

Chapter 2

Hamiltonian Systems of Few Degrees of Freedom

Abstract In Chap. 2 we provide first an elementary introduction to some simple examples of Hamiltonian systems of one and two degrees of freedom. We describe the essential features of phase space plots and focus on the concepts of periodic and quasiperiodic motions. We then address the questions of integrability and solvability of the equations, first for linear and then for nonlinear problems. We present the important integrability criteria of Painlevé analysis in *complex time* and show, on the non-integrable Hénon-Heiles model, how chaotic orbits arise on a Poincaré Surface of Section of the dynamics in phase space. Using the example of a periodically driven Duffing oscillator, we explain that chaos is connected with the intersection of invariant manifolds and describe how these intersections can be analytically studied by the perturbation approach of Mel'nikov theory.

2.1 The Case of $N = 1$ Degree of Freedom

One of the first physical systems that an undergraduate student encounters in his (or her) science studies is the harmonic oscillator. This describes the oscillations of a mass m , tied to a spring, which exerts on the mass a force that is proportional to the negative of the displacement q of the mass from its equilibrium position ($q = 0$), as shown in Fig. 2.1. The dynamics is described by Newton's second order differential equation

$$m \frac{d^2 q}{dt^2} = -kq, \quad (2.1)$$

where $k > 0$ is a constant representing the hardness (or softness) of the spring. Equation 2.1 can be easily solved by standard techniques of linear ODEs to yield the displacement $q(t)$ as an oscillatory function of time of the form

$$q(t) = A \sin(\omega t + \alpha), \quad \omega = \sqrt{k/m}, \quad (2.2)$$

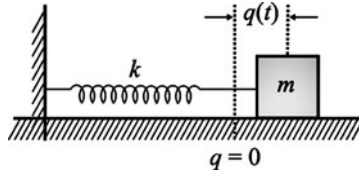
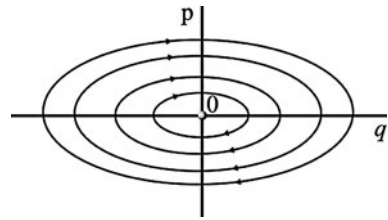


Fig. 2.1 A harmonic oscillator consists of a mass m moving horizontally on a frictionless table under the action of a force which is proportional to the negative of the displacement $q(t)$ of the mass from its equilibrium position at $q = 0$ ($k > 0$ is a proportionality constant)

Fig. 2.2 Plot of the solutions of the harmonic oscillator Hamiltonian (2.4) in the (q, p) phase space, as a one-parameter family of curves, for different values of the total energy E



where A and α are free constants corresponding to the amplitude and phase of oscillations and ω is the frequency.

If we now recall from introductory physics that $p = mdq/dt$ represents the momentum of the mass m , we can rewrite (2.1) in the form of two first order ODEs

$$\frac{dq}{dt} = \frac{p}{m} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -kq = -\frac{\partial H}{\partial q}, \quad (2.3)$$

where we have already anticipated that the dynamics is derived from the Hamiltonian function

$$H(q, p) = \frac{p^2}{2m} + k \frac{q^2}{2} = E, \quad (2.4)$$

which is an integral of the motion (since $dH/dt = 0$), whose value E represents the total (kinetic plus potential) energy of the system. Thus, this mass-spring system is our first example of a Hamiltonian system of $N = 1$ dof.

We will not attempt to solve (2.3). Had we done so, of course, we would have recovered (2.2). Instead, we will consider the solutions of the problem *geometrically* as a one-parameter family of trajectories (or orbits) given by (2.3) and plotted in the (q, p) phase space of the system in Fig. 2.2. Thus, as this figure shows, both variables $q(t)$ and $p(t)$ oscillate periodically with the same frequency ω . Their orbits are ellipses whose (semi-) major and minor axes are determined by the parameters k, m and the value of E .

Note now that the energy integral (2.4) can also be used to bring us one step closer to the complete solution of the problem, by solving (2.4) for p and obtaining an equation where the variable q is separated from the time t . Integrating both sides of this equation in terms of the respective variables, we arrive at the expression

$$\int_0^q \frac{dq}{\sqrt{2E/m - \omega^2 q^2}} = t + c, \quad (2.5)$$

which defines implicitly the solution $q(t)$ as function of t , c being the second free constant of the problem (the first one is E). The reader now recognizes the left side of (2.5) as the inverse sine function, whence we finally obtain

$$q(t) = \sqrt{\frac{2E}{k}} \sin(\omega(t + c)), \quad (2.6)$$

where the constants E and c can be related to A and α , by direct comparison with (2.2).

The process described above is called *integration by quadratures* and was made possible by the crucial existence of the integral of motion (2.4).

Of course, the harmonic oscillator is a linear system whose dynamics is very simple. It possesses a single *equilibrium or fixed point* at $(0, 0)$, where all first derivatives in (2.3) vanish. This point is called *elliptic* and all closed curves about it in Fig. 2.2 describe oscillations whose frequency ω is independent of the choice of initial conditions.

Let us turn, therefore, to a more interesting one dof Hamiltonian system representing the motion of a simple pendulum shown in Fig. 2.3. Its equation of motion according to classical mechanics is

$$ml^2 \frac{d^2\theta}{dt^2} = -mgl \sin \theta, \quad (2.7)$$

where the right side of (2.7) expresses the restoring torque due to the weight mg of the mass (g being the acceleration due to gravity) and the left side describes the time derivative of the angular momentum of the mass. If we now write this equation as a system of two first order ODEs, as we did for the harmonic oscillator, we find again that they can be cast in Hamiltonian form

$$\frac{dq}{dt} = p = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{g}{l} \sin q = -\frac{\partial H}{\partial q}, \quad (2.8)$$

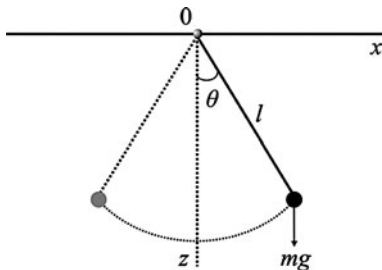
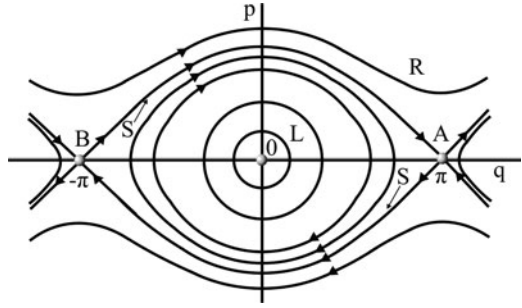


Fig. 2.3 The simple pendulum is moving in the two-dimensional plane (x, z) , but its motion is completely described by the angle θ and its derivative $d\theta/dt$

Fig. 2.4 Plot of the curves of the energy integral (2.9) in the (q, p) phase space. Note, the presence of additional equilibrium points at $(\pm\pi, 0)$, which are of the saddle type. Observe also the curve S separating oscillatory motion (L) about the central elliptic point at $(0, 0)$ from rotational motion (R)



where $q = \theta$ and l is the length of the pendulum. Of course, we cannot solve these equations as easily as we did for the harmonic oscillator. Still, as you will find out by solving Exercise 2.1, the solution of (2.8) can be obtained via the so-called Jacobi elliptic functions [6, 107, 169], which have been thoroughly studied in the literature and are directly related to the solution of an anharmonic oscillator with cubic nonlinearity, about which we will have a lot to say in later sections.

Still, we can make great progress in understanding the dynamics of the simple pendulum through its Hamiltonian function, which provides the energy integral

$$H(q, p) = \frac{p^2}{2} + \frac{g}{l}(1 - \cos q) = E. \quad (2.9)$$

Plotting this family of curves in the (q, p) phase space for different values of E we now obtain a much more interesting picture than Fig. 2.2 depicted in Fig. 2.4. Observe that, besides the elliptic fixed point at the origin, there are two new equilibria located at the points $(\pm\pi, 0)$. These are called *saddle points* and *repel* most orbits in their neighborhood along hyperbola-looking trajectories. For this reason points A and B in Fig. 2.4 are also called *unstable*, in contrast to the $(0, 0)$ fixed point, which is called *stable*. More precisely, in view of the theory presented in Chap. 1, $(0, 0)$ is characterized by what we called conditional (or neutral) stability.

Let us now discuss the *linear stability* of fixed points of a Hamiltonian system within the more general framework outlined in Sect. 1.2. According to this analysis, we first need to linearize Hamilton's system of ODEs (1.18) about a fixed point, which we assume to be located at the origin $\mathbf{0} = (0, 0, \dots, 0)$ of the $2ND$ phase space. The resulting set of linear ODEs is written in terms of the Jacobian matrix $J = \{J_{jk}\}$:

$$\dot{x}_j = \sum_{k=1}^{2N} J_{jk} x_k, \quad j = 1, 2, \dots, 2N, \quad (2.10)$$

whose elements (show this as a simple exercise starting with (1.18))

$$\begin{aligned}
j = 1, 2, \dots, N : \quad J_{jk} &= \frac{\partial^2 H}{\partial p_j \partial q_k}(\mathbf{0}), \quad k = 1, \dots, N, \\
J_{jk} &= \frac{\partial^2 H}{\partial p_j \partial p_{k-N}}(\mathbf{0}), \quad k = N+1, \dots, 2N, \\
j = N+1, \dots, 2N : \quad J_{jk} &= -\frac{\partial^2 H}{\partial q_{j-N} \partial q_k}(\mathbf{0}), \quad k = 1, \dots, N, \\
J_{jk} &= -\frac{\partial^2 H}{\partial q_{j-N} \partial p_{k-N}}(\mathbf{0}), \quad k = N+1, \dots, 2N,
\end{aligned} \tag{2.11}$$

where x_j represents small variations away from the origin ($q_j = p_j = 0$, $j = 1, \dots, N$) of the q_j variables for $j = 1, \dots, N$ and the p_j variables for $j = N+1, \dots, 2N$.

For Hamiltonian systems of $N = 1$ dof, it is clear from (2.11) that the elements of the 2×2 Jacobian matrix $J = \{J_{jk}\}$, $j, k = 1, 2$ are

$$J_{11} = -J_{22} = \frac{\partial^2 H}{\partial q \partial p}(0, 0), \quad J_{12} = \frac{\partial^2 H}{\partial p^2}(0, 0), \quad J_{21} = -\frac{\partial^2 H}{\partial q^2}(0, 0). \tag{2.12}$$

In the case of the harmonic oscillator (2.3) this matrix has two imaginary eigenvalues $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, and hence corresponds to the case of conditional (neutral) stability defined in Sect. 1.2. As a local property, it characterizes the fixed point at the origin of Fig. 2.2 as elliptic. However, since the original problem is linear, this type of stability is also *global* and implies that the solutions are oscillatory not only near the origin but in the full phase plane (q, p) .

In the case of the pendulum Hamiltonian, however, the situation is very different. First of all, there are more than one fixed points. In fact, there is an infinity of them located at $(\theta_n, 0) = (\pm n\pi, 0)$, $n = 0, 1, 2, \dots$, where $(\dot{q}, \dot{p}) = (0, 0)$ in the (q, p) phase space. The linearized equations of motion about these fixed points are:

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} \cos \theta_n & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}(-1)^n & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned} \tag{2.13}$$

Thus, for $n = 2k$ even (k any integer) the Jacobian matrix J has two complex conjugate (imaginary) eigenvalues $\lambda_1 = i\omega_0$, $\lambda_2 = -i\omega_0$ and corresponds to elliptic fixed points at $(2k\pi, 0)$, while for $n = 2k+1$, the equilibria are of the *saddle type* with real eigenvalues $\lambda_1 = \omega_0$, $\lambda_2 = -\omega_0$, where $\omega_0 = \sqrt{g/l}$ (see points A, B in Fig. 2.4). Near these saddle points, the motion is governed by the two real eigenvectors corresponding to the above two eigenvalues. Along the eigenvector

corresponding to $\lambda_1 > 0$, solutions of (2.13) move exponentially away from the fixed point, while along the $\lambda_2 < 0$ eigenvector they exponentially converge to the fixed point.

The important observation here is that, in the case of the simple pendulum, the solutions of the linearized equations of motion (2.13) are valid only *locally* within an infinitesimally small neighborhood of the fixed points. The reason for this is that the full equations of the system (2.8) are nonlinear and hence there is no guarantee that the linear dynamics near the fixed points is *topologically equivalent* to the solutions of the system when the nonlinear terms are included.

By topological equivalence, we are referring to the existence of a homeomorphism (i.e. a continuous and invertible map, with a continuous inverse) that maps orbits of (2.13) in a one-to-one way to orbits of (2.8), see [170, 265, 346]. How do we know that? How can we be sure that if we expand the right hand sides of (2.8) in powers of q , p , using e.g. $\sin q = q - q^3/6 + \dots$ as we did in Sect. 1.2, we will obtain solutions which are topologically equivalent to those of the linearized equations (2.13)?

The answer to this question is provided by the Hartman-Grobman theorem [170, 175, 257, 265, 346], which states that such an equivalence can be established provided all eigenvalues of the Jacobian matrix have real part different from zero. Now, as we learned from Lyapunov's theory, if all eigenvalues satisfy $\text{Re}(\lambda_i) < 0$, the equilibrium is asymptotically stable. However, if there is at least one eigenvalue whose real part is positive, the general solutions of the nonlinear system deviate exponentially away from the fixed point.

Well, this settles the issue about the saddle points at $((2k + 1)\pi, 0)$ of the simple pendulum. It does not help us, however, with the elliptic points at $(2k\pi, 0)$, since the linearized equations about them have $\text{Re}(\lambda_i) = 0$, $i = 1, 2$ and the Hartman-Grobman theorem does not apply. Here is where the integral of the motion represented by the Hamiltonian function (2.9) comes to the rescue. It establishes the existence of a family of closed curves globally, i.e. within a large region around these elliptic points.

In our problem, this region extends all the way to the *separatrix* curve S joining the *unstable manifold* of saddle point A to the *stable manifold* of saddle point B (and the unstable manifold of B with the stable one of A), as shown in Fig. 2.4. These manifolds are analytic curves which are tangent to the corresponding eigenvectors of the linearized equations and are called *invariant*, since all solutions starting on such a manifold remain on it for all $t > 0$ (or $t < 0$).

Invariant manifolds, however, do not need to join smoothly to form separatrices, as in Fig. 2.4. In fact, as we explain in the next sections, invariant manifolds of Hamiltonian systems of $N \geq 2$ dof generically intersect each other transversally (i.e. with non-zero angle) and give rise to chaotic regions in the $2ND$ phase space $\mathbb{R}^{2N}$. If, on the other hand, these manifolds do join smoothly, this suggests that the system has N analytic, single-valued integrals and is hence completely integrable in the sense of the LA theorem, as in the case of the simple pendulum.

2.2 The Case of $N = 2$ Degrees of Freedom

Following the above discussion, it would be natural to extend our study to Hamiltonian systems of two dof, joining at first two harmonic oscillators, as shown in Fig. 2.5 and applying the approach of Sect. 2.1. We furthermore assume that our oscillators have equal masses $m_1 = m_2 = m$ and spring constants $k_1 = k_2 = k$ and impose fixed boundary conditions to their endpoints, as shown in Fig. 2.5. Clearly, Newton's equations of motion give in this case:

$$m \frac{d^2 q_1}{dt^2} = -kq_1 - k(q_1 - q_2), \quad m \frac{d^2 q_2}{dt^2} = -kq_2 + k(q_1 - q_2), \quad (2.14)$$

where $q_i(t)$ are the particles' displacements from their equilibrium positions at $q_i = 0$, $i = 1, 2$. If we also introduce the momenta $p_i(t)$ of the two particles in terms of their velocities, as we did for the single harmonic oscillator, we arrive at the following fourth order system of ODEs

$$\frac{dq_1}{dt} = \frac{p_1}{m}, \quad \frac{dp_1}{dt} = -kq_1 - k(q_1 - q_2), \quad \frac{dq_2}{dt} = \frac{p_2}{m}, \quad \frac{dp_2}{dt} = -kq_2 + k(q_1 - q_2), \quad (2.15)$$

which is definitely Hamiltonian, since it is derived from the Hamiltonian function

$$H(q_1, p_1, q_2, p_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + k \frac{q_1^2}{2} + k \frac{q_2^2}{2} + k \frac{(q_1 - q_2)^2}{2} = E \quad (2.16)$$

(see (1.18)), expressing the integral of total energy E . Note that we may think of the two endpoints of the system as represented by positions and momenta q_0, p_0 and q_3, p_3 that vanish identically for all t .

What can we say about the solutions of this system? Are they all periodic, as in the case of the single harmonic oscillator? Is the dynamics more complicated in the four-dimensional phase space of the $q_i(t)$, $p_i(t)$, $i = 1, 2$ and in what way? Can we still use the integral (2.16) to solve the system by quadratures? Note, first of all, that, once we specify a value for the total energy E , our variables $q_i(t)$, $p_i(t)$, $i = 1, 2$ are no longer independent, as the motion actually evolves on a three-dimensional so-called *energy surface* $\Gamma(E)$. If we choose to solve (2.16) for one of them, say

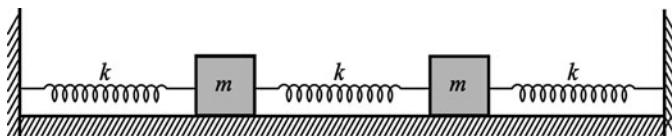


Fig. 2.5 Two coupled harmonic oscillators of equal mass m and force constant k , with displacements $q_1(t)$, $q_2(t)$ from their equilibrium position, moving horizontally on a frictionless table with their endpoints firmly attached to two immovable walls

p_1 , in terms of the other three, we do not arrive immediately at an equation whose variables can be separated and thus integration by quadratures is impossible. What would happen, however, if a *second integral* of the motion were available? Could the method of quadratures lead us to the solution in that case?

Your immediate reaction, of course, would be to say no. Two integrals could only serve to express one variable, say p_1 again, in terms of two others, say q_1 and q_2 , suggesting again that integration by quadratures is impossible. But, here is where appearances deceive. Note that the difficulty arises because the equations of motion (2.14), as given to us by Newtonian mechanics, are *coupled* in the variables q_1, q_2 . Is it perhaps possible to use a suitable coordinate transformation to separate them and write our equations as a system of two uncoupled harmonic oscillators?

2.2.1 Coordinate Transformations and Solution by Quadratures

It is not very difficult to answer this question. Indeed, if we perform the change of variables

$$Q_1 = \frac{q_1 + q_2}{\sqrt{2}}, \quad Q_2 = \frac{q_1 - q_2}{\sqrt{2}}, \quad P_1 = \frac{p_1 + p_2}{\sqrt{2}}, \quad P_2 = \frac{p_1 - p_2}{\sqrt{2}} \quad (2.17)$$

(the factor $1/\sqrt{2}$ will be explained later), we see that, adding and subtracting by sides the two equations in (2.14) (dividing also by m and introducing $\omega = \sqrt{k/m}$), splits the problem to two harmonic oscillators

$$\frac{dQ_i}{dt} = \frac{P_i}{m}, \quad \frac{dP_i}{dt} = -\omega_i^2 Q_i, \quad i = 1, 2, \quad \omega_1 = \omega, \quad \omega_2 = \sqrt{3}\omega, \quad (2.18)$$

which have *different* frequencies ω_1, ω_2 . Observe that the new Hamiltonian of the system becomes

$$K(Q_1, P_1, Q_2, P_2) = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + k \frac{Q_1^2}{2} + 3k \frac{Q_2^2}{2} = E. \quad (2.19)$$

We now see that the factor $1/\sqrt{2}$ was introduced in (2.17) to make the new equations of motion (2.18) have the same *canonical form* as two uncoupled harmonic oscillators, while the new Hamiltonian K is expressed as the sum of the Hamiltonians of these oscillators. In this way, we may say that changing from q_i, p_i to Q_i, P_i variables we have performed a canonical coordinate transformation to our system.

In this framework, we immediately realize that our problem possesses not one but *two* integrals of the motion which may be thought of as the energies of the two oscillators

$$F_i(Q_1, P_1, Q_2, P_2) = \frac{P_i^2}{2m} + k_i \frac{Q_i^2}{2} = E_i, \quad i = 1, 2, \quad (2.20)$$

with $k_1 = k$, $k_2 = 3k$, while E_i are two free parameters of the system to be fixed by the initial conditions $q_i(0)$, $p_i(0)$, $i = 1, 2$. Thus, we may now proceed to apply the method of quadratures to each of these oscillators, as described in Sect. 2.1 to obtain the general solution of the system in the form

$$Q_i(t) = \sqrt{\frac{2E_i}{k_i}} \sin(\omega_i(t + c_i)), \quad i = 1, 2, \quad (2.21)$$

c_i being the other two free parameters needed for the complete solution of the four first order ODEs (2.18). Naturally, if we wish to write our general solution in terms of the original variables of the problem, we only need to invert (2.17) to find $q_1(t)$, $q_2(t)$. We also remark that the above considerations show that the only fixed point of the system lies at the origin of phase space and is elliptic (why?).

Let us make a crucial observation at this point: Note that the successful application of the above analysis rests on the existence of two integrals of motion (2.20) associated with the free constants E_1 , E_2 connected with the *amplitudes* of the corresponding oscillations. This shows that, for a system of two dof, two integrals are necessary and sufficient for its integration by quadratures, as the remaining two arbitrary constants c_1 , c_2 merely represent phases of oscillation and are not as important as E_1 and E_2 . To understand this better, note that c_1 , c_2 are *not* single-valued and hence do not belong to the class of integrals required by the LA theorem for complete integrability.

All this suggests that it would be advisable to introduce one more transformation to the so-called *action-angle* variables I_i , θ_i , $i = 1, 2$ as follows

$$Q_i = \sqrt{\frac{2\omega_i I_i}{k_i}} \sin \theta_i, \quad P_i = \sqrt{2m\omega_i I_i} \cos \theta_i, \quad i = 1, 2 \quad (2.22)$$

and write our integrals (2.20) as $F_i = I_i \omega_i$ so that the new Hamiltonian (2.19) may be expressed in a form

$$G(I_1, \theta_1, I_2, \theta_2) = G(I_1, I_2) = I_1 \omega_1 + I_2 \omega_2 \quad (2.23)$$

that is *independent* of the angles θ_i , which also demonstrates the irrelevance of the phases of the oscillations in (2.21). Evidently, our action-angle variables also satisfy Hamilton's equations of motion

$$\frac{d\theta_i}{dt} = \frac{\partial G}{\partial I_i} = \omega_i, \quad \frac{dI_i}{dt} = -\frac{\partial G}{\partial \theta_i} = 0, \quad (2.24)$$

which imply that the new momenta I_i are constants of the motion, while the θ_i are immediately integrated to give $\theta_i = \omega_i t + \theta_{i0}$, where θ_{i0} are the two initial phases. Thus, the change to action-angle variables defines a canonical transformation and the oscillations (2.22) are called *linear normal modes* of the system.

Let us now discuss the solutions of this coupled system of linear oscillators. Using (2.17) and (2.21) we see that they are expressed, in general, as linear combinations of two trigonometric oscillations with frequencies $\omega_1 = \sqrt{k}$, $\omega_2 = \sqrt{3k}$. If these frequencies were *rationally dependent*, i.e. if their ratio were a rational number $\omega_1/\omega_2 = m_1/m_2$ (m_1, m_2 being positive integers with no common divisor) all orbits would *close* on two-dimensional *invariant tori*, like the one shown in Fig. 2.6 and the motion would be periodic (what are the m_1, m_2 of the orbit shown in Fig. 2.6?). Note that in our example, this could only happen for initial conditions such that E_1 or E_2 is zero, whence the solutions would execute in-phase or out-of-phase oscillations with frequency ω_2 , or ω_1 respectively.

In the general case, though, E_1 and E_2 are both non-zero and the oscillations are *quasiperiodic*, in the sense that they result from the superposition of trigonometric terms whose frequencies are *rationally independent*, as the ratio $\omega_2/\omega_1 = \sqrt{3}$ is an irrational number. Hence, the orbits produced by these solutions in the 4-dimensional phase space are *never closed* (periodic). Unlike the orbit shown in Fig. 2.6, they never pass by the same point, covering eventually uniformly the two-dimensional torus of Fig. 2.6 specified by the values of E_1 and E_2 .

Thus, in the spirit of the LA theorem we have verified for a Hamiltonian system of two coupled harmonic oscillators that two global, single-valued, analytic integrals are necessary and sufficient to completely integrate the equations of motion by quadratures and moreover that the general solutions lie on two-dimensional tori, since the motion on the four-dimensional phase space of the problem is bounded about an elliptic fixed point at the origin $q_i = p_i = 0$, $i = 1, 2$.

Let us now turn to the case of a system of two coupled *nonlinear* oscillators. We could choose to do so by introducing nonlinear terms in the spring forces involved in the above two linear oscillators. However, we prefer to postpone such a study for later chapters, where we will deal extensively with one-dimensional chains of $N > 2$ coupled nonlinear oscillators. Thus, we shall discuss here one of the most famous Hamiltonian two dof system, originally due to Hénon and Heiles [164]

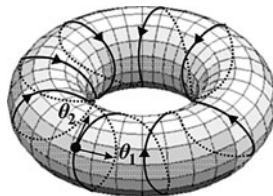


Fig. 2.6 If the ratio of the frequencies of a two dof integrable Hamiltonian system is a *rational* number, the orbits are periodic (and hence close upon themselves) on two-dimensional invariant tori, such as the one shown in the figure. However, if the ratio ω_1/ω_2 is irrational, the orbits never close and eventually cover uniformly two-dimensional tori in the four-dimensional phase space of the coordinates q_i, p_i , $i = 1, 2$

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) + q_1^2 q_2 - \frac{C}{3} q_2^3. \quad (2.25)$$

This describes the motion of a star of mass m in the axisymmetric potential of a galaxy whose lowest order (quadratic) terms are those of two uncoupled harmonic oscillators and the only higher order terms present are cubic in the q_1, q_2 variables (for a most informative recent review on the applications of nonlinear dynamics and chaos to galaxy models see [99]). Note that the value of m in (2.25) can be easily scaled to unity by appropriately rescaling time, while an additional parameter before the $q_1^2 q_2$ term in (2.25) has already been removed by a similar change of scale of all variables q_i, p_i .

Thus, introducing the more convenient variables $q_1 = x, q_2 = y, p_1 = p_x, p_2 = p_y$, we can rewrite the above Hamiltonian in the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(Ax^2 + By^2) + x^2 y - \frac{C}{3} y^3 = E, \quad (2.26)$$

where E is the total energy and we have set $\omega_1^2 = A > 0$ and $\omega_2^2 = B > 0$. It is instructive to write down Newton's equations of motion associated with this system

$$\frac{d^2 x}{dt^2} = -\frac{\partial V}{\partial x} = -Ax - 2xy, \quad \frac{d^2 y}{dt^2} = -\frac{\partial V}{\partial y} = -By - x^2 + Cy^2. \quad (2.27)$$

Before going further, let us first write down here all possible fixed points of the system, which are easily found by setting the right hand sides in (2.27) equal to zero:

$$\begin{aligned} \text{(i)} \quad (x, y) &= (0, 0), \quad \text{(ii)} \quad (x, y) = \left(0, \sqrt{\frac{B}{C}}\right), \\ \text{(iii)} \quad (x, y) &= \left(\pm \frac{\sqrt{2BA + CA^2}}{2}, -\frac{A}{2}\right) \end{aligned} \quad (2.28)$$

(the momentum coordinates corresponding to all these points are, of course, zero). The local (or linear) stability of these equilibria will be discussed later, with reference to some very important special cases of the Hénon-Heiles problem. However, there is one of them whose character can be immediately deduced from the above equations: It is, of course, the origin (i), where all nonlinear terms in (2.27) do not contribute to the analysis. Thus, infinitesimally close to this point, the equations become identical to those of two uncoupled harmonic oscillator (since $A > 0, B > 0$) and therefore the equilibrium at the origin is *elliptic*.

Let us remark now that (2.26) represents a first integral of this system. If we could also find a second one, as in the problem of the two harmonic oscillators, with all the nice mathematical properties required by the LA theorem, the problem would be completely integrable and we would be able to integrate the (2.27) by quadratures and obtain the general solution. Unfortunately, in nonlinear two dof

Hamiltonian systems, the existence of such an integral is far from guaranteed. In fact, all indications for the Hénon-Heiles system is that a second integral for all A , B , C does not exist.

“How does one know that?”, you may rightfully ask. Well, this is a very good question. A first answer may be given by the fact that in many cases that have been integrated numerically by many authors, even close to the equilibrium point (i), one finds, besides periodic and quasiperiodic orbits, a new kind of solution that appears “irregular” and “unpredictable”, one may even say random-looking [226]! These are the orbits we have called chaotic. They tend to occupy densely three-dimensional regions in the four-dimensional phase space and depend very sensitively on initial conditions, in the sense that starting at almost any other point in their immediate vicinity produces another trajectory (also chaotic), which deviates exponentially from the original (reference) orbit as time increases.

This would not happen, of course, if the problem was completely integrable, since in that case—at least close to the elliptic equilibrium point at the origin—all phase space would be occupied by two-dimensional tori, which are uniformly covered by periodic or quasiperiodic solutions. These have nothing “irregular” or “unpredictable” about them. In fact, it can be shown that the distance between nearby orbits in their vicinity deviate linearly from each other in phase space [226].

Still, if one works only numerically, it is possible to miss chaotic orbits altogether, since their associated regions can be very thin and may not even appear in the computations. To speak about complete integrability therefore (or the absence thereof) one must have some analytical theory to support rigorously one’s statements. One such theory is provided by the so-called *Painlevé analysis*, which investigates a very fundamental property of the solutions related to their *singularities in complex time* [98, 152, 279].

2.2.2 Integrability and Solvability of the Equations of Motion

It is of course very difficult to look at a pair of nonlinear coupled second order ODEs like (2.27) and try to integrate them by some clever change of variables, as we did in the case of two coupled linear oscillators (2.14). Still, the careful reader will notice (by inspection, as we say) that there is one obvious case where (2.27) can be directly integrated: $A = B$, $C = -1$. Under this condition, a transformation to sum and difference variables reduces the Hénon-Heiles equations to the simple *uncoupled* form

$$\frac{d^2 X}{dt^2} = -AX - \sqrt{2}X^2, \quad \frac{d^2 Y}{dt^2} = -AY + \sqrt{2}Y^2, \quad X = \frac{x+y}{\sqrt{2}}, \quad Y = \frac{x-y}{\sqrt{2}} \quad (2.29)$$

(see (2.17)). “This is great!”, you would exclaim: Now, we can multiply these equations respectively by dX/dt and dY/dt and integrate them to arrive at the following integrals of the motion

$$F_1 = \frac{1}{2} \left(\frac{dX}{dt} \right)^2 + \frac{A}{2} X^2 - \frac{\sqrt{2}}{3} X^3, \quad F_2 = \frac{1}{2} \left(\frac{dY}{dt} \right)^2 + \frac{A}{2} Y^2 + \frac{\sqrt{2}}{3} Y^3. \quad (2.30)$$

It is easy to see now that these two integrals are *not* independent from the Hamiltonian (2.26). Indeed, if we also define the momentum coordinates $P_X = dX/dt$, $P_Y = dY/dt$ and observe that $p_x = (P_X + P_Y)/\sqrt{2}$, $p_y = (P_X - P_Y)/\sqrt{2}$ we easily verify that $H = F_1 + F_2$ and hence that the change to new variables is a canonical transformation.

Furthermore, it is possible to prove that the two integrals (2.30) are independent *from each other*. This can be done by demonstrating that *the tangent space* spanned by the partial derivatives of F_1 with respect to X , P_X and F_2 with respect to Y , P_Y at almost all points where $F_1 = F_2$ has the full dimensionality of the phase space of the system. Moreover, the Poisson bracket $\{F_1, F_2\}$ vanishes which implies that the two integrals are *in involution* [19, 226]. We conclude, therefore, according to the LA theorem, that the system is completely integrable and hence its general solution can be obtained by quadratures.

Try it: If you go back to (2.30), solve for dX/dt and dY/dt and integrate by separation of variables, as we did for the two coupled harmonic oscillators, you will be led to elliptic integrals, which are the inverse of the so-called Jacobi elliptic functions [6, 107, 169]. The most fundamental ones of them are $\text{sn}(\mu t, \kappa^2)$ and $\text{cn}(\mu t, \kappa^2)$ and their parameters μ, κ are related to the parameter A of our problem. When their so called *modulus* $0 \leq \kappa \leq 1$ tends to 1, the sn and cn tend to trigonometric functions (sine and cosine respectively), while $\mu \rightarrow \sqrt{A}$, which is the frequency of oscillations of our variables near the origin, see (2.29). On the other hand, when $\kappa \rightarrow 0$, the period of sn and cn tends to infinity and they become hyperbolic $\sinh$ and $\cosh$ functions (see [6, 107, 169] and Exercise 2.1).

So everything is fine with the $A = B$, $C = -1$ case of the Hénon-Heiles equations. What do we do, however, for other choices of these parameters? Can we still hope to come up with some “clever” transformations of variables, such as the ones employed above, which will help us uncouple and eventually solve the nonlinear ODEs of the problem? The answer, unfortunately, is negative. As clever as we may be, resourcefulness has its limits. Inspection alone is not sufficient to make progress here. We need to develop a more systematic approach, such as the one provided by the so called Painlevé analysis of (2.27) [98, 152, 279].

Let us, therefore, think more analytically and try to understand more about the mathematics of the integrable case we discovered above. Our motivation is the following: What if integrability (and hence also solvability) of our equations is connected with the *mathematical simplicity* of the solutions of these equations? In other words, what problems can we solve using the known mathematical functions available in the literature today? And since we already discovered elliptic functions in one example of our problem, why not try to find other integrable cases of the Hénon-Heiles system, by *requiring* that their solutions belong to the same “class” as the Jacobi elliptic functions?

This is not such a crazy question. In fact, it was first asked by the famous Russian mathematician S. Kowalevskaya in the 1880s, who was thus able to discover one more integrable case of the rotating body, in addition to the other three that had already been found by Euler and Jacobi. To this day, these are the *only four* known cases, where the rotating body can be explicitly integrated. So the moral of the story is clear: If you want to become famous, find a new integrable case of a physically important system of differential equations!

To understand what Kowalevskaya did we need to start from a remarkable mathematical property of the Jacobi elliptic functions: When their independent variable $u = \mu t$ is complex, their *only singularities* in the complex t -plane are poles. In other words, these functions are not analytic everywhere in the complex domain, and thus are more general than the ordinary exponentials and trigonometric functions of elementary mathematics. They are, however, *single-valued*, in the sense that when evaluated on a closed loop around their singularities they recover their values after every turn.

The recipe, therefore, appears rather simple: First express the solutions $x(t)$, $y(t)$ of (2.27) about a singularity at $t = t_*$ as series expansions that begin with a possible pole of order r and s respectively. This gives

$$x(t) = \sum_{k=r}^{\infty} a_k \tau^k \quad y(t) = \sum_{k=s}^{\infty} b_k \tau^k, \quad \tau = t - t_*, \quad (2.31)$$

where r, s are integers and at least one of them is *negative*. These series are assumed to converge within a finite radius $0 < |\tau| = |t - t_*| < \rho$ in the complex t -plane and thus provide the complete solution of the problem near this singularity. It follows, therefore, that among the coefficients a_k, b_k three must be arbitrary, as t_* is already one of the free constants of the solution.

Substituting these series in (2.27) and balancing the most divergent terms leads to the equations

$$r(r-1)a_r \tau^{r-2} = -2a_r b_s \tau^{r+s}, \quad s(s-1)b_s \tau^{s-2} = -a_r^2 \tau^{2r} + C b_s^2 \tau^{2s}, \quad (2.32)$$

which imply that there are *two* possible dominant behaviors (and hence two such singularity types) for this problem, namely

$$\begin{aligned} \text{(i)} \quad r = s = -2 : \quad a_{-2} &= \pm \sqrt{18 + 9C}, \quad b_{-2} = -3 \\ \text{(ii)} \quad s = -2, \quad r > s : \quad a_r &= \text{arbitrary}, \quad b_{-2} = \frac{r(1-r)}{2} = \frac{6}{C}. \end{aligned} \quad (2.33)$$

Note that, in type (ii), the power r of the leading term of the expansion of $x(t)$ is directly related to the value of the parameter C . Since we have assumed that our series involve only *integer* powers of τ , the only C values that ensure this are $C = -1, -2, -6$. That's great! The first one of them is already the beginning of

the thread that leads to the integrable case discovered earlier in this section by pure inspection.

What remains to be done now is entirely algorithmic and can even be programmed on the computer: Take each one of the above cases and compute the coefficients of the Laurent series of the solutions near the corresponding singularity type. As we have already explained, at most three of the a_k, b_k should be arbitrary. And here is where the most important stumbling block of the method lies: When you try to evaluate these coefficients you will discover that they satisfy a degenerate system of linear algebraic equations that is *not* homogeneous and hence is most likely *incompatible* for an arbitrary choice of the parameters A, B, C ! So what do we do now?

Well, the first thing you can do is *choose* these parameter values in such a way that these algebraic equations do become compatible. You will thus be rewarded by all the possible integrable cases that this procedure can identify. When all is said and done, you will be pleasantly surprised to find that besides the case you already know about there are two new ones for which you can in principle now integrate the Hénon-Heiles equations completely. Thus all the known integrable cases of the problem can be summarized as follows:

$$\begin{aligned} \text{Case 1 : } & A = B, \quad C = -1, \\ \text{Case 2 : } & A, B \text{ free, } \quad C = -6, \\ \text{Case 3 : } & B = 16A, \quad C = -16. \end{aligned} \tag{2.34}$$

Well, to be completely honest, Case 3 does not arise automatically from the above procedure. In fact, when you use $C = -16$ in the condition of singularity type (ii) you find that the value of the exponent r is a half integer. But that does not mean all is lost. It may be that if we switch from x to a variable $u = x^2$, this new variable may have a true Laurent series expansion with the required number of free constants and that is precisely what happens here. Thus all three cases listed in (2.34) above are successfully identified by the Painlevé analysis.

And what about $C = -2$? Well, in this case we are not so lucky. Although everything begins promisingly, we cannot find the required free constants unless we also introduce *logarithmic terms* in our series expansions. This means that the solutions of the Hénon-Heiles system for $C = -2$ contain logarithmic singularities already at leading order and hence the above method does not apply. Indeed, when we integrate numerically the equations in this case we find that they possess chaotic solutions and hence a second global, analytic and single-valued integral most likely does not exist [56].

In conclusion, we can say that if a system of ODEs has the so-called *Painlevé property*, i.e. its solutions have *only poles* as *movable* singularities, it is expected to be completely integrable, even explicitly solvable in terms of known functions. The term “movable” implies that the location of a singularity t_* is one of the free constants to be specified by the initial conditions of the problem. It serves

to differentiate movable singularities from the so-called *fixed* ones, which appear explicitly in the equations of motion [107, 169].

Of course, identifying integrable cases by the Painlevé analysis only tells us where to look for N dof Hamiltonian systems whose solutions are globally ordered and predictable. It does not tell us how to find the N integrals that must exist (according to the LA theorem) and which are necessary to solve the equations of motion by quadratures. But do we really want to get into all this trouble?

First of all the integrals are not at all easy to find. Even for the simple-looking Hénon-Heiles system, the best one can do is assume that the second integral F is a polynomial in the momentum and position variables and solve for its coefficients by direct differentiation. What happens, however, if F turns out to be an infinite series? Well, in the Hénon-Heiles we are lucky. The second integral in Cases 2 and 3 of (2.34) truncates to a polynomial of the form [60, 156]

$$\begin{aligned}
 \text{Case 2 : } F &= x^4 + 4x^2y^2 - 4p_x^2y + 4xp_xp_y + 4Ax^2y \\
 &\quad + (4A - B)(p_x^2 + Ax^2), \\
 \text{Case 3 : } F &= 3p_x^4 + 6(A + 2y)x^2p_x^2 - 4x^3p_xp_y - 4x^4(Ay + y^2) \\
 &\quad + 3A^2x^4 - \frac{2}{3}x^6.
 \end{aligned} \tag{2.35}$$

Now what? If we wish to arrive at the complete solution of the problem we must somehow use (2.35) to uncouple the variables of the problem and solve two one dof systems as we did in integrable Case 1. But that is a far less trivial task in Cases 2 and 3. So, again the question relentlessly arises: Why do it?

After all, integrable cases, you may argue, are so rare, that they can hardly be expected to come up in realistic physical situations. Why worry about them? Why not try to find out instead what happens to the Hénon-Heiles for all other A, B, C values apart from the ones of (2.34)? Well, don't be impatient, we can always do that numerically, of course. Before resorting to this alternative, however, it is worth noting that integrable systems are especially useful for two important reasons: First, they frequently arise as close approximations of many physically realistic problems (the solar system being the most famous example) and second, their basic dynamical features do *not* change very much under small perturbations, as the KAM theorem [19] assures us and as we know by now from an abundance of examples.

Let's take for instance the best-studied case of the Hénon-Heiles system, which is both physically interesting and (most likely) non-integrable, $A = B = 1, C = 1$ [164, 226]

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3 = E. \tag{2.36}$$

Let us select a value of the total energy small enough, within the interval $0 < E < 1/6$, for which it is known that all solutions of (2.36) are bounded. To visualize these solutions, let us plot in Fig. 2.7 the intersections of the corresponding orbits with the

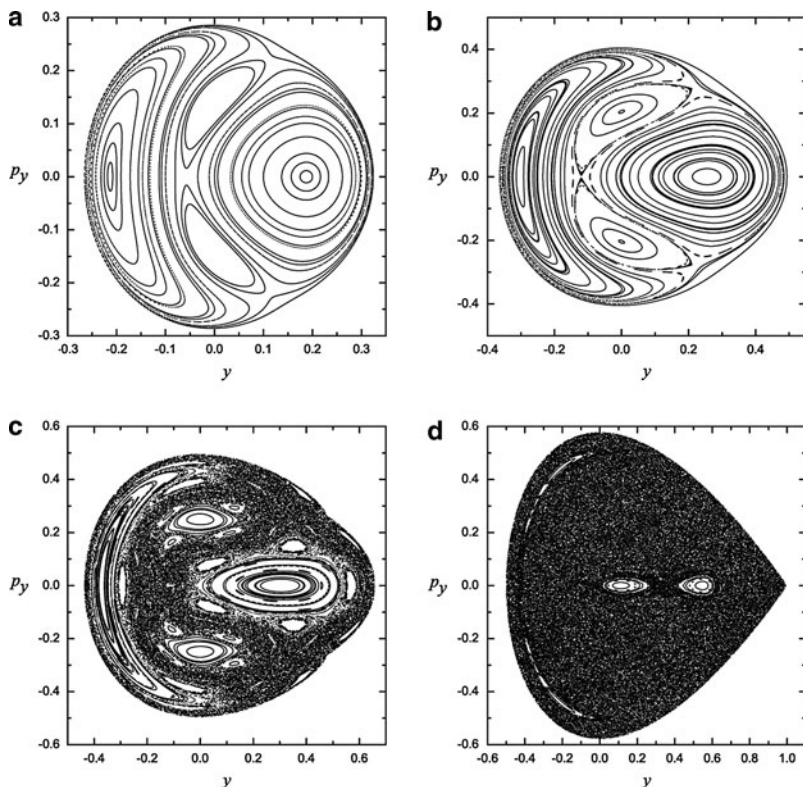


Fig. 2.7 Orbit intersections with the PSS (y, p_y) of the solutions of the Hénon-Heiles system with Hamiltonian (2.36), at times t_k , $k = 1, 2, \dots$, where $x(t_k) = 0$, $p_x(t_k) > 0$ and for fixed values of the energy E . Note in panels (a) and (b), where $E = 1/24$ and $E = 1/12$ respectively, no chaotic orbits are visible and a second integral of motion besides (2.36) appears to exist. When we raise the energy however to $E = 1/8$ in (c), this is clearly seen to be an illusion. Ordered (i.e. quasiperiodic) motion is restricted within islands surrounded by chaos, whose size diminishes rapidly as the energy increases further. Thus, in (d) where E attains the value $E = 1/6$ above which orbits escape to infinity, the domain of chaotic motion extends over most of the available energy surface

plane (y, p_y) every time $t = t_k$, $k = 1, 2, \dots$ at which $x(t_k) = 0$, $p_x(t_k) > 0$. This is what we call a *Poincaré surface of section* (PSS) for which we shall have a lot more to say in the remainder of this chapter. Let us not forget, after all, that it was Poincaré himself who first pointed out the complexity of the solutions of Hamiltonian systems such as (2.36) in his celebrated study of the non-integrability of the three-body problem of celestial mechanics [274].

Do you see any chaos in panel (a) of this figure, where $E = 1/24$? No. Yet, it is *there*, between the invariant curves that correspond to intersections of two-dimensional tori of quasiperiodic orbits, in the form of thin chaotic layers that are too small to be visible. To see these chaotic solutions, one would have to either increase the resolution of Fig. 2.7a or raise the energy. Let us do the latter. Observe in this way, in Fig. 2.7b, c, that we would have to reach energy values as high as

$E = 1/8$ before we begin to see widespread chaotic regions on the PSS. Note also that these regions grow rapidly as E approaches the escape energy $E = 1/6$, where chaos seems to spread over the full domain of allowed motion, while regular orbits are restricted within small “islands” of quasiperiodic motion that seem to vanish as the energy increases.

How can we better understand these dynamical phenomena? What is the degree of their complexity? Is Hamiltonian dynamics just a battle between order and chaos and all we need to do is find out who is “winning” by estimating their respective domains of “influence”? Well, even that would not be such a small accomplishment! Clearly, however, to make progress in this direction, we would have to know more about the “border” that separates order from chaos in Hamiltonian systems. And, as it often happens in mathematical physics, to understand a problem deeper, we need to analyze first in detail its simplest occurrences.

2.3 Non-autonomous One Degree of Freedom Hamiltonian Systems

Perhaps the simplest non-integrable Hamiltonian system in existence is a single periodically driven anharmonic oscillator, studied by Georg Duffing [116], a German engineer who worked on nonlinear vibrations. This oscillator is described by what is called nowadays Duffing’s equation of motion [160, 205, 346]

$$\ddot{q}(t) + \omega_0^2 q(t) + \alpha q^2(t) + \beta q^3(t) + \gamma \sin \Omega t = 0, \quad (2.37)$$

where we denote from now on time differentiation by a “dot” over the differentiated variable. In (2.37) $q(t)$ again represents the displacement of this one-dimensional oscillator from its zero position. Furthermore, we can also denote its momentum by $p = \dot{q}$ and write the above equation as a driven one dof Hamiltonian system of the form

$$\dot{q} = p = \frac{\partial H}{\partial p}, \quad \dot{p} = -\omega_0^2 q - \alpha q^2 - \beta q^3 - \gamma \sin \Omega t = -\frac{\partial H}{\partial q}, \quad (2.38)$$

where the Hamiltonian function

$$H = H(q, p, t) = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \frac{\alpha}{3} q^3 + \frac{\beta}{4} q^4 + \gamma q \sin \Omega t, \quad (2.39)$$

for $\gamma \neq 0$ depends explicitly on t . Thus, H is no longer an integral of the motion.

Indeed, when $\gamma \neq 0$ things quickly start to get complicated. First of all, $q = p = 0$ is no longer an equilibrium position since the sinusoidal term in (2.37) will immediately drive the oscillator away from that position. In fact, *no* equilibria exist at all, since it is clearly not possible to find *any* points q_0, p_0 that remain fixed when we set $\dot{q} = \dot{p} = 0$ in these equations. More importantly, of course, the phase

space of the system is *not* two-dimensional. Observe that it is no longer sufficient to specify a point $(q(t_0), p(t_0))$ in the (q, p) plane and expect to get a unique solution lying in that plane. You also need to specify exactly what time t_0 you are talking about, since the dynamics is no longer invariant under time translation, as in the cases of the autonomous Hamiltonian systems considered so far.

The motion, therefore, evolves in the three-dimensional space, q, p, t and (2.39) is sometimes referred to as a system of *one and a half* dof. In fact it is no different than a two dof Hamiltonian system, since, in the terminology of Sect. 2.1, we can represent the nonlinear oscillator in (2.37) by one action-angle pair I_1, θ_1 and think of q, p as coupled to a second (linear) oscillator whose action I_2 is fixed, while its angle coordinate is $\theta_2 = \Omega t$. Mathematically speaking, we say that the phase space is a *cylinder*, $\mathbb{R}^2 \times S^1$, where every point in the (q, p) plane is associated with an angle on the unit circle S^1 .

To understand how this periodic driving in (2.37) affects the dynamics we will consider the periodic driving term as a small perturbation in the two special cases $\beta = 0$ and $\alpha = 0$ separately.

2.3.1 The Duffing Oscillator with Quadratic Nonlinearity

Let us set $\beta = 0, \alpha = -1, \omega_0 = 1$ in (2.38) and examine the solutions of the system

$$\dot{q} = p, \quad \dot{p} = -q + q^2 - \gamma \sin \Omega t \quad (2.40)$$

for $0 \leq \gamma \ll 1$. When $\gamma = 0$ it is easy to see that there are two equilibria: (1) $q = p = 0$, (2) $q = 1, p = 0$. Linearizing the equations of motion about them, as described in Sect. 2.1, we easily find that (1) is a (stable) elliptic point and (2) is an (unstable) saddle. On the other hand, the dynamics of this one dof system in the full phase plane (q, p) is described by the family of curves $H = H(q, p) = (1/2)(p^2 + q^2) - (1/3)q^3 = E$ parameterized by the value of the energy E , as shown in Fig. 2.8a.

Observe that, as in the case of the simple pendulum, here also a region of oscillations exists around the origin, which extends all the way to the saddle at $(1, 0)$ and is separated from the outside part of the phase plane (where all solutions escape to infinity) by an invariant curve S called the separatrix. The only difference is that in the pendulum problem S joins two different saddles and represents a so-called *heteroclinic* solution of the equations, while in the quadratic oscillator of (2.40), S joins $(1, 0)$ to itself and is called *homoclinic* solution (these terms come from the Greek words “cline” meaning “bed” and “hetero” and “homo” which mean, of course, “different” and “same” respectively). These invariant curves, and the homoclinic (or heteroclinic) orbits that lie on them, constitute perhaps the single most important object of study in nonlinear Hamiltonian dynamics.

Let us try to understand why: As Poincaré first pointed out, when $\gamma \neq 0$, the elliptic point at $(0, 0)$ and the saddle point at $(1, 0)$ in Fig. 2.8a shift to slightly different locations P and Q on the PSS

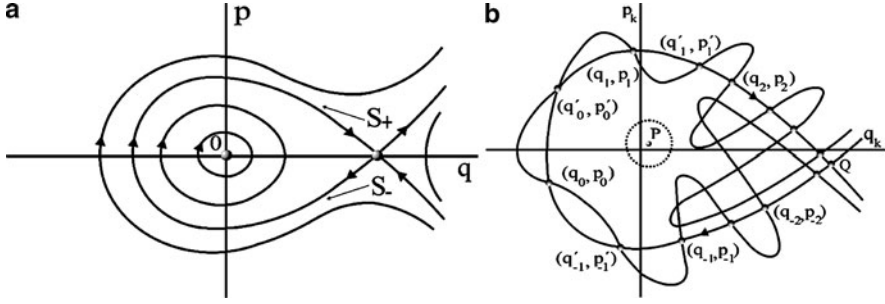


Fig. 2.8 (a) The (q, p) phase plane of the quadratic oscillator (2.40) in the $\gamma = 0$ case. Note the elliptic point at the origin and the saddle point at $(1, 0)$. The separatrix S here is a single invariant curve described by a homoclinic orbit joining the unstable and stable manifolds of the saddle at $(1, 0)$. (b) When $\gamma \neq 0$, however, these manifolds no longer join smoothly but intersect each other infinitely often on a PSS $\Sigma_{t_0} = (q(t_k), p(t_k))$, $t_k = kT + t_0$, $k = 1, 2, \dots$, where $T = 2\pi/\Omega$ is the period of the forcing term

$$\Sigma_{t_0} = \{q(t_k), p(t_k)\}, \quad t_k = kT + t_0, \quad k = \pm 1, \pm 2, \dots, \quad T = 2\pi/\Omega \quad (2.41)$$

(see Fig. 2.8b). These are no longer fixed points of the differential equations, but have become intersections of $T (= 2\pi/\Omega)$ -periodic orbits of the system (2.40), whose solutions evolve now in the three-dimensional space $\mathbb{R}^2 \times S^1$, as explained above [160, 226].

In addition, the stable and unstable manifolds S_+ and S_- of the point Q intersect transversally at a point (q_0, p_0) in Fig. 2.8b, lying furthest away from Q . This implies the simultaneous occurrence of a double infinity of such points, denoted by (q_k, p_k) , $k = \pm 1, \pm 2, \dots$, with $(q_k, p_k) \rightarrow P$ as $k \rightarrow \pm\infty$, which constitute in fact *two distinct* homoclinic orbits, one with $k = 0, \pm 2, \dots, \pm 2m, \dots$ and one with $k = \pm 1, \dots, \pm(2m+1), \dots$, m being a positive integer.

The reason for this is that the time evolution of the system on the PSS (2.41) is more conveniently described by the Poincaré map [170, 265, 346]

$$P_{t_0} : \Sigma_{t_0} \rightarrow \Sigma_{t_0}, \quad t_0 \in (0, 2\pi/\Omega), \quad (2.42)$$

defined by following the trajectories of the systems and monitoring their intersections with Σ_{t_0} . P and Q are respectively an elliptic and a saddle fixed point of this map, since they satisfy $P_{t_0}(P) = P$ and $P_{t_0}(Q) = Q$. Most importantly, the map (2.42) is a *diffeomorphism*, i.e. it is continuous and at (least once) continuously differentiable with respect to the phase space variables q, p . It is also *invertible*, i.e. $P_{t_0}^{-1}$ exists and is also a diffeomorphism. This means that it maps, forward and backward in time, neighborhoods of the PSS that exhibit diffeomorphically equivalent dynamics. Thus, an intersection point of the invariant manifolds is mapped by P_{t_0} (or $P_{t_0}^{-1}$) to the next (or previous) such intersection point, provided the manifolds in their neighborhoods have the *same orientation*.

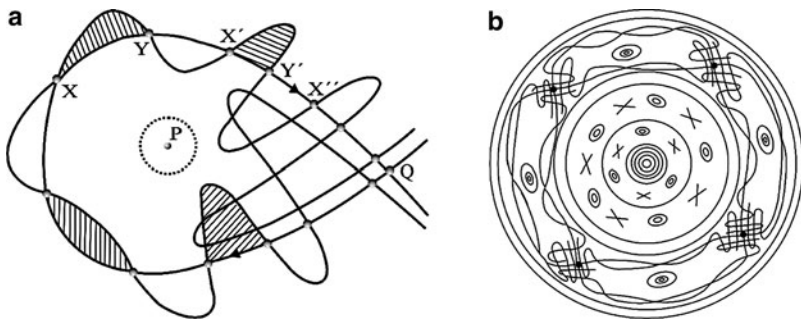


Fig. 2.9 (a) The successive action of (2.42) maps each shaded lobe of the figure to the next one, closer to the point Q , thus taking points from the region “inside” the invariant manifolds to the region “outside” and eventually to infinity. (b) Schematic picture of the dynamics around the elliptic fixed point of the Poincaré map (2.42), on the surface of section Σ_{t_0} , when $\gamma \neq 0$

This explains why, for example, as the homoclinic point X is mapped to X' , X'' , etc. (and its companion Y is mapped to Y' , Y'' , etc.) in Fig. 2.9a, a shaded lobe is mapped to the next one closer to the saddle point Q , by successive applications of the map P_{t_0} . This results in the *transport* of points which were once “inside” the manifolds S_+ , S_- to the region “outside”. On the other hand, as Fig. 2.9a also shows, there are corresponding lobes which were originally “outside” and are now mapped “inside” the domain of the elliptic point at P . Furthermore, the Hamiltonian nature of the problem implies, according to Liouville’s theorem [226,265,346], that the system is conservative in the sense that it preserves phase space volume. The consequence of this is that the areas of the lobes shown in Fig. 2.9 are equal to each other.

Meanwhile, in the *homoclinic tangle* formed by the intersecting manifolds near the saddle point Q there is a wealth of dynamical phenomena about which a lot is known. Besides the homoclinic orbits, there is a countable infinity of unstable periodic solutions, while there also exist invariant Cantor sets, for which it can be rigorously proved by the Smale horseshoe theory [160,265,346] that any two nearby points belonging to this set lead to orbits that separate from each other exponentially fast in phase space.

Of course, if instead of one saddle point our Poincaré map had two (see next subsection) the intersecting manifolds would give rise to *heteroclinic* orbits, in the same way as described above, with lobes of equal areas transporting points from “inside” the invariant manifolds “outside” and vice versa. Heteroclinic tangles would thus be formed, where invariant sets of chaotic orbits can be similarly proved to exist. The important observation is that this does not happen only near saddle fixed points of the Poincaré map (2.42). It also happens, at smaller scales, near every unstable m -periodic orbit of the system of period $T_m = mT$, $m = 2, 3, \dots$, intersecting the PSS (2.41) at m points (see Fig. 2.9b).

As we know from the KAM theorem, “most” (in the sense of positive measure) invariant curves around the origin of (2.40) at $\gamma = 0$ are preserved for sufficiently small $\gamma \neq 0$. Furthermore, a very important theorem by Birkhoff [226, 346] also

assures us that the invariant curves of the $\gamma = 0$ case corresponding to periodic orbits break up into an even number of stable and unstable m -periodic orbits, which appear on the surface of section as *elliptic* and *saddle fixed points* respectively of the m th power of the Poincaré map $P_{t_0}^m$.

This is schematically shown in Fig. 2.9b. Are you impressed? You should be. This is indeed a wonderful sign of a complex phenomenon, as it rigorously demonstrates that there are chaotic regions around saddle points present at all scales! Around every elliptic fixed point of *any power* of the Poincaré map (hence around every small island shown in Fig. 2.9b), there are smaller islands and around them even smaller ones etc., with saddles and tangles of heteroclinic chaos in between down to infinitesimally small scales!

Did you expect all this to happen, as soon as we allowed $\gamma \neq 0$? Clearly, turning on the driving force in (2.40) was not such an innocent move. The Hamiltonian ceased to be an integral and the dimensionality of the motion was increased to three, where an amazing variety of new phenomena are possible, the most important of them being the occurrence of chaotic orbits.

2.3.2 The Duffing Oscillator with Cubic Nonlinearity

Let us now repeat the above exercise for the Duffing oscillator (2.37) with only cubic nonlinearity. Setting $\alpha = 0$, the equation of motion becomes

$$\ddot{q}(t) = -\delta q(t) - \beta q^3(t) + \gamma \sin \Omega t, \quad (2.43)$$

where we have replaced the parameter before the linear term by δ , so as to be able to consider also the case $\delta < 0$. Although the phenomena we encounter here are qualitatively the same as in the example of the quadratic Duffing oscillator, we will have the chance to exploit the additional symmetry of the dynamics under reflection about the axis $q = 0$ (when $\gamma = 0$). This is relevant in many physical systems (like vibrating structures fixed on a horizontal uniform floor), where motions to the right or left are equally favorable. With (2.43), we also have the opportunity to discuss a case that has many common features with the pendulum problem and examine the interesting dynamics of the so-called double-well potential that arises frequently in many problems of physics.

Let us begin by considering the case $\delta = -\beta = 1$, where the above system is written in q, p phase space coordinates as

$$\dot{q} = p, \quad \dot{p} = -q + q^3 + \gamma \sin \Omega t. \quad (2.44)$$

Evidently, in the unforced case $\gamma = 0$, this anharmonic oscillator possesses three fixed points: (1) An elliptic one, $P = (0, 0)$ and two saddle points (2) $Q_1 = (-1, 0)$ and (3) $Q_2 = (1, 0)$. The phase space plot is thus very similar with that of the pendulum (see Fig. 2.4), only in the case of the pendulum $Q_1 = (-\pi, 0)$, $Q_2 =$

$(\pi, 0)$ and the picture is repeated about every elliptic point at $(2m\pi, 0)$ and its two saddles at $((2m \pm 1)\pi, 0)$.

Note that our separatrices here represent heteroclinic orbits joining the saddle points at $(\pm 1, 0)$. It is not difficult to show (see Exercise 2.3) that, in the unforced case, both cubic and quadratic oscillators (2.40), (2.44), possess the Painlevé property and are explicitly solvable in terms of Jacobi elliptic functions. This means that their solutions along these separatrices are given in terms of simple hyperbolic functions. We thus have the opportunity to apply to these solutions the so-called *Mel'nikov theory* [160, 346], which allows us to study exactly what happens to these heteroclinic orbits when $\gamma \neq 0$. In Problem 2.3 we ask you to carry out such a calculation, following the steps outlined below.

First of all, recall that the saddle points of our cubic oscillator (just as in the case of the quadratic oscillator) are unstable periodic orbits of (2.44) with period $T = 2\pi/\Omega$, which persist on the PSS (2.41) as saddle fixed points of the Poincaré map (2.42). These points continue to possess stable and unstable manifolds, which no longer join smoothly, as in the unforced case. Mel'nikov's approach works within the framework of perturbation theory for $|\gamma| \ll 1$ and yields expressions for the heteroclinic solutions of the perturbed problem, both for the stable as well as unstable manifolds of these saddle fixed points, as series expansions in powers of γ .

We thus obtain *different* expansions which are uniformly valid in time intervals $(-\infty, t_1)$, for the unstable manifold of the Q_1 point and (t_2, ∞) for the stable manifold of the point Q_2 , where $t_2 > t_1$. Examining these solutions at times $t_0 \in (t_1, t_2)$ where both series are valid, Mel'nikov's theory shows, *to first order in γ* , that the points at which the corresponding manifolds intersect correspond to the roots of the function

$$M(t_0) = \int_{-\infty}^{\infty} (f_1 g_2 - f_2 g_1)(\hat{\mathbf{x}}(t - t_0), t) dt, \quad (2.45)$$

which is proportional to the *distance* between the two manifolds on the PSS (2.41), as a function of t_0 (see Problem 2.3). To derive (2.45) we have written our equations of motion in the form

$$\begin{aligned} \dot{q} &= f_1(q, p) + \gamma g_1(q, p, t), & \dot{p} &= f_2(q, p) + \gamma g_2(q, p, t), \\ g_i(q, p, t) &= g_i(q, p, t + T), \end{aligned} \quad (2.46)$$

$i = 1, 2$, while $\hat{\mathbf{x}}(t - t_0) = (\hat{q}(t - t_0), \hat{p}(t - t_0))$ represents the exact heteroclinic (or homoclinic) solution of the unforced $\gamma = 0$ case.

This is fantastic! It means that we can now study analytically what happens to these stable and unstable manifolds as one turns on the perturbation of the periodic forcing in the above systems. Indeed, based on our knowledge of the exact solution on these manifolds for $\gamma = 0$ we can verify, as Poincaré predicted, that the Melnikov integral (2.45) is an *oscillatory* function of t_0 with infinitely many roots, $M(t_{0i}) = 0$, $i = 0, \pm 1, \pm 2, \dots$. These correspond to the intersection points of the manifolds at which the heteroclinic (or homoclinic) orbits of the perturbed system are located.

Let us see how all this works in the case of the cubic oscillator (2.44). As the reader can easily verify, for $\gamma = 0$, the separatrices (or heteroclinic solutions) of this problem, $S_{\pm}$, are given by the hyperbolic functions

$$\hat{\mathbf{x}}(t - t_0) = (\hat{q}(t - t_0), \hat{p}(t - t_0)) = \pm \left(\tanh \left(\frac{1}{\sqrt{2}}(t - t_0) \right), \frac{1}{\sqrt{2}} \operatorname{sech}^2(t - t_0) \right). \quad (2.47)$$

If we use the above expressions to substitute in (2.45) we obtain an integral

$$M(t_0) = \int_{-\infty}^{\infty} \hat{p}(t - t_0) \sin \Omega t dt = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \operatorname{sech}^2 t \sin \Omega(t + t_0) dt, \quad (2.48)$$

that is not elementary. Its evaluation requires that we extend the integration to a strip in the complex t -plane and use Cauchy's residue theorem (see Problem 2.3). The result is a simple formula

$$M(t_0) = \frac{\pi \Omega / 2}{\sinh(\pi \Omega / 2)} \sin \Omega t_0, \quad (2.49)$$

which, when multiplied by γ and divided by the magnitude of the unperturbed vector field $\sqrt{f_1^2(\tau) + f_2^2(\tau)}$ provides an estimate of the *distance* between the stable and unstable manifolds at $t = t_0 + \tau$ (compare with the numerical results of Exercise 2.4). The above formulas demonstrate that $M(t_0)$ has infinitely many zeros (due to the sine function in (2.49)), while the amplitude of its oscillations *grows exponentially* as τ increases and we move closer to the saddle points Q_1, Q_2 .

It is important to note that Mel'nikov's theory does *not* require that the vector fields $\hat{\mathbf{f}} = (f_1, f_2)$ or $\hat{\mathbf{g}} = (g_1, g_2)$ be Hamiltonian! All it demands is that the unperturbed system possesses a solution $\hat{\mathbf{x}}(t)$ that is either a homoclinic or heteroclinic orbit. In particular, if our Duffing oscillators involve a small *dissipative* term of the form $-\eta p$, say, with $\eta > 0$ of the same order as γ , we can include it in the periodic perturbation $\hat{\mathbf{g}} = (g_1, g_2)$ and study its crucial role in *inhibiting* the intersection of the invariant manifolds. Indeed, it can be shown that each η provides a threshold that γ must exceed for Smale horseshoe chaos to occur in the system! To understand exactly how this important effect can be studied using Mel'nikov's theory, the reader is encouraged to solve Problem 2.4.

In fact, Mel'nikov's theory does not only apply to perturbations of planar Hamiltonian systems. It can be extended to dynamical systems of arbitrary dimension, $n \geq 2$, which can be written in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \gamma \mathbf{g}(\mathbf{x}, t), \quad \mathbf{g}(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}, t + T), \quad (2.50)$$

for $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ and γ a sufficiently small real parameter. In the particular case where the unperturbed equations are derived from an integrable

Hamiltonian system, possessing n independent integrals in involution F_i , $i = 1, 2, \dots, n$, we obtain a Mel'nikov vector function, $\mathbf{M}(t_0, \nu) = (M_1, \dots, M_n)$, whose components are given by [90, 287, 288]

$$M_i(t_0, \nu) = \int_{-\infty}^{\infty} (\nabla F_i(\hat{\mathbf{x}}(t, \nu)), \mathbf{g}(\hat{\mathbf{x}}(t, \nu), t - t_0)) dt, \quad (2.51)$$

where $(\cdot)$ denotes the inner product and ν is a (generally) n -dimensional parameter entering in the homoclinic (or heteroclinic) solution $\hat{\mathbf{x}}(t, \nu)$. Now, however, matters are more complicated, as we no longer have the geometric visualization of the invariant manifolds and the distance between them that was available in the $n = 2$ case. Thus, we shall not proceed any further with the discussion of Mel'nikov's analysis and refer the interested reader to the literature for more details [90, 287, 288, 346].

Exercises

Exercise 2.1. (a) Perform the variable transformation $x = \sin(\theta/2)$ to the equation of motion of the simple pendulum (2.7) and show that this ODE becomes

$$\ddot{x} = -\left(\omega^2 + \frac{E}{2}\right)x + 2\omega^2 x^3, \quad (2.52)$$

where $\omega = \sqrt{g/l}$. Using the theory of Jacobi elliptic functions [6, 107, 169], prove that the solution of this ODE can be expressed as the Jacobi elliptic sine function

$$x(t) = C \operatorname{sn}(\mu(t - t_0), \kappa), \quad \mu^2(1 + \kappa^2) = \omega^2 + \frac{E}{2}, \quad \kappa^2 \mu^2 = \omega^2 C^2, \quad (2.53)$$

which corresponds to the initial conditions $x(t_0) = 0$, $\dot{x}(t_0) = \mu C$, while the energy equation gives $2\mu^2 C^2 = E$. Eliminating C , μ from these equations, obtain an expression relating the modulus κ of the elliptic sine function to the total energy E and show that in the limit $E \rightarrow 0$, $\kappa \rightarrow 0$, and $x(t)$ becomes the usual trigonometric sine function. Prove also that in the limit $\kappa \rightarrow 1$ the solution on the separatrix of Fig. 2.4 reduces to the hyperbolic tangent function.

(b) Show that in the oscillatory domain the period T of the pendulum is given in terms of the elliptic integral of the first kind

$$K(\kappa^2) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}}, \quad (2.54)$$

and use this result to evaluate the first 2–3 terms in an expansion of T in powers of κ^2 . Finally, use the Fourier representation of the elliptic sine function in terms of trigonometric sines to plot $\text{sn}(u, \kappa^2)$ as a function of its argument u for several values of the modulus κ^2 (see [6, 107, 169]).

Exercise 2.2. Consider a system of two coupled harmonic oscillators of the type considered in Sect. 2.2, Fig. 2.5. Call $k = k_1$ the constant of the springs by which the two masses are tied to the walls and $k = k_2$ the constant of the spring that connects them to each other. Find values k_1, k_2 such that the general solution of the system is periodic and determine the period in every case.

Exercise 2.3. Show that the unforced quadratic and cubic Duffing oscillators (2.40), (2.44) (for $\delta = \pm 1$) possess the Painlevé property and solve the two problems completely in terms of Jacobi elliptic functions. Find explicit expressions for the solutions in all parts of the phase plane corresponding to regions of bounded and unbounded motion, as well as on the separatrices between these regions.

Exercise 2.4. Consider a Duffing cubic oscillator on which periodic forcing is applied parametrically as follows

$$\dot{q} = p, \quad \dot{p} = -q(1 + \gamma \sin(\Omega t)) + q^3. \quad (2.55)$$

This is a case where the spring constant (or the length of an associated pendulum model) is taken to vary sinusoidally about its constant value. Note that the elliptic point at the origin and the two saddle points at $(\pm 1, 0)$ remain at their place when $\gamma \neq 0$.

- (a) Linearize (2.55) about the points $(\pm 1, 0)$ and obtain the corresponding linear stable and unstable manifolds $E_s^\pm, E_u^\pm$. Keeping $\gamma = 0$ locate points on these manifolds very close to the fixed points, which if propagated forward (or backward) in time by (2.55) will accurately trace the two separatrices $S_\pm$ representing the upper and lower heteroclinic orbits joining the two saddles.
- (b) When $\gamma \neq 0$, construct the associated 2×2 linearized Poincaré map L (and its inverse L^{-1}) by solving numerically (2.55) forward and backward in time and plotting points very close to $(\pm 1, 0)$ on the PSS Σ_{t_0} at $t = \pm 2\pi/\Omega$. Evaluate the linear stable and unstable manifolds $E_s^\pm, E_u^\pm$ of the 2×2 matrices L and L^{-1} .
- (c) Place many points on these linear manifolds very close to $(\pm 1, 0)$ and propagate them forward (and backward) in time by solving numerically (2.55) and plotting their iterates at $t = 2m\pi/\Omega$, $m = \pm 1, \pm 2, \dots$. Thus show that the corresponding nonlinear manifolds no longer join smoothly but intersect each other repeatedly forming heteroclinic tangles, within which dense sets of chaotic orbits can be found according to Smale horseshoe dynamics.

Problems

Problem 2.1. Perform all the calculations described in Sect. 2.2 to derive and study the general solution of the Hénon-Heiles problem in the integrable case $A = B$, $C = -1$. Show that the two integrals in (2.30) are independent and in involution. Plot the invariant curves in the (X, P_X) and (Y, P_Y) surfaces of section and relate them to the exact solutions of the problem given in terms of Jacobi elliptic functions (see Exercise 2.1). Discuss the dynamics of the system for all values of the energy $E > 0$.

Problem 2.2. Complete all steps of the singularity analysis in the complex t -plane and show that the only cases of the Hénon-Heiles Hamiltonian system possessing the Painlevé property are the ones listed in (2.34).

Problem 2.3. Use Mel'nikov's theory as described in [160, 346] to derive an expression for the distance between stable and unstable manifolds as a function of the Mel'nikov integral and the magnitude of the unperturbed vector field along the invariant manifolds. Apply this theory to the case of the cubic Duffing oscillator (2.44) as outlined in Sect. 2.3.2, evaluate the corresponding Melnikov integral and prove the result shown in (2.49). Repeat this analysis for the parametrically driven Duffing oscillator (2.55) and compare the results.

Problem 2.4. Study the cubic Duffing oscillator (2.43) in the case $\delta = -\beta = -1$, which describes a physical problem with a double well potential. Assume furthermore that the periodic perturbation is of the form

$$\dot{q} = p, \quad \dot{p} = q - q^3 + \varepsilon(-\eta p + \gamma \sin \Omega t), \quad (2.56)$$

where $-\eta p$ (with $\eta > 0$) is a term introducing dissipation. Draw the phase plane curves of the $\varepsilon = 0$ case and show that the origin is a saddle fixed point whose stable and unstable manifolds are joined by two symmetric homoclinic solutions “embracing” two elliptic points at $(0, \pm 1)$. Apply Mel'nikov's theory to show that for $\varepsilon \neq 0$ these manifolds split into homoclinic orbits when η and γ satisfy a certain inequality. Now solve (2.56) numerically, using the approach described in Exercise 2.4, to examine the validity of the inequality you derived for $0 < \varepsilon \ll 1$. How accurate are its predictions for the appearance of horseshoe chaos in the problem?



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