonic functions). Let u, v be as in Theorem 3.1, $r^* = r/c', c'$ large and for $0 \leq \tau \leq 1$, let $\tilde{u}(\cdot, \tau)$ be the p harmonic function in $\Omega \cap B(w, 4r^*)$ with continuous boundary values,

$$\tilde{u}(y, \tau) = \tau v(y) + (1 - \tau) u(y)$$

whenever $y \in \partial(\Omega \cap B(w, 4r^*))$ and $\tau \in [0, 1]$. To simplify matters assume that

$$0 \leq u \leq v/2 \text{ and } v \leq c \text{ in } \bar{\Omega} \cap B(w, 4r^*), \tag{46}$$

where c, as in the rest of this lecture, may depend only on p, n. Then from the maximum principle for p harmonic functions (Lemma 1.1) we have

$$0 \leq \frac{\tilde{u}(\cdot, \tau_2) - \tilde{u}(\cdot, \tau_1)}{\tau_2 - \tau_1} \leq c \tag{47}$$

whenever $0 \leq \tau_1, \tau_2 \leq 1$. Proceeding operationally we note that if $\tilde{u}(\cdot, \tau)$ has partial derivatives with respect to τ on $B(w, 4r^*)$, then differentiating the p Laplace equation: $\nabla \cdot (|\nabla \tilde{u}(x)|^{p-2} \nabla \tilde{u}(x, \tau)) = 0$ with respect to τ one finds that $\tilde{u}_\tau(x, \tau)$ is a solution for $x \in \Omega \cap B(w, 4r^*)$ to the PDE

$$\tilde{L}\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\tilde{b}_{ij}(x, \tau) \zeta_{x_j}(x, \tau) \right) = 0 \tag{48}$$

where $(\tilde{b}_{ij})(\cdot, \tau)$ are defined as in (9) relative to $\tilde{u}(\cdot, \tau)$. Thus, $\tilde{u}(\cdot, \tau)$ and $\tilde{u}_\tau(\cdot, \tau)$, both satisfy the same PDE. Now

$$\log \frac{v(x)}{u(x)} = \log \frac{\tilde{u}(x, 1)}{\tilde{u}(x, 0)} = \int_0^1 \frac{\tilde{u}_\tau(x, \tau)}{\tilde{u}(x, \tau)} \, d\tau \tag{49}$$

when $x \in \Omega \cap B(w, 4r^*)$. Observe also from (47) that $\tilde{u}_\tau \geq 0$ with continuous boundary value zero on $\partial\Omega \cap B(w, 4r^*)$. From this fact, (49), we see that to prove Theorem 3.1 it suffices to prove boundary Harnack inequalities for the PDE in (48) with constants independent of $\tau \in [0, 1]$. Moreover, from Steps 1–2, we see that $u(\cdot, \tau)$ is a solution to a uniformly elliptic PDE whose degeneracy is given in terms of an A_2 weight with A_2 constant independent of τ. Thus Lemma 1.8 can be applied to $\tilde{u}_\tau, \tilde{u}$ in order to conclude Theorem 3.1.

4.2 Proof of Step 1

Step 1 is stated formally as
Lemma 3.2. *Let*

$$Q = \{x : |x_i| < 1, 1 \leq i \leq n-1, 0 < x_n < 2\}.$$

Given $p, 1 < p < \infty$, *let* $u, v > 0$ *be* p *harmonic in* Q, *continuous in* $\bar{Q}$ *with* $u \equiv v \equiv 0$ *on* $\partial Q \cap \{x : x_n = 0\}$. *Then for some* $c = c(p,n) \geq 1$,

$$\left| \log\left(\frac{u(z)}{v(z)}\right) - \log\left(\frac{u(y)}{v(y)}\right) \right| \leq c|z-y|^{\sigma}$$

whenever $z, y \in Q \cap B(0, 1/16)$, *where* σ *is the exponent in Lemma 1.3.*
Proof. To begin the proof of Lemma 3.2, observe that x_n is p harmonic and vanishes when $x_n = 0$. Thus from the triangle inequality, it suffices to prove Lemma 3.2 when $v = x_n$. To prove that u/v is bounded in $Q \cap B(0, 1/2)$ we use barrier estimates. Given $x \in B(0, 1/2)$ with $x_n \leq 1/100$ let $\hat{x} = (x', 1/8)$ and let f be p harmonic in $D = B(\hat{x}, 1/8) \setminus \bar{B}(\hat{x}, 1/100)$ with continuous boundary values, $f = u(e_n/4)$ on $\partial B(\hat{x}, 1/100)$ while $f \equiv 0$ on $\partial B(\hat{x}, 1/8)$. From Lemma 1.1 we see that for some a, b,

$$f(x) = \begin{cases} a|x-\hat{x}|^{(p-n)/(p-1)} + b, \; p \neq n, \\ a \ln|x - \hat{x}| + b, \; p = n, \end{cases} \tag{50}$$

Also $f \leq cu$ in D thanks to Harnack's inequality in Lemma 1.2. Using these facts and (50) it follows from direct calculation that

$$cu(e_n/4) \leq \frac{u(x)}{x_n} \tag{51}$$

for some $c = c(p,n)$ when $x_n \in Q \cap B(0, 1/2)$ and $x_n \leq 1/100$. From Harnack's inequality we see that this inequality holds on $Q \cap B(0, 1/2)$. Next we extend u to

$$Q' = \{x : |x_i| < 1, 1 \leq i \leq n-1, |x_n| < 2\}$$

by putting $u(x', x_n) = -u(x', -x_n)$ when $x_n < 0$ (Schwarz Reflection). It is easily shown that u is p harmonic in Q'. We can now use Lemmas 1.3 and 1.5 for u to deduce for $x \in Q \cap B(0, 1/8)$, that

$$|\nabla u(x)| \leq c \max_{B(0,1/4)} u. \leq c^2 u(e_n/4). \tag{52}$$

From (52) and the mean value theorem we get

$$cu(e_n/4) \geq \frac{u(x)}{x_n} \tag{53}$$

when $x \in Q \cap B(0, 1/8)$. Combining (53), (51), we obtain

$$c^{-1} u(e_n/4) \leq u(x)/x_n \leq c u(e_n/4). \tag{54}$$

Hölder continuity of the above ratio follows from (54) and Lemma 1.3. We omit the details. □

4.3 Proof of Step 2

In the proof of (45) we shall need the following comparison lemma.

Lemma 3.3. *Let O be an open set, $w \in \partial O, r > 0$, and suppose that $\hat{u}, \hat{v}$ are positive p harmonic functions in $O \cap B(w, 4r)$. Let $a \geq 1, x \in O$, and suppose that*

$$a^{-1} \frac{\hat{v}(x)}{d(x, \partial \Omega)} \leq |\nabla \hat{v}(x)| \leq a \frac{\hat{v}(x)}{d(x, \partial \Omega)}.$$

If $\tilde{\epsilon}^{-1} = (ca)^{(1+\sigma)/\sigma}$, where σ is as in Lemma 1.3, then for $c = c(p, n)$ suitably large, the following statement is true. If

$$(1 - \tilde{\epsilon})\hat{L} \leq \frac{\hat{v}}{\hat{u}} \leq (1 + \tilde{\epsilon})\hat{L}$$

in $B(x, \frac{1}{4} d(x, \partial O))$ for some $\hat{L}, 0 < \hat{L} < \infty$, then

$$\frac{1}{ca} \frac{\hat{u}(x)}{d(x, \partial O)} \leq |\nabla \hat{u}(x)| \leq ca \frac{\hat{u}(x)}{d(x, \partial O)}$$

Proof. Using Lemmas 1.2 and 1.3, we see that if $z_1, z_2, \in B(x, t d(x, \partial O))$ and $0 < t \leq 1/100$,

$$\begin{aligned} |\nabla \hat{u}(z_1) - \nabla \hat{u}(z_2)| &\leq c t^\sigma \max_{B(x, t d(x, \partial O))} |\nabla \hat{u}(\cdot)| \\ &\leq c^2 t^\sigma \, \hat{u}(x)/d(x, \partial O). \end{aligned} \tag{55}$$

Here c depends only on p, n. From (55) we conclude that we only have to prove bounds from below for the gradient of $\hat{u}$ at x. To do this, suppose for some small $\zeta > 0$ (to be chosen) that,

$$|\nabla \hat{u}(x)| \leq \zeta \, \hat{u}(x)/d(x, \partial O). \tag{56}$$

From (55) with $z = z_1, x = z_2$ and (56) we deduce

$$|\nabla \hat{u}(z)| \leq [\zeta + c^2 t^\sigma] \; \hat{u}(x)/d(x, \partial O) \tag{57}$$

for $z \in B(x, td(x, \partial O))$. Integrating, it follows that if $y \in \partial B(x, td(x, \partial O))$, with $|x - y| = td(x, \partial O)$, $t = \zeta^{1/\sigma}$, then

$$|\hat{u}(y) - \hat{u}(x)| \leq c' \zeta^{1+1/\sigma} \, \hat{u}(x). \tag{58}$$

Constants in (57), (58) depend only on p, n. On the other hand (55) also holds with $\hat{u}$ replaced by $\hat{v}$. Let $\lambda = \frac{\nabla \hat{v}(x)}{|\nabla \hat{v}(x)|}$. Then from (55) for $\hat{v}$ and the non-degeneracy assumption on $|\nabla \hat{v}|$ in Lemma 3.3, we find

$$\langle \nabla \hat{v}(z), \lambda \rangle \geq (1 - c\, a \zeta) |\nabla \hat{v}(x)| \text{ in } \bar{B}(x, \zeta^{1/\sigma} d(x, \partial O)),$$

where $c = c(p, n)$. If $\zeta \leq (2ca)^{-1}$, where c is the constant in the above display, then we can integrate, to get for $y = x + \zeta^{1/\sigma} d(x, \partial O) \lambda$, that

$$c^* (\hat{v}(y) - \hat{v}(x)) \geq a^{-1} \zeta^{1/\sigma} \hat{v}(x) \tag{59}$$

with a constant c^* depending only on p, n. From (59), (58), we see that if $\tilde{\epsilon}$ is as in Lemma 3.3, then

$$\begin{aligned} (1 - \tilde{\epsilon}) \hat{L} \leq \frac{\hat{u}(y)}{\hat{v}(y)} \; &\leq \left(\frac{1 + c' \zeta^{1+1/\sigma}}{1 + \zeta^{1/\sigma}/(ac^*)} \right) \frac{\hat{u}(x)}{\hat{v}(x)} \\ &\leq (1 + \tilde{\epsilon}) \left(\frac{1 + c' \zeta^{1+1/\sigma}}{1 + \zeta^{1/\sigma}/(ac^*)} \right) \hat{L} < (1 - \tilde{\epsilon}) \hat{L} \end{aligned} \tag{60}$$

provided $1/(a\tilde{c})^{1/\sigma} \geq \zeta^{1/\sigma} \geq a \tilde{c} \, \tilde{\epsilon}$ for some large $\tilde{c} = \tilde{c}(p, n)$. This inequality and (59) are satisfied if $\tilde{\epsilon}^{-1} = (\tilde{c}a)^{(1+\sigma)/\sigma}$ and $\zeta^{-1} = \tilde{c}a$. Moreover, if the hypotheses of Lemma 3.3 hold for this $\tilde{\epsilon}$, then in order to avoid the contradiction in (60) it must be true that (56) is false for this choice of ζ. Hence Lemma 3.3 is true. □

As an application of Lemmas 3.2 and 3.3 we note that if $\hat{u}$ is p harmonic in $D = B(\zeta, \rho) \cap \{y : y_n > \zeta_n\}$ and $\hat{u}$ has continuous boundary value zero on $\partial D \cap \{y : y_n = \zeta_n\}$, then there exists $c = c(p, n) \geq 1$ such that

$$c^{-1} \, \frac{\hat{u}(x)}{d(x, \partial D)} \; \leq |\nabla \hat{u}(x)| \leq \; c \frac{\hat{u}(x)}{d(x, \partial D)} \tag{61}$$

in $D \cap B(\zeta, \rho/c)$. Indeed, let $\hat{v} = x_n - \zeta_n$ and put $\tilde{L} = \hat{u}(z)/\hat{v}(z)$ where z is a fixed point in $D \cap B(\zeta, \rho/c_+)$. Using Lemma 3.2 we see that

$$\left| \tilde{L} - \frac{\hat{u}(y)}{\hat{v}(y)} \right| \leq c' \, c_+^{-\sigma} \, \tilde{L}, \;\; \text{when } y \in D \cap B(\zeta, \rho/c_+). \tag{62}$$

Choosing c_+ large enough we deduce that Lemma 3.2 applies, so (61) is true. Equation (61) could also be proved more or less directly using barrier arguments, Schwarz reflection, and Lemma 1.3.

Next we use Lemma 3.3 to get the nondegeneracy property in (45). We restate this property as,

Lemma 3.4. *Let Ω be (δ, r_0) Reifenberg flat. Let $u > 0$ be p harmonic in $\Omega \cap B(w, 2r)$, continuous in $B(w, 2r)$ with $u \equiv 0$ in $B(w, 2r) \setminus \Omega$. There exists $\delta^* = \delta^*(p, n)$, $c_1 = c_1(p, n)$, such that if $0 < \delta \le \delta^*$ and $y \in \Omega \cap B(w, r/c_1)$, then*

$$c_1^{-1} \frac{u(y)}{d(y, \partial\Omega)} \le |\nabla u(y)| \le c_1 \frac{u(y)}{d(y, \partial\Omega)}.$$

Proof. Let c_* be the constant in (61) and choose $c' \ge 1000 c_*$ so that if $x \in \Omega \cap B(w, r/c')$, $s = 4c_* d(x, \partial\Omega)$, and $z \in \partial\Omega$ with $!x - z| = d(x, \partial\Omega)$, then

$$\max_{B(z,4s)} u \le cu(x) \tag{63}$$

for some $c = c(p, n)$, which is possible thanks to Lemma 1.5. From Definition B with w, r replaced by $z, 4s$, there exists a plane $P = P(z, 4s)$ with

$$\Psi(\partial\Omega \cap B(z, 4s), P \cap B(z, 4s)) \le 4\delta s.$$

Since the p Laplacian is invariant under rotations, and translations, we may assume that $z = 0$ and $P = \{y : y_n = 0\}$, Also, if

$$G = \{y \in B(0, 2s) : y_n > 8\delta s\}$$

then we may assume

$$G \subset \Omega \cap B(0, 2s).$$

Let v be the p harmonic function in G with boundary values in the Sobolev sense as follows: $v \le u$ on ∂G and

$$\begin{aligned}
&v(y) = u(y) \text{ when } y \in \partial G \text{ and } y_n > 32\delta s, \\
&v(y) = (\tfrac{y_n}{16\delta s} - 1)\, u(y) \text{ when } y \in \partial G \text{ and } 16\delta s < y_n \le 32\delta s\ . \\
&v(y) = 0 \text{ when } y \in \partial G \text{ and } y_n < 16\delta s\ .
\end{aligned}$$

From Lemma 1.1 (i.e., the maximum principle for p harmonic functions) we have $v \le u$ in G. Also, since each point of ∂G where $u(x) \ne v(x)$ lies within $100\delta s$ of a point where u is zero, it follows from (63) and Lemma 1.4 that $u \le v + c\delta^\beta u(x)$ on ∂G. Using Lemma 1.1 we conclude that

$$v \le u \le v + c\delta^\alpha u(x) \text{ in } G.$$

Thus,

$$1 \leq \frac{u(y)}{v(y)} \leq (1 - c\delta^{\alpha})^{-1}, y \in B(x, d(x, \partial G)/4). \tag{64}$$

From (64) and (61) with $\hat{u}$ replaced by v, we conclude that the hypotheses of Lemma 3.3 are satisfied with $O = G$. Applying Lemma 3.3 and using $d(x, \partial G) \approx d(x, \partial \Omega)$, we obtain Lemma 3.4. □

4.4 Proof of Step 3

In the proof of Step 3 we shall need the following lemma.

Lemma 3.5. *Let $\Omega \subset \mathbf{R}^n$ be (δ, r_0) Reifenberg flat. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose $u > 0$ is p harmonic in $\Omega \cap B(w, 2r)$, continuous in $B(w, 2r)$, with $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. Given $\epsilon > 0$, there exist $\hat{\delta} = \hat{\delta}(p, n, \epsilon) > 0$ and $c = c(p, n, \epsilon)$, $1 \leq c < \infty$, such that*

$$c^{-1} \left(\frac{\hat{r}}{r}\right)^{1+\epsilon} \leq \frac{u(a_{\hat{r}}(w))}{u(a_r(w))} \leq c \left(\frac{\hat{r}}{r}\right)^{1-\epsilon}$$

whenever $0 < \delta \leq \hat{\delta}$ and $0 < \hat{r} < r/4$.

Proof. Lemma 3.5 can be proved by a barrier type argument, using barriers which vanish on the boundary of certain cones or by an iterative type argument using Reifenberg flatness of $\partial\Omega$. We omit the details. □

Lemma 3.6. *Let Ω be (δ, r_0) Reifenberg flat. Let $w \in \partial\Omega, 0 < r < r_0$, and let $u > 0$ be p harmonic in $\Omega \cap B(w, 2r)$, continuous in $B(w, 2r)$, with $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. There exists $\delta' = \delta'(p, n), c = c(p, n, M) \geq 1$ such that if $0 < \delta < \delta'$, and $\hat{r} = r/c$, then $|\nabla u|^{p-2}$ extends to an $A_2(B(w, \hat{r}))$ weight with constant depending only on p, n.*

Proof. We use Lemma 3.5 to prove Lemma 3.6. Let $\{Q_j(x_j, r_j)\}$ be a Whitney decomposition of $\mathbf{R}^n \setminus \bar{\Omega}$ into open cubes with center at x_j and sidelength r_j. Then

$$\cup_j \bar{Q}(x_j, r_j) = \mathbf{R}^n \setminus \bar{\Omega}$$

$$Q(x_j, r_j) \cap Q(x_i, r_i) = \emptyset \; i \neq j$$

$$10^{-4n} d(Q_j, \partial\Omega) \leq r_j \leq 10^{-2n} d(Q_j, \partial\Omega).$$

Let $\hat{r} = r/c^2$, where c is so large that

$$c^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq c \frac{u(x)}{d(x, \partial\Omega)} \tag{65}$$

whenever $x \in \Omega \cap B(w, c\hat{r})$. Existence of c follows from Lemma 3.4. We also assume c is so large that if $Q_j \cap B(w, 4\hat{r}) \neq \emptyset$, then $\bar{Q}_j \subset B(w, c\hat{r})$ and there exists $w_j \in \Omega \cap B(w, c\hat{r})$ with

$$|x_j - w_j| \approx d(x_j, \partial\Omega) \approx d(w_j, \partial\Omega). \tag{66}$$

Existence follows from the fact that Ω is an NTA domain.

Let

$$\lambda(x) = |\nabla u(x)|^{p-2},\ x \in \Omega \cap B(w, 4\hat{r})$$

$$\lambda(x) = |\nabla u(w_j)|^{p-2},\ x \in Q_j \cap B(w, 4\hat{r}).$$

From (66) we see that

$$\lambda(x) = \lambda(w_j) \approx \lambda(z) \tag{67}$$

whenever $x \in Q_j$ and $z \in B(w_j, d(w_j, \partial\Omega)/2)$.

To complete the proof of Lemma 3.6 we prove that λ satisfies the A_2 condition given in Sect. 2.2. Let $\tilde{w} \in B(w, \hat{r})$ and $0 < \tilde{r} < \hat{r}$. We consider several cases. If $\tilde{r} < d(\tilde{w}, \partial\Omega)/2$, then the A_2 condition follows from (65) and Harnack's inequality. On the other hand, if $\tilde{r} \geq d(\tilde{w}, \partial\Omega)/2$ then we choose $z \in \partial\Omega$ with $d(\tilde{w}, \partial\Omega) = |\tilde{w} - z|$ and thus

$$B(\tilde{w}, \tilde{r}) \subset B(z, 3\tilde{r}) \subset B(\tilde{w}, 8\tilde{r}).$$

First suppose $p > 2$. From Hölder's inequality, Lemmas 1.2, 1.3, and (67) we see that

$$\begin{aligned} \int_{B(\tilde{w},\tilde{r})} \lambda dx \leq \int_{B(z,3\tilde{r})} \lambda dx &\leq c \int_{\Omega\cap B(z,c^*\tilde{r})} |\nabla u|^{p-2} dx \\ &\leq c \left(\int_{\Omega\cap B(z,c^*\tilde{r})} |\nabla u|^p dx \right)^{(1-2/p)} \tilde{r}^{2n/p} \\ &\leq c u(a_{\tilde{r}}(z))^{p-2} \tilde{r}^{n+2-p}. \end{aligned} \tag{68}$$

Let $\eta = \min\{1, |p-2|^{-1}\}/20$. To estimate the integral in involving λ^{-1} observe from Lemma 3.5 and once again Harnack's inequality, that if $y \in \Omega \cap B(z, c^*\tilde{r})$, and δ' is small, then

$$c u(y) \geq u(a_{\tilde{r}}(z)) \left(\frac{d(y, \partial\Omega)}{\tilde{r}} \right)^{1+\eta}. \tag{69}$$

Therefore, using (67) and (69) we obtain

$$\int_{B(\tilde{w},\tilde{r})} \lambda^{-1} dx \leq c\,\tilde{r}^{(1+\eta)(p-2)} u(a_{\tilde{r}}(z))^{2-p} \times \int_{\Omega\cap B(z,c^*\tilde{r})} d(y,\partial\Omega)^{-\eta(p-2)} dy. \tag{70}$$

To estimate the integral involving the distance function in (70) set

$$I(z,s) = \int_{\Omega\cap B(z,s)} d(y,\partial\Omega)^{-\eta(p-2)} dy$$

whenever $z \in \partial\Omega \cap B(w,r)$, $0 < s < r$. Let

$$E_k = \Omega \cap B(z,s) \cap \{y : d(y,\partial\Omega) \leq \delta^k s\}$$

for $k = 1, 2, \ldots$ We claim that

$$\int_{E_k} dy \leq c_+^k \delta^k s^n \text{ for } k = 1, 2, \ldots \tag{71}$$

where $c_+ = c_+(p,n)$. Indeed, from δ Reifenberg flatness it is easily seen that this statement holds for E_1. Moreover, E_1 can be covered by at most c/δ^{n-1} balls of radius $100\delta s$ with centers in $\partial\Omega \cap B(z,s)$. We can then repeat the argument in each ball to get that (71) holds for E_2. Continuing in this way we get (71) for all positive integers k. Using (71) and writing $I(z,s)$ as a sum over $E_k \setminus E_{k+1}, k = 1, 2, \ldots$ we get

$$I(z,s) \leq cs^{n-\eta(p-2)} + \delta^{\eta(p-2)-n} \sum_{k=1}^{\infty} (c_+^k \delta^k s)^{n-\eta(p-2)} < \tilde{c}s^{n-\eta(p-2)},$$

where $\tilde{c} = \tilde{c}(p,n)$, provided δ' is small enough. Using this estimate with $s = \tilde{r}$, we can continue our calculation in (70) and conclude that

$$\int_{B(\tilde{w},\tilde{r})} \lambda^{-1} dx \leq c\,\tilde{r}^{n+p-2}\, u(a_{\tilde{r}}(z))^{(2-p)} \tag{72}$$

To complete the proof of Lemma 3.6 in the case $p > 2$, we simply combine (68) and (72). Note that the case $p = 2$ is trivial and in case $p < 2$ the argument above can be repeated with $p - 2$ replaced by $2 - p < p$. Hence Lemma 3.6 and Step 3 are complete. $\square$

4.5 Proof of Step 4 and Theorem 3.1

To justify the claims in Step 4, first choose $c' = c'(p,n)$ so large that if $r^* = r/c'$, then

$$\max_{\Omega\cap B(w,4r^*)} h \leq ch(a_{r^*}(w)) \tag{73}$$

for some $c = c(p,n)$ whenever $h > 0$ is p harmonic in $\Omega \cap B(w,2r)$. This choice is possible as we see from Lemma 1.5 and Harnack's inequality in Lemma 1.2. We also suppose $c' > 1000c_1$, where c_1 is the constant in Lemma 3.4. Hence from Lemma 3.4 we have

$$c^{-1}\frac{k(y)}{d(y,\partial\Omega)} \leq |\nabla k(y)| \leq c\frac{k(y)}{d(y,\partial\Omega)}. \tag{74}$$

whenever $y \in \Omega \cap B(w, r^*/c')$ and $k > 0$ is p harmonic in $\Omega \cap B(w,2r^*)$, continuous in $B(w,2r^*)$, with $k \equiv 0$ on $B(w,2r^*)\setminus\Omega$. We temporarily assume that

$$0 < u \leq v/2 \leq c \text{ on } \Omega \cap B(w,4r^*). \tag{75}$$

and also that

$$c^{-1} \leq u(a_{r^*}(w))\,. \tag{76}$$

Next if $t \in [0,1]$, let $\tilde{u}(\cdot,t)$ be the p harmonic function in $\Omega \cap B(w,2r^*)$ with continuous boundary values,

$$\tilde{u}(\cdot,t) = (1-t)u + tv \tag{77}$$

on $\partial[\Omega \cap B(w,2r^*)]$. Extend $\tilde{u}(\cdot,t), t \in (0,1)$, to be Hölder continuous in $B(w,2r^*)$ by putting $\tilde{u}(\cdot,t) \equiv 0$ on $B(w,2r^*) \setminus \Omega$. Let $r' = r^*/c'$ and observe that (74) holds whenever $k = u(\cdot,t), t \in [0,1]$, on $\Omega \cap B(w,r')$. Thus from Lemma 1.3, we see that $\tilde{u}(\cdot,t)$ is infinitely differentiable in $B(w,r')$ and so

$$\nabla\left(\cdot|\nabla\tilde{u}(\cdot,t)|^{p-2}\nabla\tilde{u}(\cdot,t)\right) = 0 \tag{78}$$

in $\Omega \cap B(w,r')$. Set

$$U(x) = U(x,t,\tau) = \frac{\tilde{u}(x,t) - \tilde{u}(x,\tau)}{t-\tau}\ .$$

and note from (75), (76), for fixed $t,\tau \in [0,1]$, $t \neq \tau$, that

$$0 \leq v/2 \leq U(x) = v - u \leq v \leq c(p,n) \tag{79}$$

on $\partial(\Omega \cap B(w,2r^*))$, so by Lemma 1.1 we have

$$0 \leq U \leq c \text{ in } \Omega \cap B(w,r'),\ U \equiv 0 \text{ on } \partial\Omega \cap B(w,r'). \tag{80}$$

From (80) we see for fixed $x \in \Omega \cap B(w, 2r^*)$ that $t \to \tilde{u}(x,t)$, is Lipschitz with norm $\leq c$. Thus $\tilde{u}_\tau(x, \cdot)$ exists almost everywhere in [0,1]. Let (x_ν) be a dense sequence of $\Omega \cap B(w, 2r^*)$ and let W be the set of all $t \in [0,1]$ for which $\tilde{u}_t(x_m, \cdot)$ exists, in the sense of difference quotients, whenever $x_m \in (x_\nu)$. We note that $H^1([0,1] \setminus W) = 0$ where H^1 is one-dimensional Hausdorff measure. Next we note from (79) that for $t \in (0,1]$

$$\tilde{u}(\cdot, t)/2 \leq U(\cdot, t, \tau) \leq t^{-1} \tilde{u}(\cdot, t) \tag{81}$$

on $\partial\Omega \cap B(w, 2r^*)$, so by Lemma 1.1, this inequality also holds in $\Omega \cap B(w, 2r^*)$. To find a divergence form PDE that U satisfies let $\xi = (\xi_1, \ldots, \xi_n)$, $w = (w_1, \ldots, w_n) \in \mathbf{R}^n \setminus \{0\}$, and $1 \leq i \leq n$. Then

$$\begin{aligned}
&|\xi|^{p-2}\, \xi_i - |w|^{p-2}\, w_i \\
&\quad = \int_0^1 \tfrac{d}{d\lambda} \{ |\lambda \xi + (1-\lambda) w|^{p-2} \, [\lambda \xi_i + (1-\lambda) w_i] \} d\lambda \\
&\quad = \sum_{j=1}^n (\xi - w)_j \, \left(\int_0^1 a_{ij}[\lambda \xi + (1-\lambda) w] d\lambda \right),
\end{aligned}$$

where for $1 \leq i, j \leq n$, $\eta \in \mathbf{R}^n \setminus \{0\}$,

$$a_{ij}(\eta) = |\eta|^{p-4} \, [(p-2)\eta_i \, \eta_j \; + \; \delta_{ij} \, |\eta|^2]. \tag{82}$$

In this display δ_{ij}, once again, denotes the Kronecker delta. Using (78), (80)–(82) we find for fixed t, τ that if

$$\begin{aligned}
A_{ij}(x) &= A_{ij}(x, t, \tau) \\
&\quad = \int_0^1 a_{ij} [\lambda \nabla \tilde{u}(x,t) + (1-\lambda) \nabla \tilde{u}(x, \tau)] \, d\lambda,
\end{aligned}$$

then, for $x \in \Omega \cap B(w, r')$, $t, \tau \in [0,1]$,

$$\tilde{L} U(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} [A_{ij}(x) \, U_{x_j}] = 0. \tag{83}$$

Moreover, if $x \in \Omega \cap B(w, r')$, then

$$\begin{aligned}
&c^{-1} |\xi|^2 ||\nabla \tilde{u}(x,t)| + |\nabla \tilde{u}(x, \tau)||^{p-2} \\
&\qquad \leq \sum_{i,j=1}^n A_{ij}(x) \xi_i \xi_j \text{ whenever } \xi \in \mathbf{R}^n \setminus \{0\}.
\end{aligned} \tag{84}$$

Also,

$$\sum_{i,j=1}^{n} |A_{ij}(x)| \leq c||\nabla \tilde{u}(x,t)| + |\nabla \tilde{u}(x,\tau)||^{p-2}, \tag{85}$$

where c depends only on p, n. From Lemma 1.3, (82)–(85), and (74) for $\tilde{u}(\cdot,t), \tilde{u}(\cdot,\tau)$, we see that U is a solution on $\Omega \cap B(w,r')$ to a locally uniformly elliptic divergence form PDE with C^∞ coefficients. Since $\tilde{u}(x,t) - \tilde{u}(x,\tau) = (t-\tau)(v(x)-u(x))$ on $\partial(\Omega \cap B(w,2r))$ it follows from the maximum principle for p harmonic functions that $\tilde{u}(x,\tau) \to \tilde{u}(x,t)$ uniformly in the closure of $\Omega \cap B(x,2r^*)$ as $\tau \to t$. Also from Lemma 1.2 we deduce that $\nabla\tilde{u}(\cdot,\tau) \to \nabla\tilde{u}(\cdot,t)$ on compact subsets of $\Omega \cap B(w,2r^*)$. Using these facts and (74) we see for $1 \leq i,j \leq n$,

$$A_{i,j}(x,t,\tau) \to \tilde{b}_{ij}(x) \tag{86}$$

as $\tau \to t$ uniformly on compact subsets of $\Omega \cap B(w,r')$, where $\tilde{b}_{ij}$ are defined as in (9) relative to $\tilde{u}(\cdot,t)$. Finally we note from linear elliptic PDE theory that $U(\cdot,t,\tau)$ is locally in $W^{1,2}$ and locally Hölder continuous on $\Omega \cap B(w,r')$ with norms independent of τ. From (81) we see that $U(\cdot,t,\tau), t \in (0,1]$, has a Hölder continuous extension to $B(w,r')$ obtained by putting $U \equiv 0$ in $B(w,r') \setminus \Omega$. Also Hölder constants can be chosen independent of τ for τ near t. Using these facts we see for fixed $t \in (0,1]$, that there is a sequence $U(\cdot,t,\tau_k) \to f(\cdot,t)$ on $\Omega \cap B(w,r')$ as $\tau_k \to t$. Put $f \equiv 0$ on $B(w,r') \setminus \Omega$. Then from (80)–(86), Lemma 1.2, and Schauder estimates, we conclude that f has the following properties:

$$\begin{aligned}
&(a)\ \tilde{L}f = 0 \text{ in } \Omega \cap B(w,r') \ \text{ where } \tilde{L} \text{ is as in (48)},\\
&(b)\ f \text{ is continuous in } B(w,r') \text{ with } f \equiv 0 \text{ on } B(w,r') \setminus \Omega,\\
&(c)\ f(x_m,t) = \tilde{u}_t(x_m,t) \text{ when } x_m \in (x_\nu),\ t \in W,\\
&(d)\ u/2 \leq \tilde{u}(\cdot,t)/2 \leq f(\cdot,t) \leq c \text{ in } \Omega \cap B(w,r'),\\
&(e)\ f \in C^\infty[\Omega \cap B(w,r')].
\end{aligned} \tag{87}$$

From (87) (c) we get

$$\ln\left(\frac{v(x_m)}{u(x_m)}\right) = \ln\left(\frac{\tilde{u}(x_m,1)}{\tilde{u}(x_m,0)}\right) = \int_0^1 \frac{f(x_m,t)}{\tilde{u}(x_m,t)}\,dt \tag{88}$$

whenever $x_m \in (x_\nu)$. Since this sequence is dense in $\Omega \cap B(w,r')$, we conclude from (87), (88) that Claims (48), (89) are true. From (88) and Lemmas 3.4, 3.6 we see that $\tilde{u}_\tau(\cdot,\tau), \tilde{u}(\cdot,\tau)$ are solutions to a degenerate divergence form elliptic PDE whose degeneracy is given in terms of an A_2 weight. Thus Lemma 1.8 can be used with r replaced by $\tilde{r} = \min(r',\hat{r})$, $v_1 = f(\cdot,t)$, $v_2 = \tilde{u}(\cdot,t)$, where f is as in (87), (88). Let $r'' = \tilde{r}/c$ where c is the constant in Lemma 1.8. From (88)(d), (76), and Harnack's inequality we get

$$c^{-1} \leq \frac{f(a_{r''}(w),t)}{\tilde{u}(a_{r''}(w),t)} \leq c \tag{89}$$

where c depends only on p, n. Using this fact and Lemma 1.8, we find that

$$\frac{f(y,t)}{\tilde{u}(y,t)} \leq c.$$

whenever $y \in \Omega \cap B(w, r'')$ and thereupon

$$\left| \frac{f(z,t)}{\tilde{u}(z,t)} - \frac{f(y,t)}{\tilde{u}(y,t)} \right| \leq c' \left(\frac{|z-y|}{r} \right)^{\alpha} \tag{90}$$

whenever $y, z \in \Omega \cap B(w, r'')$. Hence,

$$\begin{aligned} &\left| \log\left(\frac{v(y)}{u(y)}\right) - \log\left(\frac{v(z)}{u(z)}\right) \right| \\ &\quad \leq \int_0^1 \left| \frac{f(z,t)}{\tilde{u}(z,t)} - \frac{f(y,t)}{\tilde{u}(y,t)} \right| dt \leq c' \left(\frac{|z-y|}{r} \right)^{\alpha}. \end{aligned} \tag{91}$$

From (91) we conclude that Theorem 3.1 is valid under assumptions (75), (76). To remove (75), (76), suppose u, v are as in Theorem 1.3. We assume as we may that

$$1 = u(a_{r^*}(w)) = v(a_{r^*}(w)) \tag{92}$$

since otherwise we divide u, v by their values at this point and use invariance of the p Laplacian under scaling. Let $\tilde{u}, \tilde{v}$ be p harmonic functions in $\Omega \cap B(w, 2r^*)$, continuous in $\bar{B}(w, 2r^*)$ with

$$\tilde{u} = \min(u,v), \tilde{v} = 2\max(u,v) \text{ on } \partial[B(w,2r^*) \setminus \Omega].$$

Let $r^{**} = r^*/c$. From Lemmas 1.1, 1.5, and (92) we see that

$$\tilde{u} \leq min(u,v) \leq \max(u,v) \leq \tilde{v}/2 \leq c \text{ in } \Omega \cap B(w, 2r^{**}).$$

Also, using (92), Harnack's inequality, and Lemmas 1.4, 1.5 we see that (76) is valid with r^* replaced by r^{**}. Hence (75), (76), are valid with r^*, replaced by r^{**} so Theorem 3.1 can be applied to get

$$\max[v/u, u/v] \leq \tilde{v}/\tilde{u} \leq c \tag{93}$$

in $\Omega \cap B(w, r'')$. It follows for suitably chosen c, that u, cv satisfy (75), (76) in $\Omega \cap B(w, r'')$. Consequently Theorem 3.1 is valid without assumptions (75), (76). $\square$

4.6 More on Boundary Harnack Inequalities

In [62] we generalize Theorem 3.1 to weak solutions u of $\nabla \cdot A(x, \nabla u(x)) = 0$, where $A = (A_1, ..., A_n) : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$. Also $A = A(x, \eta)$ is continuous in $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ and $A(x, \eta)$, for fixed $x \in \mathbf{R}^n$, is continuously differentiable in η_k, for every $k \in \{1, ..., n\}$, whenever $\eta \in \mathbf{R}^n \setminus \{0\}$. Moreover,

(i)
$$\alpha^{-1}|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(x, \eta)\xi_i\xi_j,$$

(ii)
$$\left|\frac{\partial A_i}{\partial \eta_j}(x, \eta)\right| \leq \alpha|\eta|^{p-2}, 1 \leq i, j \leq n,$$

(iii)
$$|A(x, \eta) - A(y, \eta)| \leq \beta|x - y|^{\gamma}|\eta|^{p-1},$$

(iv)
$$A(x, \eta) = |\eta|^{p-1}A(x, \eta/|\eta|).$$

Under these assumptions, we prove

Theorem 3.7. *Let $\Omega \subset \mathbf{R}^n$ be a (δ, r_0)-Reifenberg flat domain. Suppose that u, v are positive A-harmonic functions in $\Omega \cap B(w, 4r)$, that u, v are continuous in $B(w, 4r)$ and $u = 0 = v$ on $B(w, 4r) \setminus \Omega$. For fixed p, α, β, γ, there exist $\tilde{\delta}, \sigma > 0$ and $c_1 \geq 1$, all depending only on $p, n, \alpha, \beta, \gamma$ such that*

$$\left|\log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)}\right| \leq c_1 \left(\frac{|y_1 - y_2|}{r}\right)^{\sigma}$$

whenever $y_1, y_2 \in \Omega \cap B(w, r/c_1)$. Here $w \in \partial\Omega, 0 < r < r_0,$ and $0 < \delta < \tilde{\delta}$.

For completeness we now give the definition of a Lipschitz domain.

Definition C. *$\Omega \subset \mathbf{R}^n$ is said to be a bounded Lipschitz domain provided there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial\Omega$ and such that, for each i,*

$$\Omega \cap B(x_i, 4r_i) = \{y = (y', y_n) \in \mathbf{R}^n : y_n > \phi_i(y')\} \cap B(x_i, 4r_i),$$

$$\partial\Omega \cap B(x_i, 4r_i) = \{y = (y', y_n) \in \mathbf{R}^n : y_n = \phi_i(y')\} \cap B(x_i, 4r_i),$$

in an appropriate coordinate system and for a Lipschitz function ϕ_i. The Lipschitz constants of Ω are defined to be $M = \max_i |||\nabla\phi_i|||_\infty$ and $r_0 = \min r_i$.

In [52] we prove
Theorem 3.8. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u and v are positive p harmonic functions in $\Omega \cap B(w, 2r)$. Assume also that u and v are continuous in $B(w, 2r)$ and $u = 0 = v$ on $B(w, 2r) \setminus \Omega$. Under these assumptions there exist $c_1, 1 \leq c_1 < \infty$, and α, $\alpha \in (0, 1)$, both depending only on p, n, and M, such that if $y_1, y_2 \in \Omega \cap B(w, r/c_2)$ then*

$$\left|\log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)}\right| \leq c_1 \left(\frac{|y_1 - y_2|}{r}\right)^{\alpha}.$$

Remark. To prove Theorem 3.7 we argue as in Steps 1–4. The techniques and arguments are similar to the proof of Theorem 3.1.

In the proof of Theorem 3.8 we follow a somewhat similar game plan although in this case the techniques are necessarily more sophisticated. For example there is no readily available comparison function (x_n in Theorems 3.1 or 3.7) from which we can extrapolate the fundamental inequality in Step 2. Thus we study p harmonic capacitary functions in starlike Lipschitz ring domains and show these functions satisfy the fundamental inequality near boundary points where they vanish. Also, in the Lipschitz case simple examples show $|\nabla u|^{p-2}$ need not extend to an A_2 weight locally, so that Lemma 1.8 cannot be applied. Instead for given $w, r \in \partial\Omega$ we show that $|\nabla u|^{p-2}$ satisfies a Carleson measure condition in a certain starlike Lipschitz subdomain $\tilde{\Omega} \subset \Omega$. This fact and a theorem in [42] (see also [35]) can then be used to conclude that a boundary Harnack inequality (as in Lemma 1.8) holds for solutions to (8), (9) in the Lipschitz case.

Finally we note that Theorems 3.7 and 3.8 are weaker than the corresponding results for $p = 2$. Thus for example does Theorem 3.8 hold in an NTA domain? Can the assumptions on A in Theorem 3.7 be weakened?

4.7 The Martin Boundary Problem

The Martin boundary for harmonic functions was first introduced by Martin [67]. Over the years, it has been of considerable interest to researchers in potential theory. Unfortunately Martin did not receive many accolades for his contribution, as according to math. sci. net, the above is the only paper he ever wrote. In order to define the p Martin boundary of a NTA domain, we need to first define a minimal positive p harmonic function.
Definition D. *Fix $p, 1 < p < \infty$. Then $\hat{u}$ is said to be a minimal positive p harmonic function in the NTA domain Ω relative to $w \in \partial\Omega$, provided $\hat{u} > 0$ is p harmonic in Ω and $\hat{u}$ has continuous boundary value 0 on $\partial\Omega \setminus \{w\}$. $\hat{u}$ is said to be unique up to constant multiples if $\hat{v} = \lambda\hat{u}$, for some constant λ,*

whenever $\hat{v}$ is a minimal positive p harmonic function relative to $w \in \partial\Omega$. The p Martin boundary of Ω is equivalence classes of all minimal positive p harmonic functions defined relative to boundary points of Ω. Two minimal positive p harmonic functions are in the same equivalence class if and only if they are constant multiples of each other. We say that the p Martin boundary of Ω can be identified with $\partial\Omega$ provided each $w \in \partial\Omega$ corresponds to a unique (up to constant multiples) minimal positive p harmonic function.

Theorem 3.9. *Let Ω, δ, p, r_0, be as in Theorem 3.1. There exists $\delta_+ = \delta_+(p,n)$ such that if $0 < \delta < \delta_+$, then the p Martin boundary of Ω can be identified with $\partial\Omega$.*

Remark. Theorem 3.9 for $p = 2$, i.e., harmonic functions, in an NTA domain G is an easy consequence of the boundary Harnack inequality for harmonic functions in NTA domains. Indeed, if $w \in \partial G$ and if u, v are minimal harmonic functions corresponding to w, one first uses the boundary Harnack inequality for harmonic functions to show that $\gamma = \inf_G u/v > 0$. Next one applies this result to $u - \gamma v, v$ in order to conclude that $u = \gamma v$. Note however that this argument depends heavily on linearity of the Laplacian and thus the argument fails for the p Laplacian when $p \neq 2$. We also note that if $r_0 = \infty$ in Theorem 3.1, i.e., Ω is an unbounded Reifenberg flat NTA domain, and u, v are minimal positive p harmonic functions relative to ∞, then we can let $r \to \infty$ in the conclusion of Theorem 3.1, to get $u = \lambda v, \lambda =$ constant. To make this idea work when $w \in \partial\Omega \setminus \{\infty\}$, we need to prove an analogue of Theorem 3.1 for positive p harmonic functions in $\Omega \setminus B(w, r')$, ($r'$ small) vanishing on $\partial\Omega \setminus B(w, r')$. We could get this analogue, by arguing as in the proof of Theorem 3.1, except we do not know a priori, that our functions have the fundamental nondegeneracy property (45) in an appropriate domain. We overcome this deficiency, using arguments from the proof of Theorem 3.1, as well as an induction—bootstrap type argument. We start by showing that if one such p harmonic function has the fundamental nondegeneracy property then all such functions have this property.

Lemma 3.10. *Let Ω be (δ, r_0) Reifenberg flat. Let $\hat{u}, \hat{v} > 0$ be p harmonic in $\Omega \setminus B(w, r')$, continuous in $\mathbf{R}^n \setminus B(w, r')$, with $\hat{u} \equiv \hat{v} \equiv 0$ on $\mathbf{R}^n \setminus [\Omega \cup B(w, r')]$. Suppose for some $r_1, r' < r_1 < r_0$, and $A \geq 1$, that*

$$A^{-1}\,\frac{\hat{u}(x)}{d(x,\partial\Omega)} \leq |\nabla \hat{u}(x)| \leq A\frac{\hat{u}(x)}{d(x,\partial\Omega)}.$$

whenever $x \in \Omega \cap [B(w, r_1) \setminus B(w, r')]$. There exists $\alpha > 0, \lambda, c \geq 1$, depending on p, n, A, such that if $0 < \delta < \hat{\delta}$ ($\hat{\delta}$ as in Theorem 3.1),

$$\lambda^{-1}\,\frac{\hat{v}(x)}{d(x,\partial\Omega)} \leq |\nabla \hat{v}(x)| \leq \lambda\frac{\hat{v}(x)}{d(x,\partial\Omega)}.$$

for $x \in \Omega \cap [B(w, r_1/c) \setminus B(w, cr')]$. Moreover,

$$\left|\log\left(\frac{\hat{u}(z)}{\hat{v}(z)}\right) - \log\left(\frac{\hat{u}(y)}{\hat{v}(y)}\right)\right| \leq c\left(\frac{r'}{\min(r_1, |z-w|, |z-y|)}\right)^{\alpha}$$

whenever $z, y \in \Omega \setminus B(w, cr')$.

Proof. We assume that $r'/r_1 << 1$, since otherwise there is nothing to prove. Let $\tilde{r} = \hat{c}r'$. If $\hat{c} = \hat{c}(p, n)$ is large enough, we may assume

$$\hat{u} \leq \hat{v}/2 \leq \hat{c}u \text{ in } \Omega \setminus B(w, \tilde{r}), \tag{94}$$

as we see from Theorem 3.1, Harnack's inequality, and the maximum principle. As in Step 4 of Sect. 1, let $u(\cdot, t), t \in [0, 1]$, be p harmonic in $\Omega \setminus B(w, \tilde{r})$, with continuous boundary values,

$$u(\cdot, t) = (1-t)\hat{u} + t\hat{v} \text{ on } \partial[\Omega \setminus B(w, \tilde{r})]. \tag{95}$$

Extend $u(\cdot, t), t \in [0, 1]$, to be continuous on $\mathbf{R}^n \setminus [\Omega \cup B(w, \tilde{r})]$ by setting $u(\cdot, t) \equiv 0$ on this set. Next we note from Lemma 3.3 that there exists $\epsilon_0 = \epsilon_0(p, n, M, A)$ such that if $\tilde{r} \leq s_1 < \rho_1/4 \leq r_1/16$, $\tau \in (0, 1]$, and

$$(1-\epsilon_0)\tilde{L} \leq u(\cdot, \tau)/\hat{u} \leq (1+\epsilon_0)\tilde{L}, \tag{96}$$

in $\Omega \cap [B(w, 2\rho_1) \setminus B(w, s_1)]$, for some $\tilde{L}$, then

$$\hat{\lambda}^{-1}\,\frac{u(x, \tau)}{d(x, \partial\Omega)} \leq |\nabla u(x, \tau)| \leq \hat{\lambda}\frac{u(x, \tau)}{d(x, \partial\Omega)} \tag{97}$$

in $\Omega \cap [B(w, \rho_1) \setminus B(w, 2s_1)]$ where $\hat{\lambda} = \hat{\lambda}(p, n, A)$. Observe from (94), (95), that if $\tau_1, \tau_2 \in [0, 1]$,

$$\begin{aligned} c^{-1}u(\cdot, \tau_1) \leq U(\cdot, \tau_1, \tau_2) = {} & \frac{u(\cdot, \tau_2) - u(\cdot, \tau_1)}{\tau_2 - \tau_1} \\ = {} & v - u \leq c\,u(\cdot, \tau_1) \end{aligned} \tag{98}$$

on $\partial[\Omega \setminus B(w, \tilde{r})]$, so from the maximum principle this inequality also holds in $\Omega \setminus B(w, \tilde{r})$. Thus for ϵ_0 as in (96), there exists $\epsilon_0', 0 < \epsilon_0' \leq \epsilon_0$, with the same dependence as ϵ_0, such that if $|\tau_2 - \tau_1| \leq \epsilon_0'$, then

$$1 - \epsilon_0/2 \leq \frac{u(\cdot, \tau_2)}{u(\cdot, \tau_1)} \leq 1 + \epsilon_0/2 \text{ in } \Omega \setminus B(w, \tilde{r}). \tag{99}$$

Divide [0,1] into closed intervals, disjoint except for endpoints, of length $\epsilon_0'/2$ except possibly for the interval containing 1 which is of length $\leq \epsilon_0'/2$. Let $\xi_1 = 0 < \xi_2 < \ldots < \xi_m = 1$ be the endpoints of these intervals. Thus [0,1] is divided into $\{[\xi_k, \xi_{k+1}]\}_1^m$. Next suppose for some $l, 1 \leq l \leq m-1$, that (97) is valid whenever $\tau \in [\xi_l, \xi_{l+1}]$ and $x \in \Omega \cap [B(w, \rho_1) \setminus B(w, 2s_1)]$. Under this assumption we claim for some $\hat{c}_1, \hat{c}_2, \alpha$, depending only on p, n, A that

$$\left| \log \frac{u(z, \xi_{l+1})}{u(z, \xi_l)} - \log \frac{u(y, \xi_{l+1})}{u(y, \xi_l)} \right| \leq \hat{c}_1 \left(\frac{s_1}{\min\{|z-w|, |w-y|\}} \right)^{\alpha} \tag{100}$$

whenever $z, y \in \Omega \cap [B(w, \rho_1/\hat{c}_2) \setminus B(w, \hat{c}_2 s_1)]$.

Indeed we can retrace the argument in Theorem 3.1 to get for $z, y \in \Omega \cap [B(w, \rho_1/c) \setminus B(w, cs_1)]$, that there exists f as in (87) with

$$\begin{aligned} &\left| \log \frac{u(z, \xi_{l+1})}{u(z, \xi_l)} - \log \frac{u(y, \xi_{l+1})}{u(y, \xi_l)} \right| \\ &\qquad \leq \int_{\xi_l}^{\xi_{l+1}} \left| \frac{f(z,\tau)}{u(z,\tau)} - \frac{f(y,\tau)}{u(y,\tau)} \right| d\tau \leq c \left(\frac{s_1}{\min\{|z-w|, |y-w|\}} \right)^{\alpha}. \end{aligned} \tag{101}$$

The last inequality in (101) follows from a slightly more general version of Lemma 1.8.

We now proceed by induction. Observe from (99) as well as $u(\cdot, \xi_1) = \hat{u}$ that (96) holds whenever $\tau \in [\xi_1, \xi_2]$. Thus (97) and consequently (100) are true for $l = 1$ with $s_1 = \tilde{r}, \rho_1 = r_1/4$. Let $s_2 = \hat{c}_2 s_1$, $\rho_2 = \rho_1/\hat{c}_2$. By induction, suppose for some $2 \leq k < m$,

$$\left| \log \frac{u(z, \xi_k)}{\hat{u}(z)} - \log \frac{u(y, \xi_k)}{\hat{u}(y)} \right| \leq (k-1)\hat{c}_1 \left(\frac{s_k}{\min\{|z-w|, |y-w|\}} \right)^{\alpha} \tag{102}$$

whenever $z, y \in \Omega \cap [B(w, \rho_k) \setminus B(w, s_k)]$, where $\alpha, \hat{c}_1$ are the constants in (100). Choose $s'_k \geq 2s_k$, so that

$$\left| \frac{u(z, \xi_k)}{\hat{u}(z)} - \frac{u(y, \xi_k)}{\hat{u}(y)} \right| \leq \eta \frac{u(z, \xi_k)}{\hat{u}(z)}$$

whenever $z, y \in \Omega \cap [B(w, \rho_k) \setminus B(w, s'_k)]$. Fix z as above and choose $\eta > 0$ so small that

$$(1 - \epsilon_0) \frac{u(z, \xi_k)}{\hat{u}(z)} \leq \frac{u(y, \tau)}{\hat{u}(y)} \leq (1 + \epsilon_0) \frac{u(z, \xi_k)}{\hat{u}(z)} \tag{103}$$

whenever $y \in \Omega \cap [B(w, \rho_k) \setminus B(w, s'_k)]$ and $\tau \in [\xi_k, \xi_{k+1}]$. To see the size of η observe for $\tau \in [\xi_k, \xi_{k+1}]$ that

$$\begin{aligned} \frac{u(y,\tau)}{\hat{u}(y)} &= \frac{u(y,\tau)}{u(y,\xi_k)} \cdot \frac{u(y,\xi_k)}{\hat{u}(y)} \\ &\leq (1 + \epsilon_0/2)(1+\eta) \frac{u(z,\xi_k)}{\hat{u}(z)}. \end{aligned}$$

Thus if $\eta = \epsilon_0/4$ (ϵ_0 small), then the right hand inequality in (103) is valid. A similar argument gives the left hand inequality in (103) when $\eta = \epsilon_0/4$. Also since $k \leq 2/\epsilon_0'$, and ϵ_0', α depend only on p, n, M, A, we deduce from (102) that one can take $s_k' = \hat{c}_3 s_k$ for $\hat{c}_3 = \hat{c}(p, n, M, A)$ large enough. From (103) we find that (96) holds with $\tilde{L} = \frac{u(z,\xi_k)}{\hat{u}(z)}$ in $\Omega \cap [B(w, \rho_k) \setminus B(w, s_k')]$. From (97) we now get that (100) is valid for $l = k$ in $\Omega \cap [B(w, \frac{\rho_k}{2\hat{c}_2}) \setminus B(w, \hat{c}_2 s_k')]$. Let $s_{k+1} = \hat{c}_3 \hat{c}_2 s_k$ and $\rho_{k+1} = \frac{\rho_k}{2\hat{c}_2}$. Using (100) and the induction hypothesis we have

$$\begin{aligned}\left|\log \frac{u(z,\xi_{k+1})}{\hat{u}(z)} - \log \frac{u(y,\xi_{k+1})}{\hat{u}(y)}\right| &\leq \left|\log \frac{u(z,\xi_{k+1})}{u(z,\xi_k)} - \log \frac{u(y,\xi_{k+1})}{u(y,\xi_k)}\right| \\ + \left|\log \frac{u(z,\xi_k)}{\hat{u}(z)} - \log \frac{u(y,\xi_k)}{\hat{u}(y)}\right| &\leq k\hat{c}_1 \left(\frac{s_{k+1}}{\min\{|z-w|, |w-y|\}}\right)^{\alpha}\end{aligned} \tag{104}$$

whenever $z, y \in \Omega \cap [B(w, \rho_{k+1}) \setminus B(w, s_{k+1})]$. Thus by induction we get (102) with $k = m$. Since $u(\cdot, \xi_m) = \hat{v}$ and $s_m \leq cr'$, $\rho_m \geq r_1/c$, for some large $c = c(p, n, A)$, we can now argue as in (103) to first get (96) with $u(\cdot, \tau)$ replaced by $\hat{v}$ and then (97) for $\hat{v}$. We conclude that Lemma 3.10 is valid for $z, y \in \Omega \cap [B(w, r_1/c) \setminus B(w, cr')]$ provided c is large enough. Using the maximum principle it follows that the last display in Lemma 3.49 is also valid for $z, y \in \Omega \setminus B(w, r_1/c)$. □

4.8 Proof of Theorem 3.9

Note that if $\hat{u}$ is a minimal p harmonic function satisfying (45) in $\Omega \cap B(w, r_1)$, then one can let $r' \to 0$ in Lemma 3.10 to get Theorem 3.9. Thus to complete the proof of Theorem 3.9 it suffices to show the existence of a minimal positive p harmonic function $\hat{u}$ relative to $w \in \partial\Omega$ and $0 < r_1 < r_0$ for which the fundamental nondegeneracy property in (45) holds in $\Omega \cap B(w, r_1)$. To this end we introduce,

Definition E. *We call $\tilde{\Omega} \subset \Omega$ a non tangential approach region at $w \in \partial\Omega$ if for some $\tilde{\eta} > 0$, $d(x, \partial\Omega) \geq \tilde{\eta}|x - w|$ for all $x \in \tilde{\Omega}$.*

If we wish to emphasize $w, \tilde{\eta}$ in Definition E we write $\tilde{\Omega}(w, \tilde{\eta})$. Now let $\hat{u}$ be a minimal positive p harmonic function in Ω relative to $w \in \partial\Omega$ and $0 < \delta < \delta^*$. Then we can apply Lemma 3.3 to conclude for each $\hat{x} \in \partial\Omega \setminus \{w\}$ that for some $\tilde{c} = \tilde{c}(p, n) \geq 1$,

$$\tilde{c}^{-1} \frac{\hat{u}(x)}{d(x, \partial\Omega)} \leq |\nabla \hat{u}(x)| \leq \tilde{c} \frac{\hat{u}(x)}{d(x, \partial\Omega)} \tag{105}$$

whenever $x \in \partial\Omega \cap B(\hat{x}, |\hat{x} - w|/\tilde{c}) \cap B(w, r_0)$. Using this fact we see that if $0 < \delta_+ < \delta^*$ in Theorem 3.9 then there exists $\tilde{\eta}$ depending only on p, n, such that

$$\hat{u} \text{ satisfies (105) in } [\Omega \setminus \tilde{\Omega}(w, \tilde{\eta})] \cap B(w, r_0). \tag{106}$$

From (106) we see that if (105) holds in $\Omega(w, \tilde{\eta}) \cap B(w, r_1)$, then $\hat{u}$ can be used in Lemma 3.10 for each small $r' > 0$ so Theorem 3.9 is true. To prove the above statement, we first extend $\hat{u}$ continuously to $\mathbf{R}^n \setminus \{w\}$ by letting $\hat{u} = 0$ in $\mathbf{R}^n \setminus \Omega$. Let $0 < r < r_0/n$ and $\sigma = 100n\delta$. Using translation, rotation invariance of the p Laplacian and Reifenberg flatness in Definition B, we assume as we may that $w = 0$ and for given $\sigma > 0$ (sufficiently small) that

$$\begin{aligned} &B(0, nr) \cap \{y : y_n \geq \sigma r\} \subset \Omega \\ &B(0, nr) \cap \{y : y_n \leq -\sigma r\} \subset \mathbf{R}^n \setminus \Omega \end{aligned} \tag{107}$$

Extend $\hat{u}$ to a continuous function in $\mathbf{R}^n \setminus \{0\}$ by putting $\hat{u} \equiv 0$ on $\mathbf{R}^n \setminus (\Omega \cup \{0\})$. Let

$$Q = \{y : |y_i| < r, 1 \leq i \leq n-1\} \cap \{y : \sigma r < y_n < r\} \setminus B(0, \sqrt{\sigma}\, r\,)$$

and let v_1 be the p harmonic function in Q with the following continuous boundary values,

$$\begin{aligned} v_1(y) &= \hat{u}(y) \quad \text{if } y \in \partial Q \cap \{y : 2\sigma r \leq y_n\}, \\ v_1(y) &= \frac{(y_n - \sigma r)}{\sigma r}\hat{u}(y) \quad \text{if } y \in \partial Q \cap \{y : \sigma r \leq y_n < 2\sigma r\}\,. \end{aligned}$$

Comparing boundary values and using the maximum principle for p harmonic functions, we deduce

$$v_1 \leq \hat{u} \text{ in Q}. \tag{108}$$

Let $\sigma(\epsilon) = \exp(-1/\epsilon)$. To complete the proof of Theorem 3.9 we will make use of the following lemmas.

Lemma 3.11. *Let $0 < \epsilon \leq \hat{\epsilon}$, let $\sigma = \sigma(\epsilon)$ be as above and let $\tilde{\eta}$ be as in (106). If $\hat{\epsilon}$ is small enough, then there exists $\hat{\theta} = \hat{\theta}(p, n), 0 < \hat{\theta} \leq 1/4$, such that if $\hat{\rho} = \sigma^{1/2 - \hat{\theta}} r$, then*

$$1 \,\leq\, \hat{u}(y)/v_1(y) \,\leq\, 1 + \epsilon$$

whenever $y \in \tilde{\Omega}(0, \tilde{\eta}/16) \cap [B(0, \hat{\rho}) \setminus B(0, 4\sqrt{\sigma} r)]$.

Lemma 3.12. *Let $v_1, \epsilon, \hat{\epsilon}, \hat{\theta}, r, \sigma,$ be as in Lemma 3.11 and let $\tilde{\eta}$ be as in (106). If $\hat{\epsilon}$ is small enough, then there exist $\theta = \theta(p, n), 0 < \theta \leq \hat{\theta}/10$, and $c = c(p, n) > 1$ such that if $\rho = \sigma^{1/2 - 4\theta} r, b = \sigma^{-\theta}$, then*

$$c^{-1}\frac{v_1(x)}{d(x,\partial\Omega)} \le |\nabla v_1(x)| \le c\frac{v_1(x)}{d(x,\partial\Omega)}$$

whenever $x \in \tilde{\Omega}(0,\tilde{\eta}/4) \cap [B(0,b\rho) \setminus B(0,\rho/b)]$ *and* $0 < \epsilon \le \hat{\epsilon}$.

Before proving Lemmas 3.11 and 3.12 we indicate how the proof of Theorem 3.9 follows from these lemmas. Indeed, using Lemmas 3.3, 3.11, 3.12 we see for $\hat{\epsilon}$ sufficiently small and fixed, $0 < \epsilon \le \hat{\epsilon}$, that there exists $\tilde{c} > 1$, depending only on p, n, such that

$$\tilde{c}^{-1}\frac{\hat{u}(x)}{d(x,\partial\Omega)} \le |\nabla \hat{u}(x)| \le \tilde{c}\frac{\hat{u}(x)}{d(x,\partial\Omega)} \tag{109}$$

in $\tilde{\Omega}(0,\tilde{\eta}/2) \cap [B(0,b^{1/2}\rho) \setminus B(0,\rho/b^{1/2})]$. With $\epsilon > 0$ now fixed it follows from (109), (106), and arbitrariness of $\rho < r_0/c$ that $\hat{u}$ can be used in Lemma 3.10. As mentioned earlier, Theorem 3.9 follows from Lemma 3.10.

Proof of Lemma 3.11. From (108) we observe that it suffices to prove the righthand inequality in Lemma 3.11. We note that if $y \in \partial Q$ and $\hat{u}(y) \ne v_1(y)$, then y lies within $4\sigma r$ of a point in $\partial\Omega$. Also, $\max_{\partial B(0,t)} u$ is nonincreasing as a function of $t > 0$ as we see from the maximum principle for p harmonic functions. Using these notes and Lemmas 1.2–1.5 we see that

$$\hat{u} \le v_1 + c\sigma^{\alpha/2}\, u(\sqrt{\sigma} e_n) \tag{110}$$

on ∂Q. By Lemma 1.1 this inequality also holds in Q. Using Lemmas 1.2–1.5 we also find that there exist $\beta = \beta(p,n) \ge 1$ and $c = c(p,n) > 1$ such that

$$\max\{\psi(z), \psi(y)\} \le c\,(d(z,\partial Q)/d(y,\partial Q))^{\beta} \min\{\psi(z), \psi(y)\} \tag{111}$$

whenever $z \in Q$, $y \in Q \cap B(z, 4d(z,\partial Q))$ and $\psi = \hat{u}$ or v_1. Also from Lemmas 1.2–1.5 applied to v_1 we deduce

$$v_1(2\sqrt{\sigma}\, r e_n) \ge c^{-1}\, \hat{u}(\sqrt{\sigma}\, r e_n). \tag{112}$$

Let $\hat{\rho}, \hat{\theta}$ be as in Lemma 3.11. From (110)–(112) we see that if $y \in \tilde{\Omega}(0,\tilde{\eta}/16) \cap [B(0,\hat{\rho}) \setminus B(0, 4\sqrt{\sigma}\, r)]$, then

$$\hat{u}(y) \le v_1(y) + c\sigma^{\alpha/2} u(\sqrt{\sigma} e_n) \le (1 + c^2\sigma^{\alpha/2 - \hat{\theta}\beta}) v_1(y) \le (1+\epsilon) v_1(y) \tag{113}$$

provided $\hat{\epsilon}$ is small enough and $\hat{\theta}\beta = \alpha/4$. Thus Lemma 3.11 is true. □

Proof of Lemma 3.12. Using Lemmas 1.2–1.5 we note that there exist $\gamma = \gamma(p,n) > 0, 0 < \gamma \le 1/2$, and $c = c(p,n) > 1$ such that

$$\hat{u}(x) \le c(s/t)^{\gamma} \hat{u}(s e_n) \tag{114}$$

provided $x \in \mathbf{R}^n \setminus B(0,t), t \geq s$, and $se_n \in \Omega$ with $d(se_n, \partial\Omega) \geq c^{-1}s$. Using (114) with $t = r, s = \sqrt{\sigma}r$, we find that

$$v_1 \,\leq\, c\,\sigma^{\gamma/2}\,\hat{u}(\sqrt{\sigma}re_n) \text{ on } \partial Q \setminus \bar{B}(0, \sqrt{\sigma}\,r\,), \tag{115}$$

where c depends only on p, n. Let $\tilde{v}$ be the p harmonic function in Q with continuous boundary values $\tilde{v} = 0$ on $\partial Q \setminus \bar{B}(0, \sqrt{\sigma}\,r)$ and $\tilde{v} = v_1$ on $\partial Q \cap \partial B(0, \sqrt{\sigma}r)$. From Lemma 1.1 and (115) it follows that

$$0 \,\leq\, \tilde{v} \,\leq\, v_1 \,\leq\, \tilde{v} + c\sigma^{\gamma/2}u(\sqrt{\sigma}re_n) \text{ in } Q. \tag{116}$$

From Lemmas 1.2–1.5 we observe that

$$\tilde{v}(2\sqrt{\sigma}re_n) \geq c^{-1}\,v_1(\sqrt{\sigma}re_n) = c^{-1}u(\sqrt{\sigma}re_n). \tag{117}$$

Using (116), (117), and (111) applied to $\psi = \tilde{v}$ we obtain for $\rho = \sigma^{1/2-4\theta}r, \theta$ small, $b = \sigma^{-\theta}$, and $\hat{b} = 8b^2$, that

$$\tilde{v} \leq v_1 \leq (1 + c\sigma^{\gamma/2-6\theta\beta})\tilde{v} \leq (1+\epsilon)\tilde{v} \tag{118}$$

on $\tilde{\Omega}(0, \tilde{\eta}/8) \cap [B(0, \hat{b}\rho) \setminus B(0, \rho/\hat{b})]$, provided $\hat{\epsilon}$ is small enough and $\theta = \min\{\gamma/(24\beta), \hat{\theta}/10\}$.

Next let v be the p harmonic function in

$$Q' = \{y : |y_i| < r, 1 \leq i \leq n-1\} \cap \{y : \sigma r < y_n < r\} \setminus \bar{B}(2\sqrt{\sigma}\,re_n, \sqrt{\sigma}\,r\,)$$

with continuous boundary values $v = 0$ on $\partial Q' \setminus \bar{B}(2\sqrt{\sigma}e_n,\, \sqrt{\sigma}\,r)$ while $v = 1$ on $\partial B(2\sqrt{\sigma}\,re_n, \sqrt{\sigma}\,r\,)$. One can show that

$$v(x) \leq c\langle 2\sqrt{\sigma}\,re_n - x, \nabla v(x)\rangle \tag{119}$$

when $x \in Q'$ where $c = c(p,n)$. Clearly this inequality implies that there exists $c = c(p,n,\eta) \geq 1$, for given $\eta, 0 < \eta \leq 1/2$, such that

$$c^{-1}\frac{v(x)}{d(x, \partial Q')} \,\leq\, |\nabla v(x)| \,\leq c\frac{v(x)}{d(x, \partial Q')} \tag{120}$$

in $\tilde{Q}'(0,\eta)\setminus B(0, 10\sqrt{\sigma}\,r\,)$ where $\tilde{Q}'(0,\eta)$ is the non-tangential approach region defined relative to $0, \eta, Q'$. Using Theorem 1.3 and (120) for suitable $\eta = \eta(p,n)$ we conclude that (120) actually holds in $Q' \setminus B(0, 10\sqrt{\sigma}r)$. We now use Lemma 3.10 applied to $v, \tilde{v}$ with Ω, r', replaced by $Q', 10\sqrt{\sigma}r$, in order to get, for some $a = a(p,n) > 0$ and $c = c(p,n) > 1$, that

$$c^{-1}\frac{\tilde{v}(x)}{d(x, \partial\Omega)} \,\leq\, |\nabla \tilde{v}(x)| \,\leq c\frac{\tilde{v}(x)}{d(x, \partial\Omega)} \tag{121}$$

in $[B(0, r/c) \setminus B(0, c)\, r)]$. Finally, note that if $0 \leq \epsilon \leq \hat{\epsilon}$ and if $\hat{\epsilon}$ is sufficiently small, then $r/c > b^2\rho > \rho/b^2 > c\sqrt{\sigma}\, r$. Hence, if $\hat{\epsilon}$ is small enough then we can use (121), (118), and Lemma 3.3 to conclude that Lemma 3.12 is valid. The proof of Theorem 3.9 is now complete. $\square$

4.9 Further Remarks

We note that Theorem 3.9 has been generalized in [62] to weak solutions of $\nabla \cdot A(x, \nabla u) = 0$ where A is as in Theorem 3.7. Also in [52] we show that the conclusion of Theorem 3.9 holds when Ω is convex or the complement of a convex domain. Finally the conclusion of Theorem 3.9 is valid when $\Omega \subset \mathbf{R}^2$ is a Lipschitz domain. This problem for Lipschitz domains remains open when $n \geq 3$. However the same argument as in Theorem 3.9 yields that a p Martin function in a Lipschitz domain is unique (up to constant multiples) at each boundary point where a tangent plane exists.

5 Uniqueness and Regularity in Free Boundary: Inverse Type Problems

We begin this section by outlining the proof of

Theorem 4.1. *Let E be a compact convex set, $a > 0$, and p fixed, $1 < p < \infty$. If $H^{n-p}(E) > 0$, then there is a unique solution to the following free boundary problem: Find a bounded domain D with $E \subset D$ and u, p harmonic in $D \setminus E$, satisfying*

(a) u has continuous boundary values 1 on E and 0 on ∂D,
(b) u is p harmonic in $D \setminus E$,
(c) $\mu = a^{p-1}\, H^{n-1}|_{\partial D}$ where μ is the measure associated with u as in (15),
(d) For some positive c, r_0, and all $x \in \partial D$

$$\mu(B(x, r)) \leq c r^{n-1}, 0 < r \leq r_0.$$

5.1 History of Theorem 4.1

My interest in free boundary problems of the above type started in ≈1989 when Andrew Vogel (my former Ph.D. student) was a graduate student at the University of Kentucky. He went to a conference where the following problem was proposed:

5.1.1 The Ball Problem

Let g be the Green's function for a domain D with smooth boundary and pole at $0 \in D$. If $|\nabla g| = a =$ constant on ∂D, show that D is a ball with center at 0. We came up with the following proof: From the above assumption and properties of g one has

$$\omega(\partial D) = 1 = aH^{n-1}(\partial D)$$

where ω is harmonic measure on ∂D relative to 0. Thus

$$H^{n-1}(\partial D) = 1/a. \tag{122}$$

Choose $B(0,R) \subset D$ so that $y \in \partial B(0,R) \cap \partial D$. Let G be the Green's function for $B(0,R)$ with pole at 0. If $b = |\nabla G|(y)$, then since $|\nabla G| = b$ on $\partial B(0,R)$ we have, as in (122),

$$H^{n-1}(B(0,R)) = 1/b. \tag{123}$$

On the other hand from $G \leq g$ one gets $b \leq a$ or $1/a \leq 1/b$. Combining this inequality with (122), (123), we deduce

$$H^{n-1}(\partial D) \leq H^{n-1}(\partial B(0,R))$$

which in view of the isoperimetric inequality or the fact that H^{n-1} decreases under a projection implies that $B(0,R) = D$. Later we found out that numerous other authors, including H. Shahgholian, had also obtained that D is a ball, under various smoothness assumptions on $g, \partial D$.

In fact Henrot and Shahgholian in [34], generalized this problem, by proving

Theorem 4.2. *Let E be a compact convex set, $a > 0$, and p fixed, $1 < p < \infty$. If $H^{n-p}(E) > 0$, then there is a unique solution to the following free boundary problem: Find a bounded domain D with $E \subset D$ and u, p harmonic in $D \setminus E$, satisfying*

(α) *u has continuous boundary values 1 on E and 0 on ∂D,*

(β) *u is p harmonic in $D \setminus E$,*

(γ) *$|\nabla u|(x) \to a$ as $x \to \partial D$.*

Moreover D is convex and ∂D is C^∞.

To prove uniqueness in Theorem 4.2, given existence, one can argue as in the ball problem using the nearest point projection onto a convex set. In fact if u', D' are solutions to the above problem for a given a, p, one can show there exists a Henrot–Shahgholian domain $D \subset D'$ with $\partial D \cap \partial D' \neq \emptyset$. Using the nearest point projection onto a convex set as in the ball problem, it follows that $D = D'$. We note that if E is a ball in Theorem 4.2, then necessarily D is a ball, since radial solutions exist. Also, Henrot–Shahgholian domains will play the same role in the proof of Theorem 4.1, as balls did in the ball problem.

Andy and I considered other generalizations of the ball problem. We first rephrased the problem as in (c) of Theorem 4.1 by

$$\omega = aH^{n-1}|_{\partial D} \tag{124}$$

where ω is harmonic measure with respect to $0, \partial D$. However some examples of Keldysh and Larrentiev in two dimensions showed that this assumption was not enough to guarantee that D was a ball, so we also assumed that

$$|\nabla g| \leq M < \infty \text{ near } \partial D.$$

Using the Riesz representation formula for subharmonic functions one can show that boundedness of $|\nabla g|$ near ∂D, is equivalent to

$$\omega(B(x,r)) \leq cr^{n-1} \text{ for all } x \in \partial D \text{ and } 0 < r \leq r_0 \tag{125}$$

(i.e., condition (d) in Theorem 4.1). Given (124), (125), and the added assumption that

$$\begin{aligned}&H^{n-1} \text{ almost every point of } \partial D \text{ lies in the reduced boundary}\\ &\text{of } D \text{ in the sense of geometric measure theory,}\end{aligned} \tag{126}$$

we were able to show that

$$\limsup_{x \to \partial D} |\nabla g| \leq a \tag{127}$$

which then allowed us to repeat the argument in the smooth case and get that D is a ball. Our paper appeared in [56]. In this paper we listed a number of symmetry problems including whether hypothesis (126) was needed for the ball theorem as well as analogues for the p Laplacian. During this period we also wrote [55]. This paper used a technique in [72] to construct a bounded domain D in $\mathbf{R}^n, n \geq 3$ for which ∂D is homeomorphic to a sphere and (124), (126) hold, but $D \neq$ ball (so necessarily (125) is false). We improved this result in [58] where we showed that there exists quasi-spheres $\neq$ ball for which (124), (126), held.

Finally in [57] we were able to show that our generalization of the ball theorem remained valid without (126). Thus we obtained an endpoint result for harmonic functions. The key new idea in removing (126) was the following square function—Carleson measure estimate:

$$\int_{D \cap B(x,r)} g \sum_{i,j=1}^{n} (g_{y_i y_j})^2 \, dy \,\leq\, c\, r^{n-1},\, 0 < r \leq r_1, \tag{128}$$

for some positive c, r_1, and all $x \in \partial D$. Let $d(E, F)$ denote the distance between the sets E, F. Then (128) allowed us to conclude for given $\epsilon > 0$ (in a qualitative H^{n-1} sense) that there were 'lots' of tangent balls $B(x, d(x, \partial D))$ where ∇g had oscillation $\leq \epsilon$ in $B(x, (1-\epsilon) d(x, \partial D))$. Moreover using this fact and subharmonicity of $|\nabla g|$ we could also conclude that if $\Lambda = \limsup\limits_{y \longrightarrow D} |\nabla g|$, then in a certain percentage of these balls,

$$\Lambda - \epsilon \,\leq |\nabla g|(x) \,\leq \Lambda + \epsilon \tag{129}$$

Using (124), (129), and an asymptotic argument, we then concluded that $\Lambda \leq a$, which as earlier implies $D = B(0, R)$.

5.2 Proof of Theorem 4.1

To prove Theorem 4.1 for fixed $p, 1 < p < \infty$, we first showed that (d) in Theorem 4.1 implies,

$$|\nabla u| \leq M < \infty \text{ near } \partial D. \tag{130}$$

To prove (130) for x near ∂D we used the inequality

$$u(x) \leq c \int_0^{c' d(x, \partial D)} \left(\frac{\mu[B(w,t)]}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t}, \tag{131}$$

(d) of Theorem 4.1, and the interior estimate,

$$|\nabla u(x)| \leq \hat{c} u(x)/d(x).$$

Equation (131) is proved in [46]. Armed with (130) we then proved a square function estimate similar to (128) with g replaced by u. As in the $p = 2$ case, the square function estimate enabled us to conclude that ∇u had small oscillation on 'lots' of tangent balls in the H^{n-1} measure sense. In retrospect the subharmonicity of $|\nabla g|$ in the ball problem allowed us to use the Poisson integral formula on certain level sets to get a boundary type integral for which estimates could be made in terms of g, ω. To make this method work in the proof of Theorem 4.1, it was first necessary to find a suitable divergence form partial differential equation for which u is a solution and $|\nabla u|^2$ is a subsolution. For a long time we did not think there was any such PDE and that the lack of such a PDE resulted in some rather deep questions involving absolute continuity of μ with respect to H^{n-1} measure. Finally we discovered the PDE in (8), (9). It is easily checked that $|\nabla u|^2$ is a subsolution to (8), (9). Using this discovery, we were able to follow the general outline of the proof in the harmonic case and after some delicate asymptotics eventually get first (127) with g replaced by u and then Theorem 4.1. □

5.3 Further Uniqueness Results

We note that the derivation of (8), (9) depends heavily on the fact that the p Laplacian is homogeneous in ∇u so does not work for general PDE of p Laplace type. Our earlier investigations before (8), (9) led to [60] where we consider symmetry—uniqueness problems similar to those in Theorem 4.1 for non homogeneous PDE of p Laplace type. In this case $u, |\nabla u|^2$, are not a solution, subsolution, respectively of the same divergence form PDE. Thus we were forced to tackle some rather difficult questions involving absolute continuity of elliptic measure with respect to H^{n-1} measure on $\partial\Omega$. To outline our efforts, for these PDE's we could still prove a square function estimate for u similar to the one in (128). This estimate together with the stronger assumption that

$$c^{-1}\, r^{n-1} \leq \mu(B(x,r)) \leq c r^{n-1}, x \in \partial\Omega, 0 < r \leq r_0 \tag{132}$$

(where μ is the measure related to a solution u by way of an integral identity similar to (15)), enabled us to conclude that $\partial\Omega$ is uniformly rectifiable in the sense of [18]. At one time we hoped that uniform rectifiability of $\partial\Omega$ would imply absolute continuity of a certain elliptic measure with respect to H^{n-1} measure on $\partial\Omega$. Eventually however we found an illuminating example in [10]. The example showed that harmonic measure for Laplace's equation need not be absolutely continuous with respect to H^1 measure in a uniformly rectifiable domain. This example appeared to provide a negative end for our efforts.

Later we observed that in order to obtain the desired analogue of (129) it suffices to make absolute continuity type estimates for the above elliptic measure on the boundary of a certain subdomain $\Omega_1 \subset \Omega$, with $\partial\Omega_1$ uniformly rectifiable. Here Ω_1 is obtained by adding to Ω certain balls on which $|\nabla u|$ is 'small'. With this intuition we finally were able to use a rather involved stopping time argument in order to first establish the absolute continuity of our elliptic measure with respect to $H^{n-1}|_{\partial\Omega_1}$ and second get an analogue of (127).

From 2004–2006, Björn Bennewitz was my Ph.D. student. In his thesis he generalized the nonuniqueness results of Andy and I to p harmonic functions when $\Omega \subset \mathbf{R}^2$. More specifically for fixed $p, 1 < p < \infty$, he constructed an $\Omega \neq$ a Henrot–Shahgholian domain with $\partial\Omega$ a quasi circle, $E \subset \Omega$, and for which the corresponding p harmonic u satisfied all the conditions of Theorem 4.1 except (d). His construction makes important use (as in Sect. 2) of the fact that in $\mathbf{R}^2$, $v = \log|\nabla u|$ is a subsolution to (8), (9), when $p \geq 2$ and a supersolution to (8), (9), when $1 < p \leq 2$. Reference [7] is based on his thesis.

5.4 Boundary Regularity of p Harmonic Functions

Next we discuss boundary regularity and corresponding inverse problems for positive p harmonic functions vanishing on a portion of a Lipschitz domain. We first introduce some more or less standard notation for Lipschitz domains. Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with $w \in \partial\Omega$. If $0 < \rho < r_0$ let $\Delta(w, \rho) = \partial\Omega \cap B(w, \rho)$ and for given $b, 0 < b < 1$, $0 < r < r_0$, $x \in \Delta(w, r)$, define the nontangential approach region $\Gamma(x)$ relative to w, r, b by $\Gamma(x) = \Gamma_b(x) = \{y \in \Omega : d(y, \partial\Omega) > b|x - y|\} \cap B(w, 4r)\}$.

Given a measurable function k on $\cup_{x\in\Delta(w,2r)}\Gamma(x)$ we define the non tangential maximal function $N(k) : \Delta(w, 2r) \to \mathbf{R}$ for k as

$$N(k)(x) = \sup_{y\in\Gamma(x)} |k|(y) \text{ whenever } x \in \Delta(w, 2r).$$

Let σ be H^{n-1} measure on $\partial\Omega$ and let $L^q(\Delta(w, 2r))$, $1 \leq q \leq \infty$, be the space of functions which are qth power integrable, with respect to σ, on $\Delta(w, 2r)$. Furthermore, given $f : \Delta(w, 2r) \to \mathbf{R}$, we say that f is of bounded mean oscillation on $\Delta(w, r)$, $f \in BMO(\Delta(w, r))$, if there exists A, $0 < A < \infty$, such that

$$\int_{\Delta(x,s)} |f - f_\Delta|^2 d\sigma \leq A^2 \sigma(\Delta(x, s)) \tag{133}$$

whenever $x \in \Delta(w, r)$ and $0 < s \leq r$. Here f_Δ denotes the average of f on $\Delta = \Delta(x, s)$ with respect to σ. The least A for which (133) holds is denoted by $\|f\|_{BMO(\Delta(w,r))}$. Finally we say that f is of vanishing mean oscillation on $\Delta(w, r)$, $f \in VMO(\Delta(w, r))$, provided for each $\epsilon > 0$ there is a $\delta > 0$ such that (133) holds with A replaced by ϵ whenever $0 < s < \min(\delta, r)$ and $x \in \Delta(w, r)$.

Theorem 4.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with Lipschitz constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $0 < r < r_0$, suppose that u is a positive p harmonic function in $\Omega \cap B(w, 4r)$ and u is continuous in $B(w, 4r)$ with $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then*

$$\lim_{y \in \Gamma(x), y \to x} \nabla u(y) = \nabla u(x)$$

For σ almost every $x \in \Delta(w, 4r)$. Furthermore there exist $q = q(p, n, M) > p$ and a constant $c = c(p, n, M) \geq 1$, such that

(i) $N(|\nabla u|) \in L^q(\Delta(w, 2r))$

(ii) $\displaystyle\int_{\Delta(w,2r)} |\nabla u|^q d\sigma \leq c r^{(n-1)(\frac{p-1-q}{p-1})} \Bigg(\int_{\Delta(w,2r)} |\nabla u|^{p-1} d\sigma \Bigg)^{q/(p-1)}$

(iii) $\log |\nabla u| \in BMO(\Delta(w, r)), \quad \| \log |\nabla u| \|_{BMO(\Delta(w,r))} \leq c.$

Next we state

Theorem 4.4. *Let Ω, M, p, w, r and u be as in the statement of Theorem 4.3. If, in addition, $\partial\Omega$ is C^1 regular then*

$$\log |\nabla u| \in VMO(\Delta(w, r)).$$

Theorem 4.5. *Let Ω, M, p, w, r and u be as in the statement of Theorem 4.3. If $\log |\nabla u| \in VMO(\Delta(w, r))$, then the outer unit normal to $\Delta(w, r)$ is in $VMO(\Delta(w, r/2))$.*

To put these results into historical perspective, we note that for harmonic functions, i.e., $p = 2$, Theorem 4.3 was proved in [17]. Theorem 4.4 for harmonic functions was proved by [37]. This theorem for harmonic functions was generalized to vanishing chord arc domains in [43]. Also a version of Theorem 4.5 in vanishing chord arc domains was proved in [44] An improved version of this theorem in chord arc domains with small constants was proved by these authors in [45].

Currently we are in the process of generalizing Theorems 4.4, 4.5 to the more general setting of chord arc domains.

5.5 Proof of Theorem 4.3

To prove Theorem 4.3 we shall need several lemmas.

Lemma 4.6. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with constant M. Given $p, 1 < p < \infty, w \in \partial\Omega, 0 < r < r_0$, suppose $u > 0$ is p harmonic in $\Omega \cap B(w, 4r)$ and continuous in $B(w, 4r)$ with $u = 0$ on $B(w, 4r) \setminus \Omega$. Let μ be the measure corresponding to u as in (15). There exists $c = c(p, n, M)$ such that*

$$\bar{r}^{p-n} \mu(\Delta(w, \bar{r})) \approx u(a_{\bar{r}}(w))^{p-1} \text{ whenever } 0 < \bar{r} \leq r/c.$$

Proof. Lemma 4.6 was essentially proved in [20]. □

Step (ii) in Theorem 4.3 for smooth domains (i.e., the reverse Hölder inequality) follows from Lemma 4.6 and a Rellich inequality for the p Laplacian (as in the case $p = 2$). For example suppose

$$\Omega \cap B(w, 4r) = \{(x', x_n) : x_n > \psi(x')\} \cap B(w, 4r) \text{ where } \psi \in C_0^\infty(\mathbf{R}^{n-1}). \tag{134}$$

Then from results in [64] and Schauder type theory it follows that u is smooth near $\partial\Omega$ so we can apply the divergence theorem to $|\nabla u|^p \, \phi \, e_n$. Here $\phi \in C_0^\infty(B(w, 2r))$ with $\phi \equiv 1$ on $B(w, r)$ and $|\nabla \phi| \leq cr^{-1}$. Using p harmonicity of u - Lipschitzness of $\partial\Omega$, we get

$$r^{1-n} \int_{\partial\Omega \cap B(w,r)} |\nabla u|^p \, d\sigma \, \leq \, cr^{-n} \int_{\Omega \cap B(w,2r)} |\nabla u|^p dx. \tag{135}$$

Also, using Lemmas 1.4 and 4.6, and the fact that $d\mu = |\nabla u|^{p-1} d\sigma$ we find that

$$\begin{aligned} r^{-n} \int_{\Omega \cap B(w,2r)} |\nabla u|^p \, d\sigma \, &\leq \, cr^{-p} u(a_r(w))^p \\ &\leq c^2 \left(r^{1-n} \int_{\partial\Omega \cap B(w,r)} |\nabla u|^{p-1} \, d\sigma \right)^{p/(p-1)} \end{aligned} \tag{136}$$

where constants depend only on p, n, M. Combining (135), (136) we get (ii) in Theorem 4.3 with q replaced by p. However this reverse Hölder inequality has a self improving property so actually implies the higher integrability result in (ii), as follows from a theorem originally due to Gehring. Approximating u by certain p harmonic functions in smooth domains, applying (ii), and taking weak limits it follows that $d\mu = k d\sigma$ where μ is the measure corresponding to u as in (15). Moreover for some $q' > p/(p-1)$

$$\int\limits_{\Delta(w,\bar r)} k^{q'} d\sigma \; \leq c \bar r^{(n-1)(\frac{p-1-q'}{p-1})} \left(\int\limits_{\Delta(w,\bar r)} k d\sigma \right)^{1/q'}, \; 0 < \bar r \leq r/c. \tag{137}$$

Note that we still have to prove $k = |\nabla u|^{p-1}$ on $\partial\Omega$ (σ a. e.).

Lemma 4.7. *Let Ω, M, p, w, r and u be as in Theorem 4.3. Then there exists a starlike Lipschitz domain $\tilde\Omega \subset \Omega \cap B(w, 2r)$, with center at a point $\tilde w \in \Omega \cap B(w, r)$, $d(\tilde w, \partial\Omega) \geq c^{-1} r$, such that*

(a) $c\, \sigma(\partial\tilde\Omega \cap \Delta(w, r)) \geq r^{n-1}$.

(b) $c^{-1} r^{-1} u(\tilde w) \leq |\nabla u(x)| \leq c r^{-1} u(\tilde w)$, *for* $x \in \tilde\Omega$.

Proof. *Using Lemma 4.6 and (137) one can show there exists $\tilde{\Omega}$ as in Lemma 4.7 for which (a), (b) hold with $|\nabla u(x)|$ replaced by $u(x)/d(x, \partial\Omega)$. To finish off the proof of Lemma 4.7 we need to prove the fundamental inequality for u. In fact, assuming (134), the following stronger version is available in Ω : There exists $c = c(p, n, M) \geq 1$, and $z \in B(w, r/c)$ such that*

$$|\nabla u(z)| \approx u_{x_n}(z), \approx u(z)/d(z, \partial\Omega). \tag{138}$$

(138) is a consequence of a boundary Harnack inequality that was stated in Theorem 3.8. An improved version of Lemma 4.7 will be used in the proof of Theorem 4.5. □

Next from (138), we see as in (6)–(9) that $u, u_{x_i}, 1 \leq i \leq n$, both satisfy

$$\begin{aligned} &(\alpha)\, L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial \zeta}{\partial x_j}\right) \\ &(\beta)\, b_{ij}(x) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](x) \end{aligned} \tag{139}$$

and for $\xi \in \mathbf{R}^n$,

$$c^{-1}|\nabla u|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\xi_j \leq c|\nabla u|^{p-2}\,|\xi|^2\,. \tag{140}$$

From (138)–(140) we see once again that $u, u_{x_i}, 1 \leq i \leq n$, are locally solutions to a uniformly elliptic PDE in divergence form.

Lemma 4.8. Let $\Omega, \tilde{\Omega}, M, p, w, r, u$, be as in Lemma 4.7. Define, for $y \in \tilde{\Omega}$, the measure

$$d\tilde{\gamma}(y) = d(y, \partial\tilde{\Omega}) \max_{B(y, \frac{1}{2}d(y,\partial\tilde{\Omega}))} \left\{ \sum_{i,j=1}^{n} |\nabla b_{ij}(x)|^2 \right\} dy.$$

If $z \in \partial\tilde{\Omega}$ and $0 < s < r$, then

$$\tilde{\gamma}(\tilde{\Omega} \cap B(z, s)) \leq cs^{n-1}(u(\tilde{w})/r)^{2p-4}.$$

Proof. *We get Lemma 4.8 from Lemma 4.7 and integration by parts.* □

$\tilde{\gamma}$ in Lemma 4.8 is said to be a Carleson measure on $\tilde{\Omega}$. To continue the proof of Theorem 4.3, let $\tilde{\omega}(\tilde{w}, \cdot)$ be elliptic measure defined with respect to $L, \tilde{\Omega}$, and $\tilde{w}$ as above.

Lemma 4.9. Let $u, \tilde{\Omega}, \tilde{w}$, be as in Lemma 4.7 and L as in (139), (140). Then there exist $c \geq 1$ and $\theta, 0 < \theta \leq 1$, such that

$$\frac{\tilde{\omega}(\tilde{w}, E)}{\tilde{\omega}(\tilde{w}, \partial\tilde{\Omega} \cap B(z,s))} \leq c \left(\frac{\sigma(\tilde{w}, E)}{\sigma(\tilde{w}, \partial\tilde{\Omega} \cap B(z,s)} \right)^{\theta}$$

for $z \in \partial\tilde{\Omega}, 0 < s < r$, and $E \subset \partial\tilde{\Omega} \cap B(z,s)$ a Borel set. We say that $\tilde{\omega}$ is an A^∞ weight with respect to σ, on $\partial\tilde{\Omega}$. Lemma 4.9 is a direct consequence of Lemma 4.8 and a theorem in [42]. □

Next we prove by a contradiction argument that ∇u has non tangential limits for σ almost every $y \in \Delta(w, 4r)$. To begin suppose there exists a set $F \subset \Delta(w, 4r)$, $\sigma(F) > 0$, such that if $y \in F$ then the limit of $\nabla u(z)$, as $z \to y$ with $z \in \Gamma(y)$, does not exist. Let $y \in F$ be a point of density for F with respect to σ. Then $t^{1-n}\sigma(\Delta(y,t) \setminus F) \to 0$ as $t \to 0$, so if $t > 0$ is small enough, then $c\sigma(\partial\tilde{\Omega} \cap \Delta(y,t) \cap F) \geq t^{n-1}$ where $\tilde{\Omega} \subset \Omega$ is the starlike Lipschitz domain defined in Lemma 4.7 with $w, \tilde{w}, r$ replaced by $y, \tilde{y}, t$. From (139), (140), we see that $u_{x_i}, 1 \leq i \leq n$, is a weak solution to $L\zeta = 0$ in $\tilde{\Omega}$.

We now apply a theorem in [14] to deduce that $u_{x_k}, 1 \leq k \leq n$, has nontangential limits in $\tilde{\Omega}$, almost everywhere with respect to $\tilde{\omega}(\cdot, \tilde{w})$. Recall from Lemma 4.9 that $\tilde{\omega}$ and σ are mutually absolutely continuous. Thus these limits also exist almost everywhere with respect to σ. Since nontangential limits in $\tilde{\Omega}$ agree with those in Ω, for σ almost every point in F, we have reached a contradiction. Thus ∇u has nontangential limits almost everywhere in Ω. Step (i) follows from (137), Lemmas 4.6 and 1.3. Finally we use nontangential limits of ∇u, the fact that for small $t, \{u = t\}$ is Lipschitz, the implicit function theorem, as well as (i) of Theorem 4.3, to take limits as $t \to 0$ in order to conclude that $k = |\nabla u|$ in (137). The proof of Theorem 4.3 is now complete. □

5.6 *Proof of Theorem 4.4*

To prove Theorem 4.4 it suffices by way of an argument of Sarason (see [38]) to show that there exist $0 < \epsilon_0$ and $\tilde{r} = \tilde{r}(\epsilon)$, for $\epsilon \in (0, \epsilon_0)$, such that whenever $y \in \Delta(w, r)$ and $0 < s < \tilde{r}(\epsilon)$ we have

$$\fint_{\Delta(y,s)} |\nabla u|^p d\sigma \leq (1+\epsilon) \left(\fint_{\Delta(y,s)} |\nabla u|^{p-1} d\sigma \right)^{p/(p-1)}. \tag{141}$$

Here $\fint_E f d\sigma$ denotes the average of f on E with respect to σ. The proof of (141) is by contradiction. Otherwise there exist two sequences $\{y_m\}_1^\infty, \{s_m\}_1^\infty$ satisfying $y_m \in \Delta(w, r)$ and $s_m \to 0$ as $m \to \infty$ such that (141) is false with y, s replaced by y_m, s_m for $m = 1, 2, \ldots$. Let $A = e^{1/\epsilon}$ and put $y_m' = y_m + As_m n(y_m)$, where $n(y_m)$ is the inner unit normal to

Ω at y_m. Since $\partial\Omega$ is C^1 we see for $\epsilon > 0$ small and $m = m(\epsilon)$ large that if $\Omega(y'_m)$ is constructed by drawing all line segments from points in $B(y'_m, As_m/4)$ to points in $\Delta(y_m, As_m)$, then $\Omega(y'_m)$ is starlike Lipschitz with respect to y'_m. Let $D_m = \Omega(y'_m) \setminus \bar{B}(y'_m, As_m/8)$. and let u_m be the p harmonic function in D_m that is continuous in $\mathbf{R}^n$ with $u_m \equiv 1$ on $B(y'_m, As_m/8)$ and $u \equiv 0$ on $\mathbf{R}^n \setminus \Omega(y'_m)$. From the boundary Harnack inequality in Theorem 3.8 with w, r, u, v replaced by $y_m, As_m/100, u, u_m$ we deduce that if $w_1, w_2 \in \Omega \cap B(y_m, 2s_m)$ then

$$\left|\log\left(\frac{u_m(w_1)}{u(w_1)}\right) - \log\left(\frac{u_m(w_2)}{u(w_2)}\right)\right| \leq cA^{-\alpha}. \tag{142}$$

Letting $w_1, w_2 \to z_1, z_2 \in \Delta(y_m, 2s_m)$ in (142) and using Theorem 4.3, we get, σ almost everywhere, that

$$\left|\log\left(\frac{|\nabla u_m(z_1)|}{|\nabla u(z_1)|}\right) - \log\left(\frac{|\nabla u_m(z_2)|}{|\nabla u(z_2)|}\right)\right| \leq cA^{-\alpha}$$

or equivalently that

$$(1 - \tilde{c}A^{-\alpha})\frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|} \leq \frac{|\nabla u(z_1)|}{|\nabla u(z_2)|} \leq (1 + \tilde{c}A^{-\alpha})\frac{|\nabla u_m(z_1)|}{|\nabla u_m(z_2)|}. \tag{143}$$

Using (143) and the fact that (141) is false we obtain

$$\begin{aligned} &\frac{\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^p d\sigma}{\left(\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u_m|^{p-1} d\sigma\right)^{p/(p-1)}} \\ &\geq (1 - cA^{-\alpha})\frac{\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u|^p d\sigma}{\left(\displaystyle\fint_{\Delta(y_m,s_m)} |\nabla u|^{p-1} d\sigma\right)^{p/(p-1)}} \geq (1 - cA^{-\alpha})(1 + \epsilon)\,. \end{aligned} \tag{144}$$

Let T_m be a conformal affine mapping of $\mathbf{R}^n$ which maps the origin and e_n onto y_m and y'_m respectively and which maps $W = \{x \in \mathbf{R}^n : x_n = 0\}$ onto the tangent plane to $\partial\Omega$ at y_m. Let D'_m, u'_m be such that $T_m(D'_m) = D_m$ and $u_m(T_m x) = u'_m(x)$ whenever $x \in D'_m$. Since the p Laplace equation is invariant under translations, rotations, and dilations, we see that u'_m is p harmonic in D'_m. Letting $m \to \infty$ one can show that u'_m converges uniformly on $\mathbf{R}^n$ to u' where u' is continuous on $\mathbf{R}^n$ and p harmonic in $D' = \Omega' \setminus B(e_n, 1/8)$ with $u \equiv 1$ on $B(e_n, 1/8)$ and $u \equiv 0$ on $\mathbf{R}^n \setminus \Omega'$. Here Ω' is obtained by drawing all line segments connecting points in $B(0,1) \cap W$ to points in $B(e_n, 1/4)$. Changing variables in (144) and using Rellich type inequalities one gets

$$
\begin{aligned}
(1 - cA^{-\alpha})(1+\epsilon) &\leq \limsup_{m \to \infty} \frac{\displaystyle \fint_{\partial D'_m \cap B(0,1/A)} |\nabla u'_m|^p d\sigma}{\left(\displaystyle \fint_{\partial D'_m \cap B(0,1/A)} |\nabla u'_m|^{p-1} d\sigma \right)^{p/(p-1)}} \\
&\leq \frac{\displaystyle \fint_{W \cap B(0,1/A)} |\nabla u'|^p d\sigma}{\left(\displaystyle \fint_{W \cap B(0,1/A)} |\nabla u'|^{p-1} d\sigma \right)^{p/(p-1)}}
\end{aligned} \tag{145}
$$

Finally from interior estimates for p harmonic functions and Schwarz reflection one finds for $z \in B(0, 1/A)$ that

$$
(1 - cA^{-\theta})|\nabla u'(0)| \leq |\nabla u'(z)| \leq (1 + cA^{-\theta})|\nabla u'(0)|
$$

which in view of (145) yields

$$
(1 + cA^{-\theta}) \geq \frac{\displaystyle \fint_{W \cap B(0,1/A)} |\nabla u'|^p dx'}{\left(\displaystyle \fint_{W \cap B(0,1/A)} |\nabla u'|^{p-1} dx' \right)^{p/(p-1)}} \geq (1 - cA^{-\alpha})(1+\epsilon).
$$

Clearly this inequality cannot hold for ϵ small since $A = e^{1/\epsilon}$. The proof of Theorem 4.4 is now complete.

5.7 Proof of Theorem 4.5

In this section we prove Theorem 4.5. We shall need the following refined version of Lemma 4.7.

Lemma 4.10. Given Ω, w, p, n, M, u as in Theorem 4.3. If $\log |\nabla u| \in VMO(\Delta(w,r))$, then for each $\epsilon > 0$ there exists, $0 < \tilde{r} = \tilde{r}(\epsilon) < r$ and $c = c(p,n,M), 1 \leq c < \infty$, such that if $0 < r' \leq \tilde{r}$ then the following :

There is a starlike Lipschitz domain $\tilde{\Omega} \subset \Omega \cap B(w, r')$, with center at a point $\hat{w} \in \Omega \cap B(w, r')$, $d(\hat{w}, \partial\Omega) \geq r'/c$, and Lipschitz constant $\leq c$, satisfying

(a)
$$\frac{\sigma(\partial\tilde{\Omega} \cap \Delta(w, r'))}{\sigma(\Delta(w, r'))} \geq 1 - \epsilon.$$

(b)
$$(1-\epsilon)b^{p-1} \leq \frac{\mu(\Delta(y,s))}{\sigma(\Delta(y,s))} \leq (1+\epsilon)b^{p-1}$$

where $0 < s \leq r', y \in \partial\tilde{\Omega} \cap \Delta(w, r')$, and $\log b$ is the average of $\log |\nabla u|$ on $\Delta(w, 4r')$. Moreover,

(c)
$$c^{-1}\frac{u(\hat{w})}{r'} \leq |\nabla u(x)| \leq c\frac{u(\hat{w})}{r'} \text{ for all } x \in \tilde{\Omega}.$$

Proof. Lemma 4.10 is proved in [51] as Lemma 4.1. □

To begin the proof of Theorem 4.5 let n denote the outer unit normal to $\partial\Omega$ and put

$$\eta = \lim_{\tilde{r}\to 0} \sup_{\tilde{w}\in\Delta(w,r/2)} \|n\|_{BMO(\Delta(\tilde{w},\tilde{r}))}. \tag{146}$$

To prove Theorem 4.5 it is enough to prove that $\eta = 0$. To do this we argue by contradiction and assume that (146) holds for some $\eta > 0$. This assumption implies that there exist a sequence of points $\{w_j\}$, $w_j \in \Delta(w, r/2)$, and a sequence of scales $\{r_j\}$, $r_j \to 0$, such that $\|n\|_{BMO(\Delta(w_j,r_j))} \to \eta$ as $j \to \infty$. To get a contradiction we use a blow-up argument. In particular, let u be as in the statement of Theorem 4.5 and extend u to $B(w, 4r)$ by putting $u = 0$ in $B(w, 4r) \setminus \Omega$. For $\{w_j\}$, $\{r_j\}$ as above we define $\Omega_j = \{r_j^{-1}(x - w_j) : \ x \in \Omega\}$ and
$$u_j(z) = \lambda_j u(w_j + r_j z) \text{ whenever } z \in \Omega_j.$$

Using properties of Lipschitz domains, one can show that subsequences of $\{\Omega_j\}, \{\partial\Omega_j\}$ converge to $\Omega_\infty, \partial\Omega_\infty$, in the Hausdorff distance sense, where Ω_∞ is an unbounded Lipschitz domain with Lipschitz constant bounded by M. Moreover, from (15), Lemmas 1.2–1.5, and 4.10 we deduce for an appropriate choice of (λ_j), that a subsequence of (u_j) converges uniformly on compact subsets of $\mathbf{R}^n$ to u_∞, a positive p harmonic function in Ω_∞ vanishing continuously on $\partial\Omega_\infty$. If $d\mu_j = |\nabla u_j|^{p-1} d\sigma|_{\partial\Omega_j}$, it also follows that a subsequence of (μ_j) converges weakly as Radon measures to μ_∞ where

$$\int_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla\phi \rangle dx = -\int_{\partial\Omega_\infty} \phi \, d\mu_\infty \tag{147}$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n-1})$ Moreover, using Lemma 4.10, one can show that μ_∞ and u_∞, satisfy,

$$\begin{aligned} &(a)\, \mu_\infty = \sigma \text{ on } \partial\Omega_\infty, \\ &(b)\, c^{-1} \leq |\nabla u_\infty(z)| \leq 1 \text{ whenever } z \in \Omega_\infty. \end{aligned} \tag{148}$$

Finally one shows that (147), (148) imply

$$\Omega_\infty \text{ is a halfspace} \tag{149}$$

which in turn implies that $\eta = 0$, a contradiction to (146). If M is sufficiently small, then (149) follows directly from a theorem in [1]. For large M we needed to use our generalization in [53] of the work in [11].

We discuss this work further in the next section.

5.8 Regularity in a Lipschitz Free Boundary Problem

We begin our discussion of two phase free boundary problems for p harmonic functions with some notation. Let $D \subset \mathbf{R}^n$ be a bounded domain and suppose that u is continuous on D. Put

$$D^+(u) = \{x \in D : u(x) > 0\}$$

$$F(u) = \partial D^+(u) \cap D$$

$$D^-(u) = D \setminus [D^+(u) \cup F(u)].$$

$F(u)$ is called the free boundary of u in D. Let $G > 0$ be an increasing function on $[0,\infty)$ and suppose for some $N > 0$ that $s^{-N}G(s)$ is decreasing when $s \in (0,\infty)$. Let $u^+ = \max(u,0)$ and $u^- = -\min(u,0)$.

Definition F. *We say that u satisfies weakly the two sided boundary condition* $|\nabla u^+| = G(|\nabla u^-|)$ *on* $F(u)$ *provided the following holds. Assume that* $w \in F(u)$ *and there is a ball* $B(\hat w, \hat\rho) = \{y : |y - \hat w| < \hat\rho\} \subset D^+(u) \cup D^-(u)$ *with* $w \in \partial B(\hat w, \hat\rho)$*. If* $\nu = (\hat w - w)/|\hat w - w|$ *and* $B(\hat w, \hat\rho) \subset D^+(u)$*, then*

$$(*)\;\; u(x) = \alpha\langle x - w, \nu\rangle^+ - \beta\langle x - w, \nu\rangle^- + o(|x - w|),$$

as $x \to w$ *non-tangentially while if* $B(\hat w, \hat\rho) \subset D^-(u)$*, then* $(*)$ *holds with* $x - w$ *replaced by* $w - x$*, where,* $\alpha,\ \beta \in [0,\infty]$ *and* $\alpha = G(\beta)$*.*

We note that if If $F(u), u$ are sufficiently smooth in D, then at w,

$$\alpha = |\nabla u^+| = G(\beta) = G(|\nabla u^-|).$$

In [51] we prove,
Theorem 4.11. Let u be continuous in D, p harmonic in $D \setminus F(u), 1 < p < \infty, p \neq 2$, and a weak solution to $|\nabla u^+| = G(|\nabla u^-|)$ on $F(u)$. Suppose $B(0,2) \subset D, 0 \in F(u)$, and $F(u)$ coincides in B (0, 2) with the graph of a Lipschitz function. Then $F(u) \cap B(0,1)$ is $C^{1,\sigma}$ where σ depends only on p, n, N and the Lipschitz constant for the graph function.

5.9 History of Theorem 4.11

For $p = 2$, i.e., harmonic functions, Caffarelli developed a theory for general two-phase free boundary problems in [11–13]. In [11] Lipschitz free boundaries were shown to be $C^{1,\sigma}$-smooth for some $\sigma \in (0,1)$ and in [13] it was shown that free boundaries which are well approximated by Lipschitz graphs are in fact Lipschitz. Finally, in [12] the existence part of the theory was developed. We also note that the work in [11] was generalized in [70] to solutions of fully nonlinear PDEs of the form $F(\nabla^2 u) = 0$, where F is homogeneous. Further analogues of [11] were obtained for a class of nonisotropic operators and for fully nonlinear PDE's of the form $F(\nabla^2 u, \nabla u) = 0$, where F is homogeneous in both arguments, in [24, 25]. Extension of the results in [11] were made to non-divergence form linear PDE with variable coefficients in [16], and generalized in [26] to fully nonlinear PDE's of the form $F(\nabla^2 u, x) = 0$. Finally generalizations of the work in [11] (also [13]) to linear divergence form PDE's with variable coefficients were obtained in [27, 28].

5.10 Proof of Theorem 4.11

To outline the proof of Theorem 4.11 we need another definition.
Definition G. *We say that a real valued function h is monotone on an open set O in the direction of $\tau \in \mathbf{R}^n$, provided $h(x - \tau) \leq h(x)$ whenever $x \in O$.*
If $x, y \in \mathbf{R}^n$, let $\theta(x,y)$ be the angle between x and y. Given $\theta_0, 0<\theta_0<\pi, \epsilon_0>o$, and ν with $|\nu| = 1$, put

$$\Gamma(\nu, \theta_0, \epsilon_0) = \{y \in \mathbf{R}^n : |\theta(y,\nu)| < \theta_0,\ 0 < |y| < \epsilon_0\}.$$

We note from elementary geometry that if h is monotone on O with respect to the directions in $\Gamma(\nu, \theta_0, \epsilon_0)$, then

$$\sup_{B(x-\tau, |\tau| \sin(\theta_0/2))} h \leq h(x) \tag{2}$$

whenever $x \in O$ and $\tau \in \Gamma(\nu, \theta_0/2, \epsilon_0/2)$. To establish Theorem 4.11 we first show the existence of a cone of monotonicity. To this end, we assume as we may, that

$$\Omega \cap B(0,2) = \{(x', x_n) : x_n > \psi(x')\} \cap B(w, 4r), \ \psi \text{ Lipschitz on } \mathbf{R}^{n-1}. \tag{150}$$

If M is the Lipschitz norm of ψ, then as in (138) we see that Theorem 3.8 and (150) imply there exists $c = c(p, n, M) \geq 1$, such that whenever $z \in B(0, r_1), r_1 = 1/c$,

$$|\nabla u(z)| \approx u_{x_n}(z), \approx u(z)/d(z, \partial\Omega). \tag{151}$$

Clearly (151) implies the existence of $\theta_0 \in (0, \pi/2], \epsilon_0 > 0$, and $c > 1$, depending only on p, n, M, such that

$$\begin{aligned} &u \text{ is monotone in } B(0, r_1) \text{ with respect to} \\ &\text{the directions in the cone } \Gamma(e_n, \theta_0, \epsilon_0). \end{aligned} \tag{152}$$

5.11 Enlargement of the Cone of Monotonicity in the Interior

Let $\tau \in \Gamma(e_n, \theta_0/2, \epsilon_0)$ for θ_0, ϵ_0, as in (152), put $\epsilon = |\tau| \sin(\theta_0/2)$ and set

$$v_\epsilon(x) = v_{\epsilon,\tau}(x) = \sup_{y \in B(x,\epsilon)} u(y - \tau)$$

We note from (152) that $v_\epsilon(x) \leq u(x)$, when $x \in B(0, r_1)$. Next we show that if $\rho, \mu > 0$ are small enough, depending only on p, n, and the Lipschitz constant for $\partial\Omega \cap B(0,1)$, and $\nu = \nabla u(\frac{r_1 e_n}{8}))/|\nabla u(\frac{r_1 e_n}{8})|$, then

$$v_{(1+\mu\lambda)\epsilon}(x) \leq (1 - \mu\lambda\epsilon)u(x) \text{ whenever } x \in B(\tfrac{r_1 e_n}{8}, \rho r_1). \tag{153}$$

where $\lambda = \cos(\theta_0/2 + \theta(\nu, \tau))$, and $0 < |\tau| \leq \epsilon_0 \rho r_1$. The proof of (153) is essentially the same as in [11], thanks to basic interior estimates for p harmonic functions, Theorem 3.8, and (152).

5.12 Enlargement of the Cone of Monotonicity at the Free Boundary

In this part of the proof we show there exists $\bar{\mu} > 0$, depending only on p, n, M, such that if τ, ϵ are as defined above (153), then

$$v_{(1+\bar{\mu}\lambda)\epsilon}(x) \leq u(x) \text{ whenever } x \in B(0, r_1/100), \tag{154}$$

It is shown in [11] that (154) implies the existence of $\omega, |\omega| = 1, \bar{\theta} \in (0, \pi/2]$, $c_-, c_+ > 1$, such that

$$u \text{ is monotone in } \Gamma(\omega, \bar{\theta}, \epsilon_0/c_+) \tag{155}$$

where $\pi/2 - \bar{\theta} = c_-^{-1}(\pi/2 - \theta_0)$, $\Gamma(e_n, \theta_0, \epsilon_0) \subset \Gamma(\omega, \bar{\theta}, \epsilon_0/c_+)$, and all constants depend only on p, n, M.

Using (155) as well as invariance of the p Laplace equation under scalings and translations, we can replace $u(x)$ by $u(x_0 + \eta x)/\eta$ and then repeat our argument in (153), (154), in order to eventually conclude the $C^{1,\sigma}$-smoothness of $F(u) \cap B(0, 1/2)$. Hence to prove Theorem 4.11 we have to prove (154). To do this, given an $n \times n$ symmetric matrix M let

$$P(M) = \inf_{A \in A_p} \sum_{i,j=1}^{n} a_{ij} M_{ij} \,.$$

where A_p denotes the set of all symmetric $n \times n$ matrices $A = \{a_{ij}\}$ which satisfy

$$\min\{p-1, 1\} \, |\xi|^2 \, \leq \, \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \leq \max\{p-1, 1\} \, |\xi|^2$$

whenever $\xi \in \mathbf{R}^n \setminus \{0\}$. Next we state

Lemma 4.12. *Let $\phi > 0$ be in $C^2(D)$, $\|\nabla \phi\|_{L^\infty(D)} \leq 1/2$, p fixed, $1 < p < \infty$, and suppose that*

$$\phi(x) P(\nabla^2 \phi(x)) \geq \, 50pn \, |\nabla \phi(x)|^2$$

whenever $x \in D$. Let u be continuous in an open set O containing the closure of $\bigcup_{x \in D} B(x, \phi(x))$ and define

$$v(x) = \max_{\bar{B}(x, \phi(x))} u$$

whenever $x \in D$. If u is p-harmonic in $O \setminus \{u = 0\}$, then v is continuous and a p-subsolution in $\{v \neq 0\} \cap G$ whenever G is an open set with $\bar{G} \subset D$.

Theorem 4.13. *Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz graph domain with Lipschitz constant M. Given $p, 1 < p < \infty, w \in \partial\Omega$, $r > 0$, suppose that $\hat{u}$ and $\hat{v}$ are non-negative p-harmonic functions in $\Omega \cap B(w, 2r)$ with $\hat{v} \leq \hat{u}$. Assume also that $\hat{u}, \hat{v}$, are continuous in $B(w, 2r)$ with $\hat{u} \equiv 0 \equiv \hat{v}$ on $B(w, 2r) \setminus \Omega$. There exists $c \geq 1$, depending only on p, n, M, such that if $y, z \in \Omega \cap B(w, r/c)$, then*

$$\frac{\hat{u}(y) - \hat{v}(y)}{\hat{v}(y)} \leq c \frac{\hat{u}(z) - \hat{v}(z)}{\hat{v}(z)} \,.$$

Remark. Lemma 4.12 is not much more difficult than the corresponding lemma in [11], thanks to translation, dilation and rotational invariance of the p Laplacian. Theorem 4.13 uses the full toolbox developed in [49–52] and [62]. For $p = 2$ Theorem 4.13 is equivalent to a boundary Harnack inequality for harmonic functions while for $p \neq 2$, it is stronger than the boundary Harnack inequality in Theorem 3.8. That is, Theorem 4.13 implies not only boundedness of $\hat{u}/\hat{v}$, but also Hölder continuity of the ratio in $\Omega \cap B(w, r/c^*)$, for some $c^* - c^*(p, n, M) \geq 1$. Theorem 4.13 is our main contribution to the proof of Theorem 4.11. To prove (154), using Lemma 4.12 and Theorem 4.13, let

$$\tilde{v}_t(x) := \sup_{y \in B(x, \epsilon\phi_{\mu\lambda t}(x))} u(y - \tau) \text{ for } t \in [0, 1]$$

and $x \in B(0, r_1) \setminus B(r_1 e_n/8, \rho r_1)$. Here $\{\phi_t\}$, is a family of C^2 functions, each satisfying the hypotheses of Lemma 4.12 in $B(0, r_1) \setminus B(r_1 e_n/8, \rho r_1)$. Moreover,

(a) $\phi_t \equiv 1$ on $B(0, r_1) \setminus B(0, r_1/2)$
(b) $1 \leq \phi_t \leq 1 + t\gamma$ in $B(0, r_1) \setminus B(r_1 e_n/8, \rho r_1)$
(c) $\phi_t \geq 1 + ht\gamma$ in $B(0, r_1/100)$
(d) $|\nabla \phi_t| \leq \gamma t$.

In (c), $h > 0$ depends on ρ, p, n while $\gamma > 0$ is a parameter to be chosen sufficiently small. From (c) one sees that (154) holds if

$$\tilde{v}_t \leq u \text{ in } B(0, r_1) \setminus B(r_1 e_n/8, \rho r_1) \text{ whenever } t \in [0, 1]. \tag{156}$$

From Step 1 and (b) above, this inequality holds when $t = 0$. One can now use a method of continuity type argument to show that if (156) is false then there exist $t \in [0, 1]$ for which $\tilde{v}_t \leq u$, $w \in F(u) \cap \{\tilde{v}_t = 0\}$, and $\hat{w}$, $\hat{\rho} > 0$ with

$$B(\hat{w}, \hat{\rho}) \subset \{\tilde{v}_t > 0\} \subset D^+(u) \;\text{ and } w \in \partial B(\hat{w}, \hat{\rho}).$$

One then uses in this tangent ball, the asymptotic free boundary condition for u, similar asymptotics for $\tilde{v}_t$, (153), Theorem 4.13, and a Hopf maximum principle type argument to get a contradiction. Thus (156) and so (154) are valid. This completes our outline of the proof of Theorem 4.11. □

5.13 An Application of Theorem 4.11

Recall that the proof of Theorem 4.5 was by contradiction. Indeed assuming that

$$\eta = \lim_{\tilde{r} \to 0} \sup_{\tilde{w} \in \Delta(w, r/2)} \|\nu\|_{BMO(\Delta(\tilde{w}, \tilde{r}))} \neq 0 \tag{157}$$

we obtained u_∞, a positive p-harmonic function in a Lipschitz graph domain, Ω_∞, which is Hölder continuous in $\mathbf{R}^n$ with $u_\infty \equiv 0$ on $\mathbf{R}^n \setminus \Omega_\infty$. We also had

$$\int_{\mathbf{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla \psi \rangle dx = - \int_{\partial \Omega_\infty} \psi d\sigma_\infty \tag{158}$$

whenever $\psi \in C_0^\infty(\mathbf{R}^n)$ and

$$c^{-1} \leq |\nabla u_\infty(z)| \leq 1 \text{ whenever } z \in \Omega_\infty. \tag{159}$$

where σ_∞ is surface area on $\partial\Omega_\infty$. To get a contradiction to (157) we needed to show that (158), (159) imply Ω_∞ is a halfspace. This conclusion follows easily from applying the argument in Theorem 4.11 to $u_\infty(Rx)/R$ and letting $R \to \infty$, once it is shown that u_∞ is a weak solution to the free boundary problem in Theorem 4.11 with $G(s) = 1 + s$ for $s \in (0, \infty)$. That is, u is a weak solution to the 'one phase free boundary problem'

$$|\nabla u^+| \equiv 1 \text{ and } |\nabla u^-| \equiv 0 \text{ on } F(u). \tag{160}$$

To prove (160) assume $w \in F(u_\infty)$ and that there exists a ball $B(\hat{w}, \hat{\rho})$, $\hat{w} \in \mathbf{R}^n \setminus \partial\Omega_\infty$ and $\hat{\rho} > 0$, such that $w \in \partial B(\hat{w}, \hat{\rho})$. Let P be the plane through w with normal $\nu = (\hat{w} - w)/|\hat{w} - w|$. We claim that P is a tangent plane to Ω_∞ at w in the usual sense. That is given $\epsilon > 0$ there exists $\hat{r}(\epsilon) > 0$ such that

$$\Psi(P \cap B(w, r), \partial\Omega_\infty \cap B(w, r)) \leq \epsilon r \tag{161}$$

whenever $0 < r \leq \hat{r}(\epsilon)$. Once (161) is proved we can show that

$$\begin{aligned} &(i) \quad \text{If } B(\hat{w}, \hat{\rho}) \subset \Omega_\infty \text{ then } u_\infty^+(x) = \langle x - w, \nu \rangle + o(|x - w|) \text{ in } \Omega_\infty \\ &(ii) \quad \text{If } B(\hat{w}, \hat{\rho}) \subset \mathbf{R}^n \setminus \Omega_\infty \text{ then } u_\infty^+(x) = \langle w - x, \nu \rangle + o(|x - w|) \text{ in } \Omega_\infty. \end{aligned} \tag{162}$$

To prove (162) (given (161)) we assume that $w = 0, \nu = e_n$, and $\hat{\rho} = 1$. This assumption is permissible since linear functions and the p-Laplacian are invariant under rotations, translations, and dilations. Then $\hat{w} = e_n$ and either $B(e_n, 1) \subset \Omega_\infty$ or $B(e_n, 1) \subset \mathbf{R}^n \setminus \bar{\Omega}_\infty$. We assume that $B(e_n, 1) \subset \Omega_\infty$, since the other possibility, $B(e_n, 1) \subset \mathbf{R}^n \setminus \bar{\Omega}_\infty$, is handled similarly. Let $\{r_j\}$ be a sequence of positive numbers tending to 0 and let $\hat{u}_j(z) = u_\infty(r_j z)/r_j$ whenever $z \in \mathbf{R}^n$. Let $\hat{\Omega}_j = \{z : r_j z \in \Omega_\infty\}$ be the corresponding blow-up regions. Then $\hat{u}_j$ is p-harmonic in $\hat{\Omega}_j$ and Hölder continuous in $\mathbf{R}^n$ with $\hat{u}_j \equiv 0$ on $\mathbf{R}^n \setminus \hat{\Omega}_j$. Moreover, (159) is valid for each j with u_∞ replaced by $\hat{u}_j$. Using these facts, assumption (161), and Lemmas 1.2–1.5 we see that a subsequence of $\{\hat{u}_j\}$, denoted $\{u'_j\}$, converges uniformly on compact subsets of $\mathbf{R}^n$, as $j \to \infty$, to a Hölder continuous function u'_∞. Moreover, u'_∞ is a nonnegative p-harmonic function in $H = \{x : x_n > 0\}$ with $u'_\infty \equiv 0$ on $\mathbf{R}^n \setminus H$. Let $\{\Omega'_j\}$ be the subsequence of $\{\hat{\Omega}_j\}$ corresponding to $\{u'_j\}$. From (161) we see that $\Omega'_j \cap B(0, R)$ converges to $H \cap B(0, R)$ whenever $R > 0$, in the sense of Hausdorff distance as $j \to \infty$. Finally we note that $\nabla u'_j \to \nabla u'_\infty$

uniformly on compact subsets of H and hence

$$c^{-1} \leq |\nabla u'_\infty| \leq 1 \tag{163}$$

where c is the constant in (159). Next we apply the boundary Harnack inequality in Theorem 4.1 with

$$\Omega = H, \hat{u}(x) = u'_\infty(x), \text{ and } \hat{v}(x) = x_n.$$

Letting $r \to \infty$ in Theorem 4.1, it follows that

$$u'_\infty(x) = l x_n \tag{164}$$

for some nonnegative l. From (163) and the above discussion we conclude that

$$c^{-1} \leq l \leq 1. \tag{165}$$

Next using (158) we see that if σ'_j is surface area on Ω'_j, σ surface area on H, and $\phi \geq 0 \in C_0^\infty(\mathbf{R}^n)$, then

$$\begin{aligned} \int_{\partial\{u'_j>0\}} \phi d\sigma'_j &= -\int_{\mathbf{R}^n} |\nabla u'_j|^{p-2} \langle \nabla u'_j, \nabla \phi \rangle dx \\ &\to -\int_{\mathbf{R}^n} |\nabla u'_\infty|^{p-2} \langle \nabla u'_\infty, \nabla \phi \rangle dx = l^{p-1} \int_{\{x_n=0\}} \phi d\sigma \end{aligned} \tag{166}$$

as $j \to \infty$. Moreover, using the divergence theorem we find that

$$\int_{\partial\{u'_j>0\}} \phi d\sigma'_j \geq -\int_{\{u'_j>0\}} \nabla \cdot (\phi e_n) dx \to -\int_{\{u'_\infty>0\}} \nabla \cdot (\phi e_n) dx = \int_{\{x_n=0\}} \phi d\sigma \tag{167}$$

as $j \to \infty$. Combining (166), (167) we obtain first that $l \geq 1$ and thereupon from (165) that $l = 1$. Thus any blowup sequence of u_∞, relative to zero, tends to x_n^+ uniformly on compact subsets of $\mathbf{R}^n$, and the corresponding gradients tend uniformly to e_n on compact subsets of H. This conclusion is easily seen to imply (162). Hence (161) implies (162).

5.14 Proof of (161)

The proof of (161) is again by contradiction. We continue under the assumption that $w = 0, \nu = \hat{w} = e_n$, and $\hat{\rho} = 1$. First suppose that

$$B(e_n, 1) \subset \Omega_\infty. \tag{168}$$

If (161) is false, then there exists a sequence $\{s_m\}$ of positive numbers and $\delta > 0$ with $\lim_{m\to\infty} s_m = 0$ and the property that

$$\Omega_\infty \cap \partial B(0, s_m) \cap \{x : x_n \leq -\delta s_m\} \neq \emptyset \tag{169}$$

for each m. To get a contradiction we show that (169) leads to

$$\limsup_{t\to 0} t^{-1} u_\infty(te_n) = \infty \tag{170}$$

which in view of the mean value theorem from elementary calculus, contradicts (159). For this purpose let f be the p-harmonic function in $B(e_n, 1) \setminus \bar{B}(e_n, 1/2)$ with continuous boundary values,

$$f \equiv 0 \text{ on } \partial B(e_n, 1) \text{ and } f \equiv \min_{\bar{B}(e_n,1/2)} u_\infty \text{ on } \partial B(e_n, 1/2).$$

Recall that n f can be written explicitly in the form,

$$f(x) = \begin{cases} A|x - e_n|^{(p-n)/(p-1)} + B \text{ when } p \neq n, \\ -A \log|x - e_n| + B \text{ when } p = n, \end{cases}$$

where A, B are constants. Doing this we see that

$$\lim_{t\to 0} t^{-1} f(te_n) > 0. \tag{171}$$

From the maximum principle for p-harmonic functions we also have

$$u_\infty \geq f \text{ in } B(e_n, 1) \setminus \bar{B}(e_n, 1/2). \tag{172}$$

Next we show that if $0 < s < 1/4$, and $u_\infty \geq kf$ in $\bar{B}(0, s) \cap B(e_n, 1)$, for some $k \geq 1$, then there exists $\xi = \xi(p, n, M, \delta) > 0$ and $s', 0 < s' < s/2$, such that

$$u_\infty \geq (1 + \xi)kf \text{ in } \bar{B}(0, s') \cap B(e_n, 1). \tag{173}$$

Clearly (171)–(173) and an iterative argument yield (170). To prove (173) we observe from a direct calculation that

$$|\nabla f(x)| \approx f(x)/(1 - |x - e_n|) \text{ when } x \in B(e_n, 1) \setminus \bar{B}(e_n, 1/2), \tag{174}$$

where proportionality constants depend only on p, n. Also, we observe from (169) and Lipschitzness of $\partial\Omega_\infty$ that if m_0 is large enough, then there exists a sequence of points $\{t_l\}_{m_0}^\infty$ in $\Omega_\infty \cap \{x : x_n = 0\}$ and $\eta = \eta(p, n, M, \delta) > 0$ such that for $l \geq m_0$,

$$\eta s_l \leq |t_l| \leq \eta^{-1} s_l \text{ and } d(t_l, \partial\Omega_\infty) \geq \eta |t_l|. \tag{175}$$

Choose $t_m \in \{t_l\}_{m_0}^{\infty}$ such that $\eta^{-1}|t_m| \leq s/100$. If $\rho = d(t_m, \partial\Omega_\infty)$, then from (175), and Lemmas 1.2–1.5 for u_∞ we deduce for some $C = C(p, n, M, \delta) \geq 1$ that

$$C u_\infty(t_m) \geq \max_{\bar{B}(0,4|t_m|)} u_\infty. \tag{176}$$

From (176), the assumption that $kf \leq u_\infty$, Lemmas 1.2–1.5 for kf, and the fact that t_m lies in the tangent plane to $B(e_n, 1)$ through 0, we see there exists $\lambda = \lambda(p, n, M, \delta), 0 < \lambda \leq 10^{-2}$, and $m_1 \geq m_0$ such that if $m \geq m_1$ and $t'_m = t_m + 3\lambda\rho e_n$, then

$$B(t'_m, 2\rho\lambda) \subset B(e_n, 1) \text{ and } (1 + \lambda)kf \leq u_\infty \text{ on } \bar{B}(t'_m, \rho\lambda). \tag{177}$$

Let $\tilde{f}$ be the p-harmonic function in $G = B(0, 4|t_m|) \cap B(e_n, 1) \setminus \bar{B}(t'_m, \rho\lambda)$ with continuous boundary values $\tilde{f} = kf$ on $\partial[B(e_n, 1) \cap B(0, 4|t_m|)]$ while $\tilde{f} = (1 + \lambda)kf$ on $\partial B(t'_m, \rho\lambda)$. Using (174), Theorem 1.13, and Harnack's inequality for $\tilde{f} - kf, kf$ as in the proof of (154), we deduce the existence of $\tau > 0, \bar{c} \geq 1$, with

$$(1 + \tau\lambda)kf \leq \tilde{f} \tag{178}$$

in $B(e_n, 1) \cap \bar{B}(0, |t_m|/\bar{c})$ where $\tau = \tau(p, n, M, \delta), 0 < \tau < 1$, and $\bar{c} = \bar{c}(p, n, M) \geq 1$. Moreover, using the maximum principle for p-harmonic functions we see from (177) that

$$\tilde{f} \leq u_\infty \text{ in } G. \tag{179}$$

Combining (178), (179), we get (173) with $\xi = \tau\lambda$ and $s' = |t_m|/\bar{c}$. As mentioned earlier, (173) leads to a contradiction. Hence (161) is true when (168) holds. If $B(e_n, 1) \subset \mathbf{R}^n \setminus \bar{\Omega}_\infty$ we proceed similarly. That is, if (161) is false, then there exists a sequence $\{s_m\}$ of positive numbers and $\delta > 0$ with $\lim\limits_{m\to\infty} s_m = 0$ and the property that

$$\mathbf{R}^n \setminus \bar{\Omega}_\infty \cap \partial B(0, s_m) \cap \{x : x_n \leq -\delta s_m\} \neq \emptyset \tag{180}$$

for each m. To get a contradiction one shows that (180) leads to

$$\liminf_{t\to 0} t^{-1} \max_{B(0,t)} u_\infty = 0 \tag{181}$$

which in view of Lipschitzness of Ω_∞ and the mean value theorem from elementary calculus, again contradicts (159). We omit the details. □

5.15 Closing Remarks

To state our likely results in [54] we need a definition.

Definition H. *Given $\epsilon > 0$ we say that u is ϵ-monotone in $O \subset \mathbf{R}^n$, with respect to the directions in the cone $\tilde{\Gamma}(\nu, \theta_0) = \{y \in \mathbf{R}^n : |y| = 1$ and $\theta(\nu, y) < \theta_0\}$, if*

$$\sup_{B(x, \epsilon' \sin \theta_0)} u(y - \epsilon' \nu) \leq u(x)$$

whenever $\epsilon' \geq \epsilon$ and $x \in O$ with $B(x - \epsilon' \nu, \epsilon' \sin \theta_o) \subset O$. Moreover, u is said to be monotone or fully monotone in $O \subset \mathbf{R}^n$, with respect to the directions in the cone $\Gamma(\nu, \theta_0)$, provided the above inequality holds whenever $\epsilon' > 0$.

Plausible Theorem 4.14. *Let $D, u, D^+(u), D^-(u), F(u), G$ be as in Theorem 4.11. If $\bar{\theta} \in (\pi/4, \pi/2)$, then there is a $\bar{\epsilon} = \bar{\epsilon}(\bar{\theta}, p, n, N)$ such that if u is ϵ monotone on $B(0,2)$ with respect to the directions in the cone $\Gamma(e_n, \bar{\theta})$, for some $\epsilon \in (0, \bar{\epsilon})$, then u is monotone in $B(0, 1/2)$ with respect to the directions in the cone $\Gamma(e_n, \bar{\theta}_1)$, where $\bar{\theta}_1$ has the same dependence as $\bar{\epsilon}$. Equivalently, $F(u) \cap B(0, 1/2)$ is the graph of a Lipschitz function with Lipschitz norm depending on $\bar{\theta}, p, n, N$.*

We note that we can use Theorems 4.14 and 4.11 to conclude that ϵ monotonicity of u implies $F(u)$ is $C^{1,\sigma}$ provided ϵ is small enough.

Plausible Theorem 4.15. *Let $D, F(u), D^+(u), D^-(u), G$ be as in Plausible Theorem 4.14 except for the following changes:*

(a) Assume only that u^+ is ϵ monotone in $\Gamma(e_n, \bar{\theta})$,
(b) Assume $0 < \delta \leq |\nabla u| \leq \delta^{-1}$ on $D^+(u) \cap B(0, 2)$,
(c) G is also Lipschitz continuous.

There exists $\hat{\epsilon} > 0$ and $\hat{\theta} \in (\pi/4, \pi/2)$, both depending on p, n, δ, N, such that if $\hat{\theta} < \bar{\theta} \leq \pi/2$, and $0 < \epsilon \leq \hat{\epsilon}$, then u^+ is monotone in $B(0,2)$ with respect to the directions in the cone $\Gamma(e_n, \theta_1)$ for some $\theta_1 > 0$, depending on $p, n, \delta, N, \hat{\epsilon}, \hat{\theta}$.

As a corollary to Plausible Theorem 4.15 we also plan to show that

Plausible Corollary 4.16. *Replace the ϵ monotonicity assumption in Plausible Theorem 4.15 by*

$$\Psi(F(u) \cap B(0,2), \Lambda \cap B(0,2)) \leq \epsilon,$$

where Λ is the graph of a Lipschitz function with Lipschitz norm $\leq$ tan $(\pi/2 - \hat{\theta})$. Then the same conclusion holds as in Plausible Theorem 4.15.

One can also ask if Theorem 4.11 generalizes to equations of p Laplace type, as in the $p = 2$ case. In this case, the analogue of Lemma 4.12 may be difficult since the proof of this lemma made important use of the invariance of the

p Laplace equation under rotations and translations. Also, the analogue of Theorem 4.13 could be difficult.

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