

## Chapter 2

# Yang-Mills Theory and its Infrared Behavior

Yang-Mills theory [1] describes the gluonic part of the strong interaction. It is a gauge theory and consequently the choice of a gauge is required in functional approaches. This amounts to choosing one representative among all the physically equivalent gauge field configurations. In Sect. 2.1 I will describe the standard procedure for gauge fixing, as employed in perturbation theory, as well as the problems arising when going to the non-perturbative regime. An improved gauge fixing taking these into account will be presented in Chap. 6.

As in this thesis the aspects of the infrared (IR) behavior of correlation functions is investigated, I will describe in Sect. 2.2 two well known ways of connecting the IR behavior of propagators to confinement: first I explain the basic idea behind the Kugo-Ojima scenario [2, 3] and then I introduce the Gribov-Zwanziger scenario [4–7]. The idea behind the latter will be important for Chap. 6.

In the final section of this chapter I will shortly review the development of our understanding of the IR regime in the Landau gauge. This gauge is widely preferred in functional approaches because of its simplicity. Indeed calculations in all other gauges lead to one or more additional obstacles that have to be overcome. But although the Landau gauge has been intensively investigated for many years by now, there is no consensus yet about the solution for the Green functions in the IR. Two possible candidates have emerged, and even the variant that they correspond only to different non-perturbative extensions of the Landau gauge is discussed. I will describe the features of both solutions in this section. Furthermore, I will also explain the notion of positivity violation, which is a mathematical criterion for confinement.

### 2.1 The Action of Yang-Mills Theory

In this section the Lagrangian of Yang-Mills theory is introduced as the gluonic part of QCD. How to fix the gauge, the BRST symmetry [8–10] and properties of the Gribov region are described.

### 2.1.1 The Gauge Invariant Action

The Lagrangian density of QCD is [1, 11]

$$\mathcal{L}_{QCD} = \bar{q}(-\not{D} + m)q + \mathcal{L}_{YM}, \quad (2.1)$$

$$\mathcal{L}_{YM} = \frac{1}{2} \text{tr}\{F_{\mu\nu}F_{\mu\nu}\}. \quad (2.2)$$

The first term in  $\mathcal{L}_{QCD}$  describes the propagation of the massive quarks and their interaction with gluons via the covariant derivative  $D_\mu$ :

$$D_\mu = \partial_\mu + i g A_\mu. \quad (2.3)$$

The second term,  $\mathcal{L}_{YM}$ , constitutes the purely gluonic part of QCD and thus contains the propagation of gluons and their self-interactions. While it is not possible to find a gauge invariant Lagrangian for quarks without gluons, this is not true in the opposite direction, i.e.,  $\mathcal{L}_{YM}$  is gauge invariant and can be considered on its own without quarks. It is this term that is investigated in this thesis.

The quantity  $F_{\mu\nu}$  is called the field strength tensor and is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g [A_\mu, A_\nu]. \quad (2.4)$$

The gauge field  $A_\mu$  lives in an algebra defined by the hermitian generators  $T^r$  of a generic gauge group. They obey the relations

$$[T^r, T^s] = i f^{rst} T^t, \quad (2.5)$$

$$\text{tr}\{T^r T^s\} = T_f \delta^{rs}, \quad (2.6)$$

with  $T_f = \frac{1}{2}$  for  $SU(N)$ . The decomposition of the gauge field is

$$A_\mu = A_\mu^r T^r \quad (2.7)$$

and similarly for the field strength tensor:

$$F_{\mu\nu} = F_{\mu\nu}^r T^r, \quad (2.8)$$

$$F_{\mu\nu}^r = \partial_\mu A_\nu^r - \partial_\nu A_\mu^r - g f^{rst} A_\mu^s A_\nu^t. \quad (2.9)$$

In components the YM Lagrangian reads

$$\mathcal{L}_{YM} = \frac{1}{4} F_{\mu\nu}^r F_{\mu\nu}^r. \quad (2.10)$$

The transformation of the gluon field under which the QCD Lagrangian is invariant is given by

$$A_\mu^U(x) = U(x) A_\mu(x) U(x)^{-1} + \frac{i}{g} (\partial_\mu U(x)) U(x)^{-1}, \quad (2.11)$$

where  $U(x)$  is

$$U(x) = e^{i g \omega(x)} \quad (2.12)$$

with  $\omega(x)$  the Lie algebra valued gauge parameter:

$$\omega(x) = \omega^r(x) T^r. \quad (2.13)$$

Equation (2.11) is called a gauge transformation. In infinitesimal form it reads

$$A_\mu^r \rightarrow A_\mu^r + \delta A_\mu^r = A_\mu^r - \partial_\mu \omega^r - g f^{rst} \omega^s A_\mu^t = A_\mu^r - D_\mu^{rs} \omega^s, \quad (2.14)$$

where the covariant derivative in the adjoint representation  $D_\mu^{rs}$  is defined as

$$D_\mu^{rs} = \delta^{rs} \partial_\mu + g f^{rst} A_\mu^t. \quad (2.15)$$

As mentioned above the gauge group for the strong interaction is the compact simple Lie group  $SU(3)$ . In this thesis, however, I will for a large part adopt the gauge group  $SU(N)$ . The reason is that not only  $SU(3)$  per se is of interest, but also general  $SU(N)$  and especially  $SU(2)$ . In the latter case calculations can become easier and especially in the maximally Abelian gauge  $SU(2)$  is often employed as the Lagrangian simplifies considerably, see Chap. 5. General  $SU(N)$  or also other groups like  $G_2$ , see, for example, Refs. [12, 13], are investigated to determine the dependence of certain aspects of Yang-Mills theories on the gauge group. Also investigations in the so-called large  $N$  limit are performed [14].

### 2.1.2 Fixing the Gauge

To work with functional methods we cannot use the action Eq. (2.10) as problems appear at several points. For example, in the canonical quantization, see, e.g., Ref. [15], the commutator relations cannot be fulfilled, as the conjugate momentum of  $A_\mu$ , given by

$$\Pi_\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu(x))} = F_{0\mu}, \quad (2.16)$$

has no zero component  $\Pi_0$ . But this contradicts the equal-time commutation relation

$$[A_\mu(x), \Pi_\nu(y)] = i g_{\mu\nu} \delta(x - y). \quad (2.17)$$

If, however, a certain gauge is chosen,  $\Pi_0$  does no longer vanish and one can proceed as usual.

In the path integral the source of the problems is that one integrates over all gauge equivalent gauge field configurations, although the idea is only to take into account gauge non-equivalent, i.e., physically different configurations. In other words the defining symmetry of the theory makes the integration in the path integral overcomplete. Furthermore, one cannot calculate the propagator of the theory, as the gluon two-point function has zero eigenvalues. Consequently it is singular and cannot be inverted to get the propagator.

It is useful to introduce here the notion of a gauge orbit, which is the set of field configurations connected by a gauge transformation:

$$O[A] := \{A'_\mu | A'_\mu = A^U_\mu\}. \quad (2.18)$$

$A^U$  is given in Eq. (2.11). Ideally one takes only one representative per gauge orbit. The idea by Faddeev and Popov to achieve this was to restrict the integration in the path integral to a hyperplane [16]. This can be done by inserting unity, given by

$$1 = \Delta[A] \int \mathcal{D}U \delta(f[A^U]), \quad (2.19)$$

into the path integral. The delta functional defines the hyperplane  $f[A] = 0$  and  $\mathcal{D}U$  is an integration over group space. The factor  $\Delta[A]$  is the Jacobian arising from the transition from field variables  $A$  to gauge transformation variables  $U$ . The integration over  $U$  can be absorbed into the normalization of the path integral and the factor  $\Delta[A]$  can be calculated as

$$\Delta^{rs}[A] = \det \left( \frac{\delta f^r[A(x)]}{\delta \omega^s(y)} \right) =: \det M^{rs}(x, y), \quad (2.20)$$

where the color indices have been made explicit.  $M(x, y)$  is known as the Faddeev-Popov operator. Its determinant can be localized by the introduction of a pair of Lie algebra valued Grassmann fields, the so-called Faddeev-Popov ghosts  $\bar{c}$  and  $c$ :

$$\det M^{rs}(x, y) = \int \mathcal{D}[\bar{c}c] e^{\int dx dy \bar{c}^r(x) M^{rs}(x, y) c^s(y)}. \quad (2.21)$$

Although they have zero spin, they obey Fermi statistics, which underlines their status as auxiliary and not physical fields.

As an alternative to the restriction to a hyperplane one can relax this condition to a Gaussian distribution over the gauge orbit with the mean value defined as the original gauge fixing condition. One example are linear covariant gauges with a gauge fixing parameter  $\xi$  that gives the width of the distribution around the Landau gauge fixing condition  $\partial_\mu A_\mu = 0$ . In the limit of  $\xi \rightarrow 0$  the Landau gauge is recovered. One should note that such a Gaussian distribution includes all gauge copies. Since it is

normalized it does not lead to the same problems as the unnormalized integration before.

The gauge fixing part which is added to the Lagrangian density is then given by

$$\mathcal{L}_{gf} = \frac{1}{2\xi} (\partial_\mu A_\mu^r(x))^2 - \int dy \bar{c}^r(x) M^{rs}(x, y) c^s(y). \quad (2.22)$$

Also in the Landau gauge the full expression has to be added,

$$\mathcal{L}_{YM} \rightarrow \mathcal{L}_{YM} + \mathcal{L}_{gf}, \quad (2.23)$$

and only after inverting the gluon two-point function one can set  $\xi = 0$ . In the path integral the factor  $\exp(-\frac{1}{2\xi} (\partial_\mu A_\mu^r)^2)$  reduces to a delta functional in the limit  $\xi \rightarrow 0$ .

### 2.1.3 The Faddeev-Popov Operator and the Gribov Region

In the remainder of this section we will confine ourselves mostly to the Landau gauge in order to discuss some specific properties of the Faddeev-Popov operator and the Gribov region. In Landau gauge it is given by

$$M^{rs}(x, y) = M^{rs}(x) \delta(x - y) = -\partial_\mu D_\mu^{rs} \delta(x - y). \quad (2.24)$$

It is easy to demonstrate that the Landau gauge fixing is not complete as there are still gauge copies left. However, the fact that the gauge is not fixed completely by a local gauge fixing condition is not a specific property of the Landau gauge, but valid for all local gauge fixing conditions [17]. Starting with a gauge configuration fulfilling the Landau gauge condition,  $\partial_\mu A_\mu = 0$ , we can perform a gauge transformation according to Eq.(2.11) demanding that the result again fulfills the Landau gauge condition:

$$\partial_\mu A_\mu = 0 \rightarrow \partial_\mu A_\mu - \partial_\mu D_\mu \omega \stackrel{!}{=} 0 \Rightarrow M\omega \stackrel{!}{=} 0. \quad (2.25)$$

Hence if the Faddeev-Popov operator has zero modes, there are still gauge equivalent configurations left.

The first to treat this issue was Gribov [4]. He proposed to restrict the configuration in the path integral to the region  $\Omega$ , where the Faddeev-Popov operator  $M(A)$  is strictly positive:

$$\Omega := \{A | \partial_\mu A_\mu = 0, M(A) > 0\}. \quad (2.26)$$

Positivity entails that the operator is invertible, which is important as its inverse corresponds to the propagator of the Faddeev-Popov ghosts. Today the region  $\Omega$  is known as the Gribov region, which has the following properties in Landau gauge:

- The vacuum configuration,  $A_\mu = 0$ , lies in the Gribov region [18]. This can easily be verified by observing that it fulfills  $\partial_\mu A_\mu = 0$  and the Faddeev-Popov operator reduces to the Laplacian,  $M(0) = -\square$ , which is a positive operator. Since perturbation theory is an expansion around  $A = 0$ , one can understand that it yields good results, as long as the quantum fluctuations do not become large enough to feel the presence of the Gribov horizon.
- The Gribov region is bounded in all directions [18]. In this it differs decisively from the Gribov region of the maximally Abelian gauge [19, 20], which will be discussed in Sect. 5.3.3.
- It is convex, i.e., two arbitrary configurations  $A_1$  and  $A_2$  within the Gribov region can be combined to a new configuration  $A_3$  that lies again within the Gribov region as follows [18]:

$$A_3 = \alpha A_1 + (1 - \alpha) A_2, \quad 0 \leq \alpha \leq 1. \quad (2.27)$$

- Every gauge orbit passes at least once through the Gribov region [21]. This property is important so that one can restrict the integration to the Gribov region without missing any gauge orbits.
- Unfortunately there are still gauge copies within the Gribov region so that the gauge fixing is not complete [22, 23]. However, expectation values are not influenced by these additional copies [18].

The statement that every gauge orbit passes through the Gribov region is related to the fact that the Landau gauge condition can be derived by minimizing the functional

$$R[A] := \frac{1}{2} \int dx A_\mu(x) A_\mu(x) \quad (2.28)$$

with respect to infinitesimal gauge transformations, Eq. (2.14):

$$\begin{aligned} \delta R[A] &= \int dx A_\mu^r(x) \delta A_\mu^r(x) = \int dx A_\mu^r(x) (-D_\mu^{rs}(x) \omega^s(x)) = \\ &= \int dx A_\mu^r(x) (-\partial_\mu \omega^r(x) - g f^{rst} A_\mu^t(x) \omega^s(x)) = \\ &= \int dx (\omega^r(x) \partial_\mu A_\mu^r(x)). \end{aligned} \quad (2.29)$$

In order to have an extremum  $\partial_\mu A_\mu^r$  must vanish. The type of the extremum can be determined by the second derivative:

$$\delta^2 R[A] = \int dx (\omega^r(x) (-\partial_\mu D_\mu^{rs}(x) \omega^s(x))) = \int dx \omega^r(x) M^{rs}(x) \omega^s(x). \quad (2.30)$$

Here we recognize the Faddeev-Popov operator  $M^{rs}$ , i.e., if it is positive, we have a minimum of the functional  $R[A]$ . Thus the restriction to the Gribov region corresponds, as asserted above, to the minimization of the functional  $R[A]$ .

As the Gribov region is still plagued by gauge copies [22, 23], one can think about better ways of fixing the gauge. A natural choice for a unique gauge fixing is the set of gauge field configurations that corresponds to the absolute minimum of  $R[A]$ . This region is known as the fundamental modular region. It possesses a topologically non-trivial boundary where degenerate global minima exist that have to be identified. Taking the global minimum is also known as absolute Landau gauge and taking one arbitrary minimum as minimal Landau gauge. In lattice calculations one can employ algorithms that correspond to the latter or approximate the former and indeed finds an influence on the correlation functions [24, 25]. More on different Landau gauges on the lattice can be found in Sect. 6.1.3.

### 2.1.4 Replacing Gauge Symmetry by the BRST Symmetry

In the action fixed to a hyperplane in field configuration space gauge invariance is explicitly broken. However, there is another symmetry that takes its place. It is named BRST symmetry after Becchi et al. [9, 10] and Tyutin [8]. It is very useful in proving renormalizability and unitarity of a theory, see, for example, Refs. [3, 26]. The corresponding transformations can be derived from the standard gauge transformation by replacing the gauge parameter  $\omega$  by a ghost field  $c$ . Due to this choice the pure Yang-Mills part is trivially invariant. The invariance of the gauge fixing part is explained below.

It is convenient to introduce the gauge fixing condition via an auxiliary Lie-algebra valued field  $b$  that takes the role of a Lagrangian multiplier:

$$\int Db^r e^{-\int dx(ib^r f^r[A])} = N\delta(f^r[A]). \quad (2.31)$$

It is not dynamical and called Nakanishi-Lautrup field [27, 28]. Relaxing the gauge fixing condition  $f[A] = 0$  into  $f[A] = i\xi b/2$  leads to

$$\int Db^r e^{-\int dx(ib^r f^r[A] + \frac{\xi}{2}b^r b^r)} = Ne^{-\frac{1}{2\xi}(f[A])^2} \quad (2.32)$$

and corresponds to the Gaussian averaging over the gauge orbit in linear covariant gauges as described above.  $N$  is some normalization factor and  $\xi$  is called a gauge fixing parameter. It determines the width of the Gaussian distribution. Specifically  $\xi = 0$  is the original gauge fixing condition. However, one should keep in mind that this is only well defined if the solution to  $f[A] = 0$  is unique, what is not the case non-perturbatively.

In Landau gauge the off-shell BRST transformation reads:

$$s A_\mu^r = -D_\mu^{rs} c^s, \quad (2.33)$$

$$s c^r = -\frac{1}{2} g f^{rst} c^s c^t, \quad (2.34)$$

$$s \bar{c}^r = i b^r, \quad (2.35)$$

$$s b^r = 0. \quad (2.36)$$

Integrating out the Nakanishi-Lautrup field yields the on-shell form, where we just have to replace  $b^r$  by  $-i (\partial_\mu A_\mu^r)/\xi$ . A substantial property of the BRST symmetry is its nilpotency,  $s^2 = 0$ , in the off-shell case. Thus BRST can also be defined as follows: the gauge parameter of the gauge transformation is taken as the anti-commuting field  $c$ . Requiring nilpotency of this transformation, i.e.,  $s^2 A_\mu = 0$ , determines  $s c$ . Finally the fields  $\bar{c}$  and  $b$  are introduced as a BRST doublet, i.e., they have the trivial BRST transformations  $s \bar{c} = i b$  and  $s b = 0$ .

The nilpotency property allows an easy way to fix the gauge without the need for a path integral [29]. This is based on the observation that one can add any quantity that is the result of a BRST transformation, a so-called BRST exact quantity, to the Lagrangian without spoiling its BRST invariance. As the gauge fixing condition  $f[A]$  has ghost number zero and the BRST transformation itself raises the ghost number by one, we introduce the factor  $\bar{c}$  to get ghost number zero and add  $s(\bar{c} f[A])$ . In the case of the Landau gauge, where  $f[A] = \partial_\mu A_\mu$ , this prescription to fix the gauge directly leads to the known gauge fixing terms:

$$\begin{aligned} \mathcal{L}_{gf} &= s(\bar{c}^r f[A]^r) = s(\bar{c}^r \partial_\mu A_\mu^r) = \\ &= i b^r (\partial_\mu A_\mu^r) - \bar{c}^r \partial_\mu (-D_\mu^{rs} c^s) = i b^r (\partial_\mu A_\mu^r) - \bar{c}^r M^{rs} c^s. \end{aligned} \quad (2.37)$$

One minus sign stems from the anti-commutativity property of  $s$  and  $\bar{c}$ . This method can be used also for other gauges and allows the use of gauge fixing conditions depending on ghost fields [29]. It will be employed for the maximally Abelian gauge in Chap. 5, where it significantly simplifies the gauge fixing procedure. As the expectation value of any gauge invariant quantity remains unaffected by adding such a BRST exact form, one can very nicely see in this way of quantization that all physical observables are independent of the chosen gauge.

## 2.2 Aspects of the Asymptotic Infrared Regime

Solving a Dyson-Schwinger equation (DSE) numerically for the complete momentum region is a challenging task and it proves useful to know the asymptotic behavior of the correlation functions [30, 31]. For the ultraviolet (UV) perturbation theory naturally provides a good guideline. For the IR regime it is more complicated to



determine the qualitative behavior and thus it is worthwhile to develop analytic methods for its investigation. But the low momentum region is not only of interest for providing input for numerical calculations. One can also learn about the realization of different scenarios of confinement [32]. In this section I will give a short overview over confinement scenarios directly related to correlation functions. The description of an additional scenario, the dual superconductor picture, is deferred to Sect. 5.1 as it does not translate directly into conditions for the correlation functions. A purely mathematical criterion for confinement, violation of positivity, is described later in Sect. 2.3.

### 2.2.1 The Gribov-Zwanziger Confinement Scenario

As mentioned in Sect. 2.1.2, the usual gauge fixing is not sufficient for the non-perturbative regime. An improvement can be achieved by restricting the integration in field configuration space to the first Gribov region. This is explicitly realized by the so-called Gribov-Zwanziger action [4, 33]. Its derivation and IR analysis are described in Chap. 6. Here I only mention the qualitative consequences of this improved gauge fixing for the theory.

The main statements of the Gribov-Zwanziger confinement scenario in the Landau gauge are that the gluon propagator vanishes at zero momentum [7] and that the ghost propagator is IR enhanced, i.e., it diverges stronger than a simple pole [5, 6]. The interesting point is that these statements are realized in the Gribov-Zwanziger action already at the perturbative level. The reason is that the existence of the Gribov horizon, which is a non-perturbative object, is taken into account. The tree-level gluon propagator is

$$D_{A,\mu\nu}^{rs}(p^2) = \delta^{rs} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{p^2}{p^4 + 2 N g^2 \gamma^4}, \quad (2.38)$$

where  $N$  is the number of colors and  $\gamma$  a mass parameter.  $\gamma$  is not free but determined by the horizon condition, see Sect. 6.1.2. This condition has to be enforced in order to make the theory well-defined in the first place [4].

Naively the bare ghost propagator goes like  $1/p^2$ , but it turns out that the horizon condition leads to a cancelation of this term at one-loop level and the IR leading part of the ghost propagator goes like  $1/p^4$ . One can show this cancelation diagrammatically for the exact ghost propagator [34] and perturbatively it was checked up to two loops in Ref. [35].

### 2.2.2 The Kugo-Ojima Confinement Scenario

Although being based on completely different considerations as the Gribov-Zwanziger scenario, the mechanism proposed by Kugo and Ojima leads qualitatively to the same predictions for the IR behavior of the gluon and ghost propagators [2, 3]. The basis of the Kugo-Ojima construction is the existence of a global, non-perturbatively defined BRST symmetry. It was long not clear if the standard BRST, as given in Eq. (2.33), could be extended in this sense to the non-perturbative regime. Recent progress in this respect, however, leads exactly to this conclusion [36–39].

The importance of the BRST symmetry rests on the fact that the corresponding charge defines the physical state space of the underlying quantum field theory by its cohomology. In contrast to the total state space, which possesses an indefinite metric, the physical state space must have a positive metric in order to allow a probabilistic interpretation of expectation values. According to Kugo and Ojima physical states  $\phi \in \mathcal{V}_{phys}$  obey the condition

$$Q_B|\phi\rangle = 0, \quad (2.39)$$

where  $Q_B$  is the Noether charge corresponding to the BRST symmetry [3, 40]. Furthermore, one can distinguish between states with positive and zero norm,  $\Psi$  and  $\chi \in \mathcal{V}_0$ , respectively:

$$\langle\Psi|\Psi\rangle > 0, \quad \langle\chi|\chi\rangle = 0. \quad (2.40)$$

The zero norm states  $\chi$  are also called daughter states, as they are the BRST variation of unphysical states, the parent states  $|\alpha\rangle$ :

$$|\chi\rangle = Q_B|\alpha\rangle, \quad |\alpha\rangle \notin \mathcal{V}_{phys}. \quad (2.41)$$

Consequently the subspace  $\mathcal{V}_0 \in \mathcal{V}_{phys}$  is orthogonal to  $\mathcal{V}_{phys}$  and does not influence any amplitude in  $\mathcal{V}_{phys}$ . The physical Hilbert space equipped with a positive metric is then given by the completed quotient space

$$H_{phys} = \overline{\mathcal{V}_{phys}/\mathcal{V}_0}. \quad (2.42)$$

In summary states fall in one of the following three classes:

- BRST non-invariant states  $|\alpha\rangle$ : These are unphysical states, as the BRST charge does not annihilate them,  $Q_B|\alpha\rangle \neq 0$ .
- BRST exact states  $|\chi\rangle$ : They are the BRST variation of the states  $|\alpha\rangle$ .
- BRST invariant but not BRST exact states  $|\Psi\rangle$ : They define real physical states, i.e., they are BRST invariant,  $Q_B|\Psi\rangle = 0$ , and have a positive norm,  $\langle\Psi|\Psi\rangle > 0$ .

Kugo and Ojima could show that the states  $|\alpha\rangle$  and  $|\chi\rangle$  always appear in so-called quartets, which is given by a “Faddeev-Popov conjugated” pair of BRST doublets [3].

Using the notation  $|k, N\rangle$ , where  $k$  represents all quantum numbers except for the ghost number  $N$ , the four particles of a quartet can be denoted as follows:

$$\begin{aligned} |\phi_1\rangle &= |k, N\rangle, & |\phi_2\rangle &= Q_B |\phi_1\rangle = |k, N-1\rangle, \\ |\phi_3\rangle &= |k, -N\rangle, & |\phi_4\rangle &= Q_B |\phi_3\rangle = |k, -(N+1)\rangle. \end{aligned} \quad (2.43)$$

$|\phi_1\rangle$  and  $|\phi_2\rangle$  are one BRST doublet,  $|\phi_3\rangle$  and  $|\phi_4\rangle$  the other. Furthermore, the states  $|\phi_1\rangle$  and  $|\phi_3\rangle$  belong to the class of BRST non-invariant states, and  $|\phi_2\rangle$  and  $|\phi_4\rangle$  to the BRST exact states. The contributions of these four states always cancel and they will never appear as asymptotic observable states.

The only other possibility of states are states  $\Psi$  that are BRST invariant but not BRST exact, i.e., they obey  $Q_B |\Psi\rangle = 0$  and cannot be written as a BRST variation of another state. These states are called singlet states and should represent the physical states of the theory. For QCD this means that the fundamental, confined fields of quarks, gluons and ghosts should belong to some quartet, whereas physical states such as mesons, baryons and glueballs should be BRST singlets.

Based on the global color charge Kugo and Ojima derived a simple criterion for color confinement in Yang-Mills theory. Using the equation of motion of the gluon field the conserved global color current can be written as

$$J_\mu^a := \partial_\nu F_{\mu\nu}^a + \{Q_B, D_\mu^{ab} \bar{c}^b\}, \quad \partial_\nu J_\nu^a = 0. \quad (2.44)$$

The related charge is the global color charge

$$Q^a := G^a + N^a := \int d^3x (\partial_i F_{0i}^a + \{Q_B, D_0^{ab} \bar{c}^b\}), \quad (2.45)$$

where  $G^a$  and  $N^a$  correspond to the first and second terms in the integral. The first criterion of Kugo and Ojima is that  $\partial_i F_{0i}^a$  contains no discrete massless pole. The charge  $G^a$  then vanishes, as the integral is over a total derivative. If there were massless contributions,  $G^a$  would be ill-defined. The second criterion requires that  $N^a$  is zero and consequently the total color charge  $Q^a$  is also zero.

In the charge  $N^a$  the anti-ghost field appears. It belongs to the so-called elementary quartet and has a massless contribution from the asymptotic field  $\tilde{\gamma}^a$ . The asymptotic contributions due to the composite operator  $g f^{abc} A_\mu^c \bar{c}^b$  are given by  $u^{ab} \partial_\mu \tilde{\gamma}^b$  characterized by the dynamical parameter  $u^{ab}$ , which can be determined from the correlation function

$$\int dx e^{i p(x-y)} \langle (D_\mu^{ae} c^e)(x) g f^{bcd} A_\nu^d(y) \bar{c}^c(y) \rangle = \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) u^{ab}(p^2). \quad (2.46)$$

The condition for the charge  $N^a$  to be well-defined can be inferred from the asymptotic behavior of  $(D_\mu^{ab} \bar{c}^b)(x)$ , given by

$$(D_\mu^{ab} \bar{c}^b)(x) \stackrel{x_0 \rightarrow \pm\infty}{=} (\delta^{ab} + u^{ab}) \partial_\mu \bar{\gamma}^b(x), \quad (2.47)$$

as

$$u^{ab} \equiv u^{ab}(0) = -\delta^{ab}. \quad (2.48)$$

In summary the two confinement criteria due to Kugo and Ojima are:

1. There is no discrete massless pole in  $\partial_i F_{0i}^a$ .
2. The parameter  $u^{ab}$  has to be  $-\delta^{ab}$ .

The last point was taken up again by Kugo sixteen years after his article with Ojima, Ref. [3]. In Ref. [2] he derived how the Landau gauge ghost propagator, parametrized by

$$D_c^{ab}(p^2) = -\delta^{ab} \frac{c_c(p^2)}{p^2} \quad (2.49)$$

is related to the parameter  $u$ :

$$c_c(0) = \frac{1}{1+u}. \quad (2.50)$$

Hence the ghost propagator is more IR singular than a simple pole, if the criterion  $u = -1$  by Kugo and Ojima is met.

## 2.3 Solutions of Landau Gauge Yang-Mills Theory in the Infrared

The non-perturbative calculation of propagators and vertices is involved and in the course of history the emerging picture underwent some changes and improvements. The best investigated gauge in this context is Landau gauge for which I will give a short review in this section.

### 2.3.1 From Infrared Slavery to Ghost Dominance in the Landau Gauge

The Landau gauge is a preferred gauge in functional approaches for several reasons. First of all, it is the gauge with the simplest form of the action. There are only two fields which interact via three vertices. This simplicity is supported by additional information such as, for example, the fact that the system of the transverse components of Green functions is closed, see, e.g., Ref. [41], and the longitudinal terms can be discarded. Furthermore, there have been arguments that the ghost-gluon vertex stays bare in the IR [42, 43]. This was used as input at the beginning, but having understood the gauge better and better it could be proven directly from functional equations [44–46].

A well-known example of early non-perturbative calculations of the gluon propagator in the Landau gauge is due to Mandelstam [47]. Motivated by perturbation theory he neglected the ghost loop, which only contributes a small amount at high momenta, and the four-gluon vertex, which only occurs at two-loop level. With an approximated three-gluon vertex he obtained an IR divergent gluon propagator and also an IR singular running coupling. As such a gluon propagator apparently is perfectly suited to yield a linear rising potential by single gluon exchange, this picture was widely accepted and became known under the name IR slavery. The results of the Mandelstam approximation were subsequently confirmed in Refs. [48–50]. However, this is a perfect example of how assumptions based on perturbation theory can lead in the wrong direction in the non-perturbative regime. In the late 1990s it was shown in Refs. [31, 51] that the ghost propagator yields the most important contribution to the IR sector of Landau gauge Yang-Mills theory. In fact, it was found that the ghost contributions were dominating all DSEs [45, 52, 53] supporting the scenarios of ghost dominance as proposed by Gribov and Zwanziger [4, 6] and Kugo and Ojima [2, 3].

This solution is characterized by power laws for the dressing functions  $c_A(p^2)$  and  $c_c(p^2)$  of the gluon and ghost propagators, respectively:

$$D_A(p^2) = d_A \cdot (p^2)^{\delta_A}, \quad D_c(p^2) = d_c \cdot (p^2)^{\delta_c}. \quad (2.51)$$

$d_A$  and  $d_c$  are momentum independent coefficients and the qualitative behavior is determined only by the exponents, called infrared exponents (IREs). An important feature of these power laws is that the exponents depend only on one parameter, usually denoted by  $\kappa$ , as they are related by a so-called scaling relation:

$$\kappa := -\delta_c = \delta_A/2. \quad (2.52)$$

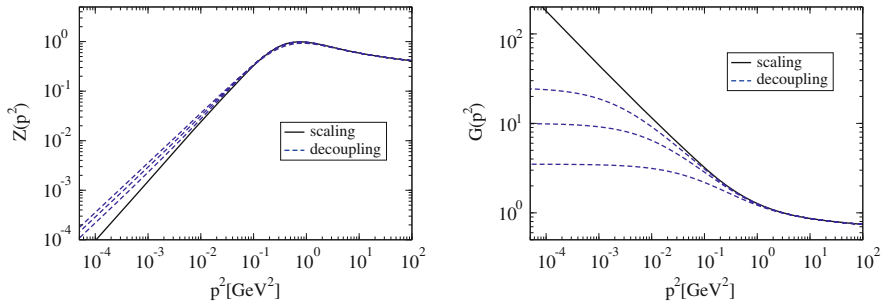
The value of  $\kappa$  can be calculated analytically. The most reliable value is  $0.5953 \dots$  [44, 54]. It is obtained for a bare ghost-gluon vertex and changes only slightly when an IR constant, but momentum dependent dressing is employed [44].

The fact that the ghost-gluon vertex is not IR enhanced was originally used as an input to solve the DSEs. It relied on an argument by Taylor [42, 43] and was confirmed by lattice [55–58] and DSEs studies [59, 60]. The self-consistency of this assumption was confirmed later on [52] and finally it was shown that this is indeed the only possible IR solution [45, 46]. This proof was possible by combining the two distinct systems of functional renormalization group equations and DSEs as suggested in [61]. A different approach with the same result was taken in Ref. [53], where as an assumption the existence of a stable skeleton expansion in the IR was used. It turned out that such an assumption necessarily holds as can be derived from functional renormalization group equations [46]. Details on the proof can be found in Sect. 4.2.3, where this emerges merely as a side result from the general analysis of the systems of functional equations.

### 2.3.2 The Decoupling Solutions and the Scaling Solution

Although the uniqueness of this solution in terms of scaling laws was established, the results did not agree with calculations on the lattice, see, for example, Refs. [56, 62–72]. For exceptions see Refs. [70, 73], the two-dimensional case [57, 68] and the strong coupling limit [74–76]. Indeed another solution was found with continuum methods that possesses quite different characteristics: the ghost propagator is not IR enhanced but stays bare in the IR, and the gluon propagator becomes finite instead of going to zero [41, 77–80]. This behavior is responsible for the name *decoupling solution*, as the gluons decouple below the scale given by their mass. One should note that a massive behavior for the gluon propagator is also possible in the formerly found solution, now called *scaling solution*, since the value of the parameter  $\kappa$  could be  $1/2$ . But then the ghost propagator would still be IR enhanced via the scaling relation Eq. (2.52). In the decoupling solution, however, the propagators do not obey an IR scaling relation and the ghost stays bare in the IR. Nevertheless there is some realization of ghost dominance in the decoupling solution, as the gluon is IR suppressed. Another qualitative difference between the two solutions is that the scaling solution is unique, while there exists a family of decoupling solutions [41]. In Fig. 2.1 both solutions are plotted.

The connection between the two solutions is the renormalization of the ghost propagator DSE [41]. Solving the DSEs one requires renormalization conditions and for the ghost propagator it is convenient to specify the renormalization condition at zero momentum. Choosing a finite value for the ghost dressing function yields a decoupling solution, where the value of the ghost dressing function at zero momentum is connected to the mass of the gluon [41]. For an infinite ghost dressing function at zero momentum the scaling solution is recovered [31, 41, 51]. One should note that it is not necessary to fix the value at zero momentum but the normalization prescription can be defined at any arbitrary momentum, since the values for different momenta are uniquely connected. Mathematically the necessary choice of the renormalization



**Fig. 2.1** Numerical solutions of the gluon and ghost two-point DSEs. Plotted are the gluon (*left*) and ghost (*right*) dressing functions for the decoupling and scaling solutions. Printed with permission from Elsevier from Ref. [41]

condition for the ghost dressing function corresponds to a boundary condition for the DSEs.

This is also related to an argument by Zwanziger that one can in principle cut the integration in field configuration space at the first Gribov horizon without modifying the form of the equations [54]. What is changed, however, are the boundary conditions. In case one cuts at the first Gribov horizon one uses as boundary condition that the ghost propagator is more enhanced than a simple pole, as derived from the horizon condition [5, 6, 34]. One can also argue for the same boundary condition with the Kugo-Ojima confinement scenario [2, 3] without referring to the Gribov region. Unfortunately it is not understood how these two arguments are related, since the implementation of the Kugo-Ojima condition  $u = -1$  into the action leads to the Gribov-Zwanziger action [81]. However, the Gribov-Zwanziger action breaks the BRST symmetry, while the Kugo-Ojima formalism relies on an unbroken BRST charge.

What speaks in favor that both solutions are valid is the fact that physical quantities do not seem to depend on the deep IR, the only region where the two solutions differ from each other. In Ref. [82] the transition temperatures of the confinement/deconfinement and the chiral transitions at vanishing chemical potential were calculated with identical results for both solutions.

In the present work I concentrate only on the scaling solution, which is more accessible to analytic approaches and offers explanations for many aspects of Yang-Mills theory. The most important one is certainly the confinement of gluons which can directly be inferred from the IR vanishing gluon propagator, as one can show analytically that the Schwinger function  $\Delta(t)$  is not positive [7]. This, however, is necessary to interpret a particle as physical [83, 84]. It boils down to show that

$$\Delta(t) := \int d^3x \int \frac{d^4p}{(2\pi)^4} e^{i x \cdot p} \sigma(p^2) \geq 0, \quad (2.53)$$

where  $\sigma(p^2)$  is a scalar function characterizing the propagator. The Fourier transformation of the gluon propagator at zero momentum is related to the Schwinger function by

$$D(p^2 = 0) = \int d^4x D(x - y) = \int dt \Delta(t). \quad (2.54)$$

Thus if  $D(p^2 = 0) = 0$ ,  $\Delta(t)$  is either zero or it has equally distributed positive and negative contributions. In this case one speaks of maximal positivity violation. But also for the decoupling solution a violation of positivity is observed both by lattice simulations [66, 85, 86] and functional equations [41].

A delicate point is the global BRST symmetry of the solutions. If a global symmetry exists that has the same form as the perturbative definition, Eq. (2.33), we know from the seminal paper of Kugo and Ojima how the propagators of the ghost and the gluon should behave in the deep IR [2, 3]: the ghost propagator should be IR enhanced and the gluon propagator should vanish at zero momentum. Only the scaling solution fulfills these criteria. However, one cannot exclude a different realization of a global BRST that leads to a different qualitative behavior in agreement with the decoupling solution.

Finally I would like to note that both the decoupling and the scaling solutions fulfill the criterion for a confining Polyakov loop potential [87] and thus quarks are confined. This is a sufficient criterion depending only on the asymptotic IR part, but in actual calculations it is found that the region responsible for confinement is the mid-momentum regime, where both solutions agree [87].

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