

Chapter 2

A Short Introduction to Critical Interfaces in 2D

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2.1 Introduction

The central goal of this chapter is to introduce stochastic Loewner evolution (SLE), but with a detailed emphasis on its interplay with statistical mechanics and conformal field-theory.

Stochastic Loewner evolutions describe growth processes, and as such they fall in the more general category of **growth phenomena**. These are ubiquitous in the real world at all scales, from crystals and plants to dunes and galaxies, and so on. They can be addressed in many ways, by deterministic or probabilistic methods, in discrete or continuous space and time. Understanding growth is usually a very difficult task. This is true even in two dimensions, the only case we mention in these notes. Yet in two dimensions, the powerful tools of complex analysis allow to tame the zoo of shapes. Indeed, many relevant growth processes involve the growth of domains (i.e. contractile open subsets of the Riemann sphere). *Riemann's uniformisation theorem* describes domains in a canonical way by conformal maps, and then growth of domains by Loewner chains. So at least the kinematic part is “easy” in two dimensions. This, and more, is explained in details in Sect. 2.3. To avoid any confusion, let us stress that being able to describe a growth process using tools from complex analysis and conformal geometry does not mean that the growth process itself is conformally invariant at all.

The mathematics in Sect. 2.2 are more down to earth. They involve mostly simple combinatorics. We concentrate on two simple examples of random geometric curves on the lattice, the *exploration process* and the *loop-erased random walk*. Both examples have been shown to have a continuum limit, moreover described by SLE.

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We explain how to make numerical simulations, and in particular how to estimate fractal dimensions. We recall how both the exploration process and the loop-erased random walk have a natural interpretation as interfaces in some (critical) statistical mechanics models called *loop models*. The exploration process leads easily to *percolation*, while loop-erased random walks lead to a more intricate combinatorics related to determinantal identities and symplectic fermions.

We also mention in passing another important but still very mysterious random growth model, *diffusion-limited aggregation* (DLA), which is expected to have a non-conformally invariant continuum limit. Simulations of DLA produce beautiful shapes, as do many other growth models. The universality classes are still debated, but DLA and its cousins, *Laplacian Growth*, the *Hele-Shaw problem*, *dielectric breakdown* are all important in various applications, though we shall not touch this subject in these notes.

Stochastic Loewner evolution is derived in Sect. 2.4, following Schramm's original argument. Its validity rests on two properties: *conformal invariance* which can be expected only in the continuum limit, and the *domain Markov property*, which holds already on the lattice in many models (including the exploration process and loop-erased random walks). The outcome if that conformally invariant measures on curves with the domain Markov property are parameterised by a single arbitrary positive number, named κ , which appears as the variance parameter of a Brownian motion.

SLE is a simple but particularly interesting example of growth process for which the growth is local and continuous so that the resulting set is a continuous curve, or at least is closely related to a continuous curve. What makes stochastic Loewner evolutions so important (at least in the author's view) is first that they are among the very few growth processes that can be studied analytically in great detail, and second that they have solved a problem that remained opened for two decades, the description of conformally invariant extended objects. For a physicist working in statistical mechanics, in particular for a conformal field-theorist, a rather startling feature of SLE is that it shows the *hidden Markovian character of critical interfaces*, which is far from obvious *a priori*. At the technical level, another striking feature of SLE is that it has turned many questions concerning interfaces that seemed just out of reach into exercises in stochastic calculus. Of course this simplicity is one of the blueprints of important discoveries. To give one illustration, we show how the locality property of the exploration process leaves $\kappa = 6$ as the sole possibility to describe its continuum limit.

The last section, Sect. 2.5 is devoted to the basics of the relationships between SLE and conformal field-theory (CFT). It is reassuring that the two subjects are closely interwoven. The basic statement is that conformal field-theory and stochastic Loewner evolutions are coupled in such away that CFT correlators are SLE martingales. The origin of this relationship is basically an instance of double counting, as we show in the last subsection. The consequence is that the Itô generator of SLE seen as a diffusion has to coincide with a particular singular vector differential operator. This allows to retrieve easily the relation between the central charge c (a CFT characteristic) and the SLE parameter κ :

$$2\kappa c = (6 - \kappa)(3\kappa - 8). \quad (2.1)$$

A more general consequence is that SLE probabilities have all the axiomatic properties of CFT correlators involving a special boundary-changing operator of dimension $\Delta = (6 - \kappa)/2\kappa$ inserted at the origin of the interface. We illustrate this by computing the simplest hitting probability either via stochastic calculus or via operator product methods. We also emphasise the importance of CFT partition functions to describe variants of SLE involving conditioning for instance, and more generally as a guiding principle to study SLE. Of course, physicists understand the relevance of partition functions because of their training in statistical mechanics, but this view is also shared by mathematicians now.

In the last paragraph, we wrote “*SLE probabilities have all the axiomatic properties of CFT correlators*” which may seem quite a twisted statement. Writing simply “SLE probabilities are CFT correlators” is OK with a nasty drawback: the CFTs of SLEs are badly behaved ones. Thirty years of CFT have made physicists comfortable with rational CFT, or unitary CFTs. The CFTs of SLEs are yet to be precisely defined, but they are definitely neither rational nor unitary. They are probably closest to certain logarithmic CFT. The operator content, the fusion algebra, etc, are unknown at the moment. The only correlators that are under control via general conformal invariance arguments are those with at most 4 boundary operators (or 1 bulk and 2 boundary operators). This may seem very negative. However, SLE is a perfectly well-defined mathematical object, and one can hope that this will give enough control to learn some general lessons on what CFT is about, outside the reassuring but limited regions of unitarity and/or minimality. There has been some recent progress in this direction.

2.2 Discrete Models

Random curves have focused the interest of physicists and mathematicians for decades. The simplest and perhaps oldest example is the symmetric **random walk** on the lattice or its continuous counterpart, **Brownian motion**. For dimensions ≤ 4 it is not a simple curve. On the other hand, polymers have a strong tendency to be *self-avoiding*, and they can be modelled crudely as simple random walks with a statistical weight giving fugacity μ to each monomer. But there is a wealth of interesting models of simple random walks. Among them are interfaces in 2D systems. Under certain circumstances, such systems are expected to have a continuum limit.

Recently a lot of progress has been achieved. Notably, *a classification of random curves in the continuum with certain special properties has been obtained*. It has received the name “**Stochastic Loewner Evolutions**” (SLE), and it is the subject of Sect. 2.4. It is hard to over-estimate the impact of SLE: it has given tools to solve formidable problems by routine computations, but moreover it has made it possible to prove that families of random simple walks and interfaces have a continuum limit.

The purpose of the examples that follow is to illustrate the connection between geometrical random curves and statistical mechanics. It turns out that partition functions under various disguises play a huge role in the study of SLE.

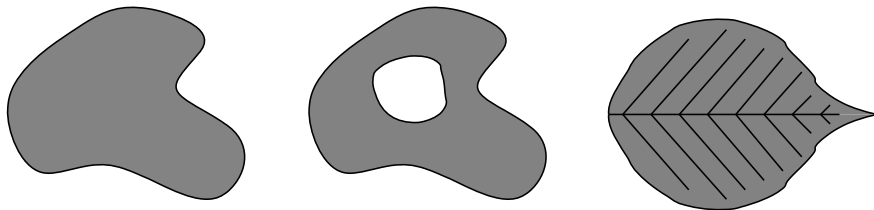


Fig. 2.1 *Left panel:* a smooth domain. *Middle panel:* a non-simply connected region is not a domain. *Right panel:* a non-smooth domain, whose topological boundary is not a simple curve

2.2.1 Discrete Domains

In what follows, a **domain** $\mathbb{D}$ is a non-empty open simply connected (i.e. no holes) strict subset of the complex plane $\mathbb{C}$. With this generality, domains and their topological boundaries can be quite complicated, as exemplified by the domain on the right in Fig. 2.1. There are mathematical theories to define a better notion of boundary suitable for our purposes. This can be achieved via the theory of “prime ends” or via the so-called *Poisson/Martin boundaries* that parameterise harmonic functions i.e. solutions h of the Laplace equation $\Delta_{\mathbf{r}} h = 0$, but we shall say only a few words about that in Sect. 2.3.1.

The complex plane admits regular tilings by hexagons, by triangles or by squares. The following definitions are given for hexagonal tilings, but they can easily be adapted for tilings by triangles and squares.

All hexagonal **tilings** can be obtained from one of them by similarities (in complex notation $z \mapsto \lambda z + \rho$). Fix such a tiling $\mathcal{T}$, for instance one whose hexagons have unit area. The plane is the disjoint union of vertices, open edges and open faces of $\mathcal{T}$: every point in the plane is either a **vertex**, or an **interior point** of an edge, or an interior point of a face.

A **hexagonal domain** $\mathbb{D}$ with reference $\mathcal{T}$ is a domain in the usual sense as defined above which is the union of vertices, open edges and open faces of $\mathcal{T}$.

This definition accommodates “smooth” domains like the left one in Fig. 2.2 whose boundary is a simple curve but also more irregular shapes like the middle one in Fig. 2.2 whose boundary is not a simple curve. If $\varepsilon > 0$ is much smaller than the size of an edge of $\mathcal{T}$, the points in the hexagonal domain $\mathbb{D}$ whose distance to the complement of $\mathbb{D}$ is ε form a simple curve, but the limit $\varepsilon \rightarrow 0^+$ is singular. The knowledge of the side from which a boundary point is approached is naively lost in the limit, but one can decide to keep track of it and this is the most useful definition of boundary in this context. For hexagonal domains we have thus a notion of **boundary** which makes it a curve even for a non-smooth domain. That such a boundary can also be defined for general domains is a non-trivial matter.

In these notes, an **admissible boundary condition** is a couple of distinct vertices (a, b) of $\mathcal{T}$, $a, b \notin \mathbb{D}$ such that there is (at least) a path from a to b in $\mathbb{D}$. A **path** (or **simple walk**) in $\mathbb{D}$ is a sequence $s_1, \dots, s_{2n+1}$, where $a = s_1$, $b = s_{2n+1}$, the s_{2m+1} , $1 \leq m < n$, (if any) are distinct vertices of $\mathcal{T}$ in $\mathbb{D}$ and the s_{2m} , $1 \leq m < n$, are

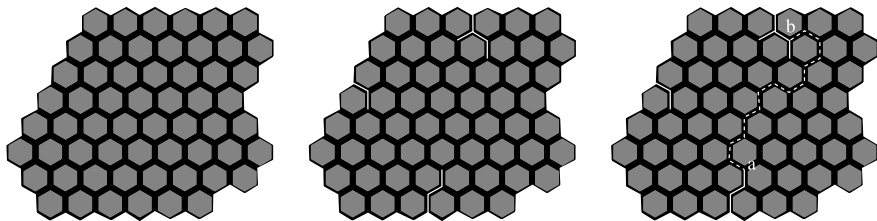


Fig. 2.2 *Left panel:* a “smooth” hexagonal domain. *Middle panel:* a non-smooth hexagonal domain. *Right panel:* an admissible boundary condition

distinct edges of $\mathcal{T}$ in $\mathbb{D}$ with boundary $\{s_{2m-1}, s_{2m+1}\}$. This is illustrated on the left of Fig. 2.2. Any such path splits $\mathbb{D}$ into a left and a right piece. The term “simple walk” is also used to mean a “path”.

If $s_1, \dots, s_{2n+1}$ is a path from a to b in $\mathbb{D}$ and $0 \leq m < n$, the set $\mathbb{D}'$ obtained by removing from $\mathbb{D}$ the sets s_l , $1 < l \leq s_{2m+1}$ is still a domain, and (s_{2m+1}, b) is an *admissible boundary condition* for $\mathbb{D}'$.

Our main interest in the next subsections will be in measures on paths from a to b in $\mathbb{D}$ when $\mathbb{D}$ is a domain and (a, b) an admissible boundary condition.

Hexagonal domains have a *special property* which is crucial for what follows. Suppose $(\mathbb{D}, a, b)$ is a hexagonal domain with admissible boundary condition. The right (resp. left) hexagons are by definition those which are on the right (resp. left) of every path from a to b in $\mathbb{D}$. Left and right hexagons are called **boundary hexagons**. The other hexagons of $\mathbb{D}$ are called **inner hexagons**.¹ Colour the left hexagons in black (say) and the right hexagons in white as in Fig. 2.3 on the left. If one colours the inner hexagons arbitrarily in black or white, then *there is a single path from a to b in $\mathbb{D}$ such that the hexagon on the left (resp. right) of any of its edges is black (resp. white)*. This is illustrated in Fig. 2.3 on the right. This path can be defined recursively because a is on the boundary of at least one left and at least one right hexagon: as a is not in $\mathbb{D}$, in any colouring there is exactly one edge in $\mathbb{D}$ with a on its boundary and bounding two hexagons of different colours. Start the path with this edge and go on.

If $\mathbb{D}$ is domain with a smooth boundary, it is easy to approximate it with high precision by hexagonal domains with reference tiling $\lambda\mathcal{T} + \rho$ by taking λ small enough. A general domain $\mathbb{D}$ may have a very complicated boundary, and approximations by hexagonal domains is not so obvious. Despite their importance, we shall remain silent on these subtleties. Also, it is useful for the general theory to have a quantitative notion of how close such an approximation is to the original domain and to have quantitative notion of *convergence* of approximations (when λ gets smaller and smaller) that guaranties that some phenomena on discrete domains (for instance some properties of certain statistical mechanics models) have an interpretation in the limit. We shall not give a formal definition of convergence, but simply mention that it exists.

¹Note that being a boundary or an inner hexagon depends on (a, b) .

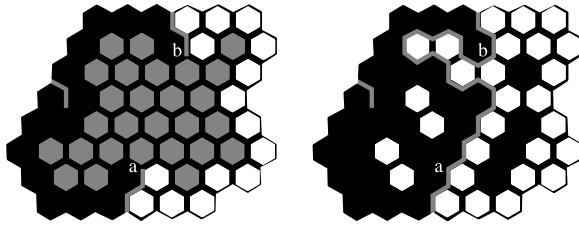


Fig. 2.3 *Left panel:* the boundary of a hexagonal lattice domain with boundary conditions. *Right panel:* the interface associated to a configuration

All the examples of **interfaces** we shall deal with in these notes can be defined on arbitrary hexagonal domains with admissible boundary condition, though sometimes we shall use square domains. Certain geometrical examples will define directly a law for the interface or a probabilistic algorithm to construct samples. Examples from statistical mechanics will give a weight for each colouring of the inner hexagons, and the law for the interface can be derived (at least in principle) from this more fundamental weight. The model of interface can depend on some parameters, called collectively p (for instance, temperature can be one of those).

Consider an interface model with parameter family p on discrete domains. Fix a domain with two marked boundary points, $(\mathbb{D}, a, b)$ and suppose it can be approximated by a “convergent” sequence of discrete domains with boundary conditions $(\mathbb{D}_n, a_n, b_n)$ whose reference tiling $\lambda_n \mathcal{T}$ has scale $\lambda_n \rightarrow 0^+$. A continuum limit exists when there is a (domain independent) function $p(\lambda)$ such that the distribution of interfaces in $(\mathbb{D}_n, a_n, b_n)$ with parameters $p(\lambda_n)$ converges to some limit. The limiting value $p(0)$ is called the **critical value** and is denoted by p_c . The choice $p(\lambda) = p_c$ leads to a scale-invariant theory.

A map $f: \mathbb{D} \rightarrow \mathbb{D}'$ between two domains sending marked boundary points to marked boundary points (i.e. $f(a) = a'$ and $f(b) = b'$) is said to be **conformal** if it preserves angles. Riemann’s theorem, to be explained in more detail in Sect. 2.3.1 asserts the existence of such maps. One can then ask, for a given interface model, whether the distribution of interfaces in $(\mathbb{D}, a, b)$ and in $(\mathbb{D}', a', b')$ are conformally equivalent. This can be checked numerically on good lattice approximations of these domains.

In two dimensions, scale-invariance plus locality is often enough to ensure conformal invariance. Thus the limiting theory of a discrete model at $p = p_c$ is a good candidate for conformal invariance. More generally, there is often a **threshold function** $p_s(\lambda)$ such that if $p(\lambda) - p_c = o(p_s(\lambda) - p_c)$ the limiting continuum theory is the same as the critical theory, if $p(\lambda) - p_c \sim p_s(\lambda) - p_c$ a limiting continuum theory exist but is not scale-invariant, and if $p_s(\lambda) - p_c = o(p(\lambda) - p_c)$ the limiting theory either does not exist or is trivial in some sense (concentrated on a single curve for instance). It is clear that only the small- λ behaviour of p_s matters and commonly $p_s(\lambda) - p_c$ can be taken to be simply a power of λ . The exponent is one of the critical exponents of the model.

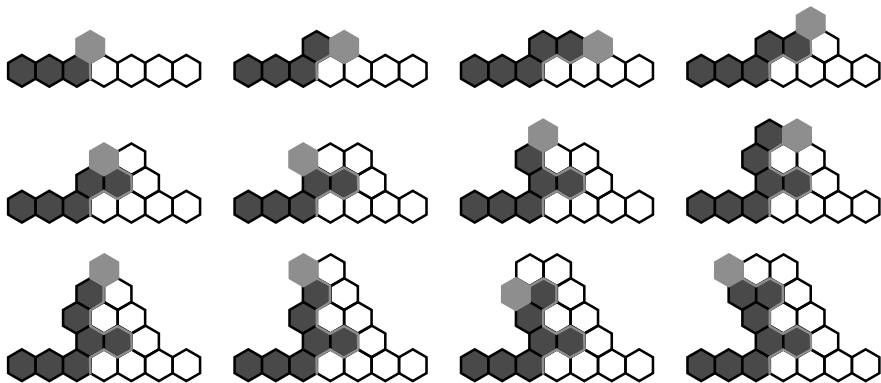


Fig. 2.4 The exploration process

2.2.2 The Exploration Process and Percolation

2.2.2.1 Definition of the Exploration Process

Let $(\mathbb{D}, a, b)$ be a hexagonal domain with admissible boundary condition. Colour the left hexagons in black (say) and the right hexagons in white. If a is incident to no inner hexagon of $\mathbb{D}$, all paths from a to b in $\mathbb{D}$ start with the same edge. Else, a is incident to exactly one inner hexagon of $\mathbb{D}$. Colour it black or white using a biased coin (say black has probability p and white $1 - p$), and make a step along the edge of $\mathbb{D}$ adjacent at a whose adjacent faces have different colours. Then remove from $\mathbb{D}$ the edge corresponding to the first step and its other end point, call it $\hat{a}$ to get a new domain $\mathbb{D}$. If $\hat{a} = b$ stop. Else $(\mathbb{D}, \hat{a}, b)$ is a new hexagonal domain with admissible boundary condition and one can iterate as shown in Fig. 2.4. Each choice of colour is made independently of the preceding ones but with the same bias. This random process is called the **exploration process**, and by construction it results in a simple path from a to b .

The fact that at some times the next step can be decided without tossing (for example, in Fig. 2.4, for the transition from the second picture to the third one, the choice of Colour for one hexagon is enough to fix two steps of the exploration path) results in a subtle interaction between the abstract independent coin tossings and their intricate effect on the geometry of the path.

There is exactly one coin toss for each inner face of $\mathbb{D}$ touching an edge of the path: this toss takes place the first time the inner face is touched by the tip of the path. In the rest of the process, this face becomes a boundary hexagon. But the path can have more than one edge along it.

The exploration process has a very important property: **locality**. It means that if $(\mathbb{D}, a, b)$ and $(\mathbb{D}', a, b)$, $\mathbb{D}' \subset \mathbb{D}$ are two hexagonal domains with the same admissible boundary condition (a, b) , the distributions of the exploration path in $\mathbb{D}$ and in $\mathbb{D}'$ coincide up to the first time the exploration path touches a boundary hexagon of

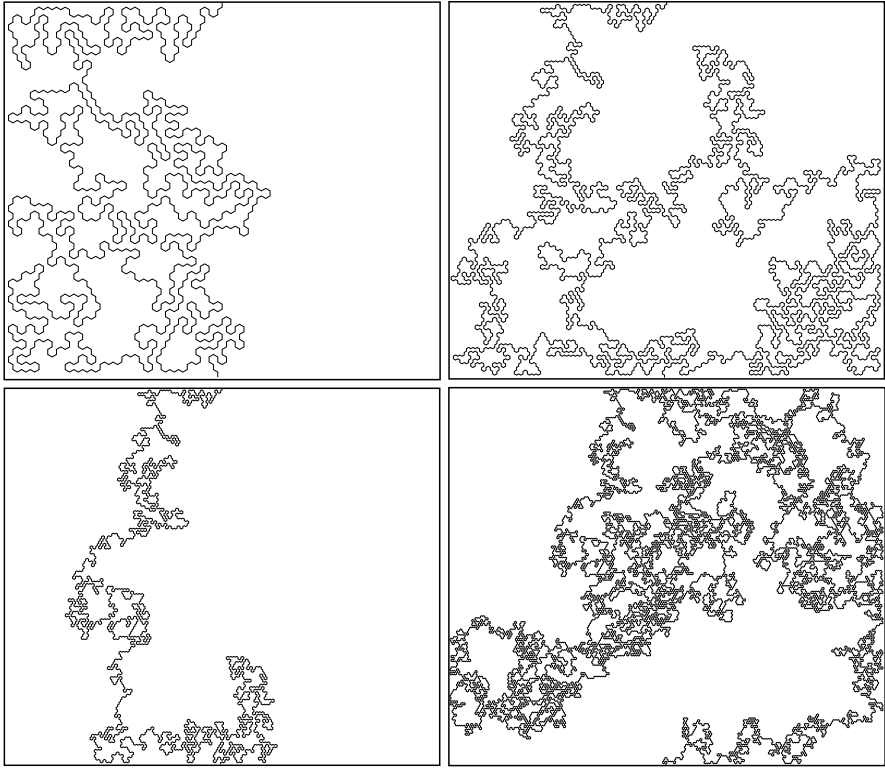


Fig. 2.5 Samples of the exploration process for increasing sizes

$\mathbb{D}'$ which is an inner hexagon of $\mathbb{D}$. This notion of locality should not be confused with the notion of locality in quantum field-theory.

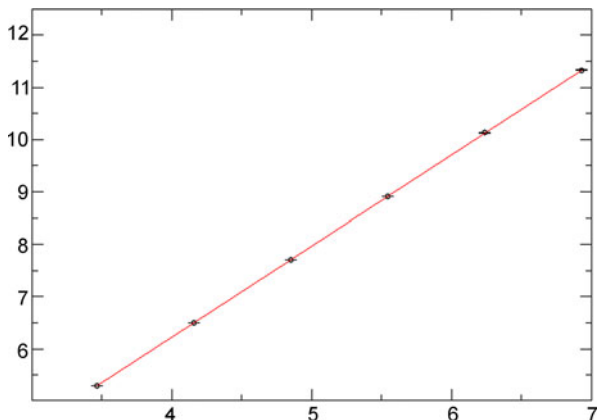
By symmetry, if there is a single value of p for which the theory is critical, it has to be $p_c = 1/2$ and the numerics confirms this intuition. Figure 2.5 shows a few samples of the symmetric exploration process. They join the middle horizontal sides of similar rectangles of increasing size. The pseudo-random sequence is the same for the four samples.

Even for small samples, the exploration process makes many twists and turns. By construction, the interface is a simple curve, but with the resolution of the figure, the exploration process for large samples does not look like a simple curve at all!

To estimate the (Hausdorff, fractal) **dimension** of the symmetric exploration process, one can generate samples in similar rectangular domains of different sizes and made the statistics of the number of steps S of the interface as a function of the size L of the rectangle domain. One observes that $S \propto L^\delta$ and a modest numerical effort (a few hour of CPU) leads to $\delta = 1.75 \pm .01$. To get an idea of how small the finite size corrections are, observe Fig. 2.6.

The exploration process is build by applying local rules involving only a few nearby sites, and we could wave our hands to argue that its scale-invariance (for

Fig. 2.6 The logarithm of the average length of the exploration path versus the logarithm of the domain size. The numerical results are the circles, the straight line is the linear regression, the error bars are shown



$p = p_c = 1/2$) should imply its conformal invariance in the continuum limit. Fortunately, hand-waving is not needed because *the exploration process* (on hexagonal domains) is one which *has been rigorously proved to have a conformally invariant distribution in the continuum limit, the fractal dimension being exactly $7/4$* . As suggested by numerical simulations, the continuum limit does not describe simple curves but curves with a dense set of double points, and in fact the—to be defined later— SLE_6 process describes not only the growth of the exploration path but also the growth of the **exploration hull**, which is the complement of the set of points that can be joined to the end point by a continuous path that does not intersect the exploration path. As we shall explain in detail in Sect. 2.4.5, among SLE_κ 's, SLE_6 is the only one that satisfies locality, so what is really to prove in this case is conformal invariance in the continuum limit (a non-trivial task), and the value of κ is for free.

2.2.2.2 Relation to Percolation and Coupling

The exploration process has been presented as a growth process, but in fact it is related to statistical mechanics in a simple way. Indeed, suppose that once the exploration sample has been constructed one tosses repeatedly (independently) the same coin to Colour also the hexagons that have not been coloured during the construction of the path. One gets a configuration in which all hexagons have been coloured independently, and from which the exploration path can be reconstructed as the sole curve joining the marked points with boarding hexagons all black on the left side and white on the right side. So one could also construct exploration samples by colouring all the hexagons independently at once and then drawing the separating curve. To summarise, *the exploration path is the interface for the statistical mechanics of percolation*.

Of course this approach is a poor idea for numerical simulations of the exploration process for a fixed p , because many hexagons are coloured for no use. But it has several advantages. First, it shows plainly that the law for the exploration

process is reversible (i.e. the choice of which of the two marked is used to start the exploration path is irrelevant). Second, percolation can be studied with other boundary conditions. Third, it makes it possible to use the powerful probabilistic tool of *coupling*, to which we turn now.

It happens frequently that on some **measure space** $(A, \mathcal{F})$ one has to deal with a family of probability laws P_u where u is some parameter. Quite often the parameter u takes only two values, but this is not mandatory. In some favourable circumstances, one can find another **probability space** $(E, \mathcal{G}, \mu)$ and a family of measurable maps $f_u : E \rightarrow A$ such that the image measure of μ by f_u is P_u : if B is a measurable subset of A (i.e. $B \in \mathcal{F}$) then $f_u^{-1}(B)$ is a measurable subset of E (i.e. $f_u^{-1}(B) \in \mathcal{G}$) and $P_u(B) = \mu(f_u^{-1}(B))$. Thus we can fix a configuration in E and by changing u see a “movie” of configurations in A . This is known as **coupling**.

Abstractly, couplings always exist, but these general constructions are of little or no use. A coupling useful to tackle a given situation does not always exist, and even it does, it may take a good amount of creative skills to discover it.

However, in the case of percolation, it is easy to find a useful coupling. Let the parameter u vary in $[0, 1]$. If H is the set of inner hexagons (the ones whose colours are not fixed by boundary conditions) of some finite hexagonal domain, set $A = \{b, w\}^H$ with $\mathcal{F} = \mathcal{P}(A)$ (all subsets of A are measurable), and set $E = [0, 1]^H$ with μ the product Lebesgue measure. So A is the set of assignments of a colour, b (lack) or w (hite), to each inner hexagon, and E is the set of assignments of a real number $\in [0, 1]^H$ to each inner hexagon. A configuration in A can be seen equivalently as a map from H to $\{b, w\}$, or as a partition of H in black and white. If $x \in [0, 1]$, set $f_u(x) = b$ if $u < x$ and $f_u(x) = w$ if $u \geq x$. Use the product structure of A and E to define $f_u : E \rightarrow A$ so that an hexagon h is white if and only if its assigned value is $\geq u$. Obviously the image measure of μ by f_u colours the inner hexagons independently, each being black with probability $1 - u$ and white with probability u .

In such a setting, a useful tool is **Russo’s formula**. Let us derive it abstractly and then interpret it. Suppose we partition A in two subsets $A = B \cup W$ in such a way that being in W is a so-called **increasing property**: if $\gamma \in W$ and if $\gamma' \in A$ is such that all hexagons which are white in the configuration γ are also white in the configuration γ' then $\gamma' \in W$. We order $\{b, w\}$ by saying that $w > b$ and use this to define a partial order: $\gamma' \geq \gamma$ if and only if all hexagons which are white in the configuration γ are also white in the configuration γ' . Viewing γ and γ' as maps from H to $\{b, w\}$, this says that if $\gamma \in W$ and $\gamma' \geq \gamma$ then $\gamma' \in W$. Then it is intuitively clear, and coupling makes it obvious, that $P_u(W)$ is an increasing function of u . If $\gamma \in A$ is a configuration, call a hexagon h **pivotal** for $A = B \cup W$ in the configuration γ either if $\gamma \in B$, h is coloured in black and changing it into white yields a configuration in W or if $\gamma \in W$, h is coloured in white and changing it into black yields a configuration in B . In the first case, we say that h is **pivotal** in γ to enter W and in the second case that h is pivotal in γ to enter B . In each configuration $\gamma \in A$ there is a certain number (possibly 0) of pivotal hexagons $\Pi(\gamma)$, and Π is thus a random variable on A . **Russo’s formula** states that

$$\frac{d}{du} P_u(W) = E_u(\Pi), \quad (2.2)$$

i.e. that the derivative of $P_u(W)$ is the expected number of pivotal points for the probability P_u .

Proof The proof is easy. We shall prove a slightly more refined identity, namely that if $\Pi_W(\gamma)$ is the number of hexagons in γ pivotal to enter W then $(1-u)\frac{d}{du}P_u(W) = E_u(\Pi_W)$. By symmetry, if $\Pi_B(\gamma)$ is the number of hexagons in γ pivotal to enter B then $u\frac{d}{du}P_u(W) = E_u(\Pi_B)$. As $\Pi = \Pi_W + \Pi_B$, the sum of these two equalities gives Russo's formula. Suppose $0 \leq u < v \leq 1$. By definition $P_v(W) - P_u(W) = \mu(f_v(X) \in W) - \mu(f_u(X) \in W)$ and by the increasing property of W this is $\mu(f_v(X) \in W \text{ and } f_u(X) \notin W)$. We can split this as a double sum to get

$$P_v(W) - P_u(W) = \sum_{\beta \in B} \sum_{\omega \in W} \mu(f_v(X) = \omega \text{ and } f_u(X) = \beta).$$

Note that the summand can be non-zero only if $\beta < \omega$ i.e. if one can go from β to ω by turning some black hexagons to white ones because this is what happens to $f_\cdot(X)$ for a fixed X by tuning the parameter from u to v . For a given X the hexagons h that change colour are those for which $X(h) \in]u, v]$, so in the above double sum only the pairs (β, ω) which disagree on a single hexagon can contribute to first order in $v - u$.

For instance we can sum first over β 's to get

$$\begin{aligned} P_v(W) - P_u(W) &= \sum_{\beta \in B} \sum_{h \text{ pivotal in } \beta} \mu(f_u(X) = \beta \text{ and } X(h) \in]u, v]) + O((v-u)^2). \end{aligned}$$

But by the definition of μ and f_u ,

$$\mu(f_u(X) = \beta \text{ and } X(h) \in]u, v]) = \mu(f_u(X) = \beta) \frac{v-u}{1-u} = P_u(\beta) \frac{v-u}{1-u}.$$

In consequence,

$$P_v(W) - P_u(W) = \frac{v-u}{1-u} \sum_{\beta \in B} P_u(\beta) \#\{\text{pivotal points in } \beta\} + O((v-u)^2).$$

The sum is just the expected number of pivotal points to enter W for P_u , and taking the limit leads to the announced result:

$$(1-u)\frac{d}{du}P_u(W) = E_u(\Pi_W).$$

Had we decided to sum first over ω 's, we would have obtained

$$u\frac{d}{du}P_u(W) = E_u(\Pi_B). \quad \square$$

Now that we have proved Russo's formula abstractly, let us apply it to a concrete decomposition of A relevant for percolation. Take a domain and split its boundary into four segments, such that the colours of the hexagons are fixed on each segment but alternate from one segment to the next as in Fig. 2.7. Then a simple topological argument shows that in any configuration either there is a black cluster connecting

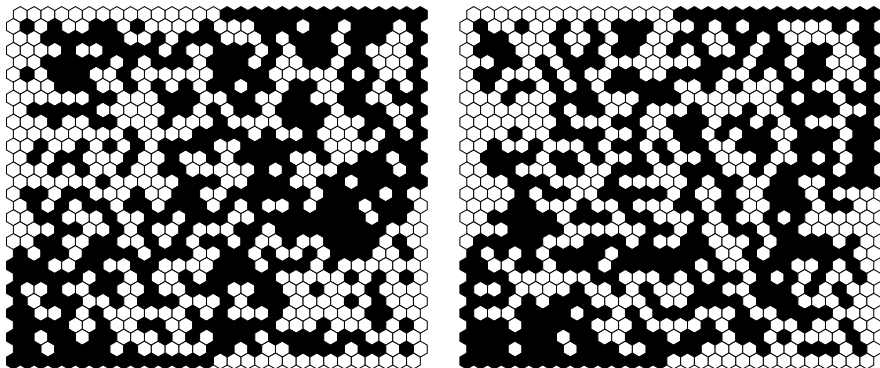


Fig. 2.7 *Left panel:* a sample of critical percolation with a black crossing and no pivotal point. *Right panel:* a sample of critical percolation with a white crossing and several pivotal points. Find these!

the two black boundary components, or there is a white cluster connecting the two white boundary components. In the first case put the configuration in B and in the second case put it in W . That being in W is an increasing property is clear. **Pivotal hexagons** are the ones which change the colour of the connecting cluster, so they have a impact on the long range properties of a configuration. Figure 2.7 shows two samples.

Such pivotal points could be called “global pivotal points” because they are defined with respect to global boundary conditions. However, in an arbitrary configuration of percolation one can look at windows of a certain size and define pivotal points with respect to that window. Anyway, at $u = p_c = 1/2$, the number of pivotal points can be shown to behave like $(L/\lambda)^{3/4}$ where L is the linear size of the system and λ is the scale of the tiling. So $p_s(\lambda) - 1/2 = \lambda^{3/4}$ is a good candidate for the threshold function: by coupling, changing u from the critical value $1/2$ to $1/2 + g\lambda^{3/4}$ just flips of order 1 pivotal points, and Russo’s formula indicates that $P_{1/2+g\lambda^{3/4}}(W) - P_{1/2}(W)$ is a finite function of g in the continuum limit. The validity of this threshold function is in fact rigorously proved (only in some weak sense at the moment, but progress is rapid).

At that point, an instructive subtlety enters the game. As physicists, we expect that a continuum field-theory describing the vicinity of the critical point exists. This theory will depend on a renormalised parameter that can be taken to be g or a correlation length. We also expect that correlation functions (involving a finite number of points) in this theory will depend smoothly on g . However, one can show in this example that in the case of percolation, *the probability measure on the interface, which is even a more global observable, does not depend smoothly on g .*

Before we consider percolation, which is already complicated, we take a glance at a simpler case, $1D$ random walks. This will allow us to explain the intuitive meaning of the smoothness statement.

Let λ be the lattice spacing, p be the probability to make a step to the right and $1 - p$ the probability to make a step to the left. Suppose we make n steps. So a

walk starting at point x has its endpoint at $x + S_n$ where $S_n = \lambda(\varepsilon_1 + \dots + \varepsilon_n)$. The random variables $\varepsilon_1, \dots, \varepsilon_n$ are independent and take value 1 or -1 with probability p or $1 - p$ respectively. It is clear that coupling can be used in this case and an analysis analogous to percolation could be carried out. One way to do so would be to consider an interval of length L containing the origin and take $S_0 = x \in [0, L]$. Then $P = P(p, \lambda, x, L)$, the probability that the first exit of $]0, L[$ is at L , plays the role of a **crossing probability** in percolation. The analysis is elementary but as the outcome is well-known, we do not go into the details. The salient features are that $p_c = 1/2$ is the critical value, that the number n of steps to exit the interval is $\sim (L/\lambda)^2$ and that $p_s = 1/2 + \lambda$ is a threshold function.

For given p , the mean (expected value) of S_n , which measures the asymmetry between steps to the right and steps to the left, is $\lambda n(2p - 1)$ and the fluctuation (variance) of S_n is $2\lambda\sqrt{np(1-p)}$. When the fluctuation is much smaller than the mean, a look at a sample will be enough evidence to decide with overwhelming confidence that it was not drawn with a symmetric distribution. However, if the asymmetry and its fluctuations are of the same order, one cannot know for sure. In the continuum limit, setting $p = 1/2 + g\lambda$ and $n \sim (L/\lambda)^2$, we find that the mean is $\sim L^2g$ and that the fluctuation is $\sim L$. Both these quantities are finite for $\lambda \rightarrow 0^+$, so it is impossible to decide whether a sample long enough to exit the interval was drawn with the symmetric distribution or not. A typical sample for the symmetric distribution is also typical for the asymmetric continuum distribution. Another way to check this statement is to look at a symmetric sample of n steps. Its probability is $(1/2)^n$. Now for the asymmetric distribution, the same sample has probability $(p(1-p))^{n/2}(p/(1-p))^{S_n/2\lambda}$. The ratio, which quantifies how less likely the symmetric sample is under the asymmetric distribution, is $(4p(1-p))^{n/2}(p/(1-p))^{S_n/2\lambda}$. Again, if $p = 1/2 + g\lambda$ and $n \sim (L/\lambda)^2$, the ratio is finite when $\lambda \rightarrow 0^+$, a fact which can be seen as a (weak) form of the statement that the ratio between the symmetric distribution and the asymmetric continuum distribution is finite: the continuum distribution depends smoothly on g .

Now we can go back to the case of percolation. We shall give first a very crude argument and then a crude one. A rigorous proof is really involved.

The typical fluctuation of the number of occupied sites in a percolation sample is $\sim L/\lambda$ because there are $\sim (L/\lambda)^2$ independent sites. However if $p = 1/2 + g\lambda^{3/4}$, the typical asymmetry is $\sim g\lambda^{3/4}(L/\lambda)^2 = gL^{3/4}(L/\lambda)^{5/4}$, which is much larger than the fluctuation $\sim L/\lambda$ when $\lambda \rightarrow 0^+$, so one can assert with certainty that an individual sample is critical or not. In fact, the same counting implies that on any set containing $\sim (L/\lambda)^d$ hexagons and chosen independently of the percolation sample, the asymmetry $\sim g\lambda^{3/4}(L/\lambda)^d$ is much larger than the fluctuation $\sim (L/\lambda)^{d/2}$ as soon as $d > 3/2$. The critical percolation interface is bounded by $\sim (L/\lambda)^{7/4}$ hexagons so it covers enough of the sample to feel a macroscopic effect of the tiny bias out of criticality. Of course, this is cheating because the interface as a set is correlated to the hexagon configuration.

To do a bit better, we need a bit more knowledge. Take an interface drawn from the symmetric distribution. We view the interface as the outcome of an exploration process. Let n be the total number of choices. It is intuitive, and can be proved, that n

scales like the total length of the interface, i.e. for a typical symmetric interface $n \sim (L/\lambda)^{7/4}$. If d is the difference between the number of black and white choices, i.e. the asymmetry, the usual rules of chance ensure that $d \sim n^{1/2}$. So $d \sim (L/\lambda)^{7/8}$. For an asymmetric distribution, the mean asymmetry for n choices is of order $n(2p-1)$, which for $p = 1/2 + g\lambda^{3/4}$ yields $\sim gL^{7/4}\lambda^{-1}$. This is much larger than $(L/\lambda)^{7/8}$ when $\lambda \rightarrow 0^+$. Hence an interface which is typical for the symmetric distribution is very atypical for an asymmetric continuum distribution.

2.2.2.3 General Remarks on Interfaces

We would like to extract some general features of interfaces that go beyond percolation. The crucial observation is the following. We have defined percolation configurations with an interface for any hexagonal domain $(\mathbb{D}, a, b)$ with admissible boundary condition. For percolation, the weight of a configuration is a product of independent weights, and partition functions are always 1. But the existence of an interface is a “topological” fact: whatever the Boltzmann weights given to a colouring of the inner hexagons in $(\mathbb{D}, a, b)$, the colouring will allow to identify a unique well-defined interface connecting a and b .

To define models generalising percolation, a crucial step is to see how the information encoded in hexagon colourings can be retrieved from another type of geometric data.

For each configuration, one paints the edges separating hexagons of different colours. In particular, the edges forming the interface are among the painted edges, but there are many more painted edges in general. The topology of the hexagonal lattice leads to interesting consequences. Consider a vertex not on the boundary. It touches three edges and three hexagons. By direct enumeration of the possible colourings of these three hexagons, one checks that either 0 or 2 painted edges contain the vertex. A boundary vertex touching only two hexagons can belong to 0 or 1 painted edge. A boundary vertex touching only one hexagon can belong to no painted edge. Consequently, if the painted edges are grouped in connected components, these components are simple curves (no branchings) that can only end at a boundary vertex. For admissible boundary conditions, this implies that apart from the interface, which ends at a and b , all the other curves are closed loops, and the painted edges form a gas of self avoiding loops, plus the interface. An example is shown in Fig. 2.8. Of course, would one fix all boundary hexagons to be of the same colour, one would have no interface but only a loop gas. At the other extreme, if the boundary conditions would allow many colour changes, one would see a gas of interfaces and loops. In any case, the curves split the domain in connected components in which all hexagons have the same colour. On the other hand, by definition, each time a curve is crossed, the hexagon colour changes. So one can proceed backwards: if one keeps the curves but erases the hexagon colours except for a single hexagon, the full colouring can be retrieved. If all hexagon colours are erased, there is a twofold degeneracy in the reconstruction of hexagon colours from the curve configuration. Of course, depending on the boundary conditions on hexagon colours, only certain curve arrangements can occur.

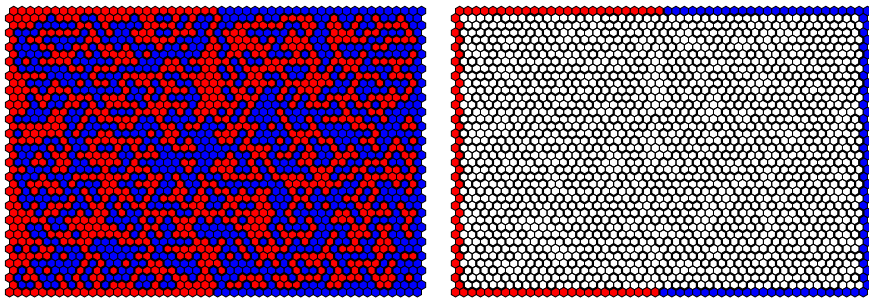


Fig. 2.8 A critical percolation sample with the hexagons colours and loops (*left*) and the same configuration with only the loops (*right*)

The **loop description** is useful for several reasons. The first one is that it leads to popular statistical mechanics models. The second is that it puts the study of the interface on some natural footing. Indeed, the loops and the interface are objects of the same nature. In particular from a physical viewpoint they are expected to share many local properties. This is especially true at a critical point when comparing large loops and the interface.

There are simple ways to associate a Boltzmann weight to a loop (or loop + interfaces) configuration. A **configuration** ℓ is made of a number of **edges**, say E_ℓ , these edges building a number, say C_ℓ , of connected components (i.e. of disjoint simple curves). We can then define a **Boltzmann weight** depending on 2 parameters, K and n by the formula

$$w(\ell) = K^{E_\ell} n^{C_\ell}. \quad (2.3)$$

Note that with this weight, the two-fold degeneracy between the hexagon colour configuration and the loop (or loop + interfaces) description is irrelevant. The parameter K is a kind of fugacity, and n (if an integer) is related to some group theory factors in another formulation of the model. Though we shall not explain why here, this is the reason why one talks generically of the **O(n) model**. **Percolation** is described by $n = 1$ and $K = 1$. Keeping $n = 1$, K can be seen as $\tan \beta$ (i.e. as a temperature-like parameter) for a very famous model, the Ising model,² as shown by a standard large temperature expansion. So there is another value of K , namely $K = 3^{-1/2}$ for which the model is again critical. A glance at Fig. 2.9, compared to Fig. 2.8 suggests that percolation leads to denser loops than the critical Ising model. This is a general feature: for $n \in [-2, 2]$ by adjusting $K = K_c^- := (2 - (2 - n)^{1/2})^{-1/2}$ one gets a critical dense loop gas and by adjusting $K = K_c^+ := (2 + (2 - n)^{1/2})^{-1/2}$ one gets a critical dilute loop gas. The section on loop-erased random walks shows that they are closely related to $n = -2$ in the dilute phase. Other values of n describe other systems of interest. For instance $n = 2$

²Warning: this relationship between percolation and the Ising model is special to hexagonal domains!

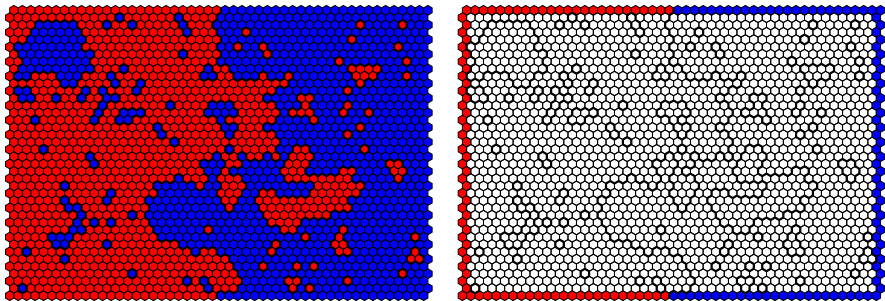


Fig. 2.9 An Ising sample with the hexagons colours and loops (*left*) and the same configuration with only the loops (*right*). The sample is only “close to critical” due to finite size effects

is related to the XY model, the Kosterlitz-Thouless transition and the Gaussian free field, $n = 0$ to self-avoiding walks (obvious with our definitions) and so on.

Let us close this short presentation by a few remarks. Keeping the domain fixed, let Z_{ab} denote the partition in $(\mathbb{D}, a, b)$ with admissible boundary conditions, and Z the same partition but with all boundary hexagons fixed to be of the same colour. The ratio Z_{ab}/Z bears some analogy with a 2-point correlation function. Observe that the inner hexagon colour configurations that contribute to Z_{ab} and Z are the same, but the loop configurations are very different. Of course the loops that do not touch the boundary hexagons are the same in both cases, but the loops that touch the boundary are sensitive to a change of boundary conditions. For analogous reasons, in general, there is no observable O_a such that $Z_{ab} = \sum_{\ell} w(\ell) O_a(\ell) O_b(\ell)$. However, one can find observables O_a^i , indexed by some parameter i such that $Z_{ab} = \sum_{\ell} w(\ell) \sum_i O_a^i(\ell) O_b^i(\ell)$. This may seem artificial, but is in fact closely related to the channels that appear in a quantum field-theory operator description of the statistical system when Z_{ab}/Z is identified with a vacuum expectation value $\langle \Omega | \psi(a) \psi(b) | \Omega \rangle$. Note also that if one attributes weight 1 instead of n to loops that touch the boundary, one can locally find observables O_a that allow to interpret Z_{ab}/Z as an almost *bona fide* statistical 2-point correlation function.

2.2.3 Loop-Erased Random Walks

This example still keeps some aspects of a growth process, in that new pieces of the process can be added recursively. Loop-erased random walks were invented by Lawler as an example of random paths more tractable than the canonical self avoiding walks. A **loop-erased random walk** (LERW) is a random walk with loops erased along as they appear.

2.2.3.1 Definition

More formally, if $X_0, X_1, \dots, X_n$ is a finite sequence of abstract objects, we define the associated **loop-erased sequence** by the following recursive algorithm.

```

Initialise counters  $l=0$  and  $m=1$  and set  $Y_0 = X_0$ 
Iterate while  $m \leq n$ 
{
  - If there is a  $k$  with  $0 \leq k \leq l$  such that  $Y_k = X_m$  set  $l = k$ 
  - Else increment  $l$  by 1 and set  $Y_l = X_m$ .
}
The loop-erased sequence is  $Y_0, \dots, Y_l$ .

```

Let us look at two examples.

1. For the “month sequence” $j, f, m, a, m, j, j, a, s, o, n, d$, the first loop is m, a, m , whose removal leads to $j, f, m, j, j, a, s, o, n, d$, then j, f, m, j , leading to j, j, a, s, o, n, d , then j, j leading to j, a, s, o, n, d where all terms are distinct.
2. For the “reverse month sequence” $d, n, o, s, a, j, j, m, a, m, f, j$, the first loop is j, j , leading to $d, n, o, s, a, j, m, a, m, f, j$, then a, j, m, a leading to d, n, o, s, a, m, f, j .

This shows that the procedure is not “time-reversal” invariant. Moreover, terms that are within a loop can survive: in the second example m, f , which stands in the j, m, a, m, f, j loop, survives because the first j is inside the loop a, j, m, a which is removed first.

The above algorithm is most useful if the sequence $X_0, X_1, \dots, X_n$ is viewed as a stream of data that is treated “on the fly”. If $X_0, X_1, \dots, X_n$ is known at once, another algorithm erases the loop in possibly fewer steps. It goes as follows:

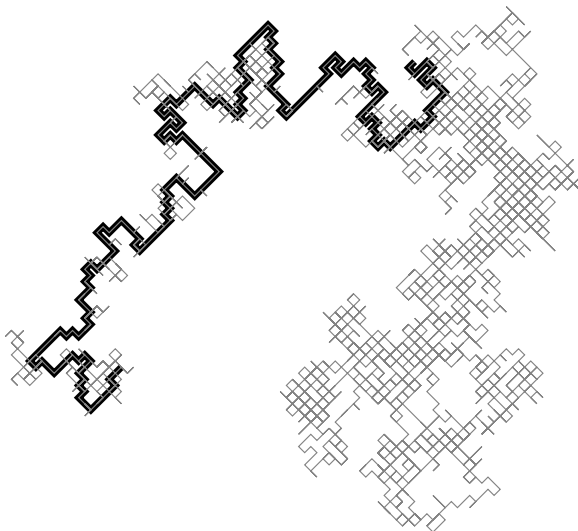
```

Initialise counters  $l=0$  and  $m=n$ 
Until  $l=m$ , iterate
{
  - Find the largest  $k \leq m$  such that  $X_k = X_l$ 
  - If  $k > l$  remove the terms with indices from  $l+1$  to  $k$ , and
    shift the indices larger than  $k$  by  $l-k$  to get a new
    sequence.
  - Decrement  $m$  by  $k-l$  and increment  $l$  by 1.
}

```

For the “month sequence”, this leads at once from $j, f, m, a, m, j, j, a, s, o, n, d$ to j, a, s, o, n, d , and then the counter l is incremented from 0 to 5 without further removals. For the “reverse month sequence”, the counter l is incremented from 0 to 4, a loop is removed leading from $d, n, o, s, a, j, j, m, a, m, f, j$ to d, n, o, s, a, m, f, j , then the counter l is incremented from 5 to 7 without further removals.

Fig. 2.10 A loop-erased random walk with its shadow



A **loop-erased random walk** arises when this procedure is applied to a (two-dimensional for our main interest) random walk. In the full plane this is very easy to do. Figure 2.10 represents a loop-erased walk of 200 steps obtained by removing the loops of a 4006 steps random walk on the square lattice. The thin grey lines build the shadow of the random walk (where **shadow** means that one does not keep track of the order and multiplicity of the visits) and the thick line is the corresponding loop-erased walk. The time-asymmetry is clearly visible and allows to assert with little uncertainty that the walk starts on the top right corner.

In this setting, it is trivial to get samples but the measure remains in the background. One possible approach is the following. Consider a symmetric random walk on the square lattice and view the successive positions as a stream of data. Remove the loops as they show up, and stop the random walk at the first time n for which the associated loop-erased walk has reached length N . The probability of the random walk is 4^{-n} . Note that the set of random walks for which the loop-erasure never reaches size N has probability 0, for instance as a subset of the set of random walks that remain in the ball of radius N centred at the origin forever. So the total probability for the set of random walks stopped when their loop-erasure reaches length N is 1. This procedure leads to a finite family of loop-erased walks, each of them can be obtained via the loop erasure of an infinite number of random walks. The probability of a given loop-erased walk is taken to be sum of the individual probabilities of its random walk ancestors.

This can be adapted to the setting of discrete domains with admissible boundary condition. Let $(\mathbb{D}, a, b)$ be such a domain, and let ν be the **coordination number** of the associated tiling, i.e. the number of edges adjacent to a vertex: ν is 6 for the triangular tilings, 4 for the square tilings, and 3 for the hexagonal tilings. Consider the set of walks from a to b in $\mathbb{D} \cup \{a, b\}$ that visit a and b only once, and give each step weight ν^{-1} , so that the weight of a walk is the usual random walk probability

on the tiling. However, the total weight of walks from a to b in $\mathbb{D} \cup \{a, b\}$ is < 1 . As before, these walks can be loop-erased, and the weight of a simple path γ from a to b in $\mathbb{D} \cup \{a, b\}$ is taken to be sum of the weights of all random walks from a to b in $\mathbb{D} \cup \{a, b\}$ whose loop erasure leads to γ . To get a probability measure, one needs to divide the weight by the total weight of all random walks from a to b in $\mathbb{D} \cup \{a, b\}$. This is easy in principle, and is closely related to the solution of the discrete Laplace equation with appropriate boundary conditions; in practice this normalisation can be computed explicitly for only a handful of examples.

In the same spirit, if we have an arbitrary weight assignment for walks from a to b in $\mathbb{D} \cup \{a, b\}$, we can use it to induce a weight on simple paths from a to b in $\mathbb{D} \cup \{a, b\}$ again by taking the weight of a simple path γ to be sum of the weights of all random walks from a to b in $\mathbb{D} \cup \{a, b\}$ whose loop erasure leads to γ . What is special about the standard random walk weight is that, as is well known, the random walk has a scale-invariant limit (Brownian motion of course), so the corresponding loop-erased random walk can be expected to have a scale-invariant limit. The loop-erased random walk is one of the first systems that has been proved to have a (not only scale—but even) conformally invariant continuum limit, the fractal dimension being $5/4$. A naive idea to get directly a continuum limit representation of loop-erased walks would be to remove the loops from a Brownian motion. This turns out to be impossible due to the proliferation of overlapping loops of small scale. However, the SLE₂ process, to be defined later, gives a direct definition. In fact, it is the consideration of loop-erased random walks that led Schramm to propose SLE as a description of interfaces.

2.2.3.2 Simulation

We have seen that it is very simple to generate loop-erased random walks of a fixed length N in the plane. We could use this technique to get a probability measure on the first N steps of loop-erased random walks of length M . However, it is unclear whether this probability measure stabilises if we fix N and let M go to infinity. One of the problems is that in two dimensions, random walks are **recurrent**: with probability one they visit every site (and then they have to do it infinitely many times). So if we erase the loops of a random walk, the resulting loop-erased walk never stabilises; if we wait long enough, the random walk comes back to the origin and at that instant the loop-erased walk starts anew from scratch.

The numerical simulation of a loop-erased random walk in domains $(\mathbb{D}, a, b)$ is not easy either, because the random walks have a tendency to leave $\mathbb{D}$. Note that it would bias the sampling if we would forbid them to leave by simply dispatching the weight of steps leaving $\mathbb{D}$ to the ones staying in $\mathbb{D}$. What one has to do is to *condition* on random walks staying in $\mathbb{D}$. So most samples would have simply to be rejected and only from time to time would a sample be a walk from a to b in $\mathbb{D} \cup \{a, b\}$.

There is one exceptional domain in which at the same time an infinite loop-erased random walk can easily be defined and simulated. It is when $\mathbb{D}$ is the square tiling of

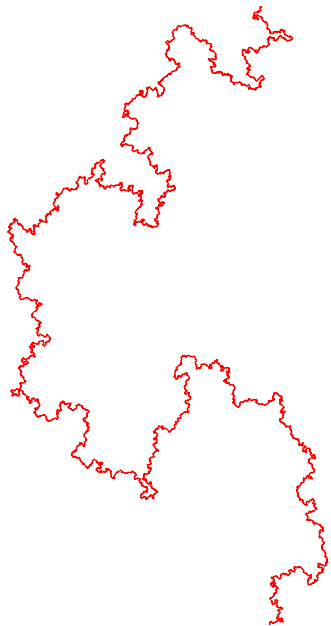
a half-space, conventionally taken to be $\mathbb{H}_{\text{int}}$, the tiling of the upper-half-plane with vertices at the points $(n, m) \in \mathbb{Z} \times \mathbb{N}$, a is $O := (0, 0)$ (by translation-invariance along the real axis, any boundary vertex would do) and b is infinity. Let us explain why random walks on the square lattice conditioned to go from O to infinity while staying in $\mathbb{H}_{\text{int}}$ have a simple description.

The horizontal steps are not an issue, and we can concentrate on vertical steps. For a simple random walk in one dimension, it is well-known that a walk started at $m \in [0, l]$ touches the boundary for the first time at the endpoint 0 with probability $1 - m/l$ and at the endpoint l with probability m/l . Indeed, if $p(m)$ is the probability to touch the boundary for the first time at the endpoint 0, then $p(0) = 1$, $p(l) = 0$ and if $m \in]0, l[$, $p(m) = \frac{1}{2}(p(m-1) + p(m+1))$ as can be seen by conditioning on the first step of the walk. So by the usual rules of conditional probabilities, if the random walk is conditioned to exit at l and is at $m \in]0, l[$ at time t , it has probability $\frac{m+1}{2m}$ to go to $m+1$ and $\frac{m-1}{2m}$ to go to $m-1$. This has three striking consequences. First, the process remains Markov and time homogeneous. Second, the transition probabilities do not depend on l , so they can be used even if l is infinite. Third, taking l infinite, the probability, starting at m , never to visit $m' < m$ is $1 - m'/m > 0$ as can be seen by conditioning on the first step of the walk.

These three properties imply that each site is visited only a finite number of times, i.e. the walk escapes to infinity. Let us explain this in more detail. Suppose the walk starts from 0, goes to 1 at the first step, and then the above transition probabilities are used. Then at the second step the walk goes to 2. With probability $1/2$ it *never* goes back to 1 again. With probability $1/2$ it comes back to 1 at some point, and then the walk starts anew. Thus the number of visits to 1 follows a geometric law: 1 is visited $k \geq 1$ times with probability $1/2^k$. In particular the probability to visit 1 at least k times is $1/2^{k-1}$ which goes to 0 (in fact exponentially). Hence with probability 1 the number of visits of point 1 is finite. The same argument generalises. First, let s be the probability that point $m \geq 1$ is never visited. Suppose the walk is at point m . With probability $\frac{m+1}{2m}$ it goes to $m+1$ and then with probability $1 - \frac{m}{m+1} = \frac{1}{m+1}$ it *never* goes back to m again. So the total probability that the walk starting from m never visits m again is $r \geq \frac{1}{2m} > 0$. It follows that the number of visits to m follows essentially a geometric law: m is visited 0 times with probability s and $k \geq 1$ times with probability $(1-s)r(1-r)^{k-1}$. Again, the probability to visit m at least $k \geq 1$ times is $(1-s)(1-r)^{k-1}$ which goes to 0 (in fact exponentially). Hence with probability 1 the number of visits of point m is finite. This is true for any m and any starting point for the walk. Hence, in particular we see recursively that if the walk is at $m' < m$, m will be visited later with probability one, because with probability 1 all points in $[0, m-1]$ are visited only finitely many times. This means that in fact $r = \frac{1}{2m}$. To summarise, if the walk starts from 0, the number of visits to $m \geq 1$ is $k \geq 1$ with probability $\frac{1}{2m}(1 - \frac{1}{2m})^{k-1}$. In particular, the walk is transient, i.e. it escapes to ∞ with probability 1.

So we use the usual random walk in the horizontal direction but the conditioned random walk in the vertical direction. Explicitly, at the first step the walk goes from $(0, 0)$ to $(-1, 1)$ or $(1, 1)$ with probability $1/2$, and later, if at (n, m) , the walk makes a step in the NE or NW directions with probability $\frac{m+1}{4m}$ and in the SE or SW direc-

Fig. 2.11 A sample of the loop-erased random walk



tions with probability $\frac{m-1}{4m}$. Call the vertical coordinate **altitude** for convenience. As explained before, the altitude of the walk goes to ∞ with probability one, and the associated loop-erased walk converges. More precisely, for $m < M$ stop the random walk the first time it reaches altitude M and stop the corresponding loop-erased random walk at altitude m . Then with probability $> 1 - m/M$ the loop-erased random walk up to altitude m will not be modified by the subsequent evolution of the random walk. This is because to close a loop, the walk has to come back to the same point, which is more stringent than to come back to the same altitude. Hence, letting M go to infinity, we get a well defined limiting distribution for loop-erased random walks from O to altitude m for any m , hence for loop-erased random walks from O to ∞ . Accurate numerical simulations are made by taking $M \gg m$. However, the process for which $m = M$ is interesting as well. It has a continuum limit which can be studied with the so-called dipolar variant of stochastic Loewner evolutions.

Figure 2.11 shows a sample of loop-erased random walk of about 10^5 steps. At first glance, one observes a simple (no multiple points) irregular curve with a fractal structure. The intuitive explanation why a loop-erased random walk has a tendency not to come back too close to itself is that if it would do so, then with large probability a few more steps of the random walk would close a loop.

2.2.3.3 Relation with Statistical Mechanics

Again, it is useful to make the connection between the purely geometric description of loop-erased random walks and more conventional statistical mechanics.

The starting point of the correspondence is a formula for the expansion of the determinant $\det(\mathbf{1} - \mathbf{A})$ when $\mathbf{A} = (A_{vv'})_{v,v' \in V}$ is a matrix with index set V . For later convenience, we call the elements of V **vertices**. A **cycle** of length $k \geq 1$ in V is sequence $(v_1, \dots, v_k)$ of distinct vertices of V modulo cyclic permutation; so that $(v_1, \dots, v_k), (v_2, \dots, v_k, v_1), \dots$ represent one and the same cycle. Cycles are said to be **disjoint** if no vertex appears in more than one of them. The subsets $\{C_1, \dots, C_n\}$ of $\mathcal{P}(V)$ made of n disjoint cycles of V form a set that we denote $\mathcal{C}_n$. The **weight** of a cycle C represented by $(v_1, \dots, v_k)$ is by definition $w(C) := A_{v_1 v_2} \dots A_{v_{k-1} v_k} A_{v_k v_1}$ (for $k = 1$, this reduces to $A_{v_1 v_1}$) which indeed is invariant under cyclic permutations.

An elementary reorganisation of Cramer's formula yields

$$\det(\mathbf{1} - \mathbf{A}) = \sum_{n \geq 0} (-1)^n \sum_{\{C_1, \dots, C_n\} \in \mathcal{C}_n} w(C_1) \dots w(C_n). \quad (2.4)$$

Similarly, for $v, v' \in V$ we define a **walk** of k steps from v to v' in V to be any sequence of vertices $(v_0, \dots, v_k)$ with $v_0 = v$ and $v_k = v'$ but with $v_1, \dots, v_{k-1}$ distinct from v and v' . Hence with this definition a walk visits its starting and end point only once. This restriction is a bit unusual, but it is not really crucial. The weight of a walk $W = (v_0, \dots, v_k)$ is taken to be $w(W) := A_{v_0 v_1} \dots A_{v_{k-1} v_k}$.

The sequence W can be loop-erased to yield a path from v to v' (remember that paths are walks in which a given vertex appears at most once). If γ is a path, we define

$$\tilde{w}(\gamma) := \sum_{W \mapsto \gamma} w(W), \quad (2.5)$$

where the sum is over all walks whose associated loop-erased walk is γ . We aim at a general formula for $\tilde{w}$.

Let $\gamma = (v_0, \dots, v_k)$ be a path from $v, v' \in V$. Let $V^{(0)} := V \setminus \{v_k, v_0\}$, $V^{(1)} := V \setminus \{v_k, v_0, v_1\}$, $\dots$. For $l = 0, \dots, k-1$ let $\mathbf{A}^{(l)}$ be the matrix $\mathbf{A}$ restricted to the vertex set $V^{(l)}$.

A walk W which yields γ after loops have been erased can be decomposed as follows (see the second loop-erasing algorithm on p. 67): the walk (v_0, v_1) , followed by an arbitrary number of walks from v_1 to v_1 in $V^{(0)}$, followed by the walk (v_1, v_2) , followed by an arbitrary number of walks from v_2 to v_2 in $V^{(1)}$ and so on. Take $1 \leq l \leq k-1$. Note that if one expands $(\frac{1}{\mathbf{1} - \mathbf{A}^{(l-1)}})_{v_l v_l}$ in formal power series in the coefficients of $\mathbf{A}$, one gets exactly the sum of the weights for the concatenation of an arbitrary number of walks from v_l to v_l in $V^{(l-1)}$. Hence we infer that

$$\begin{aligned} \tilde{w}(\gamma) &= A_{v_0, v_1} \left(\frac{\mathbf{1}}{\mathbf{1} - \mathbf{A}^{(0)}} \right)_{v_1 v_1} A_{v_1, v_2} \left(\frac{\mathbf{1}}{\mathbf{1} - \mathbf{A}^{(1)}} \right)_{v_2 v_2} A_{v_2, v_3} \\ &\quad \times \dots \times A_{v_{k-2}, v_{k-1}} \left(\frac{\mathbf{1}}{\mathbf{1} - \mathbf{A}^{(k-2)}} \right)_{v_{k-1} v_{k-1}} A_{v_{k-1}, v_k}. \end{aligned}$$

But by Cramer's formula for the inverse of a matrix,

$$\left(\frac{\mathbf{1}}{\mathbf{1} - \mathbf{A}^{(l-1)}} \right)_{v_l v_l} = \frac{\det(\mathbf{1} - \mathbf{A}^{(l)})}{\det(\mathbf{1} - \mathbf{A}^{(l-1)})} \quad \text{for } l = 0, \dots, k-1.$$

Hence the product in the above formula for $\tilde{w}(\gamma)$ is telescopic, and we get the representation we were aiming at:

$$\tilde{w}(\gamma) = w(\gamma) \frac{\det(\mathbf{1} - \mathbf{A}^{(k-1)})}{\det(\mathbf{1} - \mathbf{A}^{(0)})}. \quad (2.6)$$

A first use of this formula is that it shows clearly that if the matrix $\mathbf{A}$ is symmetric, the loop-erased random walk weight is reversible i.e. the same for a path and its opposite or time reversal. In all cases the asymmetry comes solely from the weight of γ .

It is time to interpret the formulæ obtained so far in connection with statistical mechanics.

We start with Eq. (2.4) but read from right to left. The right-hand side can be seen as a partition function for a gas of oriented loops on a graph. Indeed, if E is an arbitrary subset of $V \times V$, we can consider the corresponding oriented graph $G = (V, E)$ i.e. view E as the set of edges of G . We give each edge in $(v, v') \in E$ the weight $A_{vv'}$ and impose that $A_{vv'} = 0$ if $(v, v') \notin E$. An **oriented loop** on G is a sequence $(v_1, \dots, v_k)$ of distinct vertices of V modulo cyclic permutation, with the condition that $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v_1)$ are in E . Except for the last condition, this is what we called a cycle before: note that “cycle” reminds of the permutation context whereas “loop” reminds of geometric context. A configuration is a family of disjoint oriented loops, each oriented loop counts for a weight which is the product of the weight of the traversed edges and an overall factor (-1) . Then the partition function, i.e. the sum of the weights of all possible configurations is by definition the right-hand side of Eq. (2.4), and this reconstructs the determinant on the left-hand side. We can specialise more by assuming further that E is a symmetric subset of $V \times V$ that does not meet the diagonal, and that $\mathbf{A}$ is symmetric. Then there is no loop of length 1, and the loop $(v_1, \dots, v_k)$ has the same weight as the loop traversed in the opposite order $(v_k, \dots, v_1)$. If $k = 2$ a loop and its opposite are the same, but not if $k \geq 3$. So we get the same partition function if instead of summing over oriented loops, we sum over un-oriented loops counting each un-oriented loop of length ≥ 3 twice, i.e. giving un-oriented loop of length ≥ 3 an overall factor (-2) instead of (-1) . Finally, we could also give each edge in E the same weight K so that the weight of a loop configuration would be

$$(-1)^{\#\text{loops of length } 2} (-2)^{\#\text{loops of length } \geq 3} K^{\#\text{traversed edges}}$$

where of course loops of length 2 count for 2 traversed edges.

This statistical weight could be used as a definition of the so-called **O** (-2) **model**, where -2 reminds of the overall weight of each loop (of length ≥ 3). This model has several avatars, which are supposed to be in the same universality class, i.e. to describe the same macroscopic physics in the continuum limit. In certain versions, loops of length 2 are completely forbidden, at the price of renormalising K .

Replacing the factor (-2) by a factor n yields the general $O(n)$ model, which we have seen before.

Note that the partition function, i.e. $\det(\mathbf{1} - \mathbf{A})$ has a simple “field-theory” interpretation: if χ_v and $\bar{\chi}_v$, $v \in V$ are a collection of independent Grassmann variables, the fundamental result of Grassmann integration is

$$\det(\mathbf{1} - \mathbf{A}) = \int \prod_{v \in V} d\chi_v d\bar{\chi}_v e^{\sum_{v,v'} \chi_v (\delta_{vv'} - A_{vv'}) \bar{\chi}_{v'}}.$$

This is the clue to the quantum field-theory approach to loop-erased random walks.

Before we interpret Eq. (2.6), let us start with a general observation. Suppose $\mathcal{C}$ is a configuration space, assumed to be finite for simplicity and consider a model of statistical mechanics on $\mathcal{C}$. Each $c \in \mathcal{C}$ has a weight $w(c)$. The **partition function** is $Z := \sum_{c \in \mathcal{C}} w(c)$. Suppose $\mathcal{C}$ can be partitioned as $\mathcal{C} = \bigcup_{\gamma \in \Gamma} \mathcal{C}_\gamma$. Then we can define $Z_\gamma := \sum_{c \in \mathcal{C}_\gamma} w(c)$ for $\gamma \in \Gamma$, and Z_γ can be interpreted as the marginal weight of $\mathcal{C}_\gamma$. The probability of $\mathcal{C}_\gamma$ is simply Z_γ/Z . In concrete situations, the splitting $\mathcal{C} = \bigcup_{\gamma \in \Gamma} \mathcal{C}_\gamma$ will usually have some interpretation. For instance, in the cases we are interested in these notes, we shall look at configuration spaces $\mathcal{C}$ that describe a statistical mechanics model on domains $(\mathbb{D}, a, b)$ with boundary conditions, in such a way that in each $c \in \mathcal{C}$ we can identify unambiguously a path γ joining a to b . Of course γ depends on c , and we can use this γ to split c . Then Z_γ/Z is simply the probability to observe the path γ . The reader should have another glance at Sect. 2.2.2.2 to look at the relationship between the exploration path and percolation from this viewpoint.

Equation (2.6) can then be interpreted straightforwardly. We consider now configurations made not simply of (mutually avoiding) loops, but of (mutually avoiding) loops avoiding a path from v to v' . The total weight of configurations for a fixed path from v to v' is simply the numerator of the right-hand side of Eq. (2.6). The denominator depends on v and v' but not on the simple path between them. So from the point of view of statistical mechanics explained before, the weight the loop-erased random walk model assigns to a path γ , i.e. the left-hand side of Eq. (2.6), is proportional to the marginal weight of configurations of “loops plus that path” in the loop gas model.

Hence we have succeeded in giving an interpretation of the loop erased random walks as interfaces in a statistical mechanics model. We are cheating a bit here because even if we take a positive edge weight K , due to the $(-)$ sign associated to each loop, individual configurations may well have a negative weight, so that a straightforward probabilistic interpretation is not available.

Our interest is of course the case when the graph G is the one associated with a discrete domain $(\mathbb{D}, a, b)$ with admissible boundary conditions. If we take for the edge weight K the inverse of the coordination number ν of the tiling, $\mathbf{1} - \mathbf{A}$ is essentially the discrete Laplacian with Dirichlet boundary conditions. This suggests again that a continuum limit exists, for which (continuum) loop-erased random walks in

a (continuum) domain D from a to b are related to the field-theory of so-called symplectic fermions with measure

$$\mathcal{D}\chi \mathcal{D}\bar{\chi} \exp \int_{\mathbb{D}} \nabla \chi \cdot \nabla \bar{\chi}$$

with Dirichlet boundary conditions. This field-theory is well-known to be conformally invariant. But $K - 1/\nu = -\lambda^2$ is a scaling function which leads to the addition of a mass term to the action.

We conclude this section by noting without justification that the way to impose the existence of a path from a to b is to insert in correlation functions the observable $J(a)\bar{J}(b)$ where $J(a)$ (resp. $\bar{J}(b)$) is the normal derivative of χ (resp. $\bar{\chi}$) at a (resp. at b).

2.2.4 Another Example of Growth: DLA

Up to now, the two growth processes we have defined shared some common features. The next one, **diffusion-limited aggregation** (DLA), is of a rather different nature. It is believed to have a scale-invariant but not conformally invariant limiting distribution. Another reason to introduce DLA is that it can also be modelled via Loewner chains, a subject we touch in the next section. Finally, DLA seems to be a relevant model for a variety of phenomena in physics, for instance aggregation or deposition phenomena, but also in biology, for instance growth of bacterial colonies under certain circumstances.

DLA stands for *diffusion limited aggregation*. It refers to processes in which the domains grow by aggregating diffusing particles. Namely, one imagines building up a domain by clustering particles one by one. These particles are released from the point at infinity, or uniformly from a large circle around the growing sample, and diffuse as random walkers. They will eventually hit the sample and the first time this happens they stick to it. Then the procedure goes on. By convention, time is incremented by unity each time a particle is added to the domain. Thus at each time step the area of the domain is increased by the physical size of the particle. The position at which the particle is added depends on the probability for a random walker to visit the boundary for the first time at this position, which is essentially what is called the harmonic measure at that position. During this process the clustering domain gets ramified and develops branches and fjords of various scales. The probability for a particle to stick on the cluster is much higher on the tip of the branches than deep inside the fjords. This property, relevant at all scales, is responsible for the fractal structure of the DLA clusters.

In a discrete approach one may imagine that the particles are tiny squares whose centres move on a square lattice whose edge lengths equal that of the particles, so that particles fill the lattice when they are glued together. The centre of a particle moves as a random walker on the square tiling. The probability $Q(x)$ that a particle visits a given tile x satisfies the discrete version of the Laplace equation $\nabla^2 Q = 0$. It vanishes on the boundary of the domain, i.e. $Q = 0$ on the boundary, because the

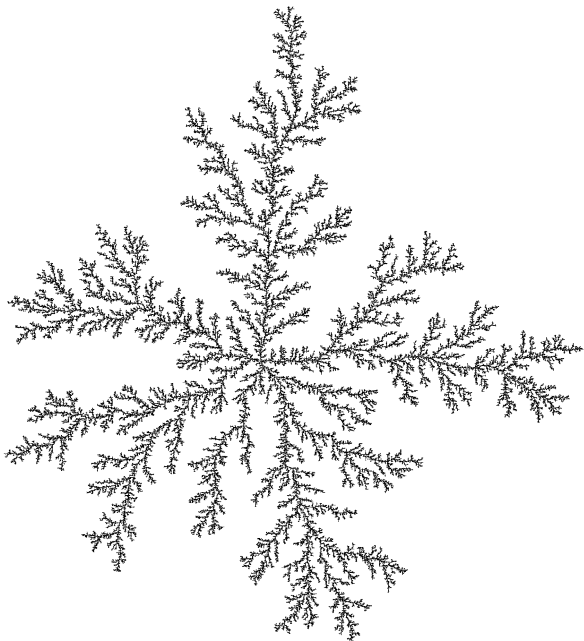
probability for a particle to visit a tile already occupied, i.e. a point of the growing cluster, is zero. The local speed at which the domain is growing is proportional to the probability for a tile next to the interface but on the outer domain to be visited. This probability is proportional to the discrete normal gradient of Q , since the visiting probability vanishes on the interface. So the local speed is $v_n = (\nabla Q)_n$. To add a new particle to the growing domain, a random walk has to wander around and the position at which it finally sticks is influenced by the whole domain. To rephrase this, for each new particle one has to solve the outer Laplace equation, a non-local problem, to know the sticking probability distribution. This is a typical example when scale-invariance is not expected to imply conformal invariance.

It is not so easy to make an unbiased simulation of DLA on the lattice. One of the reasons is that on the lattice there is no such simple boundary as a circle, for which the hitting distribution from infinity is uniform. The hitting distribution on the boundary of a square is not such a simple function. Another reason is that despite the fact that the symmetric random walk is recurrent in $2D$, each walk takes many steps to glue to the growing domain. The typical time to generate a single sample of reasonable size with an acceptable bias is comparable to the time it takes to make enough statistics on loop-erased random walks or percolation to get the scaling exponent with two significant digits. Still this is a modest time, but it is enough to reveal the intricacy of the patterns that are formed. Figure 2.12 is such a sample. The similarity with the sample in Fig. 2.13, obtained by iteration of conformal maps, is striking. But a quantitative comparison of the two models is well out of analytic control and belongs to the realm of extensive simulations. There is now a consensus that the fractal dimension of $2D$ DLA clusters is $d_{f,DLA} \simeq 1.71$. It has been long debated whether or not discrete simulations done right nevertheless do keep a remnant of the lattice at large distance. There is some consensus now that for instance the orientation of the lattice can be seen even in the large, and rotation invariant algorithms should be preferred.

2.3 Loewner Chains

There are many possible descriptions of subsets of a set. Some may look more natural than others but it is the problem at hand that decides which one is the most efficient. Growth processes in two dimensions involve time-dependent subsets of the complex plane $\mathbb{C}$. Loewner chains have proved to be an invaluable tool in this context. The simplest situation is when they are used to describe families of domains. These notes deal (almost) exclusively with that case.

Loewner chains were introduced (by Loewner!) in the context of the **Bieberbach conjecture**, now a theorem proved by de Branges in 1985. It states that if $f(z) = z + \sum_{n \geq 2} a_n z^n$ is a holomorphic function injective in the unit disc $\mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}$ then $|a_n| \leq n$ for $n \geq 2$. Bieberbach proved that $|a_2| \leq 2$ in 1912, and Loewner proved in 1923 that $|a_3| \leq 3$ using a dynamical picture of the changes of $f(\mathbb{U})$ when the a_n 's change, starting from the trivial case $f(z) = z$.

Fig. 2.12 A DLA sample

2.3.1 Around Riemann's Theorem

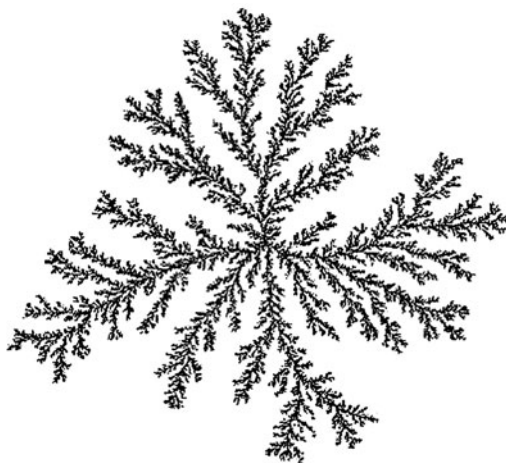
Recall that a domain $\mathbb{D}$ is a non-empty open simply connected strict subset of the complex plane $\mathbb{C}$. Simple connectedness is a notion of purely topological nature which in two dimensions asserts essentially that a domain has no holes and is contractile: a domain has the topology of a disc.

- **Riemann's theorem** states that *two domains $\mathbb{D}$ and $\mathbb{D}'$ are always conformally equivalent*, i.e. there is an invertible holomorphic map $g : \mathbb{D} \mapsto \mathbb{D}'$ between them.

Riemann stated the theorem but his argument had many gaps. This was at least partly at the origin of the formidable development of functional analysis in the twentieth century but it took decades before a proof meeting modern standards was found.

Extending g to the boundary of $\mathbb{D}$ is impossible in general if the topological boundary is used, i.e. if the boundary of $\mathbb{D}$ is taken as the complement of $\mathbb{D}$ in its closure (the notion of boundary one learns at school). As an example, take $\mathbb{D}$ to be the upper half-plane $\mathbb{H}$ with the vertical line segment $]0, ia]$ removed and $\mathbb{D}' = \mathbb{H}$. The naïve boundary of $\mathbb{D}$ is the union of $\mathbb{R}$ and $]0, ia]$. The limits of $g(z)$ when z approaches a given point of the segment $]0, ia]$ from the left or from the right must be distinct. But another notion of boundary can be defined for which a continuous extension at the boundary is always possible. Intuitively, this more involved notion keeps track of the different sides from which a naïve boundary point can be approached.

Fig. 2.13 A shape produced by iteration of random conformal maps



This is trivial in our simple example but the general situation is involved. In Sect. 2.2.1, we mentioned the idea to define an appropriate notion of boundary by duality by saying that the space of functions on the boundary is the space of harmonic functions in the domain. Riemann's theorem gives a strong incentive to do that. It is known that for any continuous function on the unit circle, there is a single function continuous in the closed unit disk, harmonic in the open unit disk and with the prescribed value on the boundary. One can extend this idea to define wilder “functions” on the unit circle by saying that they are “boundary values” of a larger class of harmonic functions in the open unit disk (for instance all bounded harmonic functions). By Riemann's theorem, any domain is conformally equivalent to the open unit disk. As holomorphic maps preserve harmonicity, one gets a notion of “function on the boundary” for an arbitrary domain and this notion is conformally invariant. Then by some kind of duality one passes from the space of functions on the boundary to the boundary itself. Serious work is needed to turn this intuition into mathematics, but it can be done. We shall freely use the word “boundary” in what follows, leaving to the reader the task of deciding from the context which kind of boundary we have in mind. In cases when there is only one way to approach naïve boundary points, the two notions coincide.

In simple cases, the map f can be found in closed form. For instance, the upper-half-plane $\mathbb{H}$ and the unit disc $\{z \in \mathbb{C}, |z| < 1\}$ centred on the origin are two domains. The conformal transformation $f(z) = i \frac{1-z}{1+z}$ maps the unit disc biholomorphically onto the upper half plane with $f(0) = i$ and $f(1) = 0$. But finding a closed formula for f in the general case is impossible.

The upper half-plane has a three-dimensional Lie group of conformal automorphisms, $PSL_2(\mathbb{R})$, that also acts on the boundary of $\mathbb{H}$. This group is made of homographic transformations $f(z) = \frac{az+b}{cz+d}$ with a, b, c, d real and $ad - bc = 1$. To specify such a map we have to impose three real conditions. Hence, there is a unique holomorphic automorphism—possibly followed by a conjugation—that maps any

triple of boundary points to any other triple of boundary points. Similarly there is unique homographic transformation that maps any pair made of a bulk point and a boundary point to another pair of bulk and boundary points. By Riemann's theorem, any domain has a Lie group of conformal automorphisms isomorphic to $PSL_2(\mathbb{R})$ and the same normalisation conditions can be used.

Riemann's theorem is used repeatedly in the rest of these notes. It is the starting point of many approaches to growth phenomena in two dimensions since it allows to code the shapes of growing domains in their uniformising conformal maps. To make the description precise, one has to choose a reference domain against which the growing domains are compared. By Riemann's theorem, we may choose any domain as reference domain—and depending on the geometry of the problem some choices are more convenient than others. The unit disc and the upper half plane are often used as reference domains.

2.3.2 Hulls

One can be more explicit when the domain $\mathbb{D}$ differs only locally from the upper half-plane $\mathbb{H}$, that is if $\mathbb{K} = \mathbb{H} \setminus \mathbb{D}$ is bounded. Such a set $\mathbb{K}$ is called a **hull**. The real points in the closure of $\mathbb{K}$ in $\mathbb{C}$ form a compact set which we call $\mathbb{K}_{\mathbb{R}}$. In that case, $\mathbb{H}$ is the convenient reference domain. Let $g : \mathbb{D} \mapsto \mathbb{H}$ be a conformal bijection. For $z \in \mathbb{D}$ define $g(\bar{z}) := \overline{g(z)}$. If z approaches a point x on the real axis while staying within $\mathbb{D}$, $g(z)$ has a real limit which we denote by $g(x)$. It follows that g extends to a holomorphic map on the connected open set $\mathbb{D} \cup \overline{\mathbb{D}} \cup (\mathbb{R} \setminus \mathbb{K}_{\mathbb{R}}) \cup \infty$ of the Riemann sphere, which contains a neighbourhood of ∞ . This is an illustration of the Schwartz reflection principle. One can use the automorphism group of $\mathbb{H}$ to ensure that $g(z) = z + O(1/z)$ for large z . This is called the **hydrodynamic normalisation**. It involves three conditions: g maps ∞ to ∞ , has unit derivative there, and has no constant term. These three conditions are real because ∞ is on the boundary of the upper half-plane seen within the Riemann sphere. There is no further freedom left. Thus any property of g is an intrinsic property of $\mathbb{K}$.

We shall denote this special representative by $g_{\mathbb{K}}$. The inverse map $f_{\mathbb{K}}$ is holomorphic on the full Riemann sphere except for cut along a compact subset of $\mathbb{R}$ across which its imaginary part has a positive discontinuity (in general this is a measure) $d\mu(x)$. Away from the cut, $f_{\mathbb{K}}$ has the standard representation

$$f_{\mathbb{K}}(w) = w - \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu(x)}{w - x}.$$

The coefficients of the expansion of $f_{\mathbb{K}}$ at infinity are essentially the moments of μ . In particular, they are real. Each of them quantifies an intrinsic property of $\mathbb{K}$. The number $C_{\mathbb{K}} := \frac{1}{\pi} \int_{\mathbb{R}} d\mu(x)$ is the **total mass** of μ . It is positive (or 0 if $\mathbb{K}$ is empty). Note that $f_{\mathbb{K}}(w) = w - C_{\mathbb{K}}/w + \dots$ at large w and by inverting, $g_{\mathbb{K}}(z) = z + C_{\mathbb{K}}/z + \dots$ at large z . The coefficient $C_{\mathbb{K}}$ plays an important role. It is called the **capacity** of $\mathbb{K}$ seen from ∞ . Its positivity is intuitively related to the fact that

one removes a piece from $\mathbb{H}$. Capacity is trivially translation-invariant, ($C_{x+\mathbb{K}} = C_{\mathbb{K}}$ where, for $x \in \mathbb{R}$, $x + \mathbb{K}$ denotes the translate of $\mathbb{K}$ by x units along the real axis) and has weight 2 under dilatations ($C_{s\mathbb{K}} = s^2 C_{\mathbb{K}}$ if s is a positive scale factor). Capacity has an additive property: simple series manipulations show that if $\mathbb{K}'$ and $\mathbb{K}''$ are two hulls and $\mathbb{K} = \mathbb{K}' \cup g_{\mathbb{K}'}^{-1}(\mathbb{K}'')$ (which is another hull) then $C_{\mathbb{K}} = C_{\mathbb{K}'} + C_{\mathbb{K}''}$.

2.3.3 Basic Examples

Example 1 (The semi-disc) Maybe the simplest example is when $\mathbb{K}$ is a semi-disc $\{z \in \mathbb{H}, |z - b| \leq r\}$ for a real b and real positive r . Then $g_{\mathbb{K}}(z) = z + r^2/(z - b)$. Expansion at large z shows that $C_{\mathbb{K}} = r^2$.

Example 2 (The vertical line segment) In the example when $\mathbb{K}$ is the vertical line segment $]0, ia]$, one gets $g_{\mathbb{K}}(z) = \sqrt{z^2 + a^2}$, a formula by which we mean the analytic continuation of the function $z\sqrt{1 + a^2/z^2}$ where the square root is defined by its usual power series around 1 when z is large. Expansion at large z shows that $2C_{\mathbb{K}} = a^2$.

Example 3 (The oblique line segment) The case when $\mathbb{K}$ is an oblique line segment $]0, ae^{i\pi b}]$ making an angle πb with respect to the real positive axis ($b \in]0, 1[$) yields

$$z = (g_{\mathbb{K}}(z) - x_+)^b (g_{\mathbb{K}}(z) - x_-)^{1-b},$$

where the real parameters $x_- < 0 < x_+$ satisfy $bx_+ + (1 - b)x_- = 0$ and $(-x_-)^b x_+^{1-b} = a$. Expansion at large z shows that $2C_{\mathbb{K}} = b(1 - b)(x_+ - x_-)^2$. The closer the line is to the real axis (i.e. the closer b is to 0 or π) and the larger a has to be to reach a given capacity.

Example 4 (Arc of circle) An instructive example is when $\mathbb{K}$ is the arc $]r, re^{i\vartheta}]$ of a circle centred at 0 of radius r . Some of the following computations require to keep a precise track of the determination of the square root that appears in the formula for $g_{\mathbb{K}}$ because it is crucial for the interpretation. The map $f(w) = (w - r)/(w + r)$ sends the arc to the vertical line segment $]0, i \tan \vartheta/2]$, so that by Example 2, $w \mapsto \sqrt{f(w)^2 + \tan^2 \vartheta/2}$ is a conformal map from $\mathbb{D}$ to $\mathbb{H}$. However, this map sends ∞ to $1/(\cos \vartheta/2)$, not to ∞ . To get the hydrodynamic normalisation, we have to compose with an appropriate automorphism of $\mathbb{H}$. This yields

$$g_{\mathbb{K}}(w) = r \frac{-(2 - \cos^2 \vartheta/2) \cos \vartheta/2 \sqrt{(\frac{z-r}{z+r})^2 + \tan^2 \vartheta/2} + 2 - 3 \cos^2 \vartheta/2}{\cos \vartheta/2 \sqrt{(\frac{z-r}{z+r})^2 + \tan^2 \vartheta/2} - 1},$$

whose expansion at ∞ starts like $g_{\mathbb{K}}(w) = w + (1 - \cos^4 \vartheta/2)r^2/w + O(1/w^2)$. Hence the capacity is $C_{\mathbb{K}} = (1 - \cos^4 \vartheta/2)r^2$.

The tip of the arc, $re^{i\vartheta}$ is mapped to $(3\cos^2\vartheta/2 - 2)r$ by $g_{\mathbb{K}}$. One checks that

$$(g_{\mathbb{K}}(w) - g_{\mathbb{K}}(re^{i\vartheta})) \frac{\partial g_{\mathbb{K}}(w)}{\partial \vartheta} = 2r^2 \sin \vartheta/2 \cos^3 \vartheta/2,$$

which is w -independent.

Moreover $\lim_{w \rightarrow r^-} g_{\mathbb{K}}(w) = r(1 - 2\sin\vartheta/2 - \sin^2\vartheta/2)$ and $\lim_{w \rightarrow r^+} g_{\mathbb{K}}(w) = r(1 + 2\sin\vartheta/2 - \sin^2\vartheta/2)$. The behaviour of $g_{\mathbb{K}}$ when $\vartheta \mapsto \pi^-$ is interesting. In this limit, $\mathbb{K}$ becomes a semicircle. Let $\tilde{\mathbb{K}} = \{w \in \mathbb{H}, |w| \leq r\}$ be the corresponding semi-disc. The points w inside $\tilde{\mathbb{K}}$ are cut away from ∞ when $\vartheta \mapsto \pi^-$, and one checks that $\lim_{\vartheta \mapsto \pi^-} g_{\mathbb{K}}(w) = -2r$ for these points, i.e. they are swallowed in the limit. However, the points $\{w \in \mathbb{H}, |w| > r\}$ are mapped to $\lim_{\vartheta \mapsto \pi^-} g_{\mathbb{K}}(w) = w + r^2/w = g_{\tilde{\mathbb{K}}}(w)$.

2.3.4 Iteration of Conformal Maps

With Riemann's theorem at our disposal, we can start to encode growth processes. Suppose that the initial domain is the upper half-plane and that a small amount of matter is removed at each time step (so that in fact it is the lower half-plane that grows). At time step n , a certain $\mathbb{K}_n$ has been removed from $\mathbb{H}$. Let $g_n := g_{\mathbb{K}_n}$ denote the corresponding map and f_n its inverse. Then $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ describes a small amount of matter removed to $\mathbb{H}$. If $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ has typical size s and is located in the neighbourhood of point x on the real axis, $\mathbb{K}_{n+1} \setminus \mathbb{K}_n$, which is what is really removed at time $n + 1$ has typical size $s|f'_n(x)|$.

Example 5 (Simple iteration) Choose a small number ε . Let b_n , $n > 0$ be an independent sequence drawn from some chosen probability distribution. At time step $n + 1$ take $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ to be the semi-disc $\{z \in \mathbb{H}, |z - b_{n+1}||f'_n(b_{n+1})| \leq \varepsilon\}$, so that

$$g_{n+1}(z) = g_n(z) + \frac{\varepsilon^2}{|f'_n(b_{n+1})|^2(g_n(z) - b_{n+1})}.$$

This defines a random growth process where at each time step a small semi-disc-like grain of matter of size $\sim \varepsilon$ is removed. Despite its simplicity, little is known (at least to the author) about this process.

Many other (probabilistic or deterministic) rules can be invented, but the resulting processes are mostly impossible to study analytically at the moment. Let us simply note to conclude that the samples obtained by methods (but using the disc geometry) look strikingly like DLA. Figure 2.13 is obtained by iteration of conformal maps, compare with Fig. 2.12.

2.3.5 Continuous Time Growth Processes

Our aim is to motivate the introduction of Loewner chains.

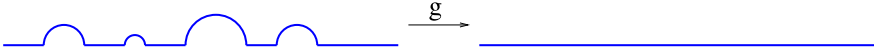


Fig. 2.14 The uniformisation of well-separated small semi-discs

If $\mathbb{K}$ is not simply a semi-disc, but a union of well-separated small semi-discs of radii r_α centred at b_α (see Fig. 2.14), a moment of thought leads to realise that

$$g_{\mathbb{K}}(z) \sim z + \sum_{\alpha} \frac{r_{\alpha}^2}{z - b_{\alpha}}.$$

The large z expansion yields $C_{\mathbb{K}} \sim \sum_{\alpha} r_{\alpha}^2$, a positive number as expected.

Taking a naïve limit, one gets that if ε is a small positive number, $v(x)$ is a non-negative function on $\mathbb{R}$ and $\mathbb{K} = \{z = x + iy \in \mathbb{H}, y \leq \varepsilon v(x)\}$ then

$$g_{\mathbb{K}}(z) \sim z + \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \frac{v(u) du}{z - u}.$$

Indeed, using that, if $v(x) \neq 0$, $\lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} m(x + i\varepsilon v(x) - u)^{-1} = \pi \delta(u - x)$ one checks that $\operatorname{Im} m \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(u) du}{x + i\varepsilon v(x) - u} \sim -v(x)$ so that to first order in ε $g_{\mathbb{K}}(z)$ is real when z is on the boundary of $\mathbb{K}$. Even more generally, one could replace the positive measure $v(u) du$ by any positive measure $d\rho(u)$. A naïve large- z expansion, certainly valid if the function v (or more generally the measure $d\rho$) has compact support and finite mass, gives $C_{\mathbb{K}} \sim \frac{\varepsilon}{\pi} \int_{\mathbb{R}} v(u) du$ (more generally $C_{\mathbb{K}} \sim \frac{\varepsilon}{\pi} \rho(\mathbb{R})$), again a positive number.

Now think about a continuous time growth process for which $\mathbb{K}_t$ has been removed from $\mathbb{H}$ at time t . Set $\mathbb{H}_t := \mathbb{H} \setminus \mathbb{K}_t$. Let $g_t := g_{\mathbb{K}_t} : \mathbb{H}_t \rightarrow \mathbb{H}$ denote the corresponding map and $f_t : \mathbb{H} \rightarrow \mathbb{H}_t$ its inverse. Fix t and a small positive ε . Then $g_t(\mathbb{K}_{t+\varepsilon} \setminus \mathbb{K}_t)$ describes a small amount of matter removed to $\mathbb{H}$. We could take as a definition of continuous time growth that the associated map $g_{t+\varepsilon} \circ f_t$ is described by a non-negative function $v_t(u)$ or more generally a positive measure $d\rho_t(u)$ as above. Taking the limit $\varepsilon \mapsto 0^+$ leads to

$$\frac{\partial g_t(z)}{\partial t} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\rho_t(u)}{g_t(z) - u}. \quad (2.7)$$

Taking the time derivative of $f_t \circ g_t(z) = z$ and substituting $w = g_t(z)$ yields

$$\frac{\partial f_t(w)}{\partial t} = -f'_t(w) p(w, t) \quad \text{where } p(w, t) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\rho_t(u)}{w - u}. \quad (2.8)$$

Note that $p(w, t)$ is holomorphic in $\mathbb{H}$ and the positive measure $\rho_t(u)$ is its boundary value in a generalised sense (hyperfunctions).

Equation (2.8) is called a **Loewner chain with reference domain $\mathbb{H}$** , though we shall use the name *Loewner chain* for (2.7) as well. The analogous equations with reference domain the unit disc can be obtained straightforwardly by the same arguments. The large- z expansion yields

$$\frac{dC_{\mathbb{K}_t}}{dt} = \frac{1}{\pi} \rho_t(\mathbb{R}) \geq 0.$$

So if hulls are constructed little by little by a growth process, the capacity increases with time (in particular it is obviously positive).

In principle, if the family of measures ρ_t is given, one can solve for $g_t(z)$ with the initial condition $g_0(z) = z$. Again, ρ_t can be random or deterministic. We should note that Loewner chains are in some sense kinematic equations that give a general framework to encode growth processes. But in a real dynamical problem ρ_t has to be specified. It may depend explicitly on g_t . For instance $d\rho_t(u) = |f'_t(u)|^{-2} du$ is related to Laplacian growth, though the unit disc geometry is the relevant one in that case. The exponent -2 , which we already interpreted for discrete iteration, ensures that the size of $\mathbb{K}_t$ grows linearly with time. But other exponents between 0 and -2 are interesting too. Note that DLA provides a discrete analogue of Laplacian growth. The particle size plays the role of an ultraviolet cutoff.

2.3.6 Geometric Interpretation

One can give the following geometric interpretation of Loewner chains. Set $g_t(z) := z_t$, view z_t as the position of a fluid particle as time goes by, and suppose for simplicity that $d\rho_t(u) = v_t(u) du$ so that the Loewner chain becomes

$$\frac{dz_t}{dt} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{v_t(u) du}{z_t - u}. \quad (2.9)$$

Hence $\frac{1}{\pi} \int_{\mathbb{R}} \frac{v_t(u) du}{z - u}$ plays the role of a time-dependent holomorphic vector field on the manifold with boundary $\mathbb{H}$. At point $z = x + i0^+$ i.e. close to the real axis (the boundary of $\mathbb{H}$) this vector field has imaginary part $-v(x)$, so that when x is away from the support of ρ_t , (that is, when $v_t(\cdot) = 0$ in a neighbourhood of x), the vector field is real, i.e. tangent to the boundary. However, if x is on the support of ρ_t the vector field has a finite negative imaginary part, which means that some fluid particles that started inside $\mathbb{H}$ can be swallowed by the boundary. In fact $\mathbb{K}_t$ is nothing but the set of fluid particles which where in $\mathbb{H}$ at $t = 0$ but have hit the boundary before time t .

The reader is urged to review Examples 1–4 in this light. For the semi-disc case, take r as time, either with $b = 0$ or with $b = r$. For the case of line segments, take a constant b and use a as time. For the arc of circle, using ϑ as time, with special care in the limit $\vartheta \mapsto \pi^-$. It is instructive to compute the measure ρ_t in each case and to check that the above interpretation of $\mathbb{K}_t$ is correct.

Another, more abstract, geometric interpretation is also possible. Let N_- be the group of series of the form $z + \sum_{m \leq -1} g_m z^{m+1}$ with real coefficients and convergent for large z (the domain of convergence may depend on the series, so N_- is made of “germs”, and is in fact the group of germs of holomorphic functions fixing ∞ and with derivative 1 at ∞). In the same spirit, let O_∞ be the space of germs of holomorphic functions at infinity. We let N_- act on O_∞ by composition, $\gamma_g \cdot F := F \circ g$. Observe that $\gamma_{g_1 \circ g_2} = \gamma_{g_2} \cdot \gamma_{g_1}$ so this is an anti-representation.

Note that the g_t 's of a Loewner chain with bounded $\mathbb{K}_t$ belong to N_- . If $F \in O_\infty$ and if z is large enough, $F(z)$ is well defined as well as $F(z_t)$ for small t (where the meaning of small may depend on z and F) and

$$\frac{dF(z_t)}{dt} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\rho_t(u)}{z_t - u} \frac{\partial F}{\partial z}(z_t),$$

which can be rewritten

$$\frac{d}{dt}(\gamma_{g_t} \cdot F) = \gamma_{g_t} \cdot (v_t \cdot F)$$

where $v_t(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\rho_t(u)}{z - u} \frac{\partial}{\partial z}$ is a germ of vector field.

So the Loewner chain equation can be viewed as a flow on N_-

$$\frac{d}{dt}\gamma_{g_t} = \gamma_{g_t} \cdot v_t.$$

The group N_- has an interesting representation theory, related to that of the Virasoro algebra, which can be used as a probe for this flow.

2.3.7 Local Growth

Suppose that as time goes by the measures ρ_s are δ -peaks of height $2\pi a_s$ (the factor 2 is purely historical) at position ξ_s : in physicist notation $d\rho_s(u) = 2\pi a_s \delta(u - \xi_s) du$. In the upper half plane reference geometry, the growth process will be described by an equation of the type

$$\frac{\partial g_s(z)}{\partial s} = \frac{2a_s}{g_s(z) - \xi_s}. \quad (2.10)$$

Note that Examples 2–4 fall in this category. The formula was given for Example 4 if $s = \vartheta$ and the other cases lead to simple computations left to the reader.

If one is interested only in the growth of the hull, but not in the way the evolution is parameterised, one can make change the time variable without arm. The statement that ξ_s changes quickly or slowly makes sense only compared with the changes in a_s . For instance, suppose that the function a_s vanishes in some interval, while ξ_s keeps on changing so that it has a different value at the beginning and at the end of the interval. During that interval g_s has not changed but when a_s starts moving again, the place at which the hull resumes growth can be far from the place where it was growing before the pause. This is a limiting case of what happens when variations of ξ_s are large with respect to those of a_s . This means that if, at s_0 , ξ_s starts to move very fast with respect to a_s , the growth takes place very near $\mathbb{K}_{s_0}$ or the real axis. This conclusion is supported by Example 3.

We also infer that to have local growth, i.e. to have the position where the hull grows vary continuously, we need to impose that ξ_s stops if a_s does. To make this statement precise, it is convenient to go to a special time parameterisation. The capacity of the hull at time s is $C_{\mathbb{K}_s} = 2 \int_0^s ds' a_{s'}$, a non-decreasing function of s .

Define $t = \int_0^s ds' a_{s'}$, take t to be the new time variable and by abuse of notation write ξ_t for $\xi_{s(t)}$, $\mathbb{K}_t$ for $\mathbb{K}_{s(t)}$ and so on. Then by construction $C_{\mathbb{K}_t} = 2t$ and the equation reads

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}. \quad (2.11)$$

We take as a definition of **local growth** that ξ_t is continuous function of t . The function ξ_t is often called the **driving function** of the Loewner evolution. It is sometimes convenient to normalise ξ_t by $\xi_0 = 0$ or what amounts to the same to impose that the hull starts growing from point 0.

A broad class of growing hulls that can be described by such an equation is given by continuous simple curves started on the boundary of $\mathbb{H}$ and staying in $\mathbb{H}$ thereafter. Let $\gamma_{[0, \infty]}$ be a parameterised simple continuous curve from 0 to ∞ in $\mathbb{H}$ and assume that the capacity parameterisations has been chosen, so that $\mathbb{K}_t := \gamma_{[0, t]}$ is a hull with capacity $2t$. When ε is small, $\mathbb{K}_{\varepsilon, t} := g_t(\gamma_{[t, t+\varepsilon]})$ is a tiny piece of a curve. The support of the discontinuity measure $d\rho_{f_{\varepsilon, t}}$ is small and becomes a point when ε goes to 0. Measures supported at a point are δ functions, so there is a point ξ_t such that, as a measure, $d\rho_{f_{\varepsilon, t}}/dx \sim 2\varepsilon\delta(x - \xi_t)$ as $\varepsilon \rightarrow 0^+$.

For a general local Loewner growth process, one defines $\gamma_t = f_t(\xi_t + i0^+) := \lim_{\varepsilon \rightarrow 0^+} f_t(\xi_t + i\varepsilon)$ (remember f_t is the inverse map of g_t). We shall often use the shorthand notation $\gamma_t = f_t(\xi_t)$. The set $\gamma_{[0, t]} := \bigcup_{s \in [0, t]} \gamma_s$ is called the **trace** of the growth process. If the hull is a simple curve, the notation is consistent. Whether the trace is a curve (simple or not) in general is highly non-obvious, but this will be the case for all examples in these notes, though proving it can be a formidable task.

At time t , growth takes place at point ξ_t in the g_t plane i.e. at point γ_t in the original “physical” plane. Thus it is tempting to conclude that $\mathbb{K}_t$ coincides with $\gamma_{[0, t]}$. Though this picture works nicely for Examples 2–3, it is slightly too naïve and fails in Example 4 when the trace, which is an arc of circle closes to a semicircle and the corresponding semi-disc completes the hull.

For a given z with $\text{Im} m z \geq 0$ and $z \neq \xi_0$, the local existence and uniqueness of solutions to Eq. (2.11) is granted by general theorems on ordinary differential equations, but problems may arise if a time τ_z (depending on z in general) exists for which $g_{\tau_z}(z) = \xi_{\tau_z}$. One possibility is to declare $g_t(z)$ undefined for $t \geq \tau_z$. But it is often the case that, as suggested by Examples 2–3, the two limits $\lim_{x \rightarrow \xi_{\tau_z}^\pm} g_t \circ f_{\tau_z}(x)$ exist, allowing to think that after τ_z , $g_t(z)$ has split in two real trajectories.

There is a **regularity criterion** on the function ξ that guaranties that if $x \neq \xi_0$ is real, τ_x is infinite. It is sufficient that for each t ,

$$\lim_{s \rightarrow t^-} \sup_{t' \in [s, t]} \frac{|\xi_t - \xi_{t'}|}{|t - t'|^{1/2}} < 4. \quad (2.12)$$

To prove this criterion, it is convenient to consider $X_t := g_t(x) - \xi_t$, a continuous function which satisfies the integral equation $X_t = x - \xi_t + \int_0^t \frac{2ds}{X_s}$. As this implies that $\xi_t - \xi_t = X_t - X_\tau + \int_t^\tau \frac{2ds}{X_s}$, we can see ξ as a functional of X . The task is to control its behaviour if X_t has a given sign, say positive, on $[0, \tau]$ and vanishes at τ . It is clear that the two terms in $X_t + \int_t^\tau \frac{2ds}{X_s}$ vary in opposite directions, in that

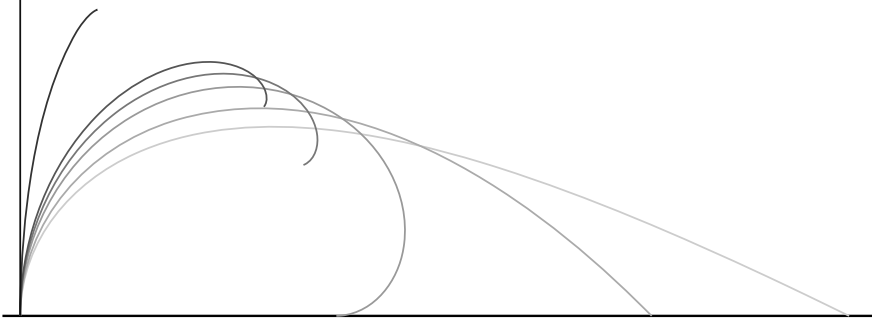


Fig. 2.15 The hull at time τ for $\alpha = 0, 1, 2, 3, 4, 5, 6$

the faster X_t goes to 0, the slower is the vanishing of $\int_t^\tau \frac{2ds}{X_s}$ at $t = \tau$. So the mildest behaviour of the sum as t goes to τ is when the two terms have a similar behaviour. A detailed analysis requires some care, but a quick and dirty way to retrieve the criterion is to impose that the two terms be equal, which gives $X_t = 2\sqrt{\tau - t}$ hence $\xi_\tau - \xi_t = 4\sqrt{\tau - t}$ as announced.

Example 6 (Square root driving term) The Loewner equation when $\xi_\tau - \xi_t = 4\alpha\sqrt{\tau - t}$ can be solved in closed form for any α though the formulae are cumbersome. We normalise ξ_t so that $\xi_0 = 0$, i.e. take $\xi_t = 4\alpha(\sqrt{\tau} - \sqrt{\tau - t})$. By left-right symmetry, we can assume that $\alpha \geq 0$. For $\alpha \in [1, +\infty[$ it is convenient to set $\alpha := \cosh \eta$, $\eta \in [0, +\infty[$. One parameterises time as

$$\frac{2e^{-\eta \coth \eta} \sinh \eta}{\sin(2\vartheta \sinh \eta)} \frac{(\sin(\vartheta e^{-\eta}))^{(\coth \eta + 1)/2}}{(\sin(\vartheta e^{-\eta}))^{(\coth \eta - 1)/2}} = \sqrt{\frac{\tau - t}{\tau}},$$

with $\vartheta \in [0, \pi e^{-\eta}]$. As a function of ϑ , the hull builds the curve

$$\left\{ 2\sqrt{\tau} \left(e^{-\eta} - \frac{2 \sinh \eta \sin(\vartheta e^{-\eta})}{\sin(2\vartheta \sinh \eta)} e^{i\vartheta e^{-\eta}} \right) \right\}_{\vartheta \in [0, \pi e^{-\eta}]}$$

For $\vartheta = \pi e^{-\eta}$ the curve closes a whole domain, just as in the arc of circle [Example 4](#), which in fact is the special case $\alpha = 3\sqrt{2}$.

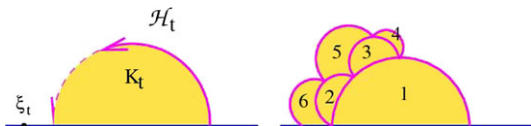
For $\alpha \in [0, 1[$ it is convenient to set $\alpha := \cos \varphi$, $\varphi \in]0, \pi/2]$. The formulae can be obtained by analytic continuation $\eta \mapsto i\varphi$, this time with a parameter $\vartheta \in [0, \infty]$. The hulls remain simple curves even for $\vartheta = \infty$.

[Figure 2.15](#) illustrates the different behaviours.

The very same criterion on the behaviour of the function ξ is also sufficient to ensure that the hull $\mathbb{K}_t$ is a simple continuous curve, say $\{\gamma_s, s \in]0, t]\}$, and $\gamma_t = f_t(\xi_t)$, i.e. that our naïve expectation $\mathbb{K}_t = \bigcup_{s \in]0, t]} f_s(\xi_s)$ is fulfilled.

The two properties—“ $g_t(x)$ for real x does not hit ξ_t ” and “the hull is a simple curve”—are in fact equivalent. The intuitive reason is the following. The fact that $g_t(x)$ for real x hits ξ_t at some time τ is the sign that at time τ the hull “swallows a whole piece of $\mathbb{H}$ ”. The previous example illustrates this relationship when the hull hits the real axis. But from the point of view of iteration, if $s \geq 0$ is fixed, it is

Fig. 2.16 A curve hitting itself or the real axis a number of times



obvious that when $t \geq 0$ varies the function $\tilde{g}_{t,s}(z) := g_{t+s} \circ f_s(z + \xi_s) - \xi_s$ satisfies the Loewner equation (2.11) with driving function $\tilde{\xi}_t := \xi_{t+s} - \xi_s$. So if the driving function $\tilde{\xi}_t := \xi_{t+s} - \xi_s$ leads to a hull hitting the real axis, the driving function ξ_t leads to a hull hitting itself or the real axis, as illustrated in Fig. 2.16. This discussion also explains why, if the trace is a continuous curve, it can have double points but no crossings.

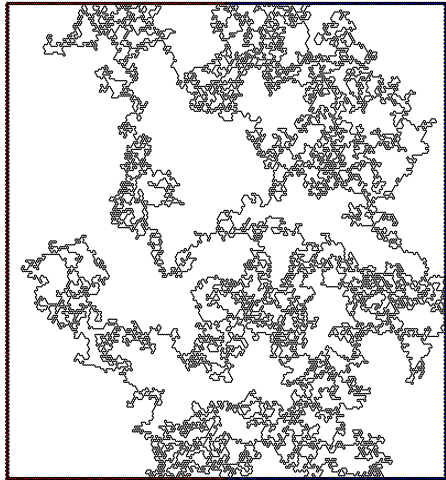
2.4 Stochastic Loewner Evolutions

Stochastic Loewner evolutions were introduced by Schramm in 1999 as a general framework to study random curves satisfying certain properties. His specific interest was to prove that loop-erased random walks on a two-dimensional lattice have a conformally invariant continuum limit. Schramm observed that these walks have on the lattice the so-called *domain Markov property* (to be defined below) a property that can be rephrased in the continuum. Though he was not able at that time to prove the existence of a conformally invariant limit of loop-erased random walks, he recognised that conformal invariance and the domain Markov property brought together would have remarkable consequences, and was able to prove that the probability measures on random curves in the continuum satisfying at the same time conformal invariance and the domain Markov property formed a one-parameter family. Crucial to the proof and the explicit description of these measures was the idea of viewing curves as hulls and to use Loewner evolutions. That in this context the most useful description of a curve is by encoding it into a growth process via a Loewner chain is at first sight very surprising and may explain why physicists who had understood the importance of conformal invariance to study many examples of random curves in the early 1980's failed to "produce Schramm's argument before Schramm".

The general idea is to impose properties relating different members in a family of probability measures on continuous curves without crossings, but possibly with multiple points. Let us note that curves here are considered modulo reparameterisations, but not simply as subsets of the plane. For simple curves, this would essentially make no difference, but curves with multiple points require more care.

In the discrete setting, it is a fact that interfaces on appropriate lattices are simple curves, so why bother to deal with non-simple curves? The answer is that even if at the scale of the lattice spacing the interface is simple, when one tries to take a continuum limit by looking at a macroscopic scale while taking a smaller and smaller lattice spacing, a curve that makes a large excursion and then comes back close to itself, say a few lattice spacings away, has a double point from the macroscopic

Fig. 2.17 A percolation sample



viewpoint. While in some models—like loop-erased random walks, Schramm’s initial motivation—the interface remain simple when the lattice spacing gets smaller, some other models—like percolation—clearly exhibit multiple points in the continuum limit. This is clearly seen on samples, see Fig. 2.17.

In the following three sections, we suppose that we are given a family of probability measures $\{\mathbf{P}_{\mathbb{D},a,b}\}$ indexed by triples consisting of a domain $\mathbb{D}$ and two distinct boundary points a, b of $\mathbb{D}$. For a given triple $(\mathbb{D}, a, b)$, $\mathbf{P}_{\mathbb{D},a,b}$ is a measure on $\Omega_{\mathbb{D},a,b}$, the set of continuous curves without crossings within $\overline{\mathbb{D}}$ —the union of $\mathbb{D}$ and its boundary (in the refined sense alluded too in Sect. 2.3.1)—joining a to b (it is understood that a and b are not multiple points).

First, we want to define what it means for the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ to be conformally invariant and to have the domain Markov property.

2.4.1 Conformal Invariance

By Riemann’s theorem, if $(\mathbb{D}, a, b)$ and $(\mathbb{D}', a', b')$ are any two triples, there is a conformal map $g : \mathbb{D} \mapsto \mathbb{D}'$ such that $g(a) = a'$ and $g(b) = b'$. It is clear that g induces a bijection, which we call $\check{g}$, from $\Omega_{\mathbb{D},a,b}$ to $\Omega_{\mathbb{D}',a',b'}$. **Conformal invariance** of the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ is the statement that $\check{g}$ is measurable and the image measure $\mathbf{P}_{\mathbb{D},a,b} \circ \check{g}^{-1}$ coincides with $\mathbf{P}_{\mathbb{D}',a',b'}$, i.e. if C' is a measurable subset of $\Omega_{\mathbb{D}',a',b'}$ then $\check{g}^{-1}(C')$ is a measurable subset of $\Omega_{\mathbb{D},a,b}$ and $\mathbf{P}_{\mathbb{D},a,b}(\check{g}^{-1}(C')) = \mathbf{P}_{\mathbb{D}',a',b'}(C')$.

Conformal invariance by itself is a rather weak constraint. Indeed, suppose that a probability $\mathbf{P}_{\mathbb{D}_0,a_0,b_0}$ on $\Omega_{\mathbb{D}_0,a_0,b_0}$ has been defined for a single triple $\mathbb{D}_0, a_0, b_0$ and that it is invariant under the conformal transformations of $\mathbb{D}_0$ fixing a_0 and b_0 . Such transformations form a group with one real parameter. Then the direct image $\mathbf{P}_{\mathbb{D}_0,a_0,b_0}$ by any conformal transformation g will define unambiguously

$\mathbf{P}_{g(\mathbb{D}_0), g(a_0), g(b_0)}$. By the Riemann mapping theorem, this defines $\mathbf{P}_{\mathbb{D}, a, b}$ for any triple, and the resulting family of probabilities is clearly conformally invariant.

To get a more rigid situation, one has to impose another constraint on the family $\{\mathbf{P}_{\mathbb{D}, a, b}\}$. Schramm translated in the continuum a property that holds for loop-erased random walks in the discrete setting: the **domain Markov property**, to which we turn our attention now.

Before doing so, let us remark that this strategy is rather typical. If continuous curves without crossings are replaced by general hulls joining a to b in $\mathbb{D}$ the notion of domain Markov property does not make sense but another one, restriction, turns out to be fruitful and allow for another complete classification. We shall have little to say about these nice “restriction measures” in these notes.

2.4.2 Domain Markov Property

Fix a triple $(\mathbb{D}, a, b)$ and consider an element $\gamma \in \Omega_{\mathbb{D}, a, b}$. If a real continuous parameter along γ is given and s is any intermediate value of the parameter, the past and the future of s split γ in two (not necessarily disjoint) curves without crossings. The curve corresponding to the past of s starts at a and is called an **initial segment** of γ . The curve corresponding to the future of s ends at b and is called a **final segment** of γ . The final segment starts at some point $c \in \mathbb{D}$ which is also the end of the initial segment. We use the notation $\gamma_{[a, c]}$ for such an initial segment with point c included and $\gamma_{[c, b]}$ for the final segment. Beware that the notation is a bit ambiguous, because of possible multiple points on γ .

Several curves γ' share the same initial segment $\gamma_{[a, c]}$, and the discussion that follows focuses on the question: if an initial segment is given, what is the distribution of the final segment?

Making sense of this question is not so obvious. First, there should be enough measurable sets in $\Omega_{\mathbb{D}, a, b}$. We shall for a while assume that this is so. But even in that case, the event “ γ' starts exactly with $\gamma_{[a, c]}$ ” is more than likely to occur with probability 0. Vaguely, what may have a non-trivial probability is the event “ γ' has an initial segment that is close (in some quantified sense) to $\gamma_{[a, c]}$ ”. Probabilists have invented so-called *conditional expectations* and regular conditional probabilities just to deal with that kind of situations. Starting from $\mathbf{P}_{\mathbb{D}, a, b}$ this allows to define new probability measures, denoted $\mathbf{P}_{\mathbb{D}, a, b}(\cdot | \gamma_{[a, c]})$, read “conditional probability given the initial segment $\gamma_{[a, c]}$ ”, that can be manipulated just as conditional probabilities when the state space is discrete.³

The set of points in $\mathbb{D}$ that cannot be joined to b by a continuous curve in $\mathbb{D}$ without hitting the initial segment form a set that we call a **hull**⁴ and denote by $\mathbb{K}_c$.

³There is a small price to pay, however. For instance, the definition of this conditional probability may fail or be ambiguous for certain $\gamma_{[a, c]}$ but these nasty initial segments form altogether a set of probability 0 for $\mathbf{P}_{\mathbb{D}, a, b}$.

⁴If $(\mathbb{D}, a, b) = (\mathbb{H}, 0, \infty)$, this is consistent with our initial definition, and with the new definition, conformal maps send hulls to hulls.

This notation is again slightly ambiguous. Note that $\mathbb{D} \setminus \mathbb{K}_c$ is again a domain. If the initial segment is $\gamma_{[a,c]}$, the final segment starts at c and never enters inside $\mathbb{K}_c$. So the support of the conditional probability $\mathbf{P}_{\mathbb{D},a,b}(\cdot | \gamma_{[a,c]})$ is included in $\Omega_{\mathbb{D} \setminus \mathbb{K}_c, c, b}$. But on this set we have another probability measure, namely $\mathbf{P}_{\mathbb{D} \setminus \mathbb{K}_c, c, b}$, and the two can be compared.

We say that a set $\{\gamma_{[a,c]}\}$ of curves in $\mathbb{D}$ without crossings starting at a is a set of distinct representatives if any curve in $\Omega_{\mathbb{D},a,b}$ has exactly one of its initial segments in $\{\gamma_{[a,c]}\}$. For instance, for the triple $(\mathbb{H}, 0, \infty)$, the initial segments whose associated hull has capacity t form a set of distinct representatives. Intuitively, to get the expectation of a random variable on $\Omega_{\mathbb{D},a,b}$, one can compute its conditional expectation on $\gamma_{[a,c]}$, and then integrate over $\gamma_{[a,c]}$ in a system of distinct representatives.

The family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ is said to have the **domain Markov property** if, for any triple $(\mathbb{D}, a, b)$, one has

$$\mathbf{P}_{\mathbb{D},a,b}(\cdot | \gamma_{[a,c]}) = \mathbf{P}_{\mathbb{D} \setminus \mathbb{K}_c, c, b}, \quad (2.13)$$

except maybe for a set of initial segments whose intersection with any system of distinct representatives is of measure 0 for $\mathbf{P}_{\mathbb{D},a,b}$.

This expression of the domain Markov property is more intuitive on the lattice in the discrete setting—because the interfaces are simple curves and because conditional probabilities have a much simpler definition—and it holds in many examples. It is vaguely related to the notion of locality in physics. The reader can check it straightforwardly for the exploration process. Equation (2.4) makes the domain Markov property plain for loop-erased random walks as well, whereas a direct proof using the original definition is more cumbersome.

2.4.3 Schramm's Argument

Our aim is to explore the interplay between conformal invariance and the domain Markov property of the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$.

First, by conformal invariance, we may concentrate on the triple $(\mathbb{H}, 0, \infty)$. We choose a parameterisation of curves in $\Omega_{\mathbb{H},0,\infty}$ in such a way that the hull $\mathbb{K}_t := \mathbb{K}_{\gamma_t}$ associated with the initial segment $\gamma_{[0,t]} := \gamma_{[0,\gamma_t]}$ of $\gamma \in \Omega_{\mathbb{H},0,\infty}$ has capacity $2t$. Because of the underlying continuous curve γ , the growth of $\mathbb{K}_t$ is local, and the associated g_t satisfies a Loewner equation $\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}$ for some continuous function ξ_t . The probability $\mathbf{P}_{\mathbb{H},0,\infty}$ on $\Omega_{\mathbb{H},0,\infty}$ induces a random process on the set of initial segments $\gamma_{[0,t]}$, hence on the set of hulls $\mathbb{K}_t$, and on the set of continuous functions ξ_t .

Our next aim is to derive consequences for the stochastic process ξ_t of the domain Markov property and conformal invariance.

First for fixed $(\mathbb{H}, 0, \infty)$ there is a remnant of conformal invariance: dilatations. Hence for $\lambda > 0$, the hull $\frac{1}{\lambda} \mathbb{K}_{\lambda^2 t}$ must have the same distribution as a $\mathbb{K}_t$. The corresponding Loewner map is $\frac{1}{\lambda} g_{\lambda^2 t}(\lambda z)$, whose driving function is $\frac{1}{\lambda} \xi_{\lambda^2 t}$. Hence the processes ξ_t and $\frac{1}{\lambda} \xi_{\lambda^2 t}$ have the same law. We say that ξ_t has dimension $1/2$.

Given K_t , the domain Markov property states that $\gamma_{[t,\infty]}$ is distributed according to $\mathbf{P}_{\mathbb{D} \setminus \mathbb{K}_t, \gamma_t, \infty}$. The conformal transformation $g_t(z) - \xi_t$ maps $\mathbb{D} \setminus \mathbb{K}_t$ to $\mathbb{H}$, γ_t to 0 and ∞ to ∞ . By conformal invariance, $g_t(\gamma_{[t,\infty]}) - \xi_t$ is distributed according to $\mathbf{P}_{\mathbb{H}, 0, \infty}$. In particular for $s \geq 0$ $g_t(\gamma_{[t, t+s]}) - \xi_t$ has the same distribution as a $\gamma_{[0, s]}$ hence is independent of $\gamma_{[0, t]}$. But the Loewner map for $g_t(\gamma_{[t, t+s]}) - \xi_t$ is $g_{s+t} \circ f_t(z + \xi_t) - \xi_t$ (remember f_t is the inverse of g_t), whose driving function is $\xi_{t+s} - \xi_t$. We infer that the random function $\xi_\cdot$ is such that for any $t, s \geq 0$, $\xi_{t+s} - \xi_t$ is independent of $\{\xi_{t'}\}, t' \in [0, t]$ and distributed like a ξ_s .

To resume our knowledge, the random process $\xi_\cdot$ has continuous samples, independent identically distributed increments and dimension 1/2. By a deep general result, a random process with continuous samples and independent identically distributed increments is of the form $\sqrt{\kappa} B_t + \rho t$ for some non-negative κ and some real ρ . Obviously it has dimension 1/2 if and only if $\rho = 0$.

To conclude, Schramm's argument shows that if a family of probabilities $\{\mathbf{P}_{\mathbb{D}, a, b}\}$ on curves without crossing indexed by triples $(\mathbb{D}, a, b)$ is conformally invariant and has the domain Markov property, the law induced by $\mathbf{P}_{\mathbb{H}, 0, \infty}$ on initial hulls of capacity $2t$ by is described by a **stochastic Loewner evolution**

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa} B_t} \quad (2.14)$$

for some $\kappa \geq 0$ and some normalised Brownian motion B_t . This process is often denoted by SLE_κ .

A priori, this does not show that each κ is realised via some family $\{\mathbf{P}_{\mathbb{D}, a, b}\}$ (because the Loewner evolution deals with hulls, not with curves).

2.4.4 Basic Properties

The first important property is a kind of converse to Schramm's result. If $\kappa \geq 0$ is a real number, and B_t a continuous realisation of a normalised Brownian motion, a deep theorem states that *the trace associated to the stochastic Loewner evolution equation (2.14) is almost surely a continuous curve joining 0 to ∞ . This curve is simple and stays in $\mathbb{H}$ if $\kappa \in [0, 4]$, has double points and hits the real axis if $\kappa \in [4, 8]$ and is space filling if $\kappa \in [8, +\infty[$.*

At the time Schramm introduced stochastic Loewner evolutions, this very hard theorem was not known (he contributed to prove it later).

As explained before, a continuous trace cannot have crossings. Thus for any $\kappa \geq 0$, the stochastic Loewner evolution defines a probability measure $\mathbf{P}_\kappa$ on continuous curves without crossings joining 0 to ∞ in $\mathbb{H}$. This measure is scale-invariant. Hence, for each κ , conformal transformations can be used to define in a consistent way a family of probabilities $\{\mathbf{P}_{\mathbb{D}, a, b}^\kappa\}$. This family is trivially conformally invariant, and it is easy to check that it satisfies the domain Markov property.

This finishes the complete classification.

Taking the existence of a curve for granted, the change of behaviour from simple curves to curves with double points at $\kappa = 4$ can be understood as follows. First, the

necessary condition (negation of Eq. (2.12)) for the existence of multiple points is fulfilled for all values of κ , though in some kind of marginal way, for if $\xi_t = \sqrt{\kappa} B_t$ where B_t is a normalised Brownian motion, the law of the iterated logarithm states that, with probability one

$$\lim_{s \rightarrow t^-} \sup_{t' \in [s, t]} \frac{|\xi_t - \xi_{t'}|}{|t - t'|^{1/2} \ln \ln |t - t'|^{-1}} = \sqrt{2\kappa}.$$

So the stochastic Loewner source is wilder by a $\ln \ln |t - t'|^{-1}$ than the criterion. The fact that for $\kappa \leq 4$ the Loewner trace is a simple curve shows that, as should be expected, the criterion is only necessary, but not sufficient. Intuitively, Brownian motion is more singular than necessary, but for $\kappa \leq 4$ with too little correlation time to behave consistently for long enough periods to produce multiple points.

This fact is related to another well studied question: recurrence of Brownian motion. If space dimension $D = 1$, Brownian motion passes infinitely many times at any point. If $D = 2$, it passes infinitely many times in the any neighbourhood of any point, but not exactly at any given point, and if $D \geq 3$, it has a non-zero probability to remain at a given finite distance of any point. So dimension 2 is somehow a marginal case. Now let R_t be the norm of a d -dimensional Brownian motion. Assume $R_0 > 0$. One can show using stochastic calculus that $W_t := -R_t + \frac{D-1}{2} \int_0^t \frac{ds}{R_s}$ is a standard 1-dimensional Brownian motion. In this equation, D appears as an explicit parameter, and one can reverse the logic: given a standard 1-dimensional Brownian motion W_t what are the properties of R_t , called the d -dimensional **Bessel process** in mathematics. Setting $\kappa = 4/(d - 1)$ one sees that $X_t := \sqrt{\kappa}(R_t + W_t)$ satisfies the equation $\frac{dX_t}{dt} = \frac{2}{X_t - \sqrt{\kappa}W_t}$. This has two important consequences: first, one can indeed retrieve R_t from W_t by solving a differential equation and second, the Bessel process is essentially a stochastic Loewner evolutions but looking only at the boundary of $\mathbb{H}$. For general D , the Bessel processes behave with respect to visits to 0 just like the recurrence properties of Brownian motion for integer d suggest: the D -dimensional Bessel process hits the origin infinitely many times if $D < 2$, but never if $D \geq 2$. Equivalently, if $\kappa \leq 4$, $X_t - \sqrt{\kappa}W_t$ never vanishes, but vanishes infinitely many times if $\kappa > 4$. But we already know that the vanishing of $X_t - \sqrt{\kappa}W_t$ is the sign that the growing curve hits itself or the real axis.

Another very hard result is the fractal dimension: the measures $\mathbf{P}_{\mathbb{D}, a, b}^\kappa$ is concentrated on curves with fractal dimension $\min\{1 + \kappa/8, 2\}$.

Recently, two important conjectures on SLE have been proven.

One of them is **reversibility**. The treatment of random curves by a Loewner evolution is quite asymmetric by definition. However interfaces between two points in physics (i.e. in statistical mechanics models) quite generally make no difference between the two ends. So it was conjectured very early that interfaces generated by an SLE process were reversible. One difficulty is with the parameterisation. Take an SLE sample in $\mathbb{H}$ from 0 to infinity, parameterise it with capacity. Apply the transformation $z \mapsto -1/z$ and parameterise the inverse sample with capacity. Now any point on the curve has two parameters attached to it. One of the troubles is that the relationship between the two parameters is extremely wild. Anyway, reversibility is now a theorem.

The second one is **duality**. Take an SLE_κ sample with $\kappa > 4$ and look at the boundary of $\mathbb{K}_t$. This is simple curve, and one can expect that its distribution is conformally invariant in some sense. So it is natural to ask if and how it fits in the SLE framework. It was conjectured by physicists that it is related in some sense to an $\text{SLE}_{16/\kappa}$, and that in particular it has dimension $1 + 2/\kappa$. Though this is correct, the precise recent theorem that gives an explicit description involves non-trivial extensions of SLE where the driving function is $\sqrt{16/\kappa} B_t$ plus some rather complicated drift terms.

2.4.5 Locality

Let us go back to percolation. Consider a domain $\mathbb{D}$ and three boundary points a, b, c such that $(\mathbb{D}, a, b)$ and $(\mathbb{D}, a, c)$ are hexagonal domains with admissible boundary conditions. On the boundary interval (b, c) the colours disagree. But both domains share the same inner hexagons, and the percolation samples are the same. For each configuration of inner coloured hexagons, it is clear that the exploration paths for $(\mathbb{D}, a, b)$ and $(\mathbb{D}, a, c)$ coincide until they hit the boundary interval (b, c) for the first time. Hence the measures on interfaces stopped when they first hit (b, c) is the same for $(\mathbb{D}, a, b)$ and $(\mathbb{D}, a, c)$.

One could even go further and consider two hexagonal domains with admissible boundary conditions $(\mathbb{D}, a, b)$ and $(\mathbb{D}', a, b')$. Say that a hexagon in $\mathbb{D} \cap \mathbb{D}'$ is special if it is inner for $\mathbb{D}$ but not for $\mathbb{D}'$ or vice versa, or if it is a boundary hexagon for both, but with a different colour. By an analogous argument, the distribution for the exploration process started at a and stopped when it hits a special hexagon is the same for $(\mathbb{D}, a, b)$ and $(\mathbb{D}', a, b')$. This is called the **locality property**.

The particular case when $b = b'$ and $\mathbb{D}' \subset \mathbb{D}$ was mentioned before.

The notion of locality can also be formulated in the continuum. Let $\mathbb{L}$ be a hull in $\mathbb{D}$ bounded away from a and b . To each curve in $\Omega_{\mathbb{D}, a, b}$ we can associate its smallest initial segment that hits the boundary of $\mathbb{L}$ (we take this initial segment to be the curve itself if it never hits $\mathbb{L}$). These initial segments form a system Σ of distinct representatives both in $\Omega_{\mathbb{D}, a, b}$ and in $\Omega_{\mathbb{D} \setminus \mathbb{L}, a, b}$. Thus both $\mathbf{P}_{\mathbb{D}, a, b}$ and $\mathbf{P}_{\mathbb{D} \setminus \mathbb{L}, a, b}$ induce a probability measure on Σ . The property of **locality** is the statement that these two measures coincide. In a more mundane way, if $\mathbb{L}$ is a hull in $\mathbb{D}$ bounded away from a and b , the distribution of curves up to the first hitting of $\mathbb{L}$ are the same in $\mathbb{D}$ and in $\mathbb{D} \setminus \mathbb{L}$.

Stochastic calculus can be used to show that the family $\{\mathbf{P}_{\mathbb{D}, a, b}^{\kappa=6}\}$ is the only one to have the locality property. Let us note that it is no surprise that a value of κ satisfying locality is > 4 . Indeed, if $\kappa \leq 4$, the traces are simple curves that do not hit the boundary. Then no trace touches $\mathbb{L}$ for $\mathbf{P}_{\mathbb{D} \setminus \mathbb{L}, a, b}$, but hitting $\mathbb{L}$ for $\mathbf{P}_{\mathbb{D}, a, b}$ has a finite probability if $\mathbb{L}$ is non-trivial, so that the supports of the two probability measures induced on Σ are not the same.

Though the amount of mathematical machinery is significantly higher than in the rest of these notes, let us give the outline of a computation that shows that *only* SLE_6 can have the locality property.

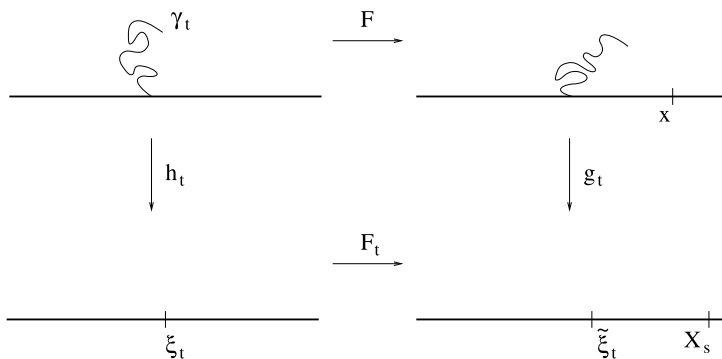


Fig. 2.18 The basic commutative diagram. (Left) Half plane, from 0 to infinity. (Right) Half plane, from 0 to x

One compares stochastic Loewner evolution in $\mathbb{H}$ from 0 to ∞ to stochastic Loewner evolution in $\mathbb{H}$ from 0 to x where x is any boundary point. By conformal invariance, any map F from $\mathbb{H}$ to $\mathbb{H}$ sending 0 to 0 and ∞ to x is such that the image measure $\mathbf{P}_{\mathbb{H},0,\infty} \circ F^{-1}$ is $\mathbf{P}_{\mathbb{H},0,x}$. We fix such an F . Uniformise SLE in the upper half-plane from 0 to ∞ with trace $\gamma_{[0,t]}$ up to time t by $h_t : \mathbb{H} \setminus \gamma_{[0,t]} \rightarrow \mathbb{H}$ in the hydrodynamical normalisation:

$$\frac{dh_t(w)}{dt} = \frac{2}{h_t(w) - \xi_t}, \quad \xi_t = \sqrt{\kappa} B_t.$$

Let g_t be the uniformising map $g_t : \mathbb{H} \setminus F(\gamma_{[0,t]}) \rightarrow \mathbb{H}$ in the hydrodynamical normalisation as above except that the time parameterisation is not at our disposal:

$$\frac{dg_t(w)}{dt} = \frac{2a_t}{g_t(w) - \tilde{\xi}_t}.$$

Finally let $F_t : \mathbb{H} \rightarrow \mathbb{H}$ be the map “closing the square”,

$$g_t \circ F = F_t \circ h_t.$$

The commutative diagram Fig. 2.18 summarises the situation.

Take the time derivative of $g_t \circ F = F_t \circ h_t$ and afterwards substitute z for h_t to get

$$\frac{2a_t}{F_t(z) - \tilde{\xi}_t} = \frac{dF_t(z)}{dt} + F'_t(z) \frac{2}{z - \xi_t}.$$

Now $F'_t(z)$ is non-singular at $z = \xi_t$ so that the pole on the right-hand side is cancelled by a pole on the left-hand side:

$$\frac{2a_t}{F_t(z) - \tilde{\xi}_t} = F'_t(\xi_t) \frac{2}{w - \xi_t} + O(1) \quad \text{when } w \rightarrow \xi_t,$$

leading to $\tilde{\xi}_t = F_t(\xi_t)$ and $a_t = F'_t(\xi_t)^2$. Continuing the expansion one step further yields

$$-a_t F''_t(\xi_t)/F'_t(\xi_t)^2 = \frac{dF_t}{dt}(\xi_t) + 2F''_t(\xi_t) \quad \text{i.e.} \quad \frac{dF_t}{dt}(\xi_t) = -3F''_t(\xi_t).$$

Now use Itô's formula to get $d\tilde{\xi}_t = d(F_t(\xi_t)) = (dF_t)(\xi_t) + F'_t(\xi_t) d\xi_t + \frac{\kappa}{2} F''_t(\xi_t) dt$, i.e.

$$d\tilde{\xi}_t = F'_t(\xi_t) d\xi_t + (\kappa/2 - 3)F''_t(\xi_t) dt.$$

Let us analyse this equation naïvely. As usual in Itô's theory, consider $\int_0^t F'_u(\xi_u) d\xi_u \simeq \sum_{n=0}^{N-1} F'_{tn/N}(\xi_{tn/N})(\xi_{t(n+1)/N} - \xi_{tn/N})$. If ξ is known up to time tn/N , the random variable $F'_{tn/N}(\xi_{tn/N})(\xi_{t(n+1)/N} - \xi_{tn/N})$ is Gaussian with mean 0 and variance $F'_{tn/N}(\xi_{tn/N})^2(\kappa t/N)$. Note that the increments $(\xi_{t(n+1)/N} - \xi_{tn/N})$ are all independent, and the scale $F'_{tn/N}(\xi_{tn/N})$ depends solely on the past. This suggests that if we would count time with a different scale, $\int_0^t F'_u(\xi_u) d\xi_u$ would become a sum of independent Gaussian random variables, and by inspection the right time scale is $s(t) := \int_0^t F'_u(\xi_u)^2 du$. This hand-waving argument is in fact confirmed by a theorem of Itô: if $\sqrt{\kappa} W_{s(t)} := \int_0^t F'_u(\xi_u) d\xi_u$ then W_s is a standard Brownian motion with parameter s . Setting $\hat{\xi}_{s(t)} := \tilde{\xi}_t$ we get that $\hat{\xi}_{s(t)} := F_t(\xi_t)$ so

$$d\hat{\xi}_s = \sqrt{\kappa} dW_s + (\kappa/2 - 3) \frac{F''_{t(s)}(\xi_{t(s)})}{F'_{t(s)}(\xi_{t(s)})^2} ds.$$

Observe that the time change is also exactly the one needed to turn $a_t = F'_t(\xi_t)^2$ into the constant 1. So if we set $\hat{h}_{s(t)} := g_t$ we find

$$\frac{d\hat{h}_s(w)}{ds} = \frac{2}{\hat{h}_s(w) - \hat{\xi}_s}.$$

This is enough to conclude that SLE_κ from 0 to ∞ is described by that same equation as SLE_κ from 0 to x if and only if $\kappa = 6$. As in the discrete case, this leads to the fact that at $\kappa = 6$ (and for no other value of κ) the measures $\mathbf{P}_{\mathbb{H},0,x}$ (resp. $\mathbf{P}_{\mathbb{H},0,y}$) on traces from 0 to x (resp. to y) induce the same measure on traces starting and 0 and stopped when they hit the interval $]x, y[$.

In the simple case at hand, one can prove straightforwardly that

$$\frac{F''_{t(s)}(\xi_{t(s)})}{F'_{t(s)}(\xi_{t(s)})^2} = \frac{2}{\tilde{\xi}_t - g_t(x)}.$$

Defining $\hat{X}_s := \hat{h}_s(x)$ we get finally:

$$d\hat{\xi}_s = \sqrt{\kappa} dW_s + (\kappa - 6) \frac{ds}{\hat{\xi}_s - \hat{X}_s}, \quad \frac{d\hat{X}_s}{ds} = \frac{2}{\hat{X}_s - \hat{\xi}_s},$$

so what looked initially like a non-local stochastic differential equation is in fact a local system of coupled stochastic differential equations.

2.5 Relation with Conformal Field-Theory

In this section, we shall crudely outline the relation between SLE and CFT. The discussion will be rather informal.

This is the only part of these notes where the author can claim to have made a contribution.

2.5.1 Motivation

The starting point is the following. If g_t is the Loewner map for SLE from 0 to ∞ , we define $h_t := g_t - \xi_t$. Then h_t satisfies the stochastic differential equation

$$dh_t(z) = \frac{2}{h_t(z)} dt - d\xi_t.$$

We can rephrase this trivially by setting $Z_t := h_t(z)$, so that

$$dZ_t = \frac{2}{Z_t} dt - d\xi_t, \quad Z_0 = z.$$

This describes the motion of particles in a time-dependent vector field v such that

$$v dt = dt \frac{2}{z} \partial_z - d\xi_t \partial_z.$$

For each $n \in \mathbb{Z}$, the generator of the transformation $z \rightarrow z + \varepsilon z^{n+1}$ is $\ell_n := -z^{n+1} \partial_z$ so that

$$v dt = -2\ell_{-2} dt + \ell_{-1} d\xi_t.$$

Now, if we consider a function of Z_t , say f , we can apply Itô's formula (or any of the techniques that are more familiar to physicists) to get the variation of the expectation value of $f(Z_t)$ namely

$$\left. \frac{d}{dt} E(f(Z_t)) \right|_{t=0} = \left(\frac{\kappa}{2} \partial_z^2 + \frac{2}{z} \partial_z \right) f(z).$$

The operator acting on the right-hand side is $\frac{\kappa}{2} \ell_{-1}^2 - 2\ell_{-2}$. Functions in the kernel of this operator describe observables $f(Z_t)$ with a time independent expectation, which in this simple case are also examples of more general probabilistic objects known as martingales.

But this differential operator is also well-known in conformal field-theory.

2.5.2 A Crash Course in Boundary CFT (BCFT)

The following is a totally unfair presentation of CFT. The first chapter in this volume, and references therein, should be consulted for any serious study.

In CFT the operators $\ell_n = -z^{n+1}\partial_z$, which satisfy the following commutation relations:

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}, \quad (2.15)$$

are represented by operators L_n which satisfy the **Virasoro algebra** vir:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n;0} \quad (2.16)$$

with the Virasoro **central charge** c , which commutes with all other generators (and is often viewed simply as a scalar multiple of the identity, though this is strictly true only in irreducible representations). The Virasoro algebra is a (in fact the only non-trivial) central extension of the algebra (2.15). This means that for non-zero c the quantum field-theory implements only a projective representation of the Lie algebra (2.15) of infinitesimal conformal transformations.

The states of a CFT are organised⁵ in **highest-weight representations** of vir. They are characterised by the facts that first, they possess a **highest-weight vector** i.e. a vector $|\Delta\rangle$ such that $L_n|\Delta\rangle = 0$ for $n > 0$ and $L_0|\Delta\rangle = \Delta|\Delta\rangle$ and second, all states in the representation of vir are linear combinations of states obtained by acting on $|\Delta\rangle$ repeatedly with Virasoro generators. The parameter Δ is called the **conformal weight** of the representation.

Due to these conditions and the commutation relations, the most general state in the representation is a finite linear combination of eigenstates of L_0 of eigenvalue $m + \Delta$ for $m = 0, 1, \dots$. An application of the **Poincaré-Birkhoff-Witt theorem** yields that these eigenstates with eigenvalue $m + \Delta$ can be written in the form $L_{-n_1}L_{-n_2}\dots L_{-n_k}|\Delta\rangle$ where $n_1 \geq n_2 \geq \dots \geq n_k \geq 1$ and $n_1 + n_2 + \dots + n_k = m$. In generic cases, these states are linearly independent. This is always true in a certain type of representations called **Verma modules** (and this can be taken as a poor man's definition/characterisation). But for certain special values of the pair (c, Δ) , Verma modules contain submodules and the corresponding quotients still satisfy the highest-weight condition. A bit more on this later.

In what follows, we shall mostly deal with boundary observables. To each state is associated a boundary operator, and the boundary operator $\varphi_\Delta(x)$ associated to $|\Delta\rangle$ for a CFT in the upper half-plane $\mathbb{H}$ (so that $x \in \mathbb{R}$) is called a **boundary primary operator** and satisfies

$$[L_n, \varphi_\Delta(x)] = \left(x^{n+1} \frac{d}{dx} + \Delta x^n \right) \varphi_\Delta(x), \quad (2.17)$$

which is the operator infinitesimal version of the rule $\varphi_\Delta(x) \mapsto \varphi_\Delta(f(x))|f'(x)|^\Delta$ under a conformal transformation f from a domain to another one, with the proviso that f should have a tangential derivative at the boundary point x . The state $|\Omega\rangle$ is associated to the identity operator, and $|\Delta\rangle = \varphi_\Delta(x)|\Omega\rangle$.

⁵This property of CFTs is in fact not general enough to cover all interesting cases, in particular for the SLE-CFT correspondence. However, the author is not aware of a simple and fully general definition of a CFT and decided to remain in the simplest setting.

2.5.3 Martingales and Singular Vectors

If one elaborates on the crude discussion of the $O(n)$ model in Sect. 2.2.2.3, one is lead to conjecture that the interface is created by inserting some boundary changing operators, say ψ sitting at the beginning and at the end of the interface. As noted there, this does not really imply that one obtains in this way a “product of local observables”. If $\langle \cdot \rangle$ denotes the average in the system without interface, we expect that the average of the observable $\mathcal{O}$ in the presence of an interface joining points a and b on the boundary will be

$$\langle \mathcal{O} \rangle_{a,b} := \frac{\langle \mathcal{O} \psi(a) \psi(b) \rangle}{\langle \psi(a) \psi(b) \rangle}. \quad (2.18)$$

This formula has the correct covariance properties under conformal transformations if and only if ψ transforms in a homogeneous way, i.e. is a density associated to some highest weight in a representation of the Virasoro algebra. We would like to understand which one.

The crucial observation is the following: if one computes the average of $\mathcal{O}$ for a fixed position of the interface and then averages over the position of the interface with the correct measure, one retrieves $\langle \mathcal{O} \rangle_{a,b}$. In the case when the interface is described by a growth process like SLE, this can be rephrased by saying that *observables are functionals of the Loewner flow whose average is time-independent*. Together with the domain Markov property, this means that *these functionals are martingales*. The discussion in the last section gives a detailed proof of this statement on the lattice.

The simplest functional has been considered above and leads to consider the kernel of $\frac{\kappa}{2} \ell_{-1}^2 - 2\ell_{-2}$.

So the natural question is: is it possible to have

$$\left(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2} \right) |h\rangle = 0?$$

The answer is that it may occur if and only if

$$c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} = 1 - 6 \frac{(\kappa-4)^2}{4\kappa}$$

and $\Delta = \frac{6-\kappa}{2\kappa}$.

One can make this heuristic discussion more rigorous and get:

- SLEs with parameter κ describe interfaces in CFTs of Virasoro central charge

$$c_\kappa = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} = 1 - 6 \frac{(\kappa-4)^2}{4\kappa}. \quad (2.19)$$

Notice that c_κ is always less than 1 and is invariant under the duality $\kappa \leftrightarrow 16/\kappa$.

- The boundary conformal operator $\psi(x)$ implementing the change of boundary condition at the point on which the interface emerges has scaling dimension⁶

$$\Delta_{1;2} = \frac{6 - \kappa}{2\kappa}. \quad (2.20)$$

It is a Virasoro primary operator degenerate at level two. In the CFT literature, this operator is often denoted by $\psi_{1;2}$.

The notation makes references to the so-called **Kac's labels** $(r; s)$ which parameterise an important family of representations, with conformal weight $\Delta_{r;s} = \frac{(\kappa r - 4s)^2 - (\kappa - 4)^2}{16\kappa}$. When r and s are positive integers, the corresponding Verma module contains non-trivial submodules. For generic κ , there is only one submodule, which is responsible for the fact that a CFT correlation functions of an operator $\psi_{r;s}$ corresponding to the quotiented Virasoro representation satisfies a differential equation of order rs . The case $(r; s) = (1; 1)$ is the one of the identity operator. The case $(r; s) = (1; 2)$ is at the heart of the **SLE-CFT correspondence**. When κ is a rational number, some conformal weights can be written as $\Delta_{r;s}$ for different values of $(r; s)$, leading to a more complicated submodule structure and to Virasoro minimal models. Note that some non-degenerate pairs (r, s) play a role in the CFTs of SLEs. For instance, a bulk operator with conformal weights $(\Delta_{0;n/2}, \Delta_{0;n/2})$ represents the insertion of n interfaces from a point in the bulk. In fact this is why, for instance, $2 - 2\Delta_{0;1} = (\kappa + 8)/8$ is the fractal dimension of SLE (for $\kappa \leq 8$). Note that $(0; n/2)$ is never in the Kac's table, and $n/2$ is not even an integer for odd n .

2.5.4 Two Examples

2.5.4.1 Hitting Probability

If $0 < u < v$, one can ask for the probability that an SLE trace from 0 to ∞ in $\mathbb{H}$ hits $[u, v]$, see Fig. 2.19. Denote this by $p(u, v)$. Suppose we let the trace grow for a certain time t and ask, then, for the same probability. If the trace is described by g_t (in the hydrodynamical normalisation), by conformal invariance, this is $p(g_t(u) - \xi_t, g_t(v) - \xi_t)$. If this is averaged over the position of the trace up to time t , $p(u, v)$ is retrieved. Writing

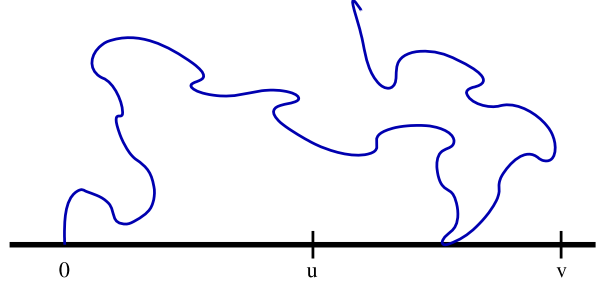
$$\left. \frac{d}{dt} E(p(g_t(u) - \xi_t, g_t(v) - \xi_t)) \right|_{t=0} = 0$$

yields a partial differential equation, obtained again routinely either via Itô's formula or any method more familiar to physicists:

$$\left(\frac{2}{u} \partial_u + \frac{2}{v} \partial_v + \frac{\kappa}{2} (\partial_u + \partial_v)^2 \right) p(u, v) = 0.$$

⁶For a boundary operator, the conformal weight is exactly the scaling dimension. Bulk operators have (possibly different) left and right conformal weights. Their sum is the scaling dimension and their difference is the spin.

Fig. 2.19 Hitting configuration



By scale invariance, $p(u, v)$ depends only on the ratio u/v and inserting boundary conditions shows that $p(u, v) = 0$ for $\kappa \leq 4$, $p(u, v) = 1$ for $\kappa \geq 8$. One also gets an explicit formula for $\kappa \in]4, 8[$. Setting $s = u/v$,

$$1 - p(u, v) = \frac{s^{\frac{\kappa-4}{\kappa}} \Gamma(\frac{4}{\kappa})}{\Gamma(\frac{\kappa-4}{\kappa}) \Gamma(\frac{8-\kappa}{\kappa})} \int_0^1 d\sigma \sigma^{-\frac{4}{\kappa}} (1 - s\sigma)^{2\frac{4-\kappa}{\kappa}}.$$

On the CFT side, one finds that $1 - p(u, v) = \langle \psi_{1;2}(\infty) \varphi_0(v) \varphi_0(u) \psi_{1;2}(0) \rangle$ where the operator φ_0 is a primary operator of weight 0 and appropriate conformal blocks are chosen. Starting from

$$\left(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2} \right) |\psi_{1;2}\rangle = 0, \quad \langle \psi_{1;2} | L_n = 0 \quad \text{for } n = -1, -2, \dots$$

plus the commutation relations between Virasoro generators and primary operators (2.17), it is a routine CFT computation to show, that $\langle \psi_{1;2}(\infty) \varphi_0(v) \varphi_0(u) \psi_{1;2}(0) \rangle$ satisfies the same partial differential equation as the one obtained above by probabilistic arguments. The boundary conditions are that it should take value 0 for $u \rightarrow 0$ and value 1 at $u \rightarrow v$. At least, this is what the probabilistic computation shows. To motivate this from the CFT viewpoint, we argue as follows.

If the trace hits between u and v , $g_t(u)$ will be close to ξ_t for t close to the hitting time, so that an OPE of $\varphi_0(g_t(u))$ with $\psi_{1;2}(\xi_t)$ will be relevant. Due to the existence of the null vector $(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2}) |\Delta_{1;2}\rangle$, only two conformal families are possible in the OPE, one with dimension $\Delta_{1;2} = \frac{6-\kappa}{2\kappa}$ and one with dimension $\Delta_{1;0} = \frac{\kappa-2}{\kappa}$. So

$$\varphi_0(x) \psi_{1;2}(0) \simeq C_{1;2} (\psi_{1;2}(0) + O(x)) + C_{1;0} x^{\frac{\kappa-4}{\kappa}} (\psi_{1;0}(0) + O(x)).$$

As hitting is forbidden in $1 - p(u, v)$, we have to choose a channel which gives a vanishing small contribution at small x , i.e. κ must be > 4 and we have to pick only the $\psi_{1;0}$ channel, i.e. $C_{1;2} = 0$. This fixes the correlation function up to multiplication by a constant.

On the other hand, if the trace does not hit between u and v , the first time it hits after v , $g_t(u)$ and $g_t(v)$ will be close to ξ_t . But, as observed in the explicit examples of hittings, when a trace hits the boundary, $g_t(u)$ and $g_t(v)$ come close to each other at an even faster rate, so that an OPE between $\varphi_0(g_t(u))$ and $\varphi_0(g_t(v))$ is relevant. This time, there is no a priori restrictions on the possible operators, but only two

conformal families have a non-vanishing three point function with the two $\psi_{1,2}$'s: one with dimension $\Delta_{1,1} = 0$ and one with dimension $\Delta_{1,3} = \frac{8-\kappa}{\kappa}$. So one picks only

$$\varphi_0(x)\varphi_0(y) \simeq C_{1,1}(\mathbf{1} + O(x-y)) + C_{1,3}(x-y)^{\frac{8-\kappa}{\kappa}}(\psi_{1,3}(y) + O(x-y)) + \dots$$

Now one can check that if $C_{1,2} = 0$ and $C_{1,0} \neq 0$ then $C_{1,3} \neq 0$ as well, so the correlation function is bounded only if $\kappa < 8$ (at $\kappa = 8$, they would be logarithms in fact).

So the block structure is dictated by the operator product expansion. In this way, the OPE encodes nicely the different phases of SLE: only for $\kappa \in]4, 8[$ is there a non-trivial hitting probability, and hitting the boundary is represented by insertion of the operator $\psi_{1,3}$.

2.5.4.2 Partition Functions

As argued before, the partition function for chordal SLE in $\mathbb{D}$ from a to b (a and b are two boundary points of $\mathbb{D}$) is quite simple:

$$Z_{\mathbb{D}}(a, b) = \langle \psi_{h_{1,2}}(a) \psi_{h_{1,2}}(b) \rangle_{\mathbb{D}}.$$

To be sure, we should multiply the right-hand side by the partition function without interface. This factor depends on $\mathbb{D}$ but plays no role in the following arguments and we omit it. Conformal invariance relates correlation functions in different domains: if $g : \mathbb{D} \rightarrow \tilde{\mathbb{D}}$ is a conformal representation, one finds, using the behaviour of boundary primary operators under conformal transformations:

$$\langle \psi_{h_{1,2}}(a) \psi_{h_{1,2}}(b) \rangle_{\mathbb{D}} = \langle \psi_{h_{1,2}}(g(a)) \psi_{h_{1,2}}(g(b)) \rangle_{\tilde{\mathbb{D}}} |g'(a)|^{\Delta_{1,2}} |g'(b)|^{\Delta_{1,2}},$$

where $\Delta_{1,2} = \frac{6-\kappa}{2\kappa}$.

In particular, taking $\mathbb{D} = \tilde{\mathbb{D}} = \mathbb{H}$ (the upper-half plane, a and b are real numbers in that case) and $g(z) = \frac{z-a}{pz+(1-p)b-a}$ where p is an arbitrary real parameter, $p \neq 1$, one gets that

$$\langle \psi_{\Delta_{1,2}}(a) \psi_{\Delta_{1,2}}(b) \rangle_{\mathbb{H}} = \langle \psi_{\Delta_{1,2}}(0) \psi_{\Delta_{1,2}}(1) \rangle_{\mathbb{H}} |a-b|^{-2\Delta_{1,2}}.$$

Taking $\mathbb{D}$ to be arbitrary again, but keeping $\tilde{\mathbb{D}} = \mathbb{H}$ one gets

$$\begin{aligned} \langle \psi_{\Delta_{1,2}}(a) \psi_{\Delta_{1,2}}(b) \rangle_{\mathbb{D}} &= \langle \psi_{\Delta_{1,2}}(g(a)) \psi_{\Delta_{1,2}}(g(b)) \rangle_{\mathbb{H}} |g'(a)|^{\Delta_{1,2}} |g'(b)|^{\Delta_{1,2}} \\ &= \langle \psi_{\Delta_{1,2}}(0) \psi_{\Delta_{1,2}}(1) \rangle_{\mathbb{H}} \left| \frac{g'(a)g'(b)}{(g(a)-g(b))^2} \right|^{\frac{(6-\kappa)}{2\kappa}}. \end{aligned}$$

As expected, these formulæ are singular if $\mathbb{D}$ is not smooth at a or b . However, comparing the cases $b = x$ and $b = y$, one is led to

$$\begin{aligned} \frac{Z_{\mathbb{D}}(a, y)}{Z_{\mathbb{D}}(a, x)} &= \frac{\langle \psi_{\Delta_{1,2}}(a) \psi_{\Delta_{1,2}}(y) \rangle_{\mathbb{D}}}{\langle \psi_{\Delta_{1,2}}(a) \psi_{\Delta_{1,2}}(x) \rangle_{\mathbb{D}}} \\ &= \left| \frac{g'(y)(g(a)-g(x))^2}{g'(x)(g(a)-g(y))^2} \right|^{\frac{(6-\kappa)}{2\kappa}}. \end{aligned}$$

This formula has two remarkable features:

- the normalising constants in the partition functions or in the two point functions have cancelled between the numerator and denominator,
- the derivative $g'(a)$ has cancelled between the numerator and denominator.

Hence, this formula is normalised in an absolute way and makes sense even if $\mathbb{D}$ is not smooth at a . We want to apply it in such a situation.

Consider chordal SLE in $\mathbb{H}$ from 0 to x . Let $\mathbb{D} = \mathbb{H}_t$ be the domain obtained by removing from $\mathbb{H}$ the SLE hull with the SLE hull at time t , and let $g_t : \mathbb{H}_t \rightarrow \mathbb{H}$ be the uniformising map in the hydrodynamical normalisation, mapping the tip γ_t of the SLE hull to ξ_t . Modulo some changes in notation, we have seen at the end of Sect. 2.4.5 that

$$d\xi_t = \sqrt{\kappa} dB_t + (\kappa - 6) \frac{dt}{\xi_t - g_t(x)}, \quad \frac{dg_t(x)}{dt} = \frac{2}{g_t(x) - \xi_t}, \quad (2.21)$$

where B_t is a standard Brownian motion.

Taking $a = \gamma_t$ (a point at which $\mathbb{D} = \mathbb{H}_t$ is certainly not smooth) we obtain for the ratio of partition functions:

$$\frac{Z_{\mathbb{H}_t}(\gamma_t, y)}{Z_{\mathbb{H}_t}(\gamma_t, x)} = \left| \frac{g'_t(y)(\xi_t - g_t(x))^2}{g'_t(x)(\xi_t - g_t(y))^2} \right|^{\frac{(6-\kappa)}{2\kappa}}.$$

We claim that this is a martingale for SLE in $\mathbb{H}$ from 0 to x , i.e. for the stochastic differential system of Eqs. (2.21). The proof is a good exercise with **Itô's formula** that we really recommend to the reader. Therefore we give some details. Note that all factors in the ratio are real, so that the modulus is only fixing a possible sign. Hence we can forget about the modulus in the computation. In the following we set $X_t := g_t(x) - \xi_t$ and $Y_t := g_t(y) - \xi_t$.

As $\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi_t}$, taking the z derivative (denoted with a $'$) yields

$$dg'_t(z) = \frac{-2g'_t(z) dt}{(g_t(z) - \xi_t)^2},$$

which gives

$$d \frac{g'_t(y)}{g'_t(x)} = -2(X_t - Y_t) \frac{g'_t(y)}{g'_t(x)} \frac{(X_t + Y_t)}{X_t^2 Y_t^2} dt.$$

Also

$$d(\xi_t - g_t(x)) = d\xi_t - dg_t(x) = d\xi_t - \frac{2 dt}{X_t}.$$

Now Itô's formula yields

$$\begin{aligned} d \frac{1}{\xi_t - g_t(y)} &= -\frac{d(\xi_t - g_t(y))}{(\xi_t - g_t(y))^2} + \frac{(d(\xi_t - g_t(y)))^2}{(\xi_t - g_t(y))^3} \\ &= -\frac{d\xi_t}{Y_t^2} - (\kappa - 2) \frac{dt}{Y_t^3} \end{aligned}$$

where we used Itô's rule $dB_t^2 = dt$, $dB_t dt = dt dt = 0$ to obtain $(d(\xi_t - g_t(y)))^2 = \kappa dt$, a consequence of (2.21). Another use of Itô's formula yields

$$\begin{aligned} d\frac{\xi_t - g_t(x)}{\xi_t - g_t(y)} &= \frac{1}{\xi_t - g_t(y)} d(\xi_t - g_t(x)) + (\xi_t - g_t(x)) d\frac{1}{\xi_t - g_t(y)} \\ &\quad + \left(d(\xi_t - g_t(x)) d\frac{1}{\xi_t - g_t(y)} \right) \\ &= (X_t - Y_t) \left(\frac{1}{Y_t^2} d\xi_t + \frac{-2Y_t + (\kappa - 2)X_t}{X_t Y_t^3} dt \right). \end{aligned}$$

Then

$$d\left(\frac{\xi_t - g_t(x)}{\xi_t - g_t(y)}\right)^2 = (X_t - Y_t) \left(\frac{2X_t}{Y_t^3} d\xi_t + \frac{-4(X_t + Y_t) + \kappa(3X_t - Y_t)}{Y_t^4} dt \right).$$

Putting all this together and setting $S_t := \frac{g'_t(y)(\xi_t - X_t)^2}{g'_t(x)(\xi_t - Y_t)^2}$, we get

$$dS_t = \frac{g'_t(y)}{g'_t(x)} (X_t - Y_t) \left(\frac{2X_t}{Y_t^3} d\xi_t + \frac{-6(X_t + Y_t) + \kappa(3X_t - Y_t)}{Y_t^4} dt \right),$$

or better

$$\frac{dS_t}{S_t} = (X_t - Y_t) \left(\frac{2}{X_t Y_t} d\xi_t + \frac{-6(X_t + Y_t) + \kappa(3X_t - Y_t)}{X_t^2 Y_t^2} dt \right).$$

Itô's formula applied once again gives, for any exponent α

$$\frac{dS_t^\alpha}{S_t^\alpha} = \alpha \frac{dS_t}{S_t} + \frac{1}{2} \alpha(\alpha - 1) \left(\frac{dS_t}{S_t} \right)^2.$$

Hence

$$\begin{aligned} \frac{dS_t^\alpha}{S_t^\alpha} &= \alpha(X_t - Y_t) \\ &\quad \times \left(\frac{2}{X_t Y_t} d\xi_t + \frac{-6(X_t + Y_t) + \kappa(3X_t - Y_t) + 2\kappa(\alpha - 1)(X_t - Y_t)}{X_t^2 Y_t^2} dt \right). \end{aligned}$$

Recalling that $d\xi_t = \sqrt{\kappa} dB_t + (\kappa - 6) \frac{dt}{\xi_t - g_t(x)} = \sqrt{\kappa} dB_t + (6 - \kappa) \frac{dt}{X_t}$, we get at last:

$$\frac{dS_t^\alpha}{S_t^\alpha} = \alpha(X_t - Y_t) \left(\frac{2}{X_t Y_t} \sqrt{\kappa} dB_t + \frac{(\kappa(1 + 2\alpha) - 6)(X_t - Y_t)}{X_t^2 Y_t^2} dt \right). \quad (2.22)$$

The drift term vanishes if and only if $\kappa(1 + 2\alpha) - 6 = 0$, i.e. $\alpha = h_{1;2} = (6 - \kappa)/(2\kappa)$. In particular, this proves that $\frac{Z_{\mathbb{H}_t}(\gamma_t, y)}{Z_{\mathbb{H}_t}(\gamma_t, x)} = |S_t|^{h_{1;2}}$ is a (local) martingale, i.e. that its Itô derivative contains no drift term.

The fact that the drift term in $d\xi_t$ is $(6 - \kappa) \frac{dt}{X_t}$ is crucial, and our computation shows that we could recover this drift term uniquely if we knew in advance that $\frac{Z_{\mathbb{H}_t}(\gamma_t, y)}{Z_{\mathbb{H}_t}(\gamma_t, x)}$ is a martingale. Moreover, if we were to take $\alpha \neq h_{1;2}$, S_t^α would be a

martingale for no choice of drift term in $d\xi_t$. These two properties should convince us that the martingale property of $\frac{Z_{\mathbb{H}_t}(\gamma_t, y)}{Z_{\mathbb{H}_t}(\gamma_t, x)}$ has little to do with chance.

In the next subsection, we give the fundamental (but totally elementary) reason why *ratios of partition functions (and in particular correlation functions) must be martingales* and give a precise meaning to this vague statement. A good part of the argument relies on a very general tautological double counting argument which then is specialised to the SLE-CFT correspondence.

To close this discussion, let us note that the above computation can be abstracted as follows: The stochastic process

$$R_t = \frac{Z_{\mathbb{H}_t}(\gamma_t, y)}{Z_{\mathbb{H}_t}(\gamma_t, x)} \quad (2.23)$$

is a **martingale** if

$$d(g_t(\gamma_t)) = \sqrt{\kappa} dB_t + \kappa \partial_b (\ln Z_{\mathbb{H}}(a, b)) \Big|_{a=g_t(\gamma_t), b=g_t(x)} dt. \quad (2.24)$$

This equation which states that the drift is the variation of the free energy with respect to a parameter is particularly nice, and in fact totally general. For instance, it is at the heart of the definition of multiples SLEs.

2.5.5 Conformal SLE Martingales

In this last subsection, we explain why *ratios of (conditional) partition functions of models of statistical mechanics (and in particular correlation functions) are martingales for appropriate stochastic processes*. This result is at the heart of the CFT approach to SLEs as outlined in the previous section. We first give the argument, which involves some technical hypotheses, and then motivate these hypotheses and their consequences in the example of interfaces. To keep the discussion simple and self-contained, we concentrate on finite or countable configuration spaces. Again, we use some terminology from probability theory (σ -algebra, martingale, filtration, ...), but try to keep it to a bare minimum. We start with some general abstract definitions.

Recall that a **partition** of a set $\mathcal{C}$ is a subset $\mathcal{Q}$ of $2^{\mathcal{C}} \setminus \{\emptyset\}$ such that each $x \in \mathcal{C}$ belongs to exactly one element of $\mathcal{Q}$. We say that a partition $\mathcal{Q}'$ is **finer** than a partition $\mathcal{Q}$, or equivalently that a partition $\mathcal{Q}$ is **coarser** than a partition $\mathcal{Q}'$ if every element of $\mathcal{Q}'$ is a subset of an element of $\mathcal{Q}$.

The link with measure theory is the following. To each partition $\mathcal{Q}$ of $\mathcal{C}$ we associate a σ -**algebra** $\mathcal{F}$ on $\mathcal{C}$ as follows: $\mathcal{F}$ is a subset of $2^{\mathcal{C}}$, and an element of $2^{\mathcal{C}}$ belongs to $\mathcal{F}$ if and only if it is a union (possibly empty) of elements of $\mathcal{Q}$. One checks immediately that $\mathcal{F}$ is indeed a σ -algebra, i.e. is stable under complementation and countable unions: if $A \in \mathcal{F}$ its complement $A^c \in \mathcal{F}$, and is $A_i \in \mathcal{F}$, $i \in I$ (a finite or countable set), then $\bigcup_{i \in I} A_i \in \mathcal{F}$. Note that one has stability under arbitrary unions in that case, but this is not required for a σ -algebra. The fact that the

partition $\mathcal{Q}'$ is finer than $\mathcal{Q}$ reads as the inclusion $\mathcal{F} \subset \mathcal{F}'$ at the level of associated σ -algebras.

Which σ -algebras on $\mathcal{C}$ arise in this way? The answer is that if $\mathcal{C}$ is finite or countable, all of them do, and there is a one to one correspondence between partitions and σ -algebras. To go back from a σ -algebra $\mathcal{F}$ to a partition $\mathcal{Q}$ when $\mathcal{C}$ is finite or countable, define an **atom** of $\mathcal{F}$ to be a non-empty element of $\mathcal{F}$ which does not contain properly any other non-empty element of $\mathcal{F}$. Then *the atoms form a partition of $\mathcal{C}$* (you can try to prove it).

We now introduce an index t (belonging to some ordered set $(T, \leq)$, typically $T = \mathbb{N}_n = \{0, 1, \dots, n\}$, $T = \mathbb{N} = \{0, 1, \dots\}$ or $T = \mathbb{R}^+ = [0, +\infty[$) which will be identified with “time”, and introduce a family $(\mathcal{Q}_t)_{t \in T}$ of partitions of $\mathcal{C}$, which get finer and finer as t increases. By convention $\mathcal{Q}_0$ is the trivial partition with $\mathcal{C}$ as its single piece.

The family of σ -algebras $(\mathcal{F}_t)_{t \in T}$ associated to the family of partitions $(\mathcal{Q}_t)_{t \in T}$ is called a **filtration** of $\mathcal{C}$: $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

After this mathematical preliminary, we can turn to more familiar constructions by assuming that $\mathcal{C}$ is the configuration space of a statistical model. Very often one works with discrete variable in finite volume, and then $\mathcal{C}$ is discrete, and finite or countable. That’s one of the reasons why physicists are not used to σ -algebras, which are the appropriate tool to deal with general configurations spaces, but which can be replaced by partitions as seen above for finite or countable configuration spaces. So we assume that $\mathcal{C}$ is finite or countable for the rest of this discussion.

Before we start, let us quote one famous instance where the above ideas are used implicitly all the time: the **renormalisation group**. If $\mathcal{C}$ describes the configurations of a spin system, one can partition $\mathcal{C}$ for instance by first partitioning “space” into blocks of spins, and then regrouping all configurations for which the magnetisation of each block is given, but forgetting the other details inside a block. Taking larger and larger blocks means taking coarser and coarser partitions leading to a filtration, and increasing t means keeping finer and finer details of small scales physics. The reader is invited to reinterpret the following constructions (conditional expectations, ...) in terms of familiar concepts (effective actions, ...).

Let $W_c, c \in \mathcal{C}$, be a family of Boltzmann weights on $\mathcal{C}$. We assume that $W_c \geq 0$ for each c and that $Z := \sum_c W_c \in]0, +\infty[$, so that $\mathbf{P}(c) = W_c/Z$ defines a probability on $\mathcal{C}$.

To start with, let $O_c, c \in \mathcal{C}$ be an observable (that is, a map from $\mathcal{C}$ to $\mathbb{R}$) and assume that $\frac{1}{Z} \sum_c W_c |O_c| < +\infty$ which ensures that $\mathbb{E}(O) := \frac{1}{Z} \sum_c W_c O_c$, the expectation of O with respect to $\mathbf{P}$, is well defined. In probability theory, we would say that O is a $\mathbf{P}$ -integrable random variable.

If $\mathcal{D}$ is a subset of $c \in \mathcal{C}$ and $\mathbf{P}(\mathcal{D}) := \frac{1}{Z} \sum_{c \in \mathcal{D}} W_c > 0$, one can do statistical mechanics on $\mathcal{D}$: we define $Z_{\mathcal{D}} := \sum_{c \in \mathcal{D}} W_c = Z\mathbf{P}(\mathcal{D})$ and $\mathbb{E}_{\mathcal{D}}(O) := \frac{1}{Z_{\mathcal{D}}} \sum_{c \in \mathcal{D}} W_c O_c$. This is a conditional expectation, and it looks trivial to physicists because statistical mechanics uses all the time partition functions, i.e. non-normalised but finite measures to compute probabilities.

A bit more generally, if $\mathcal{Q}$ is a partition of $\mathcal{C}$, with associated σ -algebra $\mathcal{F}$, we can define a new observable, denoted by $\mathbb{E}(O|\mathcal{F})$ as follows. For each $c \in \mathcal{C}$ there is a single $\mathcal{D} \in \mathcal{Q}$ such that $c \in \mathcal{D}$. If $Z_{\mathcal{D}} > 0$, then $\mathbb{E}(O|\mathcal{F})_c := \mathbb{E}_{\mathcal{D}}(O)$. If $Z_{\mathcal{D}} = 0$, give $\mathbb{E}(O|\mathcal{F})_c$ an arbitrary value (but the same for all $c \in \mathcal{D}$). So $\mathbb{E}(O|\mathcal{F})$ is still an observable on $\mathcal{C}$, but a coarse-grained one: its value on c depends only on the piece of the partition to which c belongs, and is equal to the average of O on this piece. Beware, the conditional expectation $\mathbb{E}(O|\mathcal{Q})$ is not a number. It is an observable, which moreover has a nice characterisation: it is constant on every piece of $\mathcal{Q}$ and if U is any (bounded, this is just to make sure that all formulæ are convergent) observable on $\mathcal{C}$ which is constant on every piece of $\mathcal{Q}$ then $\mathbb{E}(OU) = \mathbb{E}(\mathbb{E}(O|\mathcal{F})U)$. The reader should check that this is a characterisation. There is a small subtlety: $\mathbb{E}(O|\mathcal{F})$ is fully determined only on the pieces of $\mathcal{Q}$ with non-vanishing partition function; there can be several version of the conditional expectation, but they differ only on a sets with 0 $\mathbf{P}$ -measure.

Now we use a filtration and define $O(t) := \mathbb{E}(O|\mathcal{F}_t)$ for $t \in T$. Then, a trivial computation reveals that $\mathbb{E}(O(t)|\mathcal{F}_s) = O(s)$ for $s \leq t \in T$, i.e. $(O(t))_{t \in T}$ is a martingale. In fact, it is a special case of martingale, a closed martingale, because there is an observable, O itself, such that $O(t) := \mathbb{E}(O|\mathcal{F}_t)$. This is the content of the sentence “Observables, or correlation functions, are martingales”.

It is useful to go a bit further. Suppose that $\tilde{W}_c$, $c \in \mathcal{C}$ is another family of Boltzmann weights on $\mathcal{C}$, such that $\tilde{W}_c \geq 0$ for each c and that $\tilde{Z} := \sum_c \tilde{W}_c \in]0, +\infty[$. Let $\tilde{\mathbf{P}}$ and $\tilde{\mathbb{E}}$ be the corresponding probability measure and expectation.

Let $\mathcal{Q}$ be a partition of $\mathcal{C}$ and assume that for any piece $\mathcal{D}$ of $\mathcal{Q}$ either $Z_{\mathcal{D}} > 0$ or $\tilde{Z}_{\mathcal{D}} := \sum_{c \in \mathcal{D}} \tilde{W}_c = 0$. Then we can define an observable R on $\mathcal{C}$ as follows. For each $c \in \mathcal{C}$ there is a single $\mathcal{D} \in \mathcal{Q}$ such that $c \in \mathcal{D}$. If $Z_{\mathcal{D}} > 0$ then $R_c = \frac{\tilde{Z}_{\mathcal{D}}}{Z_{\mathcal{D}}}$. If $Z_{\mathcal{D}} = 0$, give R_c an arbitrary value (but the same for all $c \in \mathcal{D}$). If U is any $\tilde{\mathbf{P}}$ -integrable observable on $\mathcal{C}$ which is constant on every piece of $\mathcal{Q}$ then RU is a $\mathbf{P}$ -integrable observable on $\mathcal{C}$ which is constant on every piece of $\mathcal{Q}$ and $\tilde{\mathbb{E}}(U) = \mathbb{E}(RU)$. This is expressed mathematically as follows (this is mostly vocabulary): when coarse-grained down to $\mathcal{Q}$, the measure $\tilde{\mathbf{P}}$ is absolutely continuous with respect to $\mathbf{P}$, and the Radon-Nykodim derivative $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}$ is equal to R .

Now, the case of a filtration. We assume that for any $t \in T$ and any piece $\mathcal{D}$ of the partition $\mathcal{Q}_t$, either $Z_{\mathcal{D}} > 0$ or $\tilde{Z}_{\mathcal{D}} = 0$. We define $R(t)$ as before, as the ratio of partition functions for w and $\tilde{W}$ on any piece of the partition $\mathcal{Q}_t$. Then $R(t)$ is also the Radon-Nykodim derivative $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}}$ when both measures are coarse-grained down to $\mathcal{Q}_t$. The following computation will show that $\mathbb{E}(R(t)|\mathcal{F}_s) = R(s)$ for $s \leq t \in T$, i.e. that $(R(t))_{t \in T}$ is a martingale. To show that $R(t)$ is a martingale, we use the above characterisation of conditional expectations.

We need to show that, if $t \geq s$ and if U is bounded and constant on every piece of $\mathcal{Q}_s$, $\mathbb{E}(UR(t)) = \mathbb{E}(UR(s))$. When X is an observable on $\mathcal{C}$ which is constant on a subset $\mathcal{D}$ of $\mathcal{C}$, we write $X_{\mathcal{D}}$ for the value of X_c when $c \in \mathcal{D}$. We compute Z (which is non-zero) times the right-hand side $\mathbb{E}(UR(s))$. This is

$$\begin{aligned}
Z\mathbb{E}(UR(s)) &= \sum_{c \in \mathcal{C}} U_c R(s)_c W_c \\
&= \sum_{\mathcal{D} \in \mathcal{Q}_s} \sum_{c \in \mathcal{D}} U_c R(s)_c W_c \\
&= \sum_{\mathcal{D} \in \mathcal{Q}_s} U_{\mathcal{D}} R(s)_{\mathcal{D}} \sum_{c \in \mathcal{D}} W_c \\
&= \sum_{\mathcal{D} \in \mathcal{Q}_s} U_{\mathcal{D}} R(s)_{\mathcal{D}} Z_{\mathcal{D}} \\
&= \sum_{\mathcal{D} \in \mathcal{Q}_s} U_{\mathcal{D}} \tilde{Z}_{\mathcal{D}}.
\end{aligned}$$

As $t \geq s$, U is a fortiori constant on every piece of $\mathcal{Q}_t$ and a parallel computation yields

$$Z\mathbb{E}(UR(t)) = \sum_{\mathcal{D} \in \mathcal{Q}_t} U_{\mathcal{D}} \tilde{Z}_{\mathcal{D}}.$$

The two expressions are equal. Indeed, $\tilde{Z}_{\mathcal{D}}$ is an additive functional of $\mathcal{D}$: its value on a disjoint union is the sum of the values on each piece of the union. Use this to compute $\sum_{\mathcal{D} \in \mathcal{Q}_s} U_{\mathcal{D}} \tilde{Z}_{\mathcal{D}}$ by breaking each $\mathcal{D} \in \mathcal{Q}_s$ into its $\mathcal{Q}_t$ pieces.

Were the support of $\tilde{W}$ included in that of W (i.e. if $\tilde{W}_c$ vanished for all c 's for which $W_c = 0$), we could define an observable O by $O_c = \tilde{W}_c / W_c$ whenever the denominator is non-zero and keep O_c arbitrary when $W_c = 0$. We would be back to the case of observables: $R(t)$ would simply equal $O(t)$ and be a closed martingale. But in general, $R(t)$ is not a closed martingale. However, we get the general principle “Ratios of partition functions are martingales”.

The above considerations look a little bit like abstract nonsense. So let us apply them to interfaces (say on a finite lattice domain to keep a finite or countable configuration space). Suppose that to each configuration one can associate an interface, and $\mathcal{F}_t$ gives a finer description of this interface as t increases, for instance by describing completely larger and larger initial segments of the interface. Now one can be interested in certain configurations $c \in \mathcal{D}$ (resp. $\tilde{\mathcal{D}}$), which are specified by some feature of the interface they contain (for instance starting point, end point, ...). Then we consider weights W_c (resp. $\tilde{W}_c$) which vanish outside $\mathcal{D}$ (resp. $\tilde{\mathcal{D}}$). If the possible initial segments of interfaces (at least for t up to a certain value) are the same for $\mathcal{D}$ and $\tilde{\mathcal{D}}$, we get martingales by taking ratios of partial partition functions as explained above.

Going to the continuum limit requires some care: the previous argument seems to be closely related to the existence of *Radon-Nykodim derivatives*, which can be tricky when taking the continuum limit. But when this is known, the fact that ratios of partition functions are martingales is quite a powerful tool.

For instance, in the computation of locality, we obtained stochastic differential equations allowing to compare chordal SLE from 0 to ∞ to chordal SLE from 0 to a point x (say $x > 0$) at finite distance. Obviously the second measure on the full

set of curves cannot be absolutely continuous with respect to the first because the event to hit x has probability 0 for the first (at least if $\kappa < 8$) and 1 for the second. However, the two measures are absolutely continuous with respect to each other when coarse-grained to ignore what happens to a curve once it has hit the interval $]x, +\infty[$. We have already observed that the ratios of partition functions are (local) martingales in that case.

2.6 Notes and References

For a very short overview of SLE, see [25]. The lectures notes [8] concentrate on discrete models and Loewner chains. There are many reviews on SLE. Historically, the first one is [27]. The reviews [18, 24] are a valuable source of information for interfaces and SLE, but mostly seen from the CFT side. Reference [9] is a review on growth processes, Loewner chains, SLE and CFT written by physicists, but with a strong emphasis on probabilistic aspects.

Let us note that Loewner chains for certain models, for instance Laplacian growth, are related to integrable systems, see e.g. [8, 9] and references therein.

The interpretation of extended objects in 2D critical phenomena in terms of logarithmic CFT is only starting, but some interesting approaches and results can be found in [41] for crossing probabilities and [30] for SLE martingales.

The proof of convergence of the exploration process to a conformally invariant continuum limit is announced in [47] and proved along rather different lines in [15, 16].

The proof that the off-critical continuum percolation measure is singular with respect to the critical continuum percolation measure can be found in [42].

The canonical self-avoiding walk puts the uniform measure on all lattice paths of a given length. This model has proved remarkably difficult to tackle. Loop-erased random walks are introduced in [32] as an example of walks which are self avoiding but more tractable than the canonical self avoiding walk.

The seminal paper on SLE is [45], which dealt mainly with loop-erased random walks. The existence a conformal invariance of a continuum limit of the loop-erased random walk measure is proved in [38, 39].

The Riemann mapping theorem is proved in a number of textbooks, see e.g. [19]. Most proofs involve a good amount of functional analysis, related for instance to the existence of the Green function of the Laplace operator with Dirichlet boundary conditions. The deep relationship between the Laplace operator and Brownian motion can be used to give a probabilistic proof of Riemann's theorem, see e.g. [1].

The seminal paper on Loewner chains is [40]. A mathematical introduction to Loewner chains can be found in [19], or with more details in [43].

Among the physicists works that inspired mathematical work, culminating in SLE, one should quote Cardy's percolation formula [17], and intersection exponents [21]. Though physicists understood quite well conformal invariance for correlation functions of local observables since 1984 [14], they had failed to find the axiomatic framework describing conformally invariant measures on interfaces.

After the seminal millennium paper by Schramm [45], Lawler, Schramm and Werner embarked in an impressive series of contributions where they explored SLE (locality, restriction, ...) but also its applications to a number of open problem like intersection exponents, the dimension of the Brownian frontier, ... see e.g. [33–39].

The proof that the SLE trace is a curve is surprisingly deep, especially at $\kappa = 8$, see [44].

The proof that the Hausdorff dimension of SLE_κ is $\min(1 + \kappa/8, 2)$ is also very tricky. The first proof appeared in [13].

The proof of duality of chordal SLE is in [48] and that of reversibility is in [49], note also the nice (but inconclusive) approach in [31].

The seminal article on the SLE-CFT correspondence is [2]. Reference [22] deals with $\kappa = 8/3$ i.e. the case when the SLE hull is a simple curve and the conformal anomaly vanishes ($c = 0$).

The computation of the hitting probability (and others) via CFT techniques is explained in [3]. The relations between SLE martingales and the representation theory of the Virasoro algebra is explained in [4, 6] and generalised in [29] (see also [30] for applications to logarithmic CFT). Locality via CFT and partition functions is explained in [6].

A better understanding of why SLE and CFT are related the way they are came via the double counting argument explained in Sect. 2.5.5 in these notes. The case of observables is explained in [10]. More games with partition functions, conditioning and Girsanov's theorem can be found in [12, 26, 46].

The relationship between other kinds of SLE's (a subject not pursued in these notes) and CFT is explained in [5, 7, 10].

Multiple interfaces (corresponding to an arbitrary number of changes in boundary conditions) are described in [11, 20]. The first one insists a constraint called commutation, while the second exploits partition function techniques. See also [23] for a nice interpretation.

A very readable introduction to stochastic calculus is [28].

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Interfaces and Stochastic Loewner Evolution

Henkel, M.; Karevski, D. (Eds.)

2012, XVI, 189 p. 69 illus., 19 illus. in color., Softcover

ISBN: 978-3-642-27933-1