Chapter 2
Almost Periodic Solutions

In the present chapter, we shall state some basic existence and uniqueness results for almost periodic solutions of impulsive differential equations. Applications to real world problems will also be discussed.

Section 2.1 will offer the existence and uniqueness theorems for almost periodic solutions of hyperbolic impulsive differential equations.

In Sect. 2.2, using weakly non-linear integro-differential systems, the existence and exponential stability of almost periodic solutions of impulsive integro-differential equations will be discussed.

In Sect. 2.3, we shall study the existence of almost periodic solutions for forced perturbed impulsive differential equations. The example here, will state the existence criteria for impulsive differential equations of Lienard's type.

Section 2.4 will deal with sufficient conditions for the existence of almost periodic solutions of impulsive differential equations with perturbations in the linear part.

In Sect. 2.5, we shall consider the strong stability and almost periodicity of solutions of impulsive differential equations with fixed moments of impulse effect. The investigations are carried out by means of piecewise continuous Lyapunov functions.

Section 2.6 is devoted to the problem of the existence of almost periodic projektor-valued functions for dichotomous impulsive differential systems.

In Sect. 2.7, we shall investigate separated solutions of impulsive differential equations with variable impulsive perturbations and we shall give sufficient conditions for almost periodicity of these solutions.

Finally, in Sect. 2.8, the existence results for almost periodic solutions of abstract differential equations in Banach space will be given. Applications for impulsive predator–prey systems with diffusion will be considered.

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2.1 Hyperbolic Impulsive Differential Equations

In this paragraph, we shall consider the following systems of impulsive differential equations with impulses at fixed moments

$$\begin{cases} \dot{z} = A(t)z + f(t), \ t \neq t_k, \\ \Delta z(t_k) = b_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dot{z} = A(t)z + F(t,z), \ t \neq t_k, \\ \Delta z(t_k) = I_k(z(t_k)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.2}$$

where $t \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, $A : \mathbb{R} \to \mathbb{R}^{n \times n}$, $f : \mathbb{R} \to \mathbb{R}^n$, $b_k \in \mathbb{R}^n$, $F : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $I_k : \Omega \to \mathbb{R}^n$.

By $z(t) = z(t; t_0, z_0)$, we denote the solution of (2.1) or (2.2) with initial condition $z(t_0^+) = z_0$, $t_0 \in \mathbb{R}$, $z_0 \in \mathbb{R}^n$. Together with the systems (2.1) and (2.2), we shall consider the corresponding homogeneous system

$$\dot{z} = A(t)z. \tag{2.3}$$

Definition 2.1 ([71]). The system (2.3) is said to be *hyperbolic*, if there exist constants $\alpha > 0$, $\lambda > 0$ and for each $t \in \mathbb{R}$ there exist linear spaces $M^+(t)$, and $M^-(t)$, whose external direct sum is $M^+(t) \oplus M^-(t) = \mathbb{R}^n$, such that if $z_0 \in M^+(t_0)$, then for all $t \geq t_0$ the inequality

$$||z(t; t_0, z_0)|| \leq a||z_0||e^{-\lambda(t-t_0)},$$

holds true, while if $z_0 \in M^-(t_0)$ then for all $t \leq t_0$, we have

$$||z(t; t_0, z_0)|| \leq a||z_0||e^{\lambda(t-t_0)}.$$

In this part, we shall investigate the existence of almost periodic solutions of systems (2.1) and (2.2), assuming that the corresponding homogeneous system is hyperbolic.

Introduce the following conditions:

H2.1. The matrix function $A \in C[\mathbb{R}, \mathbb{R}^{n \times n}]$ is almost periodic in the sense of Bohr.
H2.2. The function $f \in PC[\mathbb{R}, \mathbb{R}^n]$ is almost periodic.
H2.3. The sequence $\{b_k\}$, $k = \pm 1, \pm 2, \ldots$, is almost periodic.
H2.4. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k = \pm 1, \pm 2, \ldots$, $j = \pm 1, \pm 2, \ldots$, is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.
H2.5. The function $F \in C[\mathbb{R} \times \Omega, \mathbb{R}^n]$ is almost periodic with respect to t uniformly in $z \in \Omega$.

H2.6. The sequence of functions $\{I_k(x)\},\ I_k \in C[\Omega, \mathbb{R}^n],\ k = \pm 1, \pm 2, \ldots,$ is almost periodic with respect to k uniformly in $z \in \Omega$.

We shall use the following lemmas:

Lemma 2.1. *Let conditions H2.1–H2.4 hold. Then for each $\varepsilon > 0$ there exist $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon$, a relatively dense set $\overline{T}$ of real numbers, and a set P of integer numbers, such that the following relations are fulfilled:*

(a) $||A(t+\tau) - A(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}.$
(b) $||f(t+\tau) - f(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}.$
(c) $||b_{k+q} - b_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots.$
(d) $|t_k^q - \tau| < \varepsilon_1,\ q \in P,\ \tau \in \overline{T},\ k = \pm 1, \pm 2, \ldots.$

The proof of Lemma 2.1 is analogous to the proof of Lemma 1.7.

Lemma 2.2. *Let the system (2.3) is hyperbolic and the condition H2.1 holds. Then there exists a non-singular transformation, defined by almost periodic matrix $S(t),\ S \in C[\mathbb{R}, \mathbb{R}^{n\times n}]$, which reduces the system (2.1.3) into the next ones*

$$\dot{x} = Q^+(t)x \tag{2.4}$$

and

$$\dot{y} = Q^-(t)y \tag{2.5}$$

where $x \in \mathbb{R}^k,\ y \in \mathbb{R}^{n-k},\ Q^+ \in C[\mathbb{R}, \mathbb{R}^{k\times k}],\ Q^- \in C[\mathbb{R}, \mathbb{R}^{(n-k)\times(n-k)}]$ and the following assertions hold true:

1. *$Q^+(t)$ and $Q^-(t)$ are almost periodic matrix-valued functions.*
2. *If $\Phi^+(t,s)$ and $\Phi^-(t,s)$ are the corresponding fundamental matrices of the systems (2.4) and (2.5), then the following inequalities hold true:*

$$||\Phi^+(t,s)|| \le \overline{a}e^{-\lambda(t-s)},\ t \ge s, \tag{2.6}$$

$$||\Phi^-(t,s)|| \le \overline{a}e^{\lambda(t-s)},\ t \le s, \tag{2.7}$$

where $s, t \in \mathbb{R},\ \overline{a} > 0$.

3. *For each $\varepsilon > 0,\ t \in \mathbb{R},\ s \in \mathbb{R}$ there exists relatively dense set $\overline{T}$ of $\varepsilon-$almost periods, such that for each $\tau \in \overline{T}$, fundamental matrices $\Phi^+(t,s)$ and $\Phi^-(t,s)$ satisfy the inequalities*

$$||\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)|| \le \varepsilon K e^{-\frac{\lambda}{2}(t-s)},\ t \ge s, \tag{2.8}$$

$$||\Phi^-(t+\tau, s+\tau) - \Phi^-(t,s)|| \le \varepsilon K e^{\frac{\lambda}{2}(t-s)},\ t \le s, \tag{2.9}$$

where $\lambda > 0,\ K > 0$.

Proof. Assertions 1 and 2 are immediate consequences of Theorem 1 in [71]. In fact, following the ideas used in [71], we define the matrix $S(t)$ to be formed by the vector-columns, which are solutions of (2.3). It follows from the condition H2.1, that $S(t)$ consists of almost periodic functions. On the other hand, the transformation $z = S(t)u$ rewrites (2.3) in the form

$$\dot{u} = Q(t)u,$$

where

$$Q(t) = S^{-1}(t)\Big(A(t)S(t) - \dot{S}(t)\Big).$$

Hence, $Q(t)$ is an almost periodic function. The estimates (2.6) and (2.7) are direct consequences of Theorem 1 in [71].

To prove Assertion 3, let $\Phi^+(t,s)$ and $\Phi^-(t,s)$ be the fundamental matrices of systems (2.4) and (2.5), respectively. Then for each $\varepsilon > 0$ the following relations hold true

$$\frac{\partial \Phi^+(t,s)}{\partial t} = Q^+(t)\Phi^+(t+\tau, s+\tau) + \Big(Q^+(t+\tau) - Q^+(t)\Big)\Phi^+(t+\tau, s+\tau),$$
$$\frac{\partial \Phi^-(t,s)}{\partial t} = Q^-(t)\Phi^-(t+\tau, s+\tau) + \Big(Q^-(t+\tau) - Q^-(t)\Big)\Phi^-(t+\tau, s+\tau)$$

and

$$\begin{aligned}\Phi^+(t+\tau, s+\tau) &= \Phi^+(t,s)\\ &\quad + \int_s^t \Phi^+(t,v)\Big(Q^+(v+\tau) - Q^+(v)\Big)\Phi^+(v+\tau, s+\tau)dv,\\ \Phi^-(t+\tau, s+\tau) &= \Phi^-(t,s)\\ &\quad + \int_s^t \Phi^-(t,v)\Big(Q^-(v+\tau) - Q^-(v)\Big)\Phi^-(v+\tau, s+\tau)dv.\end{aligned}$$

Therefore,

$$\begin{aligned}||\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)|| &\leq \int_s^t ||\Phi^+(t,v)||||(Q^+(v+\tau)\\ &\quad - Q^+(v)||||\Phi^+(v+\tau, s+\tau)||dv,\\ ||\Phi^-(t+\tau, s+\tau) - \Phi^-(t,s)|| &\leq \int_s^t ||\Phi^-(t,v)||||Q^-(v+\tau)\\ &\quad - Q^-(v)||||\Phi^-(v+\tau, s+\tau)||dv.\end{aligned}$$

It follows from (2.6), that

$$||\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)|| \leq \varepsilon K e^{-\frac{\lambda}{2}(t-s)},\ t \geq s,$$

where in this case $K = (\overline{a})^2$.

The proof of (2.9) is analogous. □

From Lemma 2.2 it follows that, by a transformation with the matrix $S(t)$, system (2.1) takes on the form

$$\begin{cases} \dot{x} = Q^+(t)x + f^+(t),\ t \neq t_k, \\ \Delta x(t_k) = b_k^+,\ k = \pm 1, \pm 2, \ldots, \\ \dot{y} = Q^-(t)y + f^-(t),\ t \neq t_k, \\ \Delta y(t_k) = b_k^-,\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.10}$$

where $x \in \mathbb{R}^k,\ y \in \mathbb{R}^{n-k},\ f^+ : \mathbb{R} \to \mathbb{R}^k,\ f^- : \mathbb{R} \to \mathbb{R}^{n-k}$, b_k^+ and b_k^- are k and $n-k$-dimensional constant vectors, respectively.

In an analogous way, the system (2.2) after a transformation with the matrix $S(t)$, goes to the form

$$\begin{cases} \dot{x} = Q^+(t)x + F^+(t,x,y),\ t \neq t_k, \\ \Delta x(t_k) = I_k^+(x(t_k), y(t_k)),\ k = \pm 1, \pm 2, \ldots, \\ \dot{y} = Q^-(t)y + F^-(t,x,y),\ t \neq t_k, \\ \Delta y(t_k) = I_k^-(x(t_k), y(t_k)),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.11}$$

where $F^+ : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k,\ F^- : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$, and $I_k^+ : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k,\ I_k^- : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$.

Theorem 2.1. *Let the following conditions hold:*

1. *Conditions H2.1–H2.4 hold.*
2. *The system (2.3) is hyperbolic.*

Then for the system (2.1) there exists a unique almost periodic solution, which is exponentially stable.

Proof. We consider the following equations

$$x(t) = \int_{-\infty}^{t} \Phi^+(t,s) f^+(s) ds + \sum_{t_k < t} \Phi^+(t, t_k) b_k^+,$$

$$y(t) = -\int_{t}^{\infty} \Phi^-(t,s) f^-(s) ds + \sum_{t_k > t} \Phi^-(t, t_k) b_k^-,$$

which are equivalent to the (2.10).

Let $\varepsilon > 0$ be an arbitrary chosen constant. It follows from Lemma 2.1 that there exist sets $\overline{T}$ and P such that for each $\tau \in \overline{T}$ and $q \in P$, the following estimates hold true:

$$\begin{aligned}||x(t+\tau)-x(t)|| = &\int_{-\infty}^{t} ||\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)||||f^+(s+\tau)||ds \\ &+ \int_{-\infty}^{t} ||\Phi^+(t,s)||||f^+(s+\tau) - f^+(s)||ds \\ &+ \sum_{t_k<t} ||\Phi^+(t+\tau, t_{k+q}) - \Phi^+(t,t_k)||||b^+_{k+q}|| \\ &+ \sum_{t_k<t} ||\Phi^+(t,t_k)||||b^+_{k+q} - b^+_k||, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}||y(t+\tau)-y(t)|| = &\int_{t}^{\infty} ||\Phi^-(t+\tau, s+\tau) - \Phi^-(t,s)||||f^-(s+\tau)||ds \\ &+ \int_{t}^{\infty} ||\Phi^-(t,s)||||f^-(s+\tau) - f^-(s)||ds \\ &+ \sum_{t>t_k} ||\Phi^-(t+\tau, t_{k+q}) - \Phi^-(t,t_k)||||b^-_{k+q}|| \\ &+ \sum_{t>t_k} ||\Phi^-(t,t_k)||||b^-_{k+q} - b^-_k||. \end{aligned} \tag{2.13}$$

From Lemma 2.2, (2.12) and (2.13), we have

$$||x(t+\tau)-x(t)|| \leq K_1\varepsilon, \tag{2.14}$$

where

$$K_1 = \frac{2K}{\lambda}\sup_{t\in\mathbb{R}}||f^+(t)|| + \frac{\overline{a}}{\lambda} + \frac{2N\overline{a}}{1-e^{-\frac{\lambda}{2}}}\sup_{k=\pm1,\pm2,\ldots}||b^+_k|| + \frac{2N\overline{a}}{1-e^{-\lambda}}.$$

In the same manner, we obtain

$$||y(t+\tau)-y(t)|| \leq K_2\varepsilon, \tag{2.15}$$

where

$$K_2 = \frac{2K}{\lambda}\sup_{t\in\mathbb{R}}||f^-(t)|| + \frac{\overline{a}}{\lambda} + \frac{2N\overline{a}}{1-e^{-\frac{\lambda}{2}}}\sup_{k=\pm1,\pm2,\ldots}||b^-_k|| + \frac{2N\overline{a}}{1-e^{-\lambda}}.$$

The number N, which is defined in the last inequalities, is from Lemma 1.2 Now, from (2.14) and (2.15), we conclude that the solution $z(t) = (x(t), y(t))$ of system (2.1) is almost periodic.

On the other hand, each solution $(x(t), y(t))$ of (2.1) can be written in the form

$$x(t) = \Phi^+(t, t_0)\chi + \int_{t_0}^{t} \Phi^+(t, s)f^+(s)ds + \sum_{s<t_k<t} \Phi^+(t, t_k)b_k^+,$$

$$y(t) = -\int_{t}^{\infty} \Phi^-(t, s)f^-(s)ds + \sum_{t_k>t} \Phi^-(t, t_k)b_k^-,$$

where χ is a constant k-dimensional vector.

It follows that, for two different solutions $z_1(t)$ and $z_2(t)$ of system (2.1) the estimate

$$||z_1(t) - z_2(t)|| \leq \overline{a}e^{-\lambda(t-t_0)}||z_1(t_0) - z_2(t_0)|| \tag{2.16}$$

holds true.

Thus, (2.16) implies that the solution $z(t)$ of (2.1) is unique and exponentially stable. □

Let $\Omega \equiv B_h$.

Theorem 2.2. *Let the following conditions hold:*

1. *Conditions H2.1, H2.4–H2.6 hold.*
2. *The system (2.3) is hyperbolic.*
3. *The functions $F(t, z)$, $I_k(z)$, $k = \pm1, \pm2, \ldots$, are Lipschitz continuous with respect to $z \in B_h$ with a Lipschitz constant $L > 0$, i.e.,*

$$||F(t, z_1) - F(t, z_2)|| + ||I_k(z_1) - I_k(z_2)|| \leq L||z_1 - z_2||,$$

and they are bounded, i.e. there exists a constant $L_1 > 0$, such that

$$max\Big(\sup_{t\in\mathbb{R}, z\in B_h} ||F(t, z)||, \sup_{k=\pm1,\pm2,\ldots, z\in B_h} ||I_k(z))||\Big) = L_1 < \infty.$$

4. *The following inequalities hold*

$$L_1\Big(\frac{\overline{a}}{\lambda} + \frac{2\overline{a}N}{1 - e^{-\lambda}}\Big) < h,$$
$$L\Big(\frac{\overline{a}}{\lambda} + \frac{2\overline{a}N}{1 - e^{-\lambda}}\Big) < 1.$$

Then for the system (2.2) there exists a unique almost periodic solution.

Proof. Denote by AP the set of all almost periodic solutions $\varphi(t)$, $\varphi \in PC[\mathbb{R}, \Omega]$, such that $||\varphi|| < h$.

We define in AP the operator SAP, such that if $\varphi \in AP$, then $\varphi = (\varphi^+, \varphi^-)$, where $\varphi^+ : \mathbb{R} \to \mathbb{R}^k$, $\varphi^- : \mathbb{R} \to \mathbb{R}^{n-k}$, $SAP\varphi = (SAP\varphi^+, SAP\varphi^-)$, $u = SAP\varphi^+$ is the almost periodic solution of

$$\begin{cases} \dot{u} = Q^+(t)u + F^+(t, \varphi(t)), t \neq t_k, \\ \Delta u(t_k) = I_k^+(\varphi(t_k)), k = \pm 1, \pm 2, \ldots, \end{cases}$$

and $v = SAP\varphi^-$ is the almost periodic solution of

$$\begin{cases} \dot{v} = Q^-(t)v + F^-(t, \varphi(t)), t \neq t_k, \\ \Delta v(t_k) = I_k^-(\varphi(t_k)), k = \pm 1, \pm 2, \ldots. \end{cases}$$

The existence of almost periodic solutions $u(t)$ and $v(t)$, is guaranteed by Theorem 2.1. In fact, the almost periodicity of the sequence $\{\varphi(t_k)\}$, $k = \pm 1, \pm 2, \ldots$ follows from Lemma 1.5, and from the method for finding of common almost periods, we obtain that the sequence $\{I_k(\varphi(t_k))\}$, $k = \pm 1, \pm 2, \ldots$, is almost periodic, also. The almost periodicity of the function $F(t, \varphi(t))$ follows from Theorem 1.17. Further on, conditions 2 and 3 imply that $SAP(AP) \subset AP$.

Let $\varphi,\ \psi \in AP$. Then, the estimate

$$||SAP\varphi - SAP\psi|| \leq L\overline{a}\Big(\frac{1}{\lambda} + \frac{2N}{1 - e^{-\lambda}}\Big)|\varphi - \psi|_\infty,$$

where $|\varphi - \psi|_\infty = \sup\limits_{t \in \mathbb{R}} ||\varphi(t) - \psi(t)||$ holds true.

It follows from condition 3, and from the last inequality, that SAP is a contracting operator on SAP. Hence, for the system (2.2) there exists a unique almost periodic solution. □

2.2 Impulsive Integro-Differential Equations

In this section, we shall present the main results on the existence of almost periodic solutions of impulsive integro-differential systems.

Consider the following linear system of impulsive integro-differential equations

$$\begin{cases} \dot{x} = A(t)x(t) + \int\limits_{t_0}^{t} K(t, s)x(s)ds + f(t),\ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.17}$$

where $t \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, $A \in PC[\mathbb{R}, \mathbb{R}^{n\times n}]$, $K \in PC[\mathbb{R}^2, \mathbb{R}^{n\times n}]$, $f \in PC[\mathbb{R}, \mathbb{R}^n]$, $B_k \in \mathbb{R}^{n\times n}, k = \pm 1, \pm 2, \ldots$.

The solution of (2.17), $x(t) = x(t; t_0, x_0)$ with initial condition $x(t_0^+) = x_0, t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^n$, is characterized at the following way:

1. For $t \neq t_k,\ k = \pm 1, \pm 2, \ldots$, the mapping point $(t, x(t))$ moves along some of the integral curves of the system

$$\dot{x} = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)ds + f(t).$$

2. At the moment $t = t_k,\ k = \pm 1, \pm 2, \ldots$, the system is subject to an impulsive effect, as a result of which the mapping point is transferred "instantly" from the position $(t_k, x(t_k))$ into a position $\big(t_k, x(t_k) + B_k x(t_k)\big)$. Afterwards, for $t_k < t < t_{k+1}$ the solution $x(t)$ coincides with the solution $y(t)$ of the system

$$\begin{cases} \dot{y} = A(t)y(t) + \int_{t_0}^{t} K(t,s)y(s)ds + f(t),\ t \neq t_k, \\ y(t_k) = x(t_k) + B_k x(t_k),\ k = \pm 1, \pm 2, \ldots, \end{cases}$$

At the moment $t = t_{k+1}$, the solution is subject to a new impulsive effect.

We shall, also, consider weakly nonlinear impulsive integro-differential systems

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)ds + F(t, x(t)),\ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.18}$$

where $F(t, x) \in PC[\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n]$, and

$$\begin{cases} \dfrac{\partial R(t,s)}{\partial t} = A(t)x(t) + \displaystyle\int_{t_0}^{t} K(t,v)R(v,s)dv,\ \ s \neq t_k,\ t \neq t_k, \\ R(t_k^+, s) = (E + B_k)R(t_k, s),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.19}$$

where $R(t, s)$ is an $n \times n$-dimensional matrix function and $R(s, s) = E,\ E$ is the identity matrix in $\mathbb{R}^n$.

Lemma 2.3 ([131]). *If $R(t, s)$ is a solution of (2.19), then the unique solution $x(t) = x(t; t_0, x_0)$ of (2.17) is given by*

$$x(t) = R(t,t_0)x(t_0) + \int_{t_0}^{t} R(t,s)f(s)ds,\ x(t_0^+) = x_0.$$

Introduce the following conditions:

H2.7. There exists an $n \times n$-dimensional matrix function $R(t,s)$, satisfying (2.19).
H2.8. $det(E + B_k) \neq 0,\ k = \pm 1, \pm 2, \ldots$.
H2.9. $\mu[A(t) - R(t,t)] \leq -\alpha,\ \alpha > 0,\ \mu[.]$ is the logarithmic norm.

Lemma 2.4 ([15]). *Let conditions H2.7–H2.9 hold.*
Then

$$||R(t,s)|| \leq K_1 e^{-\alpha(t-s)}, \tag{2.20}$$

where $K_1 > 0,\ t > s$.

Remark 2.1. In the special case, when in (2.17), $K(t,s) \equiv 0$, we obtain the linear impulsive system

$$\begin{cases} \dot{x} = A(t)x + f(t),\ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k),\ k = \pm 1, \pm 2, \ldots. \end{cases}$$

Then, from Lemma 2.3, it follows, respectively, well known variation parameters formula [94], where $R(t,s)$ is the fundamental matrix and $R(t_0,t_0) = E$.

We shall investigate the existence of almost periodic solutions of systems (2.17), (2.18), and we shall use the following conditions:

H2.10. $A(t)$ is an almost periodic $n \times n$-matrix function.
H2.11. The sequence $\{B_k\},\ k = \pm 1, \pm 2, \ldots$ is almost periodic.
H2.12. The set of sequences $\{t_k^j\},\ k = \pm 1, \pm 2, \ldots,\ j = \pm 1, \pm 2, \ldots$ is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.
H2.13. The matrix $K(t,s)$ is almost periodic along the diagonal line, i.e. for any $\varepsilon > 0$, the set $T(K,\varepsilon)$ composed from ε-almost periods τ, such that for $\tau \in T(K,\varepsilon)$, $K(t,s)$ satisfies the inequality

$$||K(t+\tau, s+\tau) - K(t,s)|| \leq \varepsilon e^{-\frac{\alpha}{2}(t-s)},$$

$t > s$, is relatively dense in $\mathbb{R}$.
H2.14. The function $f(t),\ f \in PC[\mathbb{R}, \mathbb{R}^n]$ is almost periodic.
H2.15. The function $F(t,x)$ is almost periodic along t uniformly with respect to $x \in \Omega$.

We shall use the next lemma, which is similar to Lemma 1.7.

Lemma 2.5. *Let conditions H2.10–H2.12 and H2.14 hold.*

Then for each $\varepsilon > 0$ there exist $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon$, a relatively dense set $\overline{T}$ of real numbers and a set P of integer numbers, such that the following relations are fulfilled:

(a) $||A(t+\tau) - A(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}$.
(b) $||f(t+\tau) - f(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T},\ |t - t_k| > \varepsilon,\ k = \pm 1, \pm 2, \ldots$.
(c) $||B_{k+q} - B_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots$.
(d) $|\overline{t}_k^q - \tau| < \varepsilon_1,\ q \in P,\ \tau \in \overline{T},\ k = \pm 1, \pm 2, \ldots$.

Lemma 2.6. *Let conditions H2.7–H2.13 hold.*

Then $R(t,s)$ is almost periodic along the diagonal line and the following inequality holds

$$||R(t+\tau, s+\tau) - R(t,s)|| \leq \varepsilon \Gamma e^{-\frac{\alpha}{2}(t-s)}, \tag{2.21}$$

where $t > s,\ \Gamma > 0,\ \varepsilon > 0,\ \tau$ is an almost period.

Proof. Let $\varepsilon > 0$ and τ be a common ε-almost period of $A(t)$ and $K(t,s)$.

Then, for $s \neq t'_k,\ t \neq t'_k$, we have

$$\frac{\partial R(t+\tau, s+\tau)}{\partial t} = A(t)R(t+\tau, s+\tau) + \big(A(t+\tau) - A(t)\big)R(t+\tau, s+\tau)$$
$$\times \int_s^t \big(K(t+\tau, v+\tau) - K(t,v)\big)R(v+\tau, s+\tau)dv$$
$$+ \int_s^t K(t,v)R(v+\tau, s+\tau)dv,$$

and

$$R(t'_k+\tau, s+\tau) = (E + B_k)R(t'_k+\tau, s+\tau) + (B_{k+q} - B_k)R(t_k+\tau, s+\tau),$$

where $t'_k = t_k - \tau$ and $\tau,\ q$ are the numbers from Lemma 2.5.

Hence, from (2.19), we obtain

$$R(t+\tau, s+\tau) - R(t,s)$$
$$= \int_s^t R(t,u)\big(A(u+\tau) - A(u)\big)R(u+\tau, s+\tau)du$$

$$+\int_s^t R(t,u)\Big(\int_s^u \big(K(u+\tau,v+\tau)-K(u,v)R(v+\tau,s+\tau\big)dv\Big)du$$

$$+\sum_{s\le t'_v<t} R(t,t_v^{'+})(B_{v+q}-B_v)R(t'_v+\tau,s+\tau). \tag{2.22}$$

From Lemma 2.5, it follows that, if $|t-t'_k|>\varepsilon$, $t\in\mathbb{R}$, then $t'_{k+q}<t+\tau<t'_{k+q+1}$ and from (2.20), (2.22), we obtain

$$||R(t+\tau,s+\tau)-R(t,s)||\le K_1^2\varepsilon\Big(e^{-\alpha(t-s)}(t-s)+\frac{4}{\alpha^2}e^{-\frac{\alpha}{2}(t-s)}i(s,t)e^{-\alpha(t-s)}\Big),$$

where $i(t,s)$ is the number of points t_k in the interval (t,s).

Now, from the condition H2.12 and Lemma 1.2, it follows that there exists a positive integer N, such that for any $t\in\mathbb{R}$, $s\in\mathbb{R}$ and $t>s$ the following inequality holds

$$i(s,t)\le(t-s)N+N.$$

Therefore,

$$||R(t+\tau,s+\tau)-R(t,s)||\le\varepsilon\Gamma e^{-\frac{\alpha}{2}(t-s)},$$

where $t>s$, $\Gamma=K_1^2\dfrac{2}{\alpha}\Big(1+\dfrac{2}{\alpha}N+\dfrac{N\alpha}{2}\Big)$. □

The next theorems are the main in this paragraph.

Theorem 2.3. *Let conditions H2.7–H2.14 hold.*

Then for the system (2.17), there exists a unique exponentially stable almost periodic solution $\varphi(t)$, such that

$$||\varphi(t)||\le\frac{2K_1}{\alpha}\max_{s<t}||f||. \tag{2.23}$$

Proof. Consider the function

$$\varphi(t)=\int_{-\infty}^t R(t,s)f(s)ds. \tag{2.24}$$

From (2.19), (2.24), and Fubini's theorem, it follows that

$$\dot\varphi(t)=\int_{-\infty}^t\frac{\partial R(t,s)}{\partial t}f(s)ds+f(t)$$

$$=\int_{-\infty}^t\Big(A(t)R(t,s)+\int_u^t K(t,u)R(t,u)du\Big)f(s)ds+f(t)$$

$$= A(t)\varphi(t) + \int\limits_{u}^{t} \Big(\int\limits_{-\infty}^{t} K(t,u)R(u,s)f(s)ds \Big) du$$

$$= A(t)\varphi(t) + \int\limits_{s}^{t} K(t,s)\varphi(s)ds + f(t), \tag{2.25}$$

where $s < t,\ s \neq t_k,\ t \neq t_k,\ k = \pm 1, \pm 2, \ldots$.

On the other hand, for $t = t_k$, $k = \pm 1, \pm 2, \ldots$, we have

$$\Delta\varphi(t_k) = \varphi(t_k^+) - \varphi(t_k) = B_k\varphi(t_k). \tag{2.26}$$

Then, from (2.25) and (2.26), it follows that $\varphi(t)$ is a solution of system (2.17).

From Lemma 2.4, we obtain

$$||\varphi(t)|| \leq \int\limits_{-\infty}^{t} ||R(t,s)||||f(s)||ds \leq \frac{2K_1}{\alpha} \max_{s<t} ||f(t)||.$$

Let $\tau \in \overline{T},\ q \in P$, where $\overline{T}$ and P are determined in Lemma 2.5. From Lemma 2.6, it follows that

$$||\varphi(t+\tau) - \varphi(t)|| = \int\limits_{-\infty}^{t} ||R(t+\tau, s+\tau)f(s+\tau) - R(t,s)f(s)||ds$$

$$\leq \int\limits_{-\infty}^{t} ||R(t+\tau, s+\tau) - R(t,s)||||f(s+\tau)||ds$$

$$+ \int\limits_{-\infty}^{t} ||R(t,s)||||f(s+\tau) - f(s)||ds$$

$$\leq \varepsilon\Big(\frac{2\Gamma M}{\alpha} + \frac{K_1}{\alpha}\Big), \tag{2.27}$$

where $M = \max\limits_{s<t} ||f(t)||$. The estimate (2.27) means that $\varphi(t)$ is an almost periodic function.

Let $\eta(t)$ is one other solution of (2.17). Then, from (2.20), it follows that

$$||\varphi(t) - \eta(t)|| \leq K_1 e^{-\alpha(t-t_0)} ||\varphi(t_0) - \eta(t_0)||,$$

and we obtain that the solution $\varphi(t)$ is unique and exponentially stable. □

Theorem 2.4. *Let the following conditions hold:*

1. *Conditions H2.7–H2.13, and H2.15 hold.*
2. *The function $F(t,x)$ is Lipschitz continuous with respect to $x \in B_h$ with a Lipschitz constant $L > 0$, i.e.*

$$||F(t,x_1) - F(t,x_2)|| \le L\|x_1 - x_2\|, \; x_1, x_2 \in B_h,$$

   and $F(t,x)$ is uniformly bounded, i.e. there exists a constant $G > 0$, such that

$$||F(t,x)|| \le G, \; ||x|| < h.$$

3. *The following inequalities hold*

$$\frac{K_1 G}{\alpha} < h, \; \frac{KL}{\alpha} < 1.$$

Then there exists a unique exponentially stable almost periodic solution of (2.18).

Proof. Let us denote by AP the set of all almost periodic functions $\varphi(t)$, $\varphi \in PC[\mathbb{R}, \mathbb{R}^n]$, satisfying the inequality $||\varphi(t)|| < h$, and let $|\varphi(t)|_\infty = \sup_{t \in \mathbb{R}} ||\varphi(t)||$.

In AP, we define an operator S

$$S\varphi = \int_{-\infty}^{t} R(t,s)F(t,\varphi(s))ds. \tag{2.28}$$

Let $\varphi \in AP$. From (2.28), it follows that

$$\begin{aligned} ||S\varphi|| &\le \int_{-\infty}^{t} ||R(t,s)|| ||F(t,\varphi(s))||ds \\ &\le K_1 \int_{-\infty}^{t} e^{-\alpha(t-s)} G ds \le \frac{K_1 G}{\alpha} < h. \end{aligned} \tag{2.29}$$

On the other hand, from Theorem 1.17, it follows that the function $F(t,\varphi(t))$ is almost periodic, and let τ be the common almost period of $\varphi(t)$ and $F(t,\varphi(t))$.

Then,

$$\begin{aligned} &||S\varphi(t+\tau) - S\varphi(t)|| \\ &\le \int_{-\infty}^{t} ||R(t+\tau, s+\tau)F(s,\varphi(s+\tau)) - R(t,s)F(s,\varphi(s))||ds \\ &\le \Big(\frac{2G\Gamma}{\alpha} + \frac{K_1}{\alpha}\Big)\varepsilon. \end{aligned} \tag{2.30}$$

Hence, using (2.29) and (2.30), we obtain that $S(AP) \subset AP$.

Let $\varphi \in AP$, $\eta \in AP$. From (2.28) and Lemma 2.6, we have

$$\begin{aligned} ||S\varphi(t) - S\eta(t)|| &\leq \int_{-\infty}^{t} ||R(t,s)|| ||F(s,\varphi(s)) - F(s,\eta(s))|| ds \\ &\leq \frac{K_1 L}{\alpha} |\varphi(t) - \eta(t)|_\infty. \end{aligned} \tag{2.31}$$

Therefore, the inequality (2.31) shows that S is a contracting operator in AP, and hence, there exists a unique almost periodic solution of system (2.18).

Now, let $\psi(t)$ is one other solution of (2.18). Then, Lemma 2.3 and (2.20) imply that

$$\begin{aligned} &||\varphi(t) - \psi(t)|| \\ &\quad \leq K_1 ||\varphi(t_0) - \psi(t_0)|| e^{-\alpha(t-s)} + \int_{t_0}^{t} K_1 e^{-\alpha(t-s)} L ||\varphi(s) - \psi(s)|| ds. \end{aligned} \tag{2.32}$$

Set

$$v(t) = ||\varphi(t) - \psi(t)|| e^{\alpha(t)}.$$

From (2.32) and Gronwall–Belman's inequality, we have

$$v(t) \leq K_1 v(t_0) exp\Big(\int_{t_0}^{t} K_1 L ds \Big).$$

Consequently,

$$||\varphi(t) - \psi(t)|| \leq K_1 ||\varphi(t_0) - \psi(t_0)|| e^{(K_1 L - \alpha)(t - t_0)}.$$

From the last inequality, it follows that $\varphi(t)$ is exponentially stable. □

2.3 Forced Perturbed Impulsive Differential Equations

In this part, we shall consider sufficient conditions for the existence of almost periodic solutions for forced perturbed systems of impulsive differential equations with impulsive effects at fixed moments.

We shall consider the system

$$\begin{cases} \dot{x} = A(t)x + g(t) + \mu X(t,x,\mu), \ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k) + g_k + \mu X_k(x(t_k),\mu), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.33}$$

where $t \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $g : \mathbb{R} \to \mathbb{R}^n$, $\mu \in M \subset \mathbb{R}$, $X : \mathbb{R} \times \Omega \times M \to \mathbb{R}^n$, $B_k \in \mathbb{R}^{n\times n}$, $g_k \in \mathbb{R}^n$, $X_k : \Omega \times M \to \mathbb{R}^n$, $k = \pm 1, \pm 2, \ldots$.

Denote by $x(t,\mu) = x(t;t_0,x_0,\mu)$ the solution of (2.33) with initial condition $x(t_0^+,\mu) = x_0, \ x_0 \in \Omega, \ \mu \in M$.

We shall use the following definitions:

Definition 2.2. The system

$$\begin{cases} \dot{x} = A(t)x + g(t), \ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k) + g_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.34}$$

is said to be *generating system* of (2.33).

Definition 2.3 ([56]). The matrix $A(t)$ is said to has a *column dominant with a parameter* $\alpha > 0$ *on* $[a,b]$, if

$$a_{ii}(t) + \sum_{j\neq i} |a_{ji}(t)| \leq -\alpha < 0,$$

for each $i,j = 1,\ldots,n$, and $t \in [a,b]$.

Introduce the following conditions:

H2.16. The matrix function $A \in C[\mathbb{R}, \mathbb{R}^{n\times n}]$ is almost periodic in the sense of Bohr.

H2.17. $\{B_k\}, \ k = \pm 1, \pm 2, \ldots$ is an almost periodic sequence.

H2.18. $det(E + B_k) \neq 0, \ k = \pm 1, \pm 2, \ldots$ where E is the identity matrix in $\mathbb{R}^{n\times n}$.

H2.19. The function $g \in PC[\mathbb{R}, \mathbb{R}^n]$ is almost periodic.

H2.20. $\{g_k\}, \ k = \pm 1, \pm 2, \ldots$ is an almost periodic sequence.

H2.21. The function $X \in C[\mathbb{R} \times \Omega \times M, \mathbb{R}^n]$ is almost periodic in t uniformly with respect to $(x,\mu) \in \Omega \times M$, and is Lipschitz continuous with respect to $x \in B_h$ with a Lipschitz constant $l_1 > 0$, such that

$$||X(t,x,\mu) - X(t,y,\mu)|| \leq l_1 ||x - y||, \ x,y \in B_h,$$

for any $t \in \mathbb{R}$ and $\mu \in M$.

H2.22. The sequence of functions $\{X_k(x,\mu)\}, \ k = \pm 1, \pm 2, \ldots, \ X_k \in C[\Omega \times M, \mathbb{R}^n]$ is almost periodic uniformly with respect to $(x,\mu) \in \Omega \times M$, and the functions X_k are Lipschitz continuous with respect to $x \in B_h$ with a Lipschitz constant $l_2 > 0$, such that

$$||X_k(x,\mu) - X_k(y,\mu)|| \le l_2||x-y||,\ x,y \in B_h,$$

for $k = \pm 1, \pm 2, \dots,\ \mu \in M$.

H2.23. The set of sequences $\{t_k^j\},\ t_k^j = t_{k+j} - t_k,\ k = \pm 1, \pm 2, \dots,\ j = \pm 1, \pm 2, \dots$ is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.

We shall use the next lemma, which is similar to Lemma 1.7.

Lemma 2.7. *Let conditions H2.16, H2.17, H2.19, H2.20 and H2.23 hold. Then for each $\varepsilon > 0$ there exist $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon$, a relatively dense set $\overline{T}$ of real numbers, and a set P of integer numbers, such that the following relations are fulfilled:*

(a) $||A(t+\tau) - A(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}$.
(b) $||g(t+\tau) - g(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T},\ |t - t_k| > \varepsilon,\ k = \pm 1, \pm 2, \dots$.
(c) $||B_{k+q} - B_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \dots$.
(d) $||g_{k+q} - g_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \dots$.
(e) $|t_k^q - \tau| < \varepsilon_1,\ q \in P,\ \tau \in \overline{T},\ k = \pm 1, \pm 2, \dots$.

Lemma 2.8. *Let conditions H2.19, H2.20 and H2.23 hold.*
Then there exists a positive constant C_1 such that

$$max(\sup_{t \in \mathbb{R}} ||g(t)||, \sup_{k=\pm 1, \pm 2, \dots} ||g_k||) \le C_1.$$

Proof. The proof follows from Lemma 1.7. □

Lemma 2.9 ([138]). *Let the following conditions hold:*

1. *Conditions H2.16–H2.18 and H2.23 are met.*
2. *For the Cauchy matrix $W(t,s)$ of the system*

$$\begin{cases} \dot{x} = A(t)x,\ t \ne t_k, \\ \Delta x(t_k) = B_k x(t_k),\ k = \pm 1, \pm 2, \dots, \end{cases}$$

there exist positive constants K and λ such that

$$||W(t,s)|| \le Ke^{-\lambda(t-s)},$$

where $t \ge s,\ t,s \in \mathbb{R}$.

Then for any $\varepsilon > 0,\ t \in \mathbb{R},\ s \in \mathbb{R},\ |t - t_k| > \varepsilon > 0,\ |s - t_k| > \varepsilon,\ k = \pm 1, \pm 2, \dots$, there exists a relatively dense set $\overline{T}$ of ε-almost periods of matrix $A(t)$ and a positive constant Γ, such that for $\tau \in \overline{T}$ it follows

$$||W(t+\tau, s+\tau) - W(t,s)|| \le \varepsilon \Gamma e^{-\frac{\lambda}{2}(t-s)}.$$

Now, we are ready to proof the main theorem.

Theorem 2.5. *Let the following conditions hold:*

1. *Conditions H2.16–H2.23 are met.*
2. *There exists a positive constant L_1, such that*

$$max\{\sup_{\substack{t\in\mathbb{R}\\ (x,\mu)\in\Omega\times M}} ||X(t,x,\mu)||, \sup_{\substack{k=\pm1,\pm2,\ldots\\ (x,\mu)\in\Omega\times M}} ||X_k(x,\mu)||\} \le L_1.$$

3. *For the generating system (2.34), there exists a unique almost periodic solution.*

Then there exists a positive constant μ_0, $\mu_0 \in M$ such that:

1. *For any μ, $|\mu| < \mu_0$ and $C < C_1$, where the constant C_1 is from Lemma* 2.8, *there exists a unique almost periodic solution of* (2.33).
2. *There exists a positive constant L such that*

$$||x(t,\mu_1) - x(t,\mu_2)|| \le L|\mu_1 - \mu_2|,$$

where $t \in \mathbb{R}$, $|\mu_i| < \mu_0$, $i = 1,2$.

3. *For $|\mu| \to 0$, $x(t,\mu)$ converges to the unique almost periodic solution of* (2.34).
4. *The solution $x(t,\mu)$ is exponentially stable.*

Proof of Assertion 1. Let we denote by AP, the set of all almost periodic functions $\varphi(t,\mu)$, $\varphi \in AP \in PC[\mathbb{R}\times M, \mathbb{R}^n]$ satisfying the inequality $||\varphi|| < C$, and let $|\varphi|_\infty = \sup_{t\in\mathbb{R},\ \mu\in M} ||\varphi(t,\mu)||$.

In AP, we define the operator S,

$$S\varphi = \int_{-\infty}^{t} W(t,s)\Big(g(s) + \mu X(s,\varphi(s,\mu),\mu)\Big)ds + \sum_{t_k<t} W(t,t_k)\Big(g_k + \mu X_k(\varphi(t_k,\mu),\mu)\Big). \tag{2.35}$$

From Lemma 2.8 and Lemma 2.9, it follows

$$\begin{aligned} ||S\varphi|| &= \int_{-\infty}^{t} ||W(t,s)||\Big(||g(s)|| + |\mu|||X(s,\varphi(s,\mu),\mu)||\Big)ds \\ &\quad + \sum_{t_k<t} ||W(t,t_k)||\Big(||g_k|| + |\mu|||X_k(\varphi(t_k,\mu),\mu)||\Big) \\ &\le (C_1 + |\mu|L_1)\Big(\frac{K}{\lambda} + \frac{KN}{1-e^{-\lambda}}\Big). \end{aligned}$$

Consequently, there exists a positive constant μ_1 such that for $\mu \in (-\mu_1, \mu_1)$ and $C = (C_1 + |\mu|L_1)\Big(\frac{K}{\lambda} + \frac{KN}{1-e^{-\lambda}}\Big) < C_1$, we obtain

$$||S\varphi|| \le C. \tag{2.36}$$

Now, let $\tau \in \overline{T}$, $q \in P$, where the sets $\overline{T}$ and P are determined in Lemma 2.7. From Lemma 1.5 and Theorem 1.17, we have

$$\begin{aligned}
&||S\varphi(t+\tau,\mu) - S\varphi(t,\mu)||\\
&\quad \le \int_{-\infty}^{t} ||W(t+\tau,s+\tau) - W(t,s)||\Big(||g(s+\tau)||\\
&\quad + |\mu|||X(s+\tau,\varphi(s+\tau,\mu),\mu)||\Big)ds\\
&\quad + \int_{-\infty}^{t} ||W(t,s)||\Big(||g(s+\tau) - g(s)||\\
&\quad + |\mu|||X(s+\tau,\varphi(s+\tau,\mu),\mu) - X(s,\varphi(s,\mu),\mu)||\Big)ds\\
&\quad + \sum_{t_k<t} ||W(t+\tau,t_{k+q}) - W(t,t_k)||\Big(||g_{k+q}||\\
&\quad + |\mu|||X_{k+q}(\varphi(t_{k+q},\mu),\mu)||\Big)\\
&\quad + \sum_{t_k<t} ||W(t,t_k)||\Big(||g_{k+q} - g_k||\\
&\quad + |\mu|||X_{k+q}(\varphi(t_{k+q},\mu),\mu) - X_k(\varphi(t_k,\mu),\mu)||\Big)\\
&\quad \le \varepsilon\Big((C_1 + |\mu|L_1)\Big(\frac{2\Gamma}{\lambda} + \frac{N\Gamma}{1-e^{-\lambda}}\Big) + (1+|\mu|)\Big(\frac{K}{\lambda} + \frac{NK}{1+e^{-\lambda}}\Big)\Big).
\end{aligned} \tag{2.37}$$

Thus, by (2.35) and (2.36), we obtain $S\varphi \in AP$.
Let $\varphi \in AP$, $\psi \in AP$. Then from (2.35), it follows

$$\begin{aligned}
||S\varphi - S\psi|| &\le |\mu| \int_{-\infty}^{t} ||W(t,s)||||X(s,\varphi(s,\mu),\mu) - X(s,\psi(s,\mu),\mu)||ds\\
&\quad + |\mu| \sum_{t_k<t} ||W(t,t_k)||||X_k(\varphi(t_k,\mu),\mu) - X_k(\psi(t_k,\mu),\mu)||\\
&\le |\mu|K\Big(\frac{l_1}{\lambda} + \frac{l_2}{1-e^{-\lambda}}\Big)|\varphi - \psi|_\infty.
\end{aligned}$$

Since there exists a positive constant $\mu_0 < \mu_1$ such that

$$\mu_0 K\Big(\frac{l_1}{\lambda} + \frac{l_2}{1-e^{-\lambda}}\Big) < 1,$$

we have that S is a contracting operator in AP.

Proof of Assertion 2. Let $\varphi_j = \varphi_j(t,\mu_j),\ j = 1,2$, and $|\mu_j| < \mu_0$.
Then,

$$\begin{aligned}
||\varphi_1 - \varphi_2|| &\le |\mu_1 - \mu_2|\Big(\int_{-\infty}^{t} ||W(t,s)||||X(s,\varphi_1(s,\mu_1),\mu_1)||ds\\
&+ \sum_{t_k<t} ||W(t,t_k)||||X_k(\varphi_1(t_k,\mu_1),\mu_1)||\Big)\\
&+ |\mu_2|\Big(\int_{-\infty}^{t} ||W(t,s)||||X(s,\varphi_1(s,\mu_1),\mu_1) - X(s,\varphi_2(s,\mu_2),\mu_2)||ds\\
&+ \sum_{t_k<t} ||W(t,t_k)||||X_k(\varphi_1(t_k,\mu_1),\mu_1) - X_k(\varphi_2(t_k,\mu_2),\mu_2)||\Big)\\
&\le L|\mu_1 - \mu_2|,
\end{aligned} \tag{2.38}$$

where

$$L = L_1 K\Big(\frac{l_1}{\lambda} + \frac{l_2}{1-e^{-\lambda}}\Big)(1-\mu_0 K)K\Big(\frac{l_1}{\lambda} + \frac{Nl_2}{1-e^{-\lambda}}\Big).$$

Proof of Assertion 3. Let we denote by $x(t)$ the almost periodic solution of (2.33).

From (2.35) and Lemma 2.9, it follows

$$\begin{aligned}
||x(t,\mu) - x(t)|| &\le |\mu|\Big(\int_{-\infty}^{t} ||W(t,s)||||X(s,\varphi(s,\mu),\mu)||ds\\
&\quad + \sum_{t_k<t} ||W(t,t_k)||||X_k(\varphi(t_k,\mu),\mu)||\Big)\\
&\le |\mu|L_1 K\Big(\frac{1}{\lambda} + \frac{N}{1-e^{-\lambda}}\Big).
\end{aligned}$$

Then $x(t,\mu) \to x(t)$ for $|\mu| \to 0$.

Proof of Assertion 4. Let $y(t)$ be an arbitrary solution of (2.34). Then using (2.35), we obtain

$$\begin{aligned}
y(t) - x(t,\mu) &= W(t,t_0)\big(y(t_0) - x(t_0,\mu)\big)\\
&\quad + \mu\Big(\int_{t_0}^{t} W(t,s)\big(X(s,y(s),\mu) - X(s,x(s,\mu),\mu)\big)ds\\
&\quad + \sum_{t_0<t_k<t} W(t,t_k)\big(X_k(y(t_k),\mu) - X_k(x(t_k,\mu),\mu)\big)\Big).
\end{aligned}$$

Now, we have

$$||y(t) - x(t,\mu)|| \le Ke^{-\lambda(t-t_0)}||y(t_0) - x(t_0,\mu)||$$
$$+ |\mu|\Big(\int_{t_0}^{t} Kl_1 e^{-\lambda(t-s)}||y(s) - x(s,\mu)||ds$$
$$+ \sum_{t_0<t_k<t} Kl_2 e^{-\lambda(t-t_k)}||y(t_k) - x(t_k,\mu)||\Big).$$

Set $u(t) = ||y(t) - x(t,\mu)||e^{-\lambda t}$ and from Gronwall–Bellman's inequality, it follows

$$||y(t) - x(t,\mu)|| \le K||y(t_0) - x(t_0,\mu)||(1 + |\mu|Kl_1)^{i(t_0,t)}e^{(-\lambda+|\mu|Kl_2)(t-t_0)},$$

where $i(t,s)$ is the number of points t_k in the interval (t,s). Obviously, if there exists $\mu \in M$ such that $N\ln(1+|\mu|Kl_1)+|\mu|Kl_2 < \lambda$, then the solution $x(t,\mu)$ is exponentially stable. □

Lemma 2.10. *Let the following conditions hold:*

1. *Conditions H2.16, H2.17 are met.*
2. *The matrix-valued function $A(t)$ has a column dominant with a parameter $\alpha > 0$ for $t \in \mathbb{R}$.*

Then for the Cauchy's matrix $W(t,s)$ it follows

$$||W(t,s)|| \le Ke^{-\alpha(t-s)},$$

where $t \in \mathbb{R},\ s \in \mathbb{R},\ t \ge s,\ K > 0$.

Proof. The proof follows from the definition of matrix $W(t,s)$. □

Example 2.1. We consider the following system of impulsive differential equations of Lienard's type:

$$\begin{cases} \ddot{x} + f(t)\dot{x} + q(t) = \mu h(t,x,\dot{x},\mu),\ t \ne t_k, \\ \Delta x(t_k) = b_k^1 x(t_k) + g_k^1 + \mu X_k^1(x(t_k),\dot{x}(t_k),\mu), \\ \Delta \dot{x}(t_k) = b_k^2 x(t_k) + g_k^2 + \mu X_k^2(x(t_k),\dot{x}(t_k),\mu),\ k = \pm1,\pm2,\ldots, \end{cases} \tag{2.39}$$

where $t \in \mathbb{R},\ x \in \mathbb{R},\ \mu \in M,\ \{t_k\} \in \mathcal{B}$, the functions $f \in PC[\mathbb{R},\mathbb{R}],\ q \in PC[\mathbb{R},\mathbb{R}]$ are almost periodic, the function $h \in C[\mathbb{R}^3 \times M,\mathbb{R}]$ is almost periodic in t uniformly with respect to $x,\dot{x}$ and μ, $b_k^m \in \mathbb{R},\ g_k^m \in \mathbb{R}$, the sequences $\{b_k^m\}$, $\{g_k^m\}$ are almost periodic, $X_k^m \in C[\mathbb{R}^2 \times M,\mathbb{R}]$ and the sequences $\{X_k^m\},\ k = \pm1,\pm2,\ldots,\ m = 1,2$, are almost periodic uniformly with respect to $x,\dot{x}$ and μ.

Set

$$\dot{x} = y - (f(t) - a)x,$$
$$\dot{y} = \big(af(t) - a^2 - \dot{f}(t)\big)x - ay - q(t) + \mu h(t, x, \dot{x}, \mu),$$
$$z = \begin{pmatrix} x \\ y \end{pmatrix},\ A(t) = \begin{pmatrix} -f(t) + a & 1 \\ af(t) - a^2 - \dot{f}(t) & -a \end{pmatrix},\ X = \begin{pmatrix} 0 \\ h \end{pmatrix},$$

$$X_k = \begin{pmatrix} X_k^1 \\ \big(f(t_k^+) - a\big)X_k^1 + X_k^2 \end{pmatrix},$$

$$B_k = \begin{pmatrix} b_k^1 & 0 \\ \big(f(t_k^+) - a\big)b_k^1 - b_k^2(f(t_k) - a) & b_k^2 \end{pmatrix},$$

$$g_k = \begin{pmatrix} g_k^1 \\ \big(f(t_k^+) - a\big)g_k^1 \end{pmatrix},\ g(t) = \begin{pmatrix} 0 \\ -q(t) \end{pmatrix}.$$

Then, we can rewrite system (2.39) in the form

$$\begin{cases} \dot{z} = A(t)z + g(t) + \mu X(t, z, \mu),\ t \neq t_k, \\ \Delta z(t_k) = B_k z(t_k) + g_k + \mu X_k(z(t_k), \mu),\ k = \pm 1, \pm 2, \ldots. \end{cases}$$

Now, the conditions for the column dominant of the matrix $A(t)$ are

$$1 < a < \frac{1}{2}\Big(f(t) - 1 + \sqrt{\big(f(t) - 1\big)^2 + 4f(t) - 4\dot{f}(t)}\Big),$$
$$a - f(t) + \big|af(t) - a^2 - \dot{f}(t)\big| < 0,$$

i.e.

$$\begin{aligned} &\big(f(t) - 1\big)^2 < 4\dot{f}(t) < \big(f(t) + 1\big)^2, \\ &2f(t) - \dot{f}(t) - 2 > 0. \end{aligned} \tag{2.40}$$

Theorem 2.6. *Let the following conditions hold:*

1. *Condition H2.23 and the inequalities (2.40) are met.*
2. $b_k^1 b_k^2 + b_k^1 + b_k^2 + 1 \neq 0,\ k = \pm 1, \pm 2, \ldots$.
3. *The functions $h(t, x, \dot{x}, \mu)$, $X_k(x, \dot{x}, \mu)$ are Lipschitz continuous with respect to x and $\dot{x}$ uniformly for $t \in \mathbb{R}$, $k = \pm 1, \pm 2, \ldots,$ and $\mu \in M$ respectively.*

Then there exists a positive constant μ_0, $\mu_0 \in M$ such that:

1. *For any μ, $|\mu| < \mu_0$ the system (2.39) has a unique almost periodic solution.*
2. *The almost periodic solution is exponentially stable.*
3. *For $|\mu| \to 0$ the solution is convergent to the unique almost periodic solution of the system*

$$\begin{cases} \dot{z} = A(t)z + g(t), \ t \neq t_k, \\ \Delta z(t_k) = B_k z(t_k) + g_k, \ k = \pm 1, \pm 2, \dots . \end{cases}$$

Proof. The proof follows directly from Theorem 2.5. □

Now, we shall consider the following systems

$$\begin{cases} \dot{x} = f(t, x), \ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \ k = \pm 1, \pm 2, \dots, \end{cases} \tag{2.41}$$

and

$$\begin{cases} \dot{x} = f(t, x) + g(t) + \mu X(t, x, \mu), \ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)) + g_k + \mu X_k(x(t_k), \mu), \ k = \pm 1, \pm 2, \dots . \end{cases} \tag{2.42}$$

Introduce the following conditions:

H2.24. The function $f \in C[\mathbb{R} \times \Omega, \mathbb{R}^n]$ is almost periodic in t uniformly with respect to $x \in \Omega$ and it is Lipschitz continuous with respect to $x \in B_h$ with a Lipschitz constant $l_3 > 0$, such that uniformly in $t \in \mathbb{R}$

$$||f(t, x) - f(t, y)|| \leq l_3 ||x - y||, \ x, y \in B_h.$$

H2.25. The sequence of functions $\{I_k\}, \ I_k \in C[\Omega, \mathbb{R}^n], \ k = \pm 1, \pm 2, \dots$ is almost periodic uniformly with respect to $x \in \Omega$, and the functions I_k are Lipschitz continuous with respect to $x, y \in B_h$ with a Lipschitz constant $l_4 > 0$, such that

$$||I_k(x) - I_k(y)|| \leq l_4 ||x - y||,$$

where $x, y \in B_h, \ k = \pm 1, \pm 2, \dots$.

We shall suppose that for the system (2.42) there exists an almost periodic solution $\varphi(t)$, and consider the system

$$\begin{cases} \dot{x} = \dfrac{\partial f}{\partial x}(t, \varphi(t))x, \ t \neq t_k, \\ \\ \Delta x(t_k) = \dfrac{\partial I_k}{\partial x}(\varphi(t_k)), \ k = \pm 1, \pm 2, \dots . \end{cases} \tag{2.43}$$

Let

$$L_1(\delta) = \sup_{t \in \mathbb{R}, \ z \in B_\delta} ||f(t, \varphi(t) + z) - f(t, \varphi(t))||,$$

$$L_2(\delta) = \sup_{k = \pm 1, \pm 2, \dots, \ z \in B_\delta} ||I_k(\varphi(t_k) + z) - I_k(\varphi(t_k))||.$$

Theorem 2.7. *Let the following conditions hold:*

1. *Conditions H2.19–H2.25 are met.*
2. *Condition 2 of Theorem 2.5 holds.*
3. *For the Cauchy's matrix $W_1(t,s)$ of the system (2.43), conditions of Lemma 2.3.5 are met.*
4. *There exist positive constants C_0, C_1, C_2 and μ_0 such that*

$$\frac{K}{\lambda}\Big(l_3+\mu_0 l_1+\sup_{t\in\mathbb{R}}||\frac{\partial f}{\partial x}(t,\varphi(t))||\Big)+\frac{K}{1-e^{-\lambda}}\Big(l_3+\mu_0 l_2$$
$$+\sup_{k=\pm1,\pm2,\ldots}||\frac{\partial I_k}{\partial x}(\varphi(t_k))||\Big)<1,$$
$$\frac{K}{\lambda}\Big(C_1+\mu_0 L_1+\sup_{t\in\mathbb{R}}||\frac{\partial f}{\partial x}(t,\varphi(t))||\Big)$$
$$+\frac{K}{1-e^{-\lambda}}\Big(C_2+\mu_0 L_1+\sup_{k=\pm1,\pm2,\ldots}||\frac{\partial I_k}{\partial x}(\varphi(t_k))||\Big)<C_0.$$

Then there exists a positive constant $\mu_0\in M$, and for any μ, $|\mu|<\mu_0$, system (2.42) *has a unique almost periodic solution, such that*:
1. $||x(t,\mu)-\varphi(t)||\le C_0$.
2. $\lim_{|\mu|\to0} x(t,\mu)=x(t,0)$.
3. *The solution $x(t,\mu)$ is exponentially stable.*

Proof. Set $x=z+\varphi(t)$ and from (2.43), it follows the equation

$$\begin{cases}\dot z=\frac{\partial f}{\partial x}(t,\varphi(t))z+R(t,z)+\mu X(t,z+\varphi(t),\mu),\ t\ne t_k,\\ \Delta z(t_k)=\frac{\partial I_k}{\partial z}(\varphi(t_k))+R_k(z(t_k))+\mu X(z(t_k)+\varphi(t_k,\mu),\mu),\\ \qquad k=\pm1,\pm2,\ldots,\end{cases}\tag{2.44}$$

where

$$R(t,z)=f(t,\varphi(t)+z)-f(t,\varphi(t))+g(t)-\frac{\partial f}{\partial z}(t,\varphi(t))z,$$
$$R_k(z)=I_k(\varphi(t_k)+z)-I_k(\varphi(t_k))+g_k-\frac{\partial I_k}{\partial z}(\varphi(t_k)).$$

Let AP, $AP\subset PC[\mathbb{R}\times M,\mathbb{R}^n]$ is the set of all almost periodic functions $\varphi(t,\mu)$, satisfying the inequality $||\varphi||<C_0$.

Let us define in AP an operator S_μ,

$$S_\mu z=\int_{-\infty}^{t}W_1(t,s)\Big(R(t,z(s))+\mu X(s,z(s)+\varphi(s),\mu)\Big)ds$$
$$+\sum_{t_k<t}W_1(t,t_k)\Big(R_k(z(t_k))+\mu X_k(z(t_k)+\varphi(t_k))\Big).\tag{2.45}$$

From (2.45), Lemma 1.5, Theorem 1.17, Lemma 2.8 and the conditions of Theorem 2.7 it follows that the operator S_μ is contracting in AP. Hence, there exists a unique almost periodic solution $z(t,\mu)$ of system (2.44). Moreover, $x(t,\mu) = z(t,\mu) + \varphi(t)$ is an almost periodic solution of (2.42). The proof of Assertions 1–3 are analogous to the proof of Theorem 2.5. □

2.4 Perturbations in the Linear Part

In this paragraph, sufficient conditions for the existence of almost periodic solutions of differential equations with perturbations in the linear part, are obtained.

We shall consider the system of impulsive differential equations

$$\begin{cases} \dot{x} = A(t)x + f(t), \ t \neq t_k, \\ \Delta x(t_k) = A_k x(t_k) + l_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.46}$$

where $t \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$, $A_k \in \mathbb{R}^{n\times n}$, $l_k \in \mathbb{R}^n$, $k = \pm 1, \pm 2, \ldots$. By $x(t) = x(t; t_0, x_0)$ we denote the solution of (2.46) with initial condition $x(t_0^+) = x_0$, $t_0 \in \mathbb{R}$, $x_0 \in \Omega$.

Together with the system (2.46), we shall consider the following systems of impulsive differential equations with perturbations in the linear part:

$$\begin{cases} \dot{x} = \big(A(t) + B(t)\big)x + f(t), \ t \neq t_k, \\ \Delta x(t_k) = \big(A_k + B_k\big)x(t_k) + l_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.47}$$

and

$$\begin{cases} \dot{x} = \big(A(t) + B(t)\big)x + F(t,x), \ t \neq t_k, \\ \Delta x(t_k) = \big(A_k + B_k\big)x(t_k) + I_k(x(t_k)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.48}$$

where $B : \mathbb{R} \to \mathbb{R}^{n\times n}$, $F : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $B_k \in \mathbb{R}^{n\times n}$, and $I_k : \Omega \to \mathbb{R}^n$, $k = \pm 1, \pm 2, \ldots$.

Introduce the following conditions:

H2.26. The matrix function $A \in C[\mathbb{R}, \mathbb{R}^{n\times n}]$ is almost periodic in the sense of Bohr.

H2.27. $det(E + A_k) \neq 0$, where E is the identity matrix in $\mathbb{R}^n$, and the sequence $\{A_k\}$, $k = \pm 1, \pm 2, \ldots$ is almost periodic.

H2.28. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k = \pm 1, \pm 2, \ldots$, $j = \pm 1, \pm 2, \ldots$ is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.

H2.29. The function $f \in PC[\mathbb{R}, \mathbb{R}^n]$ is almost periodic.

H2.30. The sequence $\{l_k\}$, $k = \pm 1, \pm 2, \ldots$ is almost periodic.

H2.31. The matrix function $B \in C[\mathbb{R}, \mathbb{R}^{n\times n}]$ is almost periodic in the sense of Bohr.
H2.32. The sequence $\{B_k\},\ k = \pm 1, \pm 2, \ldots$ is almost periodic.

Let us denote with $W(t,s)$ the Cauchy matrix for the linear impulsive system

$$\begin{cases} \dot{x} = A(t)x,\ t \neq t_k, \\ \Delta x(t_k) = A_k x(t_k),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.49}$$

and with $Q(t,s)$ the Cauchy matrix for the linear perturbed impulsive system

$$\begin{cases} \dot{x} = \big(A(t) + B(t)\big)x,\ t \neq t_k, \\ \Delta x(t_k) = \big(A_k + B_k\big)x(t_k),\ k = \pm 1, \pm 2, \ldots . \end{cases}$$

In this part, we shall use the following lemmas:

Lemma 2.11 ([138]). *For the system (2.46) there exists only one almost periodic solution, if and only if:*

1. *Conditions H2.26–H2.30 hold.*
2. *The matrix $W(t,s)$ satisfies the inequality*

$$||W(t,s)|| \leq Ke^{-\alpha(t-s)}, \tag{2.50}$$

where $s < t,\ K \geq 1,\ \alpha > 0$.

Lemma 2.12 ([148]). *Let the following conditions hold:*

1. *Conditions H2.26–H2.28, H2.31 and H2.32 hold.*
2. *For $K \geq 1,\ \alpha > 0$ and $s < t$, it follows*

$$||W(t,s)|| \leq Ke^{-\alpha(t-s)}.$$

Then:

1. *If there exists a constant $d > 0$ such that*

$$\sup_{t\in(t_0,\infty)} ||B(t)|| < d, \qquad \sup_{t_k\in(t_0,\infty)} ||B_k|| < d,$$

then

$$||Q(t,s)|| \leq Ke^{-(\alpha - Kd)(t-s)+i(s,t)}, \tag{2.51}$$

where $s < t$.

2. *If there exists a constant* $D > 0$ *such that*

$$\int_{t_0}^{\infty} ||B(\sigma)||d\sigma + \sum_{t_0 \le t_k} ||B_k|| \le D,$$

then

$$||Q(t,s)|| \le Ke^{KD}e^{-\alpha(t-s)}, \tag{2.52}$$

where $s < t$.

The proof of the next lemma is similar to the proof of Lemma 1.7.

Lemma 2.13. *Let the conditions H2.26–H2.32 hold. Then for each* $\varepsilon > 0$ *there exist* $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon$, *a relatively dense set* $\overline{T}$ *of real numbers and a set* P *of integer numbers, such that the following relations are fulfilled:*

(a) $||A(t+\tau) - A(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}$.
(b) $||B(t+\tau) - B(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}$.
(c) $||f(t+\tau) - f(t)|| < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T}$.
(d) $||A_{k+q} - A_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots$.
(e) $||B_{k+q} - B_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots$.
(f) $||l_{k+q} - l_k|| < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots$.
(g) $|t_k^q - \tau| < \varepsilon_1,\ q \in P,\ \tau \in \overline{T},\ k = \pm 1, \pm 2, \ldots$.

Lemma 2.14 ([148]). *Let the conditions H2.31 and H2.32 hold. Then there exist positive constants* d_1, *and* d_2, *such that*

$$\sup_{t \in (t_0,\infty)} ||B(t)|| < d_1, \quad \sup_{t_k \in (t_0,\infty)} ||B_k|| < d_2.$$

Lemma 2.15. *Let the following conditions hold:*

1. *Conditions H2.26–H2.28, H2.31 and H2.32 are met.*
2. *The following inequalities hold*
   (a) $||W(t,s)|| \le Ke^{-\alpha(t-s)}$, *where* $s < t,\ K \ge 1$ *and* $\alpha > 0$,
   (b) $\nu = -\alpha - Kd - N(1+Kd) > 0$,

where $d = max(d_1, d_2)$, d_1 *and* d_2 *are from Lemma 2.14,* N *is the number of the points* t_k *lying in the interval* (s,t).

Then for each $\varepsilon > 0,\ t \in \mathbb{R},\ s \in \mathbb{R}$ *there exists a relatively dense set* T *of* ε*-almost periods, common for* $A(t)$ *and* $B(t)$ *such that for each* $\tau \in T$ *the following inequality holds*

$$||Q(t+\tau, s+\tau) - Q(t,s)|| < \varepsilon\Gamma e^{-\frac{\nu}{2}(t-s)}, \tag{2.53}$$

where $\Gamma = \dfrac{1}{\nu}2Ke^{N\ln(1+Kd)}(1 + N + \dfrac{Nd}{2})$.

Proof. Let $\overline{T}$ and P be the sets, defined in Lemma 2.13.

Then for $\tau \in \overline{T}$ and $q \in P$ the matrix $Q(t+\tau, s+\tau)$ is a solution of the system

$$\begin{cases} \dfrac{\partial Q}{\partial t} = \big(A(t) + B(t)\big)Q(t+\tau, s+\tau) \\ \quad +\big(A(t+\tau) + B(t+\tau) - A(t) - B(t)\big)Q(t+\tau, s+\tau),\ t \neq t'_k, \\ \Delta Q(t'_k) = \big(A_k + B_k\big)(Q(t'_k+\tau, s+\tau)) \\ \quad +\big(A_{k+q} + B_{k+q} - A_k - B_k\big)Q(t'_k+\tau, s+\tau), \end{cases}$$

where $k = \pm 1, \pm 2, \ldots,\ t'_k = t_k - \tau$.

Then

$$\begin{aligned} Q(t+\tau, s+\tau) - Q(t,s) = \int_s^t & Q(t,s)\big(A(\sigma+\tau) + B(\sigma+\tau) - A(\sigma) \\ & - B(\sigma)\big)Q(\sigma+\tau, s+\tau)d\sigma + \sum_{s \le t'_k < t} Q(t, t_v'^{+}) \\ & \times \big(A_{k+q} + B_{k+q} - A_k - B_k\big)Q(t'_\nu + \tau, s+\tau). \end{aligned}$$

From Lemmas 1.2 and 2.13, we have

$$\begin{aligned} ||Q(t+\tau, s+\tau) - Q(t,s)|| \le \varepsilon K e^{N\ln(1+Kd)}(e^{-\nu(t-s)}(t-s) \\ + i(s,t)e^{-\nu(t-s)}) \le \varepsilon \Gamma e^{-\frac{\nu}{2}(t-s)}. \end{aligned}$$ □

The proof of the next lemma is analogously.

Lemma 2.16. *Let the following conditions hold:*

1. *Conditions H2.26–H2.28, H2.31 and H2.32 are met.*
2. *The following inequalities hold*
   (a) $||W(t,s)|| \le Ke^{-\alpha(t-s)}$, *where* $s < t,\ K \ge 1,\ \alpha > 0$,
   (b) $\displaystyle\int_{t_0}^{\infty} ||B(\sigma)||d\sigma + \sum_{t_0 < t_k} ||B_k|| \le D,\ D > 0,$ *where* $s < t,\ D > 0$.

Then for each $\varepsilon > 0,\ t \in \mathbb{R},\ s \in \mathbb{R}$ *there exists a relatively dense set* $\overline{T}$ *of* ε*-almost periods, common for* $A(t)$ *and* $B(t)$ *such that for each* $\tau \in \overline{T}$ *the following inequality holds*

$$||Q(t+\tau, s+\tau) - Q(t,s)|| < \varepsilon \overline{\Gamma} e^{-\frac{\alpha}{2}(t-s)}, \tag{2.54}$$

where $\overline{\Gamma} = Ke^{KD}\dfrac{2}{\alpha}\Big(1 + N + \dfrac{2N}{\alpha}\Big)$.

Now, we are ready to proof the main results in this paragraph.

Theorem 2.8. *Let the following conditions hold:*

1. *Conditions H2.26–H2.32 are met.*
2. *For the system (2.46), there exists a unique almost periodic solution.*

Then there exists a constant d_0 such that for $d \in (0, d_0]$ for the system (2.47) there exists a unique almost periodic solution $\varphi(t)$, and

$$||\varphi(t)|| \leq Cmax\big(\sup_{t\in\mathbb{R}} ||f||, \sup_{k=\pm1,\pm2,\ldots} ||l_k||\big), \tag{2.55}$$

where $C > 0$.

Proof. Let the inequalities (2.50) and (2.51) hold, and let we consider the function

$$\varphi(t) = \int_{-\infty}^{t} Q(t,s)f(s)ds + \sum_{t_k<t} Q(t,t_k^+)l_k.$$

A straightforward verification yields, that $\varphi(t)$ is a solution of (2.47). □

Then, from Lemma 2.15 it follows that there exists a constant $d_0 > 0$ such that for any $d \in (0, d_0]$, we have

$$\nu = \alpha - Kd - N\ln(1 + Kd) > 0.$$

Now, we obtain

$$||\varphi(t)|| \leq \frac{K}{\nu} \sup_{t\in\mathbb{R}} ||f(t)|| + Ke^{N\ln(1+K_1d)} \sup_{k=\pm1,\pm2,\ldots} ||l_k|| \sum_{t_k<t} e^{-\nu(t-t_k)}. \tag{2.56}$$

Then, from the relations

$$\sum_{t_k<t} e^{-\nu(t-t_k)} = \sum_{k=0}^{\infty} \sum_{t-k-1<t_k<t-k} e^{-\nu(t-t_k)} \leq \frac{2N}{1-e^{-\nu}},$$

and (2.56), we obtain

$$||\varphi(t)|| \leq Cmax\Big(\sup_{t\in\mathbb{R}} ||f(t)||, \sup_{k=\pm1,\pm2,\ldots} ||l_k||\Big),$$

where $C = Ke^{N\ln(1+Kd)}\Big(\frac{1}{\nu} + \frac{2N}{1-e^{-\nu}}\Big)$.

Let $\varepsilon > 0$ be an arbitrary chosen constant. It follows from Lemma 2.13, that there exist sets $\overline{T}$ and P, such that for each $\tau \in \overline{T}$, $q \in P$, and $d \in (0, d_0]$

the following estimates hold:

$$\begin{aligned}||\varphi(t+\tau)-\varphi(t)|| \leq & \int_{-\infty}^{t} ||Q(t+\tau,\sigma+\tau)-Q(t,\sigma)|| \, ||f(\sigma+\tau)||d\sigma \\ & + \int_{-\infty}^{t} ||Q(t,\sigma)|| \, ||f(\sigma+\tau)-f(\sigma)||d\sigma \\ & + \sum_{t_k<t} ||Q(t+\tau,t_{k+q}^{+})-Q(t,t_k^{+})|| \, ||l_{k+q}|| \\ & + \sum_{t_k<t} ||Q(t,t_k^{+})|| \, ||l_{k+q}-l_k|| \leq M\varepsilon,\end{aligned}$$

where $M > 0, \ |t - t_k| > \varepsilon$.

The last inequality implies, that the function $\varphi(t)$ is almost periodic.

The uniqueness of this solution follows from the fact that the homogeneous part of system (2.47) has only the zero bounded solution under conditions H2.26, H2.27, H2.31 and H2.32, and from the estimate (2.50). □

Theorem 2.9. *Let the following conditions hold:*

1. *Conditions H2.26–H2.32 are met.*
2. *For the system (2.46), there exists a unique almost periodic solution.*
3. *There exists a constant $D_0 > 0$, such that*

$$\int_{t_0}^{\infty} ||B(\sigma)||d\sigma + \sum_{t_0<t_k} ||B_k|| < D_0.$$

Then, for $D \in (0, D_0]$ for the system (2.47), *there exists a unique almost periodic solution $\varphi(t)$ such that*

$$||\varphi(t)|| \leq Cmax\Big(\sup_{t\in\mathbb{R}} ||f||, \sup_{k=\pm1,\pm2,\dots} ||l_k||\Big),$$

where $C > 0$.

Proof. Using Lemma 2.16 and (2.52), the proof of Theorem 2.9 is carried out in the same way as the proof of Theorem 2.8. □

Theorem 2.10. *Let the following conditions hold:*

1. *Conditions H2.26–H2.30 are met.*
2. *For the system (2.46), there exists a unique almost periodic solution.*
3. *$B(t) = B, \ B_k = \Lambda$, where B and Λ are constant matrices such that*

$$||B|| + ||\Lambda|| \leq d_1, \ d_1 > 0.$$

Then there exists a constant $d_0 > 0, \ d_0 \leq d_1$, such that for $d \in (0, d_0]$ for the system (2.47), *there exists a unique almost periodic solution.*

Proof. The proof of Theorem 2.10 is carried out in the same way as the proof of Theorem 2.8. □

Example 2.2. We shall consider the systems

$$\begin{cases} \dot{x} = -x + f(t), \ t \neq t_k, \\ \Delta x(t_k) = l_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.57}$$

and

$$\begin{cases} \dot{x} = \big(b(t) - 1\big)x + f(t), \ t \neq t_k, \\ \\ \Delta x(t_k) = l_k + g_k, \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.58}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, the function $b \in C[\mathbb{R}, \mathbb{R}]$ is almost periodic in the sense of Bohr, the function $f \in PC[\mathbb{R}, \mathbb{R}]$ is almost periodic, $b_k \in \mathbb{R}$, $l_k \in \mathbb{R}$ and $\{b_k\}$, $\{l_k\}$, $k = \pm 1, \pm 2, \ldots$, are almost periodic sequences.

Let condition H2.28 holds. From [138] it follows that for the system (2.57) there exists a unique almost periodic solution.

Then, the conditions of Theorem 2.8. are fulfilled, and hence, there exists a constant d_0 such that for any $d \in (0, d_0]$ for the system (2.58), there exists a unique almost periodic solution in the form

$$x(t) = \int_{-\infty}^{t} Q(t, \sigma) f(\sigma) d\sigma + \sum_{t_k < t} Q(t, t_k^+) l_k,$$

where

$$Q(t, s) = \prod_{s \leq t_k < t} (1 + b_k) exp\Big\{ \int_s^t b(\sigma) d\sigma - (t - s) \Big\}.$$

Now, we shall investigate the existence of almost periodic solutions for the system (2.48).

Introduce the following conditions:

H2.33. The function $F \in C[\mathbb{R} \times \Omega, \mathbb{R}^n]$ is almost periodic in t uniformly with respect to $x \in \Omega$, and it is Lipschitz continuous with respect to $x \in B_h$ with a Lipschitz constant $L > 0$,

$$||F(t, x) - F(t, y)|| \leq L||x - y||, \ x, y \in B_h, \ t \in \mathbb{R}.$$

H2.34. The sequence of functions $\{I_k(x)\}$, $I_k \in C[\Omega, \mathbb{R}^n]$ is almost periodic uniformly with respect to $x \in \Omega$, and the functions $I_k(x)$ are Lipschitz continuous with respect to $x \in B_h$ with a Lipschitz constant $L > 0$,

$$||I_k(x) - I_k(y)|| \leq L||x - y||, \ x, y \in B_h, \ k = \pm 1, \pm 2, \ldots.$$

Theorem 2.11. *Let the following conditions hold:*

1. Conditions H2.26–H2.28, H2.31–H2.34 are met.

2. *For the functions* $F(t,x)$ *and* $I_k(x)$, $k = \pm1, \pm2, \dots$, *there exists a constant* $L_1 > 0$ *such that*

$$max\Big(\sup_{t\in\mathbb{R}, x\in B_h} ||F(t,x)||, \sup_{k=\pm1,\pm2,\dots,\ x\in B_h} ||I_k(x))||\Big) \le L_1.$$

3. *The inequalities (2.50) and*

$$CL_1 < h, \quad CL < 1. \tag{2.59}$$

hold.

Then there exists a constant $d_0 > 0$ *such that for any* $d \in (0, d_0]$, *for the system* (2.48) *there exists a unique almost periodic solution.*

Proof. Let we denote by AP the set of all almost periodic solutions $\varphi(t)$, $\varphi \in PC[\mathbb{R}, \mathbb{R}^n]$, satisfy the inequality $||\varphi|| < h$, and let $|\varphi|_\infty = \sup\limits_{t\in\mathbb{R}} ||\varphi(t)||$.

We define in AP the operator S, such that if $\varphi \in AP$, then $y = S\varphi(t)$ is the almost periodic solution of the system

$$\begin{cases} \dot{y} = \big(A(t) + B(t)\big)y + F(t, \varphi(t)),\ t \ne t_k, \\ \Delta y(t_k) = \big(A_k + B_k\big)y(t_k) + I_k(\varphi(t_k)),\ k = \pm1, \pm2, \dots, \end{cases}$$

determined by Theorem 2.8.

We shall note that the almost periodicity of the sequence $\{\varphi(t_k)\}$, the function $F(t, \varphi(t))$ and the sequence $\{I_k(\varphi(t_k))\}$ follows from Lemma 1.5 and Theorem 1.17.

On the other hand, there exists a positive constant $d_0 > 0$ such that for any $d \in (0, d_0]$,

$$\alpha - Kd - N\ln(1 + Kd) > 0.$$

From the last inequality and (2.59), it follows that (2.51) and conditions of Lemma 2.15 hold.

Then $S(AP) \subset AP$.

If $\varphi \in AP$, $\psi \in AP$, then from (2.51) and condition 2 of Theorem 2.11, we get

$$||S\varphi(t) - S\psi(t)|| \le CL|\varphi - \psi|_\infty. \tag{2.60}$$

Finally, from (2.59) and (2.60,) it follows that S is contracting in AP, i.e. there exists a unique almost periodic solution of system (2.48). □

2.5 Strong Stable Impulsive Differential Equations

In this section, conditions for strong stability and almost periodicity of solutions of impulsive differential equations with impulsive effect at fixed moments will be proved. The investigations are carried out by means of piecewise continuous Lyapunov functions.

We shall consider the system of impulsive differential equations

$$\begin{cases} \dot{x} = f(t, x), \ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.61}$$

where $t \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $I_k : \Omega \to \mathbb{R}^n, \ k = \pm 1, \pm 2, \ldots$.

Set

$$\rho(x, y) = ||x - y||, \ x, y \in \mathbb{R}^n,$$

$$B_h(a) = \{x \in \mathbb{R}^n, \ ||x - a|| < h\}, \ h > 0, \ a \in \mathbb{R}^n,$$

$$\Psi_h = \{(t, x) \in \mathbb{R} \times B_h, \ x \in B_h, \text{ if } (t, x) \in G \text{ and } x + I_k(x) \in B_h, \\ \text{if } t = t_k\},$$

where G is the set from Sect. 1.1.

Introduce the following conditions:

H2.35. The function $f \in C[\mathbb{R} \times B_h, \mathbb{R}^n]$, and has continuous partial derivatives of the first order with respect to all components of $x \in B_h$.

H2.36. The functions $I_k \in C[B_h, \mathbb{R}^n], \ k = \pm 1, \pm 2, \ldots$ and have continuous partial derivatives of the first order with respect to all components of $x \in B_h$.

H2.37. There exists h_0, $0 < h_0 < h$ such that if $x \in B_{h_0}$, then $x + I_k(x) \in B_h, \ k = \pm 1, \pm 2, \ldots$.

H2.38. The functions $L_k(x) = x + I_k(x), \ k = \pm 1, \pm 2, \ldots$ are such that $L_k^{-1}(x) \in B_h$ for $x \in B_h$.

From [138] if the conditions H2.35–H2.38 are satisfied, then for each point $(t_0, x_0) \in \mathbb{R} \times B_h$, there exists a unique solution $\overline{x}(t) = \overline{x}(t; t_0, x_0)$ of system (2.61), which satisfies the initial condition $x(t_0^+) = x_0$.

We need the following condition in our subsequent analysis:

H2.39. $f(t, 0) = 0$, $I_k(0) = 0$ for $t \in \mathbb{R}$ and $k = \pm 1, \pm 2, \ldots$, respectively.

If the conditions H2.35–H2.39 hold, then there exists a zero solution for system (2.61).

Definition 2.4 ([90]). The zero solution $x(t) \equiv 0$ of system (2.61) is said to be *strongly stable*, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R})(\forall x_0 \in B_\delta \;:\; (t_0, x_0) \in \Psi_\delta)$$
$$(\forall t \in \mathbb{R}) : \rho(\overline{x}(t; t_0, x_0), 0) < \varepsilon.$$

Definition 2.5 ([90]). An arbitrary solution $\overline{x}(t) = \overline{x}(t; t_0, x_0)$ of (2.61) is said to be *strongly stable*, if

$$(\forall \varepsilon > 0)(\ \forall \eta > 0)(\exists \delta > 0)(\forall \tau_1 \in \mathbb{R},\ \forall \tau_2 \in \mathbb{R},\ \rho(\overline{x}(\tau_1), \overline{x}(\tau_2)) < \delta)$$
$$(\forall t \in \mathbb{R}) : \rho(\overline{x}(t + \tau_1), \overline{x}(t + \tau_2)) < \varepsilon.$$

Definition 2.6. The function $V \in V_0$ belongs to the class V_0^*, if V has continuous partial derivatives on the sets G_k.

For each function $V \in V_0^*$, we define the function

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x)}{\partial x_i} f_i(t, x)$$

for $(t, x) \in G$.

If $\overline{x}(t)$ is a solution of system (2.61), then

$$\frac{d}{dt} V(t, \overline{x}(t)) = \dot{V}(t, \overline{x}(t)),\ t \in \mathbb{R},\ t \neq t_k.$$

Definition 2.7. The function $V \in V_0$ belongs to the class V_0^{**}, if V has continuous partial derivatives of the second order in the sets G_k.

Let $V \in V_0^{**}$. If the function $f(t, x)$ satisfies condition H2.35 and has a continuous partial derivative with respect to t, we can define the function

$$\ddot{V}(t, x) = \frac{\partial \dot{V}(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \dot{V}(t, x)}{\partial x_i} f_i(t, x)$$

for $(t, x) \in G$.

In the further considerations, we shall use the next class K of functions

$$K = \big\{a \in C[\mathbb{R}, \mathbb{R}^+],\ \text{a is strictly increasing and } a(0) = 0\big\}.$$

Introduce the following conditions:

H2.40. The function $f(t, x)$ is almost periodic in t uniformly with respect to $x,\ x \in B_h$.

H2.41. The sequence $\{I_k(x)\},\ k = \pm 1, \pm 2, \ldots$, is almost periodic uniformly with respect to $x,\ x \in B_h$.

H2.42. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k = \pm 1, \pm 2, \ldots$, $j = \pm 1, \pm 2, \ldots$, is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.

Definition 2.8 ([114]). The set S, $S \subset \mathbb{R}$ is said to be:

(a) $\Delta - m$ set, if from every $m+1$ real numbers $\tau_1, \tau_2, \ldots, \tau_{m+1}$ one can find $i \neq j$, such that $\tau_i - \tau_j \in S$.
(b) *symmetric* $\Delta - m$ set, if S is $\Delta - m$ set symmetric with respect to the number 0.

Lemma 2.17 ([114]). *Every symmetric $\Delta - m$ set is relatively dense.*

Theorem 2.12. *Let conditions H2.35–H2.42 hold. Then any strongly stable bounded solution of (2.61) is almost periodic.*

Proof. Let $\overline{x} = \overline{x}(t; t_0, x_0)$ be a unique bounded solution of system (2.61) with initial condition $\overline{x}(t_0) = x_0$. Let $\varepsilon > 0$ be given, $\delta(\varepsilon) > 0$, and the points $a_1,\ a_2, \ldots, a_{N+1}$, $a_l \in \mathbb{R}^n$, $l = 1, 2, \ldots, N+1$, are such that for $t \in \mathbb{R},\ t \geq t_0$, it follows that $\overline{x}(t) \in B_{\frac{\delta}{2}}(a_l)$. If $t_0, \ldots, t_{N+1}$ are given real numbers, then for some $i \neq j$ and some $l \in \{1, \ldots, N+1\}$, we get

$$\rho(\overline{x}(\tau_i), a_l) < \frac{\delta(\varepsilon)}{2},\ \rho(\overline{x}(\tau_j), a_l) < \frac{\delta(\varepsilon)}{2}.$$

Consequently, $\rho(\overline{x}(t_i), \overline{x}(t_j)) < \delta(\varepsilon)$.

On the other hand, the solution $\overline{x}(t)$ is strongly stable, i.e. it follows that $\rho(\overline{x}(t + \tau_i), \overline{x}(t + \tau_j)) < \varepsilon$, where $t \in \mathbb{R}$.

Then, for $t \in \mathbb{R}$ we have $\rho(\overline{x}(t + \tau_i - \tau_j), \overline{x}(t)) < \varepsilon$ and consequently, $\tau_i - \tau_j$ is an ε-almost period of the solution $\overline{x}(t)$.

Let $\overline{T}$ be the set of all ε-almost periods of $x(t)$. Then, for any sequence of numbers $\tau_0, \ldots, \tau_N$ from above, it follows that there exists $i \neq j$, such that $\tau_i - \tau_j \in \overline{T}$.

From Definition 2.8, we get that $\overline{T}$ is a symmetric $\Delta - N$ set, and from Lemma 2.17, it follows that $\overline{T}$ is a relatively dense set. Then, $\overline{x}(t)$ is an almost periodic function. □

Let $\overline{x}(t)$ be a solution of the system (2.61). Set $z = x - \overline{x}(t)$, and consider the system

$$\begin{cases} \dot{z} = g(t, z),\ t \neq t_k, \\ \Delta z(t_k) = J_k(z(t_k)),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.62}$$

where $g(t, z) = f(t, z + \overline{x}(t)) - f(t, \overline{x}(t)),\ J_k(z) = I_k(z + \overline{x}) - I_k(\overline{x})$.

Theorem 2.13. *Let the following conditions hold:*

1. *Conditions H2.35–H2.42 are met.*
2. *There exist functions $V \in V_0^*$ and $a,\ b \in K$ such that:*

(a) $a(||z||) \leq V(t,z) \leq b(||z||),\ (t,z) \in \mathbb{R} \times B_h.$
(b) $\dot{V}(t,z) \equiv 0,\ for\ (t,z) \in \mathbb{R} \times B_h,\ t \neq t_k.$
(c) $V(t_k^+, z + I_k(z)) = V(t_k, z),\ k = \pm 1, \pm 2, \ldots,\ z \in B_h.$

Then the solution $\overline{x}(t)$ *of* (2.61) *is almost periodic.*

Proof. Let $0 < \varepsilon < h,\ 0 < \mu < h$ be given, and let

$$\delta = \delta(\varepsilon) < min\big\{\varepsilon,\ b^{-1}(a(\varepsilon)),\ b^{-1}(a(\mu))\big\},$$

where $a, b \in \mathcal{K}$. If $z(t) = z(t;t_0,z_0)$ be a solution of (2.62) such that $t_0 \in \mathbb{R},\ (t_0,x_0) \in S_\delta$, then from condition 2 of Theorem 2.13, it follows that

$$a(||z||) \leq V(t,z(t)) = V(t_0^+, z_0) \leq b(||z_0||) < b(\delta(\epsilon)) < min\big\{a(\varepsilon), a(\mu)\big\}.$$

Consequently, $||z(t;t_0,z_0)|| < min(\varepsilon,\mu)$ for $t \in \mathbb{R}$, i.e. the zero solution of (2.62) is strongly stable. Then, $\overline{x}(t)$ is strongly stable, and from conditions H2.40–H2.42, and Theorem 2.12, it follows that $\overline{x}(t)$ is almost periodic. □

Definition 2.9 ([90]). The zero solution of system (2.62) is said to be *uniformly stable to the right (to the left)*, if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that if $t_0 \in \mathbb{R}$ and $(t_0,z_0) \in \mathbb{R} \times B_{\delta(\varepsilon)}$, then $||z(t;t_0,z_0)|| < \varepsilon$ for all $t \geq t_0$ (for all $t \leq t_0$), where $z(t;t_0,z_0)$ is a solution of (2.62) such that $z(t_0^+) = z_0$.

Lemma 2.18 ([90]). *The zero solution of system (2.62) is uniformly stable to the left if and only if for any* $\varepsilon > 0$ *the following inequality holds:*

$$\gamma(\varepsilon) = inf\big\{||z(t;t_0,z_0)||\ :\ t_0 \in \mathbb{R},\ ||z_0|| \geq \varepsilon\big\} > 0.$$

Lemma 2.19 ([90]). *The zero solution of system (2.62) is strongly stable if and only if it is stable to the left and to the right at the same time.*

Example 2.3. We shall consider the linear impulsive system of differential equations

$$\begin{cases} \dot{x} = A(t)x,\ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k),\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.63}$$

where $A(t)$ is a square matrix, the elements of which are almost periodic continuous functions for $t \in \mathbb{R}$, $\{B_k\}$ is an almost periodic sequence of constant matrices such that $det(E + B_k) \neq 0$, and for the points t_k the condition H2.42 is fulfilled. Let $W(t,s)$ be the Cauchy matrix of system (2.63).

Since the nontrivial solution of (2.63) is given by the formula $x(t;t_0,x_0) = W(t,t_0)x_0$, then $x_0 = W^{-1}(t,t_0)x(t;t_0,x_0)$. Hence, for any $\varepsilon > 0$ and $||x_0\| \geq \varepsilon$, we have

$$\varepsilon \leq ||x_0|| \leq ||W^{-1}(t,t_0)||||x(t;t_0,x_0)||,$$

and

$$||x(t;t_0,x_0)|| \geq \varepsilon ||W^{-1}(t,t_0)||^{-1}.$$

However, for $t=t_0$ and $||x_0||=\varepsilon$, we have

$$||x(t;t_0,x_0)|| = \varepsilon ||W^{-1}(t,t_0)||^{-1}.$$

Hence,

$$\gamma(\varepsilon) = inf\Big\{\varepsilon||W^{-1}(t,t_0)||^{-1} \ : \ t \geq t_0\Big\} > 0$$

and, applying Lemma 2.18, we conclude that the zero solution of system (2.63) is uniformly stable to the left if and only if the function $||W^{-1}(t,s)||$ is bounded on the set $s \leq t < \infty$. Moreover, it is clear that the zero solution of (2.63) is uniformly stable to the right if and only if the function $||W(t,s)||$ is bounded on the set $s \leq t < \infty$. Then, by virtue of Lemma 2.18, the zero solution of system (2.63) is strongly stable if and only if the functions $||W(t,t_0)||$ and $||W^{-1}(t,t_0)||$ are bounded for $t \in \mathbb{R}$. Consequently, an arbitrary solution $x(t)$ of the system (2.63) is bounded and strongly stable. From Theorem 2.7, it follows that the solution $z(t)$ is almost periodic.

Now, we consider the following scalar impulsive differential equations:

$$\begin{cases} \dot{u} = \omega_1(t,u), \ t \neq t_k, \\ \Delta u(t_k) = P_k(u(t_k)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.64}$$

where $\omega_1 : [t_0 - T, t_0] \times \chi \to \mathbb{R}, \ \chi$ is an open interval in $\mathbb{R}$, and t_0 and T are constants such that $t_0 > T, \ P_k : \chi \to \chi$;

$$\begin{cases} \dot{v} = \omega_2(t,v), \ t \neq t_k, \\ \Delta v(t_k) = P_k(v(t_k)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.65}$$

where $\omega_2 : [t_0, t_0 + T] \times \chi \to \mathbb{R}$;

$$\begin{cases} \ddot{u} = \omega(t,u,\dot{u}), \ t \neq t_k, \\ \Delta u(t_k) = A_k(u(t_k)), \ k = \pm 1, \pm 2, \ldots, \\ \Delta \dot{u}(t_k) = B_k(u(t_k), \dot{u}(t_k)), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.66}$$

where $\omega : [t_0 - T, t_0 + T] \times \chi_1 \times \chi_2 \to \mathbb{R}$, $A_k : \chi_2 \to \chi_1$, $B_k : \chi_1 \times \chi_2 \to \chi_2$, χ_1 and χ_2 are open intervals in $\mathbb{R}$.

Theorem 2.14. *Let the following conditions hold:*

1. *Conditions H2.35–H2.42 are met.*
2. *The zero solution $u(t) \equiv 0$, $(v(t) \equiv 0)$ of (2.64), (2.65) is uniformly stable to the left (to the right).*
3. *The functions $u + P_k(u)$, $k = \pm 1, \pm 2, \ldots$, are monotone increasing in $\mathbb{R} \times B_h$.*

4. *There exist functions* $V \in V_0^*$ *and* $a, b \in K$ *such that*
   (a) $a(||z||) \leq V(t,z) \leq b(||z||),\ (t,z) \in \mathbb{R} \times B_h$.
   (b) $\omega_1(t, V(t,z)) \leq \dot{V}(t,z) \leq \omega_2(t, V(t,z))\ (t,z) \in \mathbb{R} \times B_h$.
   (c) $V(t_k^+, z + J_k(z)) = V(t_k, z) + P_k(V(t_k, z)),\ k = \pm 1, \pm 2, \ldots$.
5. *The solution* $\overline{x}(t)$ *of system (2.61) is bounded.*

Then the solution $\overline{x}(t)$ *of system* (2.61) *is almost periodic.*

Proof. From conditions of the theorem and [90], it follows that the zero solution of system (2.61) is strongly stable, i.e. the solution $\overline{x}(t)$ is strongly stable. Then, from H2.40–H2.42 and Theorem 2.12, it follows that $\overline{x}(t)$ is almost periodic. □

Definition 2.10 ([90]). The zero solution $x(t) \equiv 0$ of (2.66) is said to be *u-strongly stable*, if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R})(\forall u_0 : 0 \leq u_0 < \delta(\varepsilon))(\forall \dot{u}_0 \in \mathbb{R} : |\dot{u}_0| < \delta(\varepsilon))$$
$$(\forall t \in \mathbb{R}) : 0 \leq u(t; t_0, u_0, \dot{u}_0) < \varepsilon.$$

Theorem 2.15. *Let the following conditions hold:*

1. *Conditions H2.35–H2.42 are met.*
2. *The function* $g(t,x)$ *has continuous partial derivative of the first kind with respect to* t.
3. *There exist functions* $V \in V_0^{**}$ *and* $a, b \in K$, *such that*
   (a) $a(||z||) \leq V(t,z) \leq b(||z||),\ (t,z) \in \mathbb{R} \times B_h$.
   (b) $\dot{V}(t,z) \leq c\|z\|,\ c = const > 0,\ (t,z) \in G$.
   (c) $\ddot{V}(t,z) \leq \omega(t, V(t,z), \dot{V}(t,z))$ *for* $(t,z) \in \mathbb{R} \times B_h,\ t \neq t_k$, *where* $\omega(t, u_1, u_2),\ \omega : \mathbb{R}^3 \to \mathbb{R}^+$ *is continuous and monotone increasing on* u_1 *and* $\omega(t,0,0) = 0$ *for* $t \in \mathbb{R}$.
   (d) $V(t_k^+, z + J_k(z)) \leq V(t_k, z) + A_k(\dot{V}(t_k, z))$.
   (e) $\dot{V}(t_k^+, z + J_k(z)) \leq \dot{V}(t_k, z) + B_k(V(t_k, z), \dot{V}(t_k, z)), k = \pm 1, \pm 2, \ldots, z \in B_h$.
4. *The following inequalities hold*

$$u_1 + A_k(v_1) \leq u_2 + A_k(v_2),$$
$$v_1 + B_k(u_1, v_1) \leq v_2 + B_k(u_2, v_2)$$

for $u_1 \leq u_2,\ v_1 \leq v_2$, *where* $u_1, u_2 \in \chi_1,\ v_1, v_2 \in \chi_2,\ k = \pm 1, \pm 2, \ldots$.

5. *The zero solution of equation (2.66) is strongly u-stable.*
6. *The solution $\overline{x}(t)$ of system (2.61) is bounded.*

Then the solution $\overline{x}(t)$ of system (2.61) *is almost periodic.*

Proof. The proof of Theorem 2.15 is analogous to the proof of Theorem 2.14. □

2.6 Dichotomies and Almost Periodicity

In this part, the existence of an almost periodic projector-valued function of dichotomous impulsive differential systems with impulsive effects at fixed moments is considered.

First, we shall consider the linear system of impulsive differential equations

$$\begin{cases} \dot{x} = A(t)x, \ t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.67}$$

where $t \in \mathbb{R}$, $\{t_k\} \in \mathcal{B}$, $A : \mathbb{R} \to \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times n}$, $k = \pm 1, \pm 2, \ldots$.

By $x(t) = x(t; t_0, x_0)$ we denote the solution of (2.67) with initial condition $x(t_0^+) = x_0, \ x_0 \in \mathbb{R}^n$.

Introduce the following conditions:

H2.43. The matrix-valued function $A \in PC[\mathbb{R}, \mathbb{R}^{n \times n}]$ is almost periodic.
H2.44. $\{B_k\}, \ k = \pm 1, \pm 2, \ldots$ is an almost periodic sequence.
H2.45. $det(E + B_k) \neq 0, \ k = \pm 1, \pm 2, \ldots$ where E is the identity matrix in $\mathbb{R}^n$.
H2.46. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k, \ k = \pm 1, \pm 2, \ldots, \ j = \pm 1, \pm 2, \ldots$ is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.

Let $W(t, s)$ be the Cauchy matrix of system (2.67). From conditions H2.43–H6.46, it follows that the solutions $x(t)$ are written down in the form

$$x(t; t_0, x_0) = W(t, t_0)x_0.$$

It is easy to verify, that the equalities $W(t, t) = E$ and $W(t, t_0) = X(t)X^{-1}(t_0)$ are valid, $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is some non degenerate matrix solution of (2.67).

Definition 2.11. The linear system (2.67) is said to has an *exponential dichotomy in* $\mathbb{R}$, if there exist a projector P and positive constants K, L, α, β such that

$$\begin{aligned} &||X(t)PX^{-1}(s)|| \leq K \ e^{-\alpha(t-s)}, \ t \geq s, \\ &||X(t)(E - P)X^{-1}(s)|| \leq L \ e^{-\beta(t-s)}, \ s \geq t. \end{aligned} \tag{2.68}$$

Lemma 2.20. *Let the system (2.67) has an exponential dichotomy in* $\mathbb{R}$. *Then any other fundamental matrix of the form $X(t)C$ satisfies inequalities*

(2.68) with the same projector P if and only if the constant matrix C commutes with P.

Proof. The proof of this lemma does not use the particular form of the matrix $X(t)$, and is analogous to the proof of a similar lemma in [46]. $\square$

Definition 2.12. The functions $f \in PC[\mathbb{R}, \Omega]$, $g \in PC[\mathbb{R}, \Omega]$ are said to be *ε-equivalent*, and denoted $f \overset{\varepsilon}{\sim} g$, if the following conditions hold:

(a) The points of possible discontinuity of these functions can be enumerated t_k^f, t_k^g, admitting a finite multiplicity by the order in $\mathbb{R}$, so that $|t_k^f - t_k^g| < \varepsilon$.
(b) There exist strictly increasing sequences of numbers $\{t'_k\}$, $\{t''_k\}$, $t'_k < t'_{k+1}$, $t''_k < t''_{k+1}$, $k = \pm 1, \pm 2, \ldots$, for which we have

$$\sup_{t \in (t'_k, t'_{k+1}),\ t' \in (t''_k, t''_{k+1})} ||f(t) - g(t)|| < \varepsilon,\ |t'_k - t''_k| < \varepsilon,\ k = \pm 1, \pm 2, \ldots .$$

By $\rho(f, g) = inf \varepsilon$ we denote the distance between functions $f \in PC[\mathbb{R}, \Omega]$ and $g \in PC[\mathbb{R}, \Omega]$, and by $PC\varphi$ the set of all functions $\varphi \in PC[\mathbb{R}, \Omega]$, for which $\rho(f, \varphi)$ is a finite number. It is easy to verify, that $PC\varphi$ is a metric space.

Definition 2.13 ([9]). The function $\varphi \in PC[\mathbb{R}, \Omega]$ is said to be *almost periodic*, if for any ε the set

$$\overline{T} = \{\tau :\ \rho(\varphi(t + \tau), \varphi(t)) < \varepsilon,\ t, \tau \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$.

By $D = \{M_i\}, i \in I$, we denote the family of countable sets of real numbers unbounded below an above and not having limit points, where I is a countable index set. Let M_1 and M_2 be sets of D.

Lemma 2.21 ([9]). *The function $\varphi \in PC[\mathbb{R}, \Omega]$ is almost periodic if and only if for an arbitrary sequence $\{s_n\}$ the sequence $\{\varphi(t + s_n)\}$ is compact in $PC\varphi$.*

Definition 2.14. The sets M_1 and M_2 are said to be *ε–equivalent*, if their elements can be renumbered by integers m_k^1, m_k^2, admitting a finite multiplicity by their order in $\mathbb{R}$, so that

$$\sup_{k = \pm 1, \pm 2, \ldots} |m_k^1 - m_k^2| < \varepsilon.$$

Definition 2.15. The number $\rho_D(M_1, M_2) = \inf\limits_{M_1 \overset{\varepsilon}{\sim} M_2} \varepsilon$ is said to be a *distance* in D.

Throughout the rest of this paragraph, the following notation will be used:

Let conditions H2.43–H2.46 hold and let $\{s'_m\}$ be an arbitrary sequence of real numbers. Analogously to the process from Chap. 1, it follows that there exists a subsequence $\{s_n\},\ s_n = s'_m$ such that the system (2.67) moves to the system

$$\begin{cases} \dot{x} = A^s(t)x,\ t \neq t_k^s, \\ \Delta x(t_k^s) = B_k^s x(t_k^s),\ k = \pm 1, \pm 2, \dots . \end{cases} \tag{2.69}$$

The systems of the form (2.69), we shall denote by E^s, and in this meaning we shall denote (2.67) by E_0. From [127], it follows that, each sequence of shifts E^{s_n} of system E_0 is compact, and let denote by $H(A, B_k, t_k)$ the set of shifts of E_0 for an arbitrary sequence $\{s_n\}$.

Now, we shall consider the following scalar impulsive differential equation

$$\begin{cases} \dot{v} = p(t)v,\ t \neq t_k, \\ \Delta v(t_k) = b_k v(t_k),\ k = \pm 1, \pm 2, \dots, \end{cases} \tag{2.70}$$

where $p \in PC[\mathbb{R}, \mathbb{R}]$, $b_k \in \mathbb{R}$.

Lemma 2.22. *Let the following conditions hold:*

1. *Condition H2.46 holds.*
2. *The function $p(t)$ is almost periodic.*
3. *The sequence b_k is almost periodic.*
4. *The function $v(t)$ is a nontrivial almost periodic solution of (2.70).*

Then $\inf_{t\in\mathbb{R}} |v(t)| > 0$ and the function $1/v(t)$ is almost periodic.

Proof. Suppose that $\inf_{t\in\mathbb{R}} |v(t)| = 0$. Then, there exists a sequence $\{s'_m\}$ of real numbers such that $\lim_{n\to\infty} v(s_n) = 0$. From the almost periodicity of $p(t)$ and $v(t)$ it follows that, the sequences of shifts $p(t+s_n)$ and $v(t+s_n)$ are compact in the sets PC_p and PC_v, respectively. Hence, from Ascoli's diagonal process, it follows that there exists a subsequence $\{s_{n_k}\}$, common for $p(t)$ and $v(t)$ such that the limits

$$\lim_{k\to\infty} p(t + s_{n_k}) = p^s(t),$$

and

$$\lim_{k\to\infty} v(t + s_{n_k}) = v^s(t)$$

exist uniformly for $t \in \mathbb{R}$. Analogously, it is proved that for the sequences of shifts $\{t_k + n_k\}$ and $\{b_k + n_k\}$ there exists a subsequence of $\{n_k\}$, for which there exist the limits $\{t_k^s\}$ and $\{b_k^s\}$. Consequently, for the system

$$\begin{cases} \dot{v}^s = p^s(t)v^s,\ t \neq t_k^s, \\ \Delta v^s(t_k^s) = b_k^s v^s(t_k^s), \quad k = \pm 1, \pm 2, \dots, \end{cases}$$

with initial condition $v^s(0) = 0$ it follows that there exists only the trivial solution.

Then,

$$v(t) = \lim_{k\to\infty} v^{\alpha}(t - s_{n_k}) = 0$$

for all $t \in \mathbb{R}$, which contradicts the conditions of Theorem 1.20. Hence, $\inf_{t\in\mathbb{R}} |v(t)| > 0$, and from Lemma 2.21 it follows that $1/v(t)$ is an almost periodic solution. □

Theorem 2.16. *Let the following conditions hold:*

1. *Conditions H2.43–H2.46 are met.*
2. *The fundamental matrix $X(t)$, $X \in PC[\mathbb{R}, \mathbb{R}^n]$ is almost periodic.*

Then $X^{-1}(t)$ is an almost periodic matrix-valued function.

Proof. From the representation of $W(t,s)$ in Sect. 1.1, we have that $X(t) = W(t,t_0)X(t_0)$, hence

$$\begin{aligned} X^{-1}(t) &= X^{-1}(t_0)W^{-1}(t,t_0) \\ &= X^{-1}(t_0)\Big(\det W(t,t_0)\Big)^{-1}\Big(adj\ W(t,t_0)\Big)^{T}, \end{aligned}$$

where by *adj* $W(t,t_0)$ we denote the matrix of cofactors of matrix $W(t,t_0)$.

Then, $X(t)$ will be almost periodic when the following function

$$\Big(v(t)\Big)^{-1} = \Big(\det W(t,t_0)\Big)^{-1}$$

is almost periodic.

From

$$\det W(t,t_0) = \begin{cases} \prod\limits_{t_0\le t_k<t} \det(E+B_k)exp\Big(\displaystyle\int_{t_0}^{t} Tr\ A(s)ds\Big),\ t>t_0, \\ \prod\limits_{t\le t_k<t_0} \det(E+B_k)exp\Big(\displaystyle\int_{t_0}^{t} Tr\ A(s)ds\Big),\ t\le t_0, \end{cases}$$

where $TrA(t)$ is the trace of the matrix A, and a straightforward verification, it follows that the function $v(t) = \det W(t,t_0)$ is a nontrivial almost periodic solution of the system

$$\begin{cases} \dot{v} = Tr\ A(t)v,\ t \ne t_k, \\ \Delta v(t_k) = b_k v(t_k),\ k = \pm1, \pm2, \ldots. \end{cases}$$

Then, from Lemma 2.22 it follows that $1/v(t)$ is an almost periodic function. □

Theorem 2.17. *Let the following conditions hold:*

1. *Conditions H2.43–H2.46 are met.*
2. *The fundamental matrix $X(t)$ satisfies inequalities (2.68).*

Then the fundamental matrix $X^s(t)$ of system (2.70) also satisfies inequalities (2.68).

Proof. Let we denote by H the square root of the positively definite Hermite matrix

$$H^2 = PX^* XP + (E-P)X^* X(E-P).$$

Since P commutes with H^2, then P commutes with H and H^{-1}.

The matrix $X(t)$ is continuously differentiable for $t \neq t_k$ and with points of discontinuity at the first kind at $t = t_k$. Hence, the matrices $H,\ XH^{-1},\ HX^{-1}$ enjoy the properties of $X(t)$, and let $\{s_n\}$ be an arbitrary sequence of real numbers. By a straightforward verification we establish that the matrix $X_n = x(t+s_n)H^{-1}(s_n)$ is a fundamental matrix of system (2.69).

On the other hand, the matrix $H^{-1}(s_n)$ commutes with P, consequently, from Lemma 2.20 it follows that the matrix $X_n(t)$ satisfies inequalities (2.68).

Hence, the matrices $X_n(0),\ X_n^{-1}(0)$ are bounded, and then there exists a subsequence, common for both matrix sequences such that $X_n(0) \to X_0^s$, where X_0^s is invertible. Then, from the continuous dependence of the solution on initial condition and on parameter, it follows that $X_n(t)$ tends, uniformly on each compact interval, to the matrix solution $X^s(t)$ of (2.69). Since $n \to \infty$, we obtain that $X(t)$ satisfies (2.68). □

Theorem 2.18. *Let the following conditions hold:*

1. *Conditions H2.43–H2.46 are met.*
2. *For the system (2.67) there exists an exponential dichotomy with an hermitian projector P and fundamental matrix $X(t)$.*

Then, the projector-valued function $P(t) = X(t)X^{-1}(t)$ is almost periodic.

Proof. Let $\{s'_m\}$ be an arbitrary sequence of real numbers, which moves the system (2.67) to the system (2.69).

Since the function $P(t) = X(t)X^{-1}(t)$ is bounded and uniformly continuous in the intervals of the form $(t_k, t_{k+1}]$, hence the sequence $\{P(t + s'_m)\}$ is uniformly bounded and uniformly continuous on the intervals $(t_k - s'_m, t_{k+1} - s'_m]$. From Ascoli's diagonal process it follows that there exists a subsequence $\{s_n\}$ of the sequence $\{s'_m\}$ such that the sequence $\{P(t+s_n)\}$ is convergent at each compact interval, and let we denote its limit by $Y(t)$. If $\{s_n\}$ is a subsequence of $\{s'_m\}$, such that $X(s_n)H^{-1}(s_n) \to X_0^s$ is invertible, then from Theorem 2.17 it follows that the sequence $\{X(t+s_n)H^{-1}(s_n)\}$ tends uniformly in each compact interval to the fundamental matrix $X^s(t)$ of system (2.69) and $X^s(t)$ satisfies $Y(t) = X^s(t)P\Big(X^s(t)\Big)^{-1}$.

From Theorem 2.17 it follows that each uniformly convergent in a compact interval subsequences of $\{P(t+s_n)\}$ tends to one and the same limit. Thus, the sequence $\{P(t+s_n)\}$ tends uniformly to $Y(t)$ on each compact interval.

Further on, we shall show that this convergence is uniform in $\mathbb{R}$. Suppose that this is not true. Then, for some $\gamma > 0$ there exists a sequence $\{h_n\}$ of real numbers and a subsequence $\{s'_n\}$ of $\{s_n\}$ such that

$$||P(h_n + s'_n) - Y(h_n)|| \geq \gamma, \tag{2.71}$$

for each n. It is easily to verify that $E^{h_n+s'_n}$ and E^{h_n} are uniformly convergent in $H(A, B_k, t_k)$. From the almost periodicity and from the process of the construction of E^s it follows that the limit of such system in $H(A, B_k, t_k)$ is one and the same, and let we denote it by E^r. Analogously, $\{P(t+h_n+s'_n)\}$ tends uniformly on each compact interval to $Z(t)PZ^{-1}(t)$, where $Z(t)$ is the fundamental matrix of system E^r, for which there exists an exponential dichotomy with a projector P. Hence, $Y(t+h_n)$ tends to $Z(t)PZ^{-1}(t)$. Then

$$||P(h_n + s'_n) - Y(h_n)|| \to 0,$$

which contradicts the assumption (2.71). □

2.7 Separated Solutions and Almost Periodicity

In the present paragraph, by using the notion of separated solutions, sufficient conditions for the existence of almost periodic solutions of impulsive differential equations with variable impulsive perturbations are obtained. Amerio, formulated in [12] the concept of separated solutions, in order to give sufficient conditions for the existence of almost periodic solutions to ordinary differential equations.

The objective of this section is to extend the notion of separated solutions for impulsive differential equations.

Consider the system of impulsive differential equations with variable impulsive perturbations

$$\begin{cases} \dot{x} = f(t,x), \ t \neq \tau_k(x), \\ \Delta x = I_k(x), \ t = \tau_k(x), \ k = \pm 1, \pm 2, \dots, \end{cases} \tag{2.72}$$

where $t \in \mathbb{R}$, $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\tau_k : \Omega \to \mathbb{R}$, and $I_k : \Omega \to \mathbb{R}^n$, $k = \pm 1, \pm 2 \dots$.

Introduce the following conditions:

H2.47. The function $f \in C^1[\mathbb{R} \times \Omega, \mathbb{R}^n]$.
H2.48. The functions $I_k \in C^1[\Omega, \mathbb{R}^n]$, $k = \pm 1, \pm 2 \dots$.

H2.49. If $x \in \Omega$, then $x + I_k(x) \in \Omega,\ L_k(x) = x + I_k(x)$ are invertible on Ω and $L_k^{-1}(x) \in \Omega$ for $k = \pm 1, \pm 2 \ldots$.

H2.50. $\tau_k(x) \in C^1(\Omega, \mathbb{R})$ and $\lim\limits_{k \to \pm\infty} \tau_k(x) = \pm\infty$ uniformly on $x \in \Omega$.

H2.51. The following inequalities hold:

$$sup\Big\{\|f(t,x)\| \ :\ (t,x) \in \mathbb{R} \times \Omega\Big\} \le A < \infty,$$
$$sup\Big\{\|\tfrac{\partial \tau_k(x)}{\partial x}\| \ :\ x \in \Omega,\ k = \pm 1, \pm 2, \ldots\Big\} \le B < \infty,\ AB < 1,$$
$$sup\Big\{\langle \tfrac{\partial \tau_k}{\partial x}(x + sI_k(x)), I_k(x)\rangle \ :\ s \in [0,1],\ x \in \Omega,\ k = \pm 1, \pm 2, \ldots\Big\} \le 0.$$

From Chap. 1, it follows that, if conditions H2.47–H2.51 are satisfied, then system (2.72) has a unique solution $x(t) = x(t; t_0, x_0)$ with the initial condition

$$x(t_0^+) = x_0.$$

Assuming that conditions H2.48–H2.51 are fulfilled, we consider the hypersurfaces:

$$\sigma_k = \Big\{(t,x) :\ t = \tau_k(x),\ x \in \Omega\Big\},\quad k = \pm 1, \pm 2, \ldots.$$

Let t_k be the moments in which the integral curve $(t, x(t; t_0, x_0))$ meets the hypersurfaces σ_k, $k = \pm 1, \pm 2, \ldots$.

Introduce the following conditions:

H2.52. The function $f(t,x)$ is almost periodic in t uniformly with respect to $x \in \Omega$.

H2.53. The sequences $\{I_k(x)\}$ and $\{\tau_k(x)\}$, $k = \pm 1, \pm 2, \ldots$, are almost periodic uniformly with respect to $x \in \Omega$.

H2.54. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k = \pm 1, \pm 2, \ldots$, $j = \pm 1, \pm 2, \ldots$, is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.

Let conditions H2.47–H2.54 hold, and let $\{s'_m\}$ be an arbitrary sequence of real numbers. Then, there exists a subsequence $\{s_n\}$, $s_n = s'_{m_n}$, so that analogous to the process in Chap. 1, the system (2.72) moves to the system

$$\begin{cases} \dot{x} = f^s(t,x),\ t \ne \tau_k^s, \\ \Delta x = I_k(x),\ t = \tau_k^s,\ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.73}$$

and in this case, the set of systems in the form (2.73) we shall denote by $H(f, I_k, \tau_k)$.

We shall introduce the following operator notation. Let $\alpha = \{\alpha_n\}$ be a subsequence of the sequence $\alpha' = \{\alpha_n\}_{n=0}^{\infty}$, and denote $\alpha \subset \alpha'$. Also with $\alpha + \beta$ we shall denote $\{\alpha_n + \beta_n\}$ of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

By $\alpha > 0$ we mean $\alpha_n > 0$ for each n. If $\alpha \subset \alpha'$ and $\beta \subset \beta'$, then α and β are said to have matching subscripts, if $\alpha = \{\alpha'_{n_k}\}$ and $\beta = \{\beta'_{n_k}\}$.

Let we denote by $S_{\alpha+\beta}\phi$ and $S_\alpha S_\beta\phi$ the limits $\lim\limits_{n\to\infty}\theta_{\alpha_n+\beta_n}(\phi)$ and $\lim\limits_{n\to\infty}\theta_{\alpha_n}(\lim\limits_{m\to\infty}\theta_{\beta_m}\phi)$, respectively, where the number θ_{α_n} is defined in Chap. 1, and $\phi = \big(\varphi(t), T\big),\ \phi \in PC[\mathbb{R}, \Omega] \times UAPS$.

Lemma 2.23. *The function $\varphi(t)$ is almost periodic if and only if from every pair of sequences α', β' one can extracts common subsequences $\alpha \subset \alpha'$, $\beta \subset \beta'$ such that*

$$S_{\alpha+\beta}\varphi = S_\alpha S_\beta \varphi, \tag{2.74}$$

exists pointwise.

Proof. Let (2.74) exists pointwise, γ' be a sequence, such that for $\gamma \subset \gamma'$, $S_\gamma\varphi$ exists. If $S_\gamma\phi$ is uniform, we are done. If not, we can find $\varepsilon > 0$ and sequences $\beta \subset \gamma$, $\beta' \subset \gamma$ such that

$$\rho(T_n^\beta, T_n^{\beta'}) < \varepsilon,$$

but

$$\sup_{t\in\mathbb{R}\setminus F_\varepsilon(s(T_n^\beta \cup T_n^{\beta'}))} ||\varphi(t+\beta_n) - \varphi(t+{\beta'}_n)|| \geq \varepsilon > 0,$$

where T_n^β and $T_n^{\beta'}$ are the points of discontinuity of functions $\varphi(t+\beta_n)$, $\varphi(t+\beta'_n)$, $n = 0, 1, 2, \ldots$, respectively.

From the intermediate value theorem for the common intervals of continuity of functions $\varphi(t+\beta_n)$ and $\varphi(t+\beta'_n)$, and the fact that

$$\lim_{n\to\infty} ||\varphi(\beta_n) - \varphi(\beta'_n)|| = 0,$$

it follows that there exists a sequence α such that

$$\sup_{t\in\mathbb{R}\setminus F_\varepsilon(s(T_n^\beta \cup T_n^{\beta'}))} ||\varphi(\alpha_n+\beta_n) - \varphi(\alpha_n+{\beta'}_n)|| \geq \varepsilon > 0. \tag{2.75}$$

Then, for the sequence α there exist common subsequences $\alpha_1 \subset \alpha$, $\beta_1 \subset \beta$, $\beta_2 \subset \beta$ such that

$$S_{\alpha_1+\beta_1}\phi = R_1,\ S_{\alpha_1+\beta_2}\phi = R_2,$$

where $R_j = (r_j(t), P_j),\ r_j \in PC,\ P_j \in UAPS$, $j = 1, 2$, exist pointwise.

From (2.74), we get

$$\begin{aligned} R_1 = S_{\alpha_1+\beta_1}\phi &= S_{\alpha_1}S_{\beta_1}\phi = S_{\alpha_1}S_\gamma\phi \\ &= S_{\alpha_1}S_{\beta_2}\phi = S_{\alpha_1+\beta_2}\phi = R_2, \end{aligned} \tag{2.76}$$

for $t \in \mathbb{R} \setminus F_\varepsilon(s(P_1 \cup P_2))$.

On the other hand, from (2.75) it follows that

$$||r_1(0) - r_2(0)|| > 0,$$

which is a contradiction of (2.76).

Let $\varphi(t)$ be almost periodic and if α' and β' are given, we take subsequences $\alpha \subset \alpha'$, $\beta \subset \beta'$ successively, such that they are common subsequences and $S_\alpha\phi = \phi_1$, $S_\beta\phi_1 = \phi_2$ and $S_{\alpha+\beta}\phi = \phi_3$, where $\phi_j = (\phi_j, T_j)$, $\phi_j \in PC[\mathbb{R}, \Omega]\times UAPS$, $j = 1, 2, 3$, exist uniformly for $t \in \mathbb{R} \setminus F_\varepsilon(s(T_1 \cup T_2 \cup T_3))$.

If $\varepsilon > 0$ is given, then

$$||\varphi(t + \alpha_n + \beta_n) - \varphi_3(t)|| < \frac{\varepsilon}{3},$$

for n large and for all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_{n,n} \cup T_3))$, where $T_{n,n}$ is the set of points of discontinuity of functions $\varphi(t + \alpha_n + \beta_n)$.

Also,

$$||\varphi(t + \alpha_n + \beta_m) - \varphi_1(t + \beta_n)|| < \frac{\varepsilon}{3},$$

for n, m large and for all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_{n,m} \cup T_{1,n}))$, where $T_{n,m}$ is the set of points of discontinuity of functions $\varphi(t + \alpha_n + \beta_m)$ and $T_{1,n}$ is formed by the points of discontinuity of functions $\varphi_1(t + \beta_n)$.

Finally,

$$||\varphi_1(t + \beta_m) - \varphi_2(t)|| < \frac{\varepsilon}{3},$$

for m large and all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_{1,m} \cup T_2))$, where $T_{1,m}$ is the set of points of discontinuity of functions $\varphi_1(t + \beta_m)$.

By the triangle inequality for $n = m$ large, we have $||\varphi_2(t) - \varphi_3(t)|| < \varepsilon$ for all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_2 \cup T_3))$.

Since ε is arbitrary, we get $\varphi_2(t) = \varphi_3(t)$ for all $t \in \mathbb{R} \setminus F_\varepsilon(s(T_{1,m} \cup T_2))$, i.e. (2.74) holds. $\square$

Definition 2.16. The function $\varphi(t)$, $\varphi \in PC[\mathbb{R}, \Omega]$, is said to *satisfy the condition SG*, if for a given sequence γ', $\lim_{n\to\infty} \gamma'_n = \infty$ there exist $\gamma \subset \gamma'$ and a number $d(\gamma) > 0$ such that $S_\gamma\phi$, $\phi = \big(\varphi(t), T\big)$, $T \in UAPS$ exists pointwise for each $\varepsilon > 0$. If α is a sequence with $\alpha > 0$, $\beta' \subset \gamma$ and $\beta'' \subset \gamma$ are such that $S_{\alpha+\beta'}\phi = (r_1(t), P_1)$, $S_{\alpha+\beta''}\phi = (r_2(t), P_2)$, then either $r_1(t) = r_2(t)$ or $||r_1(t) - r_2(t)|| > d(\gamma)$ hold for $t \in \mathbb{R} \setminus F_\varepsilon(s(P_1 \cup P_2))$.

Definition 2.17. Let $K \subset \Omega$ be a compact. The solution $x(t)$ of system (2.72) with points of discontinuity in the set T is said to be *separated in* K, if for any other solution $y(t)$ of (2.72) in Ω with points of discontinuity in the set T there exists a number $d(y(t))$ such that $||x(t) - y(t)|| > d(y(t))$ for $t \in \mathbb{R} \setminus F_\varepsilon(s(T))$. The number $d(y(t))$ is said to be a *separated constant*.

Theorem 2.19. *The function $\varphi(t), \varphi \in PC[\mathbb{R}, \Omega]$, is almost periodic if and only if φ satisfies the condition SG.*

Proof. Let φ satisfies the condition SG, and let γ' be a sequence such that $\lim\limits_{n\to\infty} \gamma'_n = \infty$. Then there exists $\gamma \subset \gamma'$ such that $S_\gamma \phi,\ \phi = \big(\varphi(t), T\big)$ exists pointwise. If the convergence is not uniformly in $\mathbb{R}$, then there exist sequences $\delta' > 0$, $\alpha' \subset \gamma$, $\beta' \subset \gamma$, and a number $\varepsilon > 0$ such that $||\varphi(\alpha'_n + \delta'_n) - \varphi(\beta'_n + \delta'_n)|| \geq \varepsilon$, where we may pick $\varepsilon < d(\gamma)$. Since $S_\gamma(\varphi(0), T)$ exists, we have

$$||\varphi(\alpha'_n) - \varphi(\beta'_n)|| < d(\gamma), \tag{2.77}$$

for large n.

Consequently, $k(t) = \varphi(t + \alpha'_n) - \varphi(t + \beta'_n)$ satisfies $||k(0)|| < d(\gamma)$ and $||k(\delta'_n)|| \geq \varepsilon$ for large n. Hence, there exists δ''_n such that $\delta''_n \subset \delta'_n$ and $\varepsilon \leq ||k(\delta''_n)|| < d(\gamma)$.

We shall consider the sequences $\alpha' + \delta''$ and $\beta' + \delta''$. By SG there exist sequences $\alpha + \delta \subset \alpha' + \delta''$ and $\beta + \delta \subset \beta' + \delta''$ with matching subscripts such that $S_{\alpha+\delta}\phi = \phi_1$, $S_{\alpha+\delta}\phi = \phi_2$, $\phi_j = (\varphi_j, T_j)$ exist pointwise, and $\varphi_1(t) = \varphi_2(t)$ or $||\varphi_1(t) - \varphi_2(t)|| > 2d(\gamma)$, for $t \in \mathbb{R} \setminus F_\varepsilon(s(T_1 \cup T_2))$.

On the other hand,

$$||\varphi_1(0) - \varphi_2(0)|| = \lim_{n\to\infty} ||\varphi(\alpha_n + \delta_n) - \varphi(\beta_n + \delta_n)||,$$

and from (2.77), it follows that $||\varphi_1(0) - \varphi_2(0)|| \leq d(\gamma)$. The contradiction shows that $S_\gamma\varphi$ exists uniformly on $t \in \mathbb{R} \setminus F_\varepsilon(s(T))$.

Conversely, if $\varphi(t)$ is an almost periodic function, and γ' be given with $\lim\limits_{n\to\infty} \gamma'_n = \infty$ then, there exists $\gamma \subset \gamma'$ such that $S_\gamma\phi$ exists uniformly on $t \in \mathbb{R} \setminus F_\varepsilon(s(T))$ and $S_\gamma\varphi = (k(t), Q),\ (k(t), Q) \in PC[\mathbb{R}, \Omega] \times UAPS$.

Let the subsequences $\beta' \subset \gamma$, $\beta'' \subset \gamma$, and $\alpha > 0$ be such that $S_{\alpha+\beta'}\phi = (r_1(t), P_1)$, $S_{\alpha+\beta''}\phi = (r_2(t), P_2)$, $(r_j(t), P_j) \in PC[\mathbb{R}, \Omega] \times UAPS$.

From Lemma 2.23 it follows that there exist $\alpha' \subset \alpha$, $\overline{\beta}' \subset \beta'$, $\overline{\beta}'' \subset \beta''$ such that

$$\begin{aligned}(r_1(t), P_1) &= S_{\alpha'+\overline{\beta'}}(p(t), T) = S_{\alpha'}S_{\overline{\beta'}}(p(t), T) = S_{\alpha'}S_\gamma(p(t), T)\\ &= S_{\alpha'}(k(t), Q) = S_\alpha(k(t), Q),\end{aligned} \tag{2.78}$$

$$\begin{aligned}(r_2(t), P_2) &= S_{\alpha'+\overline{\beta''}}(p(t), T) = S_{\alpha'}S_{\overline{\beta''}}(p(t), T) = S_{\alpha'}S_\gamma(p(t), T)\\ &= S_{\alpha'}(k(t), Q) = S_\alpha(k(t), Q).\end{aligned} \tag{2.79}$$

Hence, from (2.78) and (2.79), we get $r_1(t) = r_2(t)$ for $t \in \mathbb{R} \setminus F_\epsilon(s(P_1 \cup P_2))$. Then, $(\varphi(t), T)$ satisfies SG. $\square$

Now, let $K \subset \Omega$ be a compact. We shall consider the system of impulsive differential equations

$$\begin{cases} \dot{x} = g(t,x), \ t \neq \sigma_k(x), \\ \Delta x = G_k(x), \ t = \sigma_k(x), \ k = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.80}$$

where $(g, G_k, \sigma_k) \in H(f, I_k, \tau_k)$.

Theorem 2.20. *Let the following conditions hold:*

1. *Conditions H2.47–H2.54 are met.*
2. *Every solution of system (2.80) in K is separated.*

Then every system in $H(f, I_k, \tau_k)$ has only a finite number of solutions and the separated constant d may be picked to be independent of solutions.

Proof. The fact that each system has only a finite number solutions in K is a consequence of a compactness of K and the resulting compactness of the solutions in K. But no solution can be a limit of others by the separated condition. Consequently, the number of solutions of any system from $H(f, I_k, \tau_k)$ is finite and d may be picked as a function of the system.

Let $(h, L_k, l_k) \in H(f, I_k, \tau_k)$ and $S_{\alpha'}(g, G_k, \sigma_k) = (h, L_k, l_k)$, with $\lim\limits_{n\to\infty} \alpha'_n = \infty$.

Let $(\varphi(t), T)$, $(\varphi_0(t), T_0)$ be two solutions in K, and let $\alpha \subset \alpha'$ be such that $S_\alpha(\varphi(t), T)$ and $S_\alpha(\varphi_0(t), T_0)$ exist uniformly on K, and are solutions of (2.80).

Then,

$$||S_\alpha(\varphi(t), T) - S_\alpha(\varphi_0(t), T_0)|| \geq d(g, G_k, \sigma_k).$$

So, if $\varphi_1, \ldots, \varphi_n$ are solutions of (2.80) in K, then $S_\alpha(\varphi_j(t), T_j)$, $j = 1, 2, \ldots, n$, are distinct solutions of (2.80) in K such that

$$||S_\alpha(\varphi_j(t), T_j) - S_\alpha(\varphi_i(t), T_i)|| \geq d(g, G_k, \sigma_k), \ i \neq j.$$

Hence, the number of solutions of (2.80) in K is greater or equal than n. By “symmetry” arguments the reverse is true, hence each system has the same number of solutions.

On the other hand, $S_\alpha(\varphi_i, T_i)$ exhaust the solutions of (2.80) in K, so that $d(g, G_k, \sigma_k) \leq d(h, L_k, l_k)$. Again by symmetry, $d(h, L_k, l_k) \geq d(g, G_k, \sigma_k)$. □

Theorem 2.21. *Let the following conditions hold:*

1. *Conditions H2.47–H2.54 are met.*
2. *For every system in $H(f, I_k, \tau_k)$ there exist only separated solutions on K.*

Then:

1. *If for some system in $H(f, I_k, \tau_k)$ there exists a solution in K, then for every system in $H(f, I_k, \tau_k)$ there exists a solution in K.*

2. *All such solutions in* K *are almost periodic and for every system in* $H(f, I_k, \tau_k)$ *there exists an almost periodic solution in* K.

Proof. The first statement has been proved in Theorem 2.20. Let $\varphi(t)$ be a solution of system (2.80) in K and δ be the separation constant.

Let γ' be a sequence such that $\lim_{n\to\infty} \gamma' = \infty$ and $\gamma \subset \gamma'$, $S_\gamma(g, G_k, \sigma_k) = (h, L_k, l_k)$, and $S_\gamma(\varphi(t), T)$ exists.

Let $\beta' \subset \gamma$, $\beta'' \subset \gamma$ and $\alpha > 0$ are such that

$$S_{\alpha+\beta'}(\varphi(t), T) = (\varphi_1(t), T_1),$$
$$S_{\alpha+\beta''}(\varphi(t), T) = (\varphi_2(t), T_2).$$

Again, take further subsequences with matching subscripts, so that (without changing notations)

$$\begin{aligned} S_{\alpha+\beta'}(g, G_k, \sigma_k) &= S_\alpha S_{\beta'}(g, G_k, \sigma_k) \\ &= S_\alpha S_\gamma(g, G_k, \sigma_k) = S_\alpha(h, L_k, l_k), \end{aligned}$$

and

$$S_{\alpha+\beta''}(g, G_k, \sigma_k) = S_\alpha(h, L_k, l_k).$$

Consequently, $\varphi_1(t)$ and $\varphi_2(t)$ are solutions of the same system and for $\varepsilon > 0$, $\varphi_1 \equiv \varphi_2$, for $\mathbb{R} \setminus F_\varepsilon(s(T_1 \bigcup T_2))$ or $||\varphi_1(t) - \varphi_2(t)|| \geq \delta = 2d$ on $\mathbb{R} \setminus F_\varepsilon(s(T_1 \bigcup T_2))$.

Therefore, $\varphi(t)$ satisfies the SG, and from Theorem 2.19 it follows that $\varphi(t)$ is an almost periodic function.

Let now $\varphi(t)$ be a solution of (2.80) in K which by the above is an almost periodic function, and let we choice $\alpha'_n = n$. Then, there exists $\alpha \subset \alpha'$ such that the limits $S_\alpha(g, G_k, \sigma_k) = (h, L_k, l_k)$, $S_{-\alpha}(h, L_k, l_k) = (g, G_k, \sigma_k)$ exist uniformly and $S_\alpha(\varphi(t), T) = (r(t), P)$, $S_{-\alpha}(r(t), P)$ exist uniformly on K, where $S_{-\alpha}(r(t), P)$ is the solution of (2.80).

From condition 2 of Theorem 2.21 it is easy to see that $(r(t), P) = S_\alpha(\varphi(t), T)$ and thus $S_{-\alpha}(r(t), P)$ exists uniformly and $\varphi(t)$ is almost periodic. □

2.8 Impulsive Differential Equations in Banach Space

The abstract differential equations arise in many areas of applied mathematics, and for this reason these equations have received much attention in the resent years. Natural generalizations of the abstract differential equations are impulsive differential equations in Banach space.

In this paragraph, we shall investigate the existence of almost periodic solutions of these equations.

Let $(X, ||.||_X)$ be an abstract Banach space.

Consider the impulsive differential equation

$$\dot{x}(t) = Ax + F(t, x) + \sum_{k=\pm1,\pm2,\ldots} \big[Bx + H_k(x)\big]\delta(t - t_k), \tag{2.81}$$

where $A : \mathcal{D}(A) \subset X \to X$, $B : \mathcal{D}(B) \subset X \to X$ are linear bounded operators with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, respectively. The function $F : \mathcal{D}(\mathbb{R}\times X) \to X$ is continuous with respect to $t \in \mathbb{R}$ and with respect to $x \in X$, $H_k : \mathcal{D}(H_k) \subset X \to X$ are continuous impulse operators, $\delta(.)$ is the Dirac's delta-function, $\{t_k\} \in \mathcal{B}$.

Denote by $x(t) = x(t; t_0, x_0)$, the solution of (2.81) with the initial condition $x(t_0^+) = x_0,\ t_0 \in \mathbb{R},\ x_0 \in X$.

The solutions of (2.81) are piecewise continuous functions [16], with points of discontinuity at the moments t_k, $k = \pm1, \pm2, \ldots$ at which they are continuous from the left, i.e. the following relations are valid:

$$x(t_k^-) = x(t_k),\ x(t_k^+) = x(t_k) + Bx(t_k) + H_k(x(t_k)),\ k = \pm1, \pm2, \ldots.$$

Let $PC[\mathbb{R}, X] = \{\varphi : \mathbb{R} \to X,\ \varphi$ is a piecewise continuous function with points of discontinuity of the first kind at the moments t_k, $\{t_k\} \in \mathcal{B}$ at which $\varphi(t_k^-)$ and $\varphi(t_k^+)$ exist, and $\varphi(t_k^-) = \varphi(t_k)\}$.

With respect to the norm $||\varphi||_{PC} = \sup\limits_{t\in\mathbb{R}} ||\varphi(t)||_X$, $PC[\mathbb{R}, X]$ is a Banach space [16].

Denote by $PCB[\mathbb{R}, X]$ the subspace of $PC[\mathbb{R}, X]$ of all bounded piecewise continuous functions, and together with (2.81) we consider the respective linear non-homogeneous impulsive differential equation

$$\dot{x} = Ax + f(t) + \sum_{k=\pm1,\pm2,\ldots} \big[Bx + b_k\big]\delta(t - t_k), \tag{2.82}$$

where $f \in PCB[\mathbb{R}, X]$, $b_k : \mathcal{D}\ (b_k) \subset X \to X$, and the homogeneous impulsive differential equation

$$\dot{x}(t) = Ax + \sum_{k=\pm1,\pm2,\ldots} Bx\delta(t - t_k). \tag{2.83}$$

Introduce the following conditions:

H2.55. The operators A and B commute with each other, and for the operator $I + B$ there exists a logarithm operator $L_n(I + B)$, I is the identity operator on the space X.

H2.56. The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k = \pm1, \pm2, \ldots$, $j = \pm1, \pm2, \ldots$, is uniformly almost periodic, and $inf_k t_k^1 = \theta > 0$.

Following [16], we denote by $\Phi(t,s)$, the Cauchy evolutionary operator for (2.83),

$$\Phi(t,s) = e^{\Lambda(t-s)}(I+B)^{-p(t-s)+i(t,s)},$$

where $\Lambda = A + pL_n(I+B)$, $i(t,s)$ is the number of points t_k in the interval (t,s), and $p > 0$ is defined in Lemma 1.1.

Lemma 2.24. *Let conditions H2.55–H2.56 hold, and the spectrum $\sigma(\Lambda)$ of the operator Λ does not intersect the imaginary axis, and lying in the left half-planes.*

Then for the Cauchy evolutionary operator $\Phi(t,s)$ of (2.83) there exist positive constants K_1 and α such that

$$||\Phi(t,s)||_X \leq K_1 e^{-\alpha(t-s)}, \tag{2.84}$$

where $t \geq s,\ t,s \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then

$$||(I+B)^{-p(t-s)+i(s,t)}||_X \leq \delta(\varepsilon) exp\big\{\varepsilon ||Ln(I+B)||_X (t-s)\big\},$$

where $\delta(\varepsilon) > 0$ is a constant.

On the other hand [50], if $\alpha_1 > 0$ and

$$\delta_1 \in (\alpha_1, \lambda^*(\alpha_1)),\ \lambda^*(\alpha_1) = inf\big\{|Re\lambda|,\ \lambda \in \sigma(\Lambda)\big\},$$

then,

$$||e^{\Lambda(t-s)}||_X \leq K_1 e^{-\alpha_1(t-s)},\ t > s$$

and (2.84) follows immediately. □

The next definition is for almost periodic functions in a Banach space of the form $PC[\mathbb{R}, X]$. □

Definition 2.18. The function $\varphi \in PC[\mathbb{R}, X]$ is said to be *almost periodic*, if:

(a) The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $k = \pm1, \pm2, \ldots$, $j = \pm1, \pm2, \ldots$, $\{t_k\} \in \mathcal{B}$ is uniformly almost periodic.
(b) For any $\varepsilon > 0$ there exists a real number $\delta(\varepsilon) > 0$ such that, if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $||\varphi(t') - \varphi(t'')||_X < \varepsilon$.
(c) For any $\varepsilon > 0$ there exists a relatively dense set $\overline{T}$ such that, if $\tau \in \overline{T}$, then $||\varphi(t+\tau) - \varphi(t)||_X < \varepsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - t_k| > \varepsilon$, $k = \pm1, \pm2, \ldots$.

The elements of $\overline{T}$ are called $\varepsilon -$ *almost periods*.

Introduce the following conditions:

H2.57. The function $f(t)$ is almost periodic.
H2.58. The sequence $\{b_k\},\ k = \pm 1, \pm 2, \ldots$ is almost periodic.

We shall use the next lemma, similar to Lemma 1.7.

Lemma 2.25. *Let conditions H2.56–H2.58 hold.*

Then for each $\varepsilon > 0$ there exist $\varepsilon_1,\ 0 < \varepsilon_1 < \varepsilon$, a relatively dense set $\overline{T}$ of real numbers, and a set P of integer numbers such that the following relations are fulfilled:

(a) $||f(t+\tau) - f(t)||_X < \varepsilon,\ t \in \mathbb{R},\ \tau \in \overline{T},\ |t - t_k| > \varepsilon,\ k = \pm 1, \pm 2, \ldots$.
(b) $||b_{k+q} - b_k||_X < \varepsilon,\ q \in P,\ k = \pm 1, \pm 2, \ldots$.
(c) $|\tau_k^q - \tau| < \varepsilon_1,\ q \in P,\ \tau \in \overline{T},\ k = \pm 1, \pm 2, \ldots$.

We shall prove the next theorem.

Theorem 2.22. *Let the following conditions hold:*

1. *Conditions H2.55–H2.58 are met.*
2. *The spectrum $\sigma(\Lambda)$ of the operator Λ does not intersect the imaginary axis, and lying in the left half-planes.*

Then:

1. *There exists a unique almost periodic solution $x(t) \in PCB[\mathbb{R}, X]$ of* (2.82).
2. *The almost periodic solution $x(t)$ is asymptotically stable.*

Proof. We consider the function

$$x(t) = \int_{-\infty}^{t} \Phi(t,s) f(s) ds + \sum_{t_k < t} \Phi(t, t_k) b_k. \tag{2.85}$$

It is immediately verified, that the function $x(t)$ is a solution of (2.82). From conditions H2.57 and H2.58, it follows that $f(t)$ and $\{b_k\}$ are bounded and let

$$max\big\{||f(t)||_{PC}, ||b_k||_X\big\} \leq C_0,\ C_0 > 0.$$

Using Lemmas 1.1 and 2.24, we obtain

$$\begin{aligned} ||x(t)||_{PC} &= \int_{-\infty}^{t} ||\Phi(t,s)||_{PC} ||f(s)||_{PC} ds + \sum_{t_k < t} ||\Phi(t, t_k)||_{PC} ||b_k||_X \\ &\leq \int_{-\infty}^{t} K_1 e^{-\alpha(t-s)} ||f(s)||_{PC} ds + \sum_{t_k < t} K e^{-\alpha(t - t_k)} ||b_k||_X \\ &\leq K_1 \Big(\frac{C_0}{\alpha} + \frac{C_0 N}{1 - e^{-\alpha}} \Big) = \overline{K}. \end{aligned} \tag{2.86}$$

From (2.86) it follows that $x(t) \in PCB[\mathbb{R}, X]$.

Let $\varepsilon > 0,\ \tau \in T,\ q \in Q$, where the sets T and P are from Lemma 2.25.

Then,

$$||x(t+\tau) - x(t)||_{PC} \\ \leq \int_{-\infty}^{t} ||\Phi(t,s)||_{PC}||f(s+\tau) - f(s)||_{PC} ds \\ + \sum_{t_k < t} ||\Phi(t,t_k)||_{PC}||b_{k+q} - b_k||_X \leq M\varepsilon,$$

where $|t - t_k| > \varepsilon,\ M > 0$.

The last inequality implies that the function $x(t)$ is almost periodic. The uniqueness of this solution follows from the fact that the (2.83) has only the zero bounded solution under conditions H2.55 and H2.56.

Let $\tilde{x} \in PCB[\mathbb{R}, X]$ be an arbitrary solution of (2.82), and $y = \tilde{x} - x$. Then $y \in PCB[\mathbb{R}, X]$ and

$$y = \Phi(t, t_0)y(t_0). \tag{2.87}$$

The proof that $x(t)$ is asymptotically stable follows from (2.87), the estimates from Lemma 2.24, and the fact that $i(t_0, t) - p(t - t_0) = o(t)$ for $t \to \infty$. □

Now, we shall investigate almost periodic solutions of (2.81).

Theorem 2.23. *Let the following conditions hold:*

1. *Conditions H2.55–H2.58 are met.*
2. *The spectrum $\sigma(\Lambda)$ of the operator Λ does not intersect the imaginary axis, and lying in the left half-planes.*
3. *The function $F(t,x)$ is almost periodic with respect to $t \in \mathbb{R}$ uniformly at $x \in \Omega$ and the sequence $\{H_k(x)\}$ is almost periodic uniformly at $x \in \Omega$, Ω is every compact from X, and*

$$||x||_X < h,\ h > 0.$$

4. *The functions $F(t,x)$ and $H_k(x)$ are Lipschitz continuous with respect to $x \in \Omega$ uniformly for $t \in \mathbb{R}$ with a Lipschitz constant $L > 0$,*

$$||F(t,x) - F(t,y)||_X \leq L||x-y||_X,\ ||H_k(x) - H_k(y)||_X \leq L||x-y||_X.$$

5. *The functions $F(t,x)$ and $H_k(x)$ are bounded,*

$$max\Big\{||F(t,x)||_X,\ ||H_k(x)||_X\Big\} \leq C,$$

where $C > 0,\ x \in \Omega$.

Then, if:

$$\overline{K}C < h \text{ and } \overline{K}L < 1,$$

where $\overline{K}$ *was defined by* (2.86), *it follows*:

1. *There exists a unique almost periodic solution* $x(t) \in PCB[\mathbb{R}, X]$ *of* (2.81).
2. *The almost periodic solution* $x(t)$ *is asymptotically stable.*

Proof. We denote by $D^* \subset PCB[\mathbb{R}, X]$ the set of all almost periodic functions with points of discontinuity of the first kind $t_k,\ k = \pm1, \pm2, \ldots$, satisfying the inequality $||\varphi||_{PC} < h$.

In D^*, we define an operator S in the following way. If $\varphi \in D^*$, then $y = S\varphi(t)$ is the almost periodic solution of the system

$$\dot{y}(t) = Ay + F(t, \varphi(t)) + \sum_{k=\pm1,\pm2,\ldots} \big[By + H_k(\varphi(t_k))\big]\delta(t - t_k), \tag{2.88}$$

determined by Theorem 2.22. Then, from (2.86) and the conditions of Theorem 2.23, it follows that $\mathcal{D}\ (S) \subset D^*$.

Let $\varphi,\ \psi \in D^*$. Then, we obtain

$$||S\varphi(t) - S\psi(t)||_{PC} \le \overline{K}L.$$

From the last inequality, and the conditions of the theorem, it follows that the operator S is a contracting operator in D^*. □

Example 2.4. In this example, we shall investigate materials with fading memory with impulsive perturbations at fixed moments of time.

We shall investigate the existence of almost periodic solutions of the following impulsive differential equation

$$\begin{cases} \ddot{x}(t) + \beta(0)\dot{x}(t) = \gamma(0)\Delta x(t) + f_1(t) f_2(x(t)),\ t \ne t_k, \\ x(t_k^+) = x(t_k) + b_k^1, \\ \dot{x}(t_k^+) = \dot{x}(t_k) + b_k^2,\ k = \pm1, \pm2, \ldots, \end{cases} \tag{2.89}$$

where $t_k = k + l_k,\ l_k = \frac{1}{4}|cosk - cosk\sqrt{2}|,\ k = \pm1, \pm2, \ldots$.

If $y(t) = \dot{x}(t)$ and

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ \gamma(0)\Delta & -\beta(0) \end{bmatrix},\ \dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix},$$

$$F(t, z) = \begin{bmatrix} 0 \\ f_1(t) f_2(x) \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ b_k = \begin{bmatrix} b_k^1 \\ b_k^2 \end{bmatrix},\ k = \pm1, \pm2, \ldots,$$

then the (2.89) rewrites in the form

$$\dot{z}(t) = Az + F(t,z) + \sum_{k=\pm 1,\pm 2,\dots}^{\infty} \big[Bz + b_k\big]\delta(t - t_k). \tag{2.90}$$

From [138], it follows that the set of sequences $\{t_k^j\},\ k = \pm 1, \pm 2, \dots,\ j = \pm 1, \pm 2, \dots$, is uniformly almost periodic and for the (2.90) the conditions of Lemma 1.2 hold.

Let $X = H_0^1(\omega) \times L^2(\omega)$, where $\omega \subset R^3$ is an open set with smooth boundary of the class $C^\infty,\ \beta(t),\ \gamma(t)$ are bounded and uniformly continuous $\mathbb{R}$ valued functions of the class C^2 on $[0,\infty),\ \beta(0) > 0,\ \gamma(0) > 0$.

If $A:\ \mathcal{D}\ (A) = H^2(\omega) \cap H_0^1(\omega) \times H_0^1(\omega) \to X$ is the operator from (2.90) and Δ is Laplacian on ω with boundary condition $y|_{\partial\omega} = 0$, then it follows that A is the infinitesimal generator of a C_0-semigroup and the conditions of Lemma 2.24 hold.

By Theorem 2.23 and similar arguments, we conclude with the following theorem.

Theorem 2.24. *Let for (2.89) the following conditions hold:*

1. *The sequences $\{b_k^i\},\ k = \pm 1, \pm 2, \dots,\ i = 1, 2$, are almost periodic.*
2. *The function $f_1(t)$ is almost periodic in the sense of Bohr.*
3. *The function $f_2(x)$ is Lipschitz continuous with respect to $||x||_X < h$ with a Lipschitz constant $L > 0$,*

$$||f_2(x_1) - f_2(x_2)||_X \le L||x_1 - x_2||_X,\ ||x_i||_X < h,\ i = 1, 2.$$

4. *The function $f_2(x)$ is bounded, $||f_2(x)||_X \le C$, where $C > 0$ and $x \in \omega$.*

Then, if

$$\overline{K}C < h \text{ and } \overline{K}L < 1,$$

where $\overline{K}$ was defined by (2.86), *it follows*:

1. *There exists a unique almost periodic solution $x \in PCB[\mathbb{R}, X]$ of* (2.89).
2. *The almost periodic solution $x(t)$ is asymptotically stable.*

Now, we shall study the existence and uniqueness of almost periodic solutions of impulsive abstract differential equations out by means of the infinitesimal generator of an analytic semigroup and fractional powers of this generator.

Let the operator A in (2.81)–(2.83) be the infinitesimal operator of analytic semigroup $S(t)$ in Banach space X. For any $\alpha > 0$, we define the fractional power $A^{-\alpha}$ of the operator A by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)dt,$$

where $\Gamma(\alpha)$ is the Gamma function. The operators $A^{-\alpha}$ are bounded, bijective and $A^{\alpha} = (A^{-\alpha})^{-1}$, is a closed linear operator such that $\mathcal{D}(A^{\alpha}) = \mathcal{R}(A^{-\alpha})$, where $\mathcal{R}(A^{-\alpha})$ is the range of $A^{-\alpha}$. The operator A^0 is the identity operator in X and for $0 \leq \alpha \leq 1$, the space $X_{\alpha} = \mathcal{D}(A^{\alpha})$ with norm $||x||_{\alpha} = ||A^{\alpha}x||_X$ is a Banach space [50, 58, 68, 115, 126].

We shall use the next lemmas.

Lemma 2.26 ([115, 126]). *Let A be the infinitesimal operator of an analytic semigroup $S(t)$.*

Then:

1. $S(t) : X \to \mathcal{D}(A^{\alpha})$ *for every* $t > 0$ *and* $\alpha \geq 0$.
2. *For every* $x \in \mathcal{D}(A^{\alpha})$ *it follows that* $S(t)A^{\alpha}x = A^{\alpha}S(t)x$.
3. *For every* $t > 0$ *the operator* $A^{\alpha}S(t)$ *is bounded, and*

$$||A^{\alpha}S(t)||_X \leq K_{\alpha}t^{-\alpha}e^{-\lambda t},\ K_{\alpha} > 0,\ \lambda > 0.$$

4. *For* $0 < \alpha \leq 1$ *and* $x \in \mathcal{D}(A^{\alpha})$, *we have*

$$||S(t)x - x||_X \leq C_{\alpha}t^{\alpha}||A^{\alpha}x||_X,\ C_{\alpha} > 0.$$

Lemma 2.27. *Let conditions H2.56–H2.58 hold, and A be the infinitesimal operator of an analytic semigroup $S(t)$.*

Then:

1. *There exists a unique almost periodic solution* $x(t) \in PCB[\mathbb{R}, X]$ *of* (2.82).
2. *The almost periodic solution* $x(t)$ *is asymptotically stable.*

Proof. We consider the function

$$x(t) = \int_{-\infty}^{t} S(t-s)f(s)ds + \sum_{t_k<t} S(t-t_k)b_k. \tag{2.91}$$

First, we shall show that the right hand of (2.91) is well defined.

From H2.57 and H2.58, it follows that $f(t)$ and $\{b_k\}$ are bounded, and let

$$max\Big\{||f(t)||_{PC}, ||b_k||_X\Big\} \leq M_0,\ M_0 > 0.$$

Using Lemma 2.26 and the definition for the norm in X^{α}, from (2.91), we obtain

$$\begin{aligned} ||x(t)||_{\alpha} = &\int_{-\infty}^{t} ||A^{\alpha}S(t-s)||_X||f(s)||_{PC}ds \\ &+ \sum_{t_k<t} ||A^{\alpha}S(t-t_k)||_X||b_k||_X \end{aligned}$$

$$\leq \int_{-\infty}^{t} K_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} ||f(s)||_{PC} ds$$
$$+ \sum_{t_k<t} K_\alpha (t-t_k)^{-\alpha} e^{-\lambda(t-t_k)} ||b_k||_X. \quad (2.92)$$

We can easy to verify, that

$$\int_{-\infty}^{t} K_\alpha (t-s)^{-\alpha} e^{-\lambda(t-s)} ||f(s)||_{PC} ds$$
$$\leq K_\alpha M_0 \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\lambda(t-s)} ds$$
$$\leq K_\alpha M_0 \frac{\Gamma(1-\alpha)}{\lambda^{1-\alpha}}. \quad (2.93)$$

Let $m = min\{t-t_k,\ 0 < t-t_k \leq 1\}$. Then from H2.58 and Lemma 1.2, the sum of (2.92) can be estimated as follows

$$\sum_{t_k<t} K_\alpha (t-t_k)^{-\alpha} e^{-\lambda(t-t_k)} ||b_k||_X$$
$$\leq K_\alpha M_0 \sum_{t_k<t} (t-t_k)^{-\alpha} e^{-\lambda(t-t_k)}$$
$$= K_\alpha M_0 \Big[\sum_{0<t-t_k\leq 1} (t-t_k)^{-\alpha} e^{-\lambda(t-t_k)}$$
$$+ \sum_{j=1}^{\infty} \sum_{j<t-t_k\leq j+1} (t-t_k)^{-\alpha} e^{-\lambda(t-t_k)} \Big]$$
$$\leq 2K_\alpha M_0 N \Big(\frac{m^{-\alpha}}{e^{-\lambda}} + \frac{1}{e^{\lambda}-1} \Big). \quad (2.94)$$

From (2.93), (2.94), and equality

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{sin\pi\alpha},\ 0 < \alpha < 1,$$

we have

$$||x(t)||_\alpha \leq K_\alpha M_0 \Big[\frac{\pi}{\Gamma(\alpha) sin\pi\alpha \lambda^{1-\alpha}} + 2N \Big(\frac{m^{-\alpha}}{e^{-\lambda}} + \frac{1}{e^{\lambda}-1} \Big) \Big],$$

and $x \in PCB[\mathbb{R}, X]$.

On the other hand, it is easy to see that the function $x(t)$ is a solution of (2.82).

Let $\varepsilon > 0,\ \tau \in T,\ q \in P$, where the sets T and P are from Lemma 2.25.

Then,

$$\begin{aligned}||\varphi(t+\tau)-\varphi(t)||_\alpha &= ||A^\alpha(x(t+\tau)-x(t))||_{PC}\\ &\le \int_{-\infty}^{t} ||A^\alpha S(t-s)||_X ||f(s+\tau)-f(s)||_{PC} ds\\ &+ \sum_{t_k<t} ||A^\alpha S(t-t_k)||_X ||b_{k+q}-b_k||_X \le M_\alpha \varepsilon,\end{aligned}$$

where $|t-t_k| > \varepsilon,\ M_\alpha > 0$.

The last inequality implies, that the function $x(t)$ is almost periodic. The uniqueness of this solution follows from conditions H2.56–H2.58 [126].

Let now, $\tilde{x} \in PCB[\mathbb{R}, X]$ be an arbitrary solution of (2.82), and $y = \tilde{x} - x$. Then, $y \in PCB[\mathbb{R}, X]$ and

$$y = S(t-t_0)y(t_0). \tag{2.95}$$

The proof that $x(t)$ is asymptotically stable follows from (2.95), the estimates from Lemma 2.26 and the fact that $i(t_0, t) - p(t-t_0) = o(t)$ for $t \to \infty$. □

Now, we shall investigate the almost periodic solutions of (2.81).

Introduce the following conditions:

H2.59. The function $F(t, x)$ is almost periodic with respect to $t \in \mathbb{R}$ uniformly at $x \in \Omega$, Ω is compact from X, and there exist constants $L_1 > 0,\ 1 > \kappa > 0,\ 1 > \alpha > 0$ such that

$$||F(t_1, x_1) - F(t_2, x_2)||_X \le L_1(|t_1 - t_2|^\kappa + ||x_1 - x_2||_\alpha),$$

where $(t_i, x_i) \in \mathbb{R} \times \Omega,\ i = 1, 2$.

H2.60. The sequence of functions $\{H_k(x)\},\ k = \pm1, \pm2, \dots$ is almost periodic uniformly at $x \in \Omega$, Ω is every compact from X, and there exist constants $L_2 > 0,\ 1 > \alpha > 0$ such that

$$||H_k(x_1) - H_k(x_2)||_X \le L_2 ||x_1 - x_2||_\alpha\ ,$$

where $x_1,\ x_2 \in \Omega$.

Theorem 2.25. *Let the following conditions hold:*

1. *Conditions H2.58–H2.60 hold.*
2. *A is the infinitesimal generator of the analytic semigroup $S(t)$.*
3. *The functions $F(t, x)$ and $H_k(x)$ are bounded:*

$$max\Big\{||F(t,x)||_X,\ ||H_k(x)||_X\Big\} \le M,$$

where $t \in \mathbb{R},\ k = \pm1, \pm2, \dots,\ x \in \Omega,\ M > 0$.

Then if $L = max\{L_1, L_2\},\ L > 0$ is sufficiently small it follows that:

1. *There exists a unique almost periodic solution $x \in PCB[\mathbb{R}, X]$ of* (2.81).

2. *The almost periodic solution $x(t)$ is asymptotically stable.*

Proof. We denote by $D^* \subset PCB[\mathbb{R}, X]$ the set of all almost periodic functions with points of discontinuity of the first kind $t_k,\ k = \pm 1, \pm 2, \ldots$, satisfying the inequality $||\varphi||_{PC} < h,\ h > 0$.

In D^*, we define the operator S^* in the following way

$$S^*\varphi(t) = \int_{-\infty}^{t} A^\alpha S(t-s) F(t, A^{-\alpha}\varphi(s))ds + \sum_{t_k < t} A^\alpha S(t-t_k) H_k(A^{-\alpha}\varphi(t_k)). \tag{2.96}$$

The facts that S^* is well defined, and $S^*\varphi(t)$ is almost periodic function follow in the same way as in the proof of Lemma 2.27. Now, we shall show, that S^* is a contracting operator in D^*.

Let $\varphi,\ \psi \in D^*$. Then, we obtain

$$\begin{aligned} &||S^*\varphi(t) - S^*\psi(t)||_X \\ &\quad\le \int_{-\infty}^{t} ||A^\alpha S(t-s)||_X ||F(t, A^{-\alpha}\varphi(t)) - F(t, A^{-\alpha}\psi(t))||_X ds \\ &\qquad + \sum_{t_k<t} ||A^\alpha S(t-t_k)||_X ||H_k(A^{-\alpha}\varphi(t_k)) - H_k(A^{-\alpha}\psi(t_k))||_X \\ &\quad\le LK_\alpha ||\varphi(t) - \psi(t)||_X \Big[\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\lambda(t-s)} ds \\ &\qquad + \sum_{t_k<t} (t-t_k)^{-\alpha} e^{-\lambda(t-t_k)} \Big]. \end{aligned}$$

With similar arguments like in (2.94), for the last inequality, we have

$$\begin{aligned} ||S^*\varphi(t) - S^*\psi(t)||_X \le LK_\alpha \Big[&\frac{\Gamma(1-\alpha)}{\lambda^{1-\alpha}} \\ &+ 2N\Big(\frac{m^{-\alpha}}{e^{-\lambda}} + \frac{1}{e^\lambda - 1}\Big)\Big] ||\varphi(t) - \psi(t)||_X. \end{aligned}$$

Then, if L is sufficiently small and

$$L \le \Big(K_\alpha \Big[\frac{\pi}{\Gamma(\alpha) sin\pi\alpha\lambda^{1-\alpha}} + 2N\Big(\frac{m^{-\alpha}}{e^{-\lambda}} + \frac{1}{e^\lambda - 1}\Big)\Big]\Big)^{-1},$$

it follows that the operator S^* is a contracting operator in D^*.

Consequently, there exists $\varphi \in D^*$ such that

$$\varphi(t) = \int_{-\infty}^{t} A^{\alpha} S(t-s) F(t, A^{-\alpha}\varphi(s)) ds$$
$$+ \sum_{t_k < t} A^{\alpha} S(t - t_k) H_k(A^{-\alpha}\varphi(t_k)). \tag{2.97}$$

On the other hand, since A^{α} is closed, we get

$$A^{-\alpha}\varphi(t) = \int_{-\infty}^{t} S(t-s) F(t, A^{-\alpha}\varphi(s)) ds$$
$$+ \sum_{t_k < t} S(t - t_k) H_k(A^{-\alpha}\varphi(t_k)). \tag{2.98}$$

Now, let $h \in (0, \theta)$, where θ is the constant from H2.56, and $t \in (t_k, t_{k+1} - h]$.

Then,

$$||\varphi(t+h) - \varphi(t)||_{\alpha}$$
$$\leq || \int_{-\infty}^{t} (S(h) - I) A^{\alpha} S(t-s) F(t, A^{-\alpha}\varphi(s)) ds||_{\alpha}$$
$$+ || \int_{t}^{t+h} A^{\alpha} S(t+h-s) F(t, A^{-\alpha}\varphi(s)) ds||_{\alpha}. \tag{2.99}$$

From Lemma 2.26 for (2.99), it follows that

$$||\varphi(t+h) - \varphi(t)||_{\alpha} \leq K_{\alpha+\beta} M C_{\beta} h^{\beta} + K_{\alpha} M \frac{h^{1-\alpha}}{1-\alpha}.$$

Then, there exists a constant $C > 0$ such that

$$||\varphi(t+h) - \varphi(t)||_{\alpha} \leq C h^{\beta}.$$

On the other hand, from H2.59 it follows that $F(t, A^{-\alpha}\varphi(t))$ is locally Hölder continuous. From H2.60 and the conditions of the theorem, $H_k(A^{-\alpha}\varphi(t_k))$ is a bounded almost periodic sequence.

Let $\varphi(t)$ be a solution of (2.97), and let consider the equation

$$\dot{x}(t) = Ax + F(t, A^{-\alpha}\varphi(t)) + \sum_{k=-\infty}^{\infty} H_k(A^{-\alpha}\varphi(t_k))\delta(t - t_k). \tag{2.100}$$

Using the condition H2.60 and Lemma 2.27, it follows that for (2.100) there exists a unique asymptotically stable solution in the form

$$\psi(t) = \int_{-\infty}^{t} S(t-s)F(s, A^{-\alpha}\varphi(s))ds + \sum_{t_k<t} S(t-t_k)H_k(A^{-\alpha}\varphi(t_k)),$$

where $\psi \in \mathcal{D}(A^{\alpha})$.

Then,

$$\begin{aligned} A^{\alpha}\psi(t) = &\int_{-\infty}^{t} A^{\alpha}S(t-s)F(s, A^{-a}\varphi(s))ds \\ &+ \sum_{t_k<t} A^{\alpha}H_k(A^{-\alpha}\varphi(t_k)) = \varphi(t). \end{aligned}$$

The last equality shows that $\psi(t) = A^{-\alpha}\varphi(t)$ is a solution of (2.81), and the uniqueness follows from the uniqueness of the solution of (2.97), (2.100) and Lemma 2.27. □

Example 2.5. Here, we shall consider a two-dimensional impulsive predator–prey system with diffusion, when biological parameters assumed to change in almost periodical manner. The system is affected by impulses, which can be considered as a control.

Assuming that the system is confined to a fixed bounded space domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, non-uniformly distributed in the domain $\overline{\Omega} = \Omega \times \partial\Omega$ and subjected to short-term external influence at fixed moment of time. The functions $u(t,x)$ and $v(t,x)$ determine the densities of predator and pray, respectively, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator and $\frac{\partial}{\partial n}$ is the outward normal derivative.

The system is written in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu_1 \Delta u + u\Big[a_1(t,x) - b(t,x)u - \dfrac{c_1(t,x)v}{r(t,x)v+u}\Big],\ t \neq t_k, \\ \dfrac{\partial v}{\partial t} = \mu_2 \Delta v + v\Big[-a_2(t,x) + \dfrac{c_2(t,x)u}{r(t,x)u+v}\Big],\ t \neq t_k, \\ u(t_k^+, x) = u(t_k^-, x)I_k(x, u(t_k,x), v(t_k,x)),\ k = \pm1, \pm2, \ldots, \\ v(t_k^+, x) = v(t_k^-, x)J_k(x, u(t_k,x), v(t_k,x)),\ k = \pm1, \pm2, \ldots, \\ \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0,\quad \dfrac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0. \end{cases} \tag{2.101}$$

The boundary condition characterize the absence of migration, $\mu_1>0$, $\mu_2 > 0$ are diffusion coefficients. We assume that, the predator functional response has the form of the ratio function $\dfrac{c_1 v}{rv+u}$. The ratio function $\dfrac{c_2 u}{rv+u}$ represents the conversion of prey to predator, a_1, a_2, c_1 and c_2 are positive functions that stand for prey intrinsic growth rate, capturing rate of the predator, death rate of the predator and conversion rate, respectively, $\dfrac{a_1(t,x)}{b(t,x)}$ gives the carrying capacity of the prey, and $r(t,x)$ is the half saturation function.

We note that the problems of existence, uniqueness, and continuability of solutions of impulsive differential equations (2.101) have been investigated in [7].

Introduce the following conditions:

H2.61. The functions $a_i(t,x),\ c_i(t,x),\ i=1,2,\ b(t,x)$ and $r(t,x)$ are almost periodic with respect to t, uniformly at $x \in \overline{\Omega}$, positive-valued on $\mathbb{R} \times \overline{\Omega}$ and locally Hölder continuous with points of discontinuity at the moments $t_k,\ k=\pm1,\pm2,\ldots$, at which they are continuous from the left.

H2.62. The sequences of functions $\{I_k(x,u,v)\}, \{J_k(x,u,v)\},\ k=\pm1,\pm2,\ldots$ are almost periodic with respect to k, uniformly at $x,u,v \in \overline{\Omega}$.

Set $w=(u,v)$, and

$$A = \begin{bmatrix} \lambda - \mu_1 \Delta & 0 \\ 0 & \lambda - \mu_2 \Delta \end{bmatrix},$$

$$F(t,w) = \begin{bmatrix} u\Big[a_1(t,x) - b(t,x)u - \dfrac{c_1(t,x)v}{r(t,x)v+u}\Big] + \lambda u \\ v\Big[-a_2(t,x) + \dfrac{c_2(t,x)u}{r(t,x)u+v}\Big] + \lambda v \end{bmatrix},$$

$$H_k(w(t_k)) = \begin{bmatrix} u(t_k,x)I_k(x,u(t_k,x),v(t_k,x)) - u(t_k,x) \\ v(t_k,x)J_k(x,u(t_k,x),v(t_k,x)) - v(t_k,x) \end{bmatrix},$$

where $\lambda > 0$.

Then, the system (2.101) moves to the equation

$$\dot{w}(t) = Aw + F(t,w) + \sum_{k=\pm1,\pm2,\ldots} G_k(w)\delta(t-t_k). \tag{2.102}$$

It is well-known [68], that the operator A is sectorial, and $Re\sigma(A) \le -\lambda$, where $\sigma(A)$ is the spectrum of A. Now, the analytic semigroup of the operator A is e^{-At}, and

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt.$$

Theorem 2.26. *Let for the equation (2.102) the following conditions hold:*

1. *Conditions H2.56, H2.61 and H2.62 are met.*
2. *For the functions* $F(t, w)$ *there exist constants* $L_1 > 0,\ 1 > \kappa > 0,\ 1 > \alpha > 0$ *such that*

$$||F(t_1, w_1) - F(t_2, w_2)||_X \leq L_1\big(|t_1 - t_2|^\kappa + ||w_1 - w_2||_\alpha\big),$$

   where $(t_i, w_i) \in \mathbb{R} \times X_\alpha,\ i = 1, 2.$
3. *For the set of functions* $\{H_k(w)\},\ k = \pm 1, \pm 2, \ldots$ *there exist constants* $L_2 > 0,\ 1 > \alpha > 0$ *such that*

$$||H_k(w_1) - H_k(w_2)||_X \leq L_2||w_1 - w_2||_\alpha.$$

   where $w_1,\ w_2 \in X_\alpha$
4. *The functions* $F(t, w)$ *and* $H_k(w)$ *are bounded for* $t \in \mathbb{R}, w \in X_\alpha$ *and* $k = \pm 1, \pm 2, \ldots$*.*

Then, if $L = max\{L_1, L_2\}$ *is sufficiently small, it follows:*

1. *There exists a unique almost periodic solution* $x \in\ PCB[\mathbb{R}, X]\ of$ (2.101).
2. *The almost periodic solution* $x(t)$ *is asymptotically stable.*

Proof. From conditions H2.61, H2.62 and conditions of the theorem, it follows that all conditions of Theorem 2.25 hold. Then, for (2.102) and consequently for (2.101) there exists a unique almost periodic solution of (2.101), which is asymptotically stable. □

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