Symmetries and Covariance of the Maxwell Equations

2

2.1 Introduction

Already within a given, fixed division of four-dimensional spacetime into the space where experiments are performed, and the laboratory time variable, Maxwell's equations show interesting transformation properties under continuous and discrete spacetime transformations. However, only the action of the whole Lorentz group on them reveals their full symmetry structure. A good example that illustrates the covariance of Maxwell's equations is provided by the electromagnetic fields of a point charge uniformly moving along a straight line.

A reformulation of Maxwell theory in the language of exterior forms over $\mathbb{R}^4$, on the one hand, sheds light on some of its properties which are less transparent in the framework of the older vector analysis. On the other hand, it reveals the geometric character of this example of a simple gauge theory and, hence, prepares the ground for the understanding of non-Abelian gauge theories which are essential for the description of the fundamental interactions of nature.

2.2 The Maxwell Equations in a Fixed Frame of Reference

In a fixed inertial system in which $\boldsymbol{x}$ are coordinates in ordinary space $\mathbb{R}^3$, t the coordinate time that an observer at rest reads on his clock, Maxwell's equations (1.44a–1.44d) read

$$\nabla \cdot \boldsymbol{B}(t, \boldsymbol{x}) = 0 \,, \tag{2.1a}$$

$$\nabla \times \boldsymbol{E}(t, \boldsymbol{x}) + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(t, \boldsymbol{x}) = 0 \,, \tag{2.1b}$$

$$\nabla \cdot \boldsymbol{D}(t, \boldsymbol{x}) = 4\pi\varrho(t, \boldsymbol{x}) \,, \tag{2.1c}$$

$$\nabla \times \boldsymbol{H}(t, \boldsymbol{x}) - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{D}(t, \boldsymbol{x}) = \frac{4\pi}{c}\boldsymbol{j}(t, \boldsymbol{x}) \,. \tag{2.1d}$$

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They are supplemented by the relationships

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \,, \qquad \boldsymbol{B} = \mu \boldsymbol{H} \tag{2.2}$$

between the displacement field and the electric field, and between the induction field and the magnetic field, respectively, ε being the dielectric constant, and μ the magnetic permeability. (In vacuum and using Gaussian units both constants are equal to 1.) The force that acts on a particle carrying the charge q and moving with the velocity $\boldsymbol{v}$ relative to the observer, is the Lorentz force (1.44e)

$$\boldsymbol{F}(t, \boldsymbol{x}) = q \left[\boldsymbol{E}(t, \boldsymbol{x}) + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B}(t, \boldsymbol{x}) \right] , \tag{2.3}$$

the second, velocity dependent, term of which is particularly remarkable. Finally, we note the relation between the current density in a given medium and the applied electric field

$$\boldsymbol{j}(t, \boldsymbol{x}) = \sigma \boldsymbol{E}(t, \boldsymbol{x}) \,, \tag{2.4}$$

where σ describes the conductivity of the medium.

The frame of reference in $\mathbb{R}^3 \times \mathbb{R}_t$ with respect to which these equations are formulated, for the time being, is defined by the observer who interprets his position as the origin of the frame, chooses appropriate coordinates in $\mathbb{R}^3$ and uses his clock for measuring time. An experimenter measures *electric* fields with specific instruments which differ from those he uses for measuring *magnetic* fields. In this sense the specific nature of the two types of vector fields is established empirically. This remark which seems to be a matter of course, will be important when one asks whether an electric field and a magnetic field, when measured by a second observer who moves relative to the first at constant velocity, will continue to be an electric or a magnetic field, respectively.

2.2.1 Rotations and Discrete Spacetime Transformations

Before following up the question raised above let us remain for a while in the inertial frame chosen by the observer and let us analyze the covariance of the equations (2.1a–2.4) with respect to rotations, space reflection, and time reversal, as well as charge conjugation.

Rotations of the Frame of Reference in $\mathbb{R}^3$

Rotations $\mathsf{R} \in \mathrm{SO}(3)$ of the frame of reference are coordinate transformations

$$(t, \boldsymbol{x})^T \longmapsto (t' = t, \boldsymbol{x}' = \mathsf{R}\boldsymbol{x})^T \,, \quad \text{with} \quad \mathsf{R}^T \mathsf{R} = \mathbb{1} \,,\ \det \mathsf{R} = +1 \,.$$

A scalar field φ, by definition, stays invariant,

$$\varphi(t, \boldsymbol{x}) \longmapsto \varphi'(t', \boldsymbol{x}') = \varphi(t, \boldsymbol{x}) \,, \tag{2.5a}$$

while a vector field transforms according to

$$\boldsymbol{A}(t,\boldsymbol{x}) \longmapsto \boldsymbol{A}'(t',\boldsymbol{x}') = \mathsf{R}\boldsymbol{A}(t,\boldsymbol{x})\,. \tag{2.5b}$$

(Here we have made use of the fact that in the orthogonal group SO(3) the inverse of the transposed equals the original matrix, $(\mathsf{R}^T)^{-1} = \mathsf{R}$.) If instead of SO(3) one admits the full group O(3) then also transformations $\tilde{\mathsf{R}} \in \mathrm{O}(3)$ must be studied whose determinant equals -1. These can be written as the product of a proper rotation $\mathsf{R} \in \mathrm{SO}(3)$ and space reflection Π. There are fields $\tilde{\varphi}$ of the first kind (2.5a) which though invariant under rotations, obtain a factor $\det \tilde{\mathsf{R}} = -1$ under space reflection. Likewise, in the second category there are fields $\tilde{\boldsymbol{A}}$ which beyond the transformation behaviour (2.5b) receive the same factor $\det \tilde{\mathsf{R}}$. Thus, with $\mathsf{R} \in \mathrm{SO}(3)$ and $\tilde{\mathsf{R}} = \mathsf{R}\Pi$ they transform according to

$$\tilde{\varphi}(t,\boldsymbol{x}) \mapsto \tilde{\varphi}'(t',\boldsymbol{x}') = \left(\det \tilde{\mathsf{R}}\right) \tilde{\varphi}(t,\boldsymbol{x})\,, \tag{2.6a}$$

$$\tilde{\boldsymbol{A}}(t,\boldsymbol{x}) \mapsto \tilde{\boldsymbol{A}}'(t',\boldsymbol{x}') = \left(\det \tilde{\mathsf{R}}\right) \mathsf{R}\tilde{\boldsymbol{A}}(t,\boldsymbol{x})\,. \tag{2.6b}$$

Although in geometric terms $\tilde{\varphi}$ is not a scalar field, and $\tilde{\boldsymbol{A}}$ is not a vector field, the customary nomenclature in physics for them is *pseudoscalar field* for $\tilde{\varphi}(t,\boldsymbol{x})$, and *axial vector field* for $\tilde{\boldsymbol{A}}(t,\boldsymbol{x})$. A few examples over the space $\mathbb{R}^3$ will illustrate these definitions:
(i) The velocity $\boldsymbol{v}$, very much like the momentum $\boldsymbol{p}$, is a genuine vector, i.e. it transforms under rotations $\mathsf{R} \in \mathrm{SO}(3)$ as indicated in (2.5b). If these vectors are defined as smooth functions over $\mathbb{R}^3$ they become vector fields. In contrast, the orbital angular momentum $\boldsymbol{\ell} = \boldsymbol{x} \times \boldsymbol{p}$ is an axial vector. Indeed, under a space reflection both $\boldsymbol{x}$ and $\boldsymbol{p}$ change sign, while $\boldsymbol{\ell}$ does not.
(ii) The scalar product $\boldsymbol{x} \cdot \boldsymbol{p}$ is a scalar. Likewise the scalar product of a spin and an orbital angular momentum $\boldsymbol{s} \cdot \boldsymbol{\ell}$ is a genuine scalar. However, the products $\boldsymbol{x} \cdot \boldsymbol{\ell}$ and $\boldsymbol{x} \cdot \boldsymbol{s}$ are pseudoscalars.

The geometric interpretation of the quantities (2.6a) and (2.6b) in the language of exterior forms will be clarified in Sect. 2.5.3 below. For the moment we will stick to the terminology defined above.

Inspection of Maxwell's equations (2.1a–2.1b) shows that they are covariant under rotations from SO(3) provided the fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$, and the current density $\boldsymbol{j}$ transform according to (2.5b) and the charge density ϱ transforms like in (2.5a). The first equation (2.1a) contains the divergence of $\boldsymbol{B}$ and is a scalar with respect to $\mathsf{R} \in \mathrm{SO}(3)$. Regarding the second equation (2.1b) we have for the first term

$$\left(\nabla' \times \boldsymbol{E}'\right) = \left(\mathsf{R}\nabla\right) \times \left(\mathsf{R}\boldsymbol{E}\right) = \mathsf{R}\left(\nabla \times \boldsymbol{E}\right),$$

and, obviously, for the second term

$$\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}' = \frac{1}{c}\frac{\partial}{\partial t}\left(\mathsf{R}\boldsymbol{B}\right) = \mathsf{R}\left(\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}\right),$$

so that covariance of (2.1b) is established. A similar reasoning proves the covariance of the two inhomogeneous Maxwell equations (2.1c) and (2.1d). All terms which are related by Maxwell's equations exhibit the same transformation behaviour.

Space Reflection of the Frame of Reference

The behaviour of Maxwell's equations under a reflection of the spatial coordinates about the origin,

$$(t, \boldsymbol{x})^T \longmapsto (t' = t, \boldsymbol{x}' = -\boldsymbol{x})^T$$

is less obvious. In a first step one asserts that the curl of a genuine vector field (in $\mathbb{R}^3$) is an axial vector field,

$$\boldsymbol{A}'(t', \boldsymbol{x}') = -\boldsymbol{A}(t, \boldsymbol{x}) \Longleftrightarrow \nabla' \times \boldsymbol{A}'(t', \boldsymbol{x}') = +\nabla \times \boldsymbol{A}(t, \boldsymbol{x}),$$

while the curl of an axial vector field is again a vector field. Equipped with this knowledge one sees that the Maxwell equations are invariant under space reflection if

$\boldsymbol{E}$, $\boldsymbol{D}$, and $\boldsymbol{j}$ are *vector* fields,
$\boldsymbol{B}$ and $\boldsymbol{H}$ are *axial vector* fields,
ϱ is a *scalar* field.

This becomes plausible if one recalls some concrete experimental situations involving electric and magnetic fields. For instance, the electric field of a point charge at rest

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{q}{r^2}\hat{\boldsymbol{r}}$$

is proportional to the position vector $\boldsymbol{r}$, up to factors which are invariant under Π, and, hence, is a vector field. A current density $\boldsymbol{j}$ may be thought of as a flux of point-like charged particles which flow through space with the velocity $\boldsymbol{v}$. This is a genuine vector field, too. The magnetic dipole density (1.120a) is proportional to the cross product of $\boldsymbol{x}$ and $\boldsymbol{j}(\boldsymbol{x})$ and, therefore, is an axial vector field. The same statement holds for the induction field (1.124b). Finally, the charge density must be a scalar, be it only for the reason that the continuity equation (1.21) relates the time derivative of ϱ with the divergence of the current density and, as a whole, must be invariant.

Once more we refer the reader to the geometric formulation of Maxwell theory if he or she wishes to work out more clearly the noted difference between the electric quantities $\boldsymbol{E}$ and $\boldsymbol{D}$ on the one hand, and the magnetic quantities $\boldsymbol{B}$ and $\boldsymbol{H}$ on the other. One will then find out that the first two are equivalent to exterior *one*-forms, while the second group are equivalent to exterior *two*-forms over $\mathbb{R}^3$.

Behaviour Under Time Reversal

It is certainly reasonable to expect that the charge density $\varrho(t, \boldsymbol{x})$ does not depend on the direction of time, whether time runs towards the future or towards the past. That is to say to require that it be invariant under time reversal T,

$$\varrho'(t', \boldsymbol{x}') = \varrho(t, \boldsymbol{x}), \qquad t' = -t, \; \boldsymbol{x}' = \boldsymbol{x}.$$

The continuity equation which contains the first derivative of the charge density with respect to time, then implies that the current density must be *odd*, $\boldsymbol{j}'(t', \boldsymbol{x}') = -\boldsymbol{j}(t, \boldsymbol{x})$. Note that this property was to be expected on the basis of the simple model developed above. In order for the two Maxwell equations (2.1c) and (2.1d) to be invariant, one must have

$$\boldsymbol{H}'(t', \boldsymbol{x}') = -\boldsymbol{H}(t, \boldsymbol{x}), \qquad \boldsymbol{D}'(t', \boldsymbol{x}') = +\boldsymbol{D}(t, \boldsymbol{x}).$$

The electric field $\boldsymbol{E}$ transforms like the displacement field $\boldsymbol{D}$, the induction field $\boldsymbol{B}$ transforms like the magnetic field $\boldsymbol{H}$.

Charge Conjugation

A particularly interesting question which is new as compared to mechanics concerns the behaviour of Maxwell's equations when the signs of all charges are reversed. This is the operation of *charge conjugation* C which plays a central role in the quantum dynamics of elementary systems. For example, when applied to a hydrogen atom this means that the proton p is replaced by an antiproton $\overline{p}$, the electron e^- is replaced by a positron e^+.

By their definition both the charge density and the current density reverse their signs. Written symbolically, $\mathsf{C}\varrho(t, \boldsymbol{x}) = -\varrho(t, \boldsymbol{x})$, $\mathsf{C}\boldsymbol{j}(t, \boldsymbol{x}) = -\boldsymbol{j}(t, \boldsymbol{x})$. From (2.1c) and from the first of these relations one concludes that the displacement field $\boldsymbol{D}$ changes sign, too. This behaviour then also applies to the electric field. The second relation, together with (2.1d), requires that $\boldsymbol{H}$ and thus also $\boldsymbol{B}$ be odd as well. In summary,

$$\begin{aligned} \mathsf{C}\boldsymbol{D}(t, \boldsymbol{x}) &= -\boldsymbol{D}(t, \boldsymbol{x}), \qquad & \mathsf{C}\boldsymbol{E}(t, \boldsymbol{x}) &= -\boldsymbol{E}(t, \boldsymbol{x}), \\ \mathsf{C}\boldsymbol{H}(t, \boldsymbol{x}) &= -\boldsymbol{H}(t, \boldsymbol{x}), \qquad & \mathsf{C}\boldsymbol{B}(t, \boldsymbol{x}) &= -\boldsymbol{B}(t, \boldsymbol{x}). \end{aligned}$$

As before these transformation rules are plausible: If the charges which are the sources of the electric field change sign (without modifying their absolute value) the electric field changes everywhere from $\boldsymbol{E}(t, \boldsymbol{x})$ to $-\boldsymbol{E}(t, \boldsymbol{x})$. As all current densities change sign, too, this applies also to the magnetic fields they give rise to.

In summary, we note that Maxwell's equations are covariant under rotations in the given frame of reference, as well as under the discrete transformations Π, T, and C. However, whether or not the discrete transformations are symmetries in the sense of quantum mechanics is a question about the interactions other than electrodynamics which are acting between the building blocks of matter. The electromagnetic interaction, taken in isolation, is indeed invariant under space reflection and time reversal, as well as under charge conjugation. In a world where all protons are replaced by antiprotons, all electrons are replaced by positrons, the atoms have the same bound states and the spectral lines of atomic physics are the same as in our familiar environment.

2.2.2 Maxwell's Equations and Exterior Forms

This section deals for the first, but not the last, time with the geometric nature of the physical quantities that are involved in Maxwell's equations. In particular, we elucidate what in the intuitive language of physics is called pseudoscalar and axial vector. We do this by means of a short summary of exterior differential calculus on Euclidean spaces $\mathbb{R}^n$, but refer to [ME], Chap. 5, for an extensive and more general presentation.

Exterior Forms on $\mathbb{R}^n$

Exterior one-forms $\overset{1}{\omega}$ in the point $x \in M = \mathbb{R}^n$ are linear maps of tangent vectors on M in x, i.e. of elements of the tangent space $T_x M$, into the reals,

$$\overset{1}{\omega} \; : T_x M \longrightarrow \mathbb{R} \, : \, v \longmapsto \overset{1}{\omega} (v) \, . \tag{2.7a}$$

An important example which illustrates well this notion is the total differential $\mathrm{d}f$ of a smooth function on $\mathbb{R}^n$, in which case

$$\mathrm{d}f(v)|_x = v(f)(x) = \sum_{i=1}^{n} v^i \left.\frac{\partial f}{\partial x^i}\right|_x \equiv \sum_{i=1}^{n} v^i \partial_i f\big|_x \tag{2.7b}$$

represents the directional derivative of the function f at the point x along the direction of v. The action of $\mathrm{d}f$ on the tangent vector v is equal to the action $v(f)$ of this vector on the function and coincides with the derivative of f in the direction defined by v. Indeed, the directional derivative is a real number. In the formulation of (2.7b) we introduced the compact notation

$$\partial_i f := \frac{\partial f}{\partial x^i} \tag{2.7c}$$

for the derivative by the contravariant component x^i which, in turn, is *co*variant.

The set of linear maps from $T_x M$ to $\mathbb{R}$ (by definition) spans the dual vector space $T_x^* M$, called *cotangent space* which is attached to the point x, like $T_x M$ is attached to x.

▶ **Remark**
If M is a smooth n-manifold which is not a Euclidean space $\mathbb{R}^n$, one must construct a complete atlas composed of local charts (φ, U) (also called coordinate

systems) where U is an open neighbourhood of the point $p \in M$ and

$$\varphi : M \to \mathbb{R}^n : U \mapsto \varphi(U)$$

is a homeomorphism from U on M to the image $\varphi(U) \subset \mathbb{R}^n$. Denoting local coordinates in this chart by $\{x^i\}$, $i = 1, \ldots, n$, the partial derivative of a function f is given by

$$\partial_i^{(\varphi)}\Big|_p (f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\big(\varphi(p)\big) . \tag{2.8}$$

Only the composition of $\varphi^{-1} : \mathbb{R}^n \to M$ and of $f : M \to \mathbb{R}$ is a real function on $\mathbb{R}^n$ which can be differentiated according to the rules of analysis. In case the manifold is an $\mathbb{R}^n$ matters simplify: For $M = \mathbb{R}^n$ only one single chart $U = M$ is needed, up to diffeomorphisms, and the corresponding map can be chosen to be $\varphi = \mathrm{id}$, the identical mapping. In this case the formerly local expression (2.8) holds on the whole of M and reduces to the usual partial derivative (2.7c) well known from real analysis.

With $v(f)$ the directional derivative of the function f in the point x, and $\partial_i f$ the partial derivative with respect to the coordinate x^i the expression

$$v = \sum_{i=1}^{n} v^i \partial_i$$

is the decomposition of the vector v in terms of the base fields $\{\partial_i\}$, $i = 1, \ldots, n$. These base fields span the tangent space $T_x M$. In the case of an $\mathbb{R}^n$, however, one can identify all tangent spaces with one another and with the base manifold. This means that every smooth vector field V on $M = \mathbb{R}^n$ can be decomposed

$$V = \sum_{i=1}^{n} v^i(x) \partial_i \ , \tag{2.9}$$

with coefficients $v^i(x)$ which are smooth functions.

Of course, the coordinates x^i are smooth functions on M themselves: x^i associates to the point $x \in M$ its i-th coordinate. The differentials $\mathrm{d}x^i$ of these functions are one-forms and are called *base one-forms.* The set of all $\{\mathrm{d}x^i\}$, $i = 1, \ldots, n$ is dual to the basis $\{\partial_i\}$, since

$$\mathrm{d}x^i(\partial_k) = \partial_k(x^i) \equiv \frac{\partial}{\partial x^k} x^i = \delta^i_k .$$

Therefore, every one-form $\overset{1}{\omega} \in T_x M$ can be expanded in this basis, $\overset{1}{\omega} = \sum \omega_i \, \mathrm{d}x^i$.

A one-form $\overset{1}{\omega}$ is said to be smooth if it is defined on all of M and if $\overset{1}{\omega}(V)$ is a smooth function for all smooth vector fields $V \in \mathcal{V}(M)$. When applied to $M = \mathbb{R}^n$ this means that every one-form $\overset{1}{\omega}$ can be written as an expansion

$$\overset{1}{\omega} = \sum_{i=1}^{n} \omega_i(x)\,\mathrm{d}x^i \tag{2.10}$$

where the coefficients $\omega_i(x)$ are smooth functions. The coefficient functions are calculated from the action of the one-form on the base vector fields, viz.

$$\omega_i(x) = \overset{1}{\omega}(\partial_i)\,.$$

Thus, the action on an arbitrary smooth vector field is given by

$$\overset{1}{\omega}(V) = \sum_{i=1}^{n} V^i(x)\omega_i(x)\,,$$

where

$$V = \sum_j V^j(x)\partial_j \quad \text{and} \quad \overset{1}{\omega} = \sum_k \omega_k(x)\,\mathrm{d}x^k\,.$$

There exists a skew-symmetric, associative product of exterior forms, called *exterior product*, or *wedge product*, whose definition is most simply given for base one-forms and their action on vectors as follows

$$\left(\mathrm{d}x^i \wedge \mathrm{d}x^j\right)(v, w) = v^i w^j - v^j w^i = \det\begin{pmatrix} v^i & w^i \\ v^j & w^j \end{pmatrix}\,, \tag{2.11a}$$

where use was made of its antisymmetry,

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i\,. \tag{2.11b}$$

The following example shows that this definition is the direct generalization of the well-known cross product in $\mathbb{R}^3$:
In the space $\mathbb{R}^3$ there are three base one-forms, $\mathrm{d}x^1$, $\mathrm{d}x^2$ and $\mathrm{d}x^3$. If one applies the wedge product of the second and the third of these to two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\left(\mathrm{d}x^2 \wedge \mathrm{d}x^3\right)(\boldsymbol{a}, \boldsymbol{b}) = a^2 b^3 - a^3 b^2 = (\boldsymbol{a} \times \boldsymbol{b})_1 \quad (\text{on } \mathbb{R}^3)\,,$$

the result is seen to be the first component of the cross product. Adding to this formula the two formulae obtained by cyclic permutation of the indices one obtains the full cross product $\boldsymbol{a} \times \boldsymbol{b}$.

The exterior product is easily extended to three or more factors. For example, with three base one-forms and three tangent vectors one has

$$\left(\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\right)(u, v, w) = \det\begin{pmatrix} u^i & v^i & w^i \\ u^j & v^j & w^j \\ u^k & v^k & w^k \end{pmatrix} \tag{2.11c}$$

This formula illustrates the associativity of the exterior product. No parentheses need be written in a product of more than two factors. For example, the product $(\mathrm{d}x^i \wedge \mathrm{d}x^j) \wedge \mathrm{d}x^k$ is the same as $\mathrm{d}x^i \wedge (\mathrm{d}x^j \wedge \mathrm{d}x^k)$. (Note that in the second example the position of parentheses corresponds to the expansion of the determinant along the first row.)

The products $\mathrm{d}x^i \wedge \mathrm{d}x^j$ with $i < j$, of which there are $n(n-1)/2 = \binom{n}{2}$, are elements of $T_x^* \times T_x^*$, which, in addition, are antisymmetric. The whole set for all i and j provides a basis for arbitrary smooth *two-forms* so that

$$\overset{2}{\omega} = \sum_{i<j} \omega_{ij}(x)\ \mathrm{d}x^i \wedge \mathrm{d}x^j\ . \tag{2.12}$$

The coefficients $\omega_{ij}(x)$ are smooth functions on $M = \mathbb{R}^n$. In the language of classical tensor analysis such an object $\overset{2}{\omega}$ is a tensor field of type $(0, 2)$

$$\overset{2}{\omega} \in \mathfrak{T}_2^0(M)\ ,$$

which, in addition, is antisymmetric. The set of all coefficients ω_{ij} gives its representation in coordinates and in the form of an antisymmetric tensor of degree 2.

The chain of base forms can be continued, in a finite number of steps, up to the wedge product of n base one-forms. This procedure yields the base k-forms $\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, $k = 3, \ldots, n$, of which there are $\binom{n}{k}$ for every k. With these tools at hand one can construct smooth k-forms

$$\overset{k}{\omega} = \sum_{i_1<\cdots<i_k} \omega_{i_1\ldots i_k}(x)\ \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}\ , \tag{2.13}$$

with coefficients $\omega_{i_1\ldots i_k}(x)$ which are again smooth functions on $\mathbb{R}^n$. The exterior form $\overset{k}{\omega}$ is an element of $\mathfrak{T}_k^0(M)$, the space of covariant tensor fields of degree k, and is antisymmetric in the k vector fields to which it is applied. It is customary to denote the space of antisymmetric, covariant tensor fields of degree k by

$$\overset{k}{\omega} \in \Lambda^k(M)\ . \tag{2.14}$$

It is not difficult to determine the dimension of theses spaces. By counting the base elements $\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k}$ one finds that the dimension of $\Lambda^k(M)$ is (s. Exercise 2.1)

$$\dim \Lambda^k(M) = \binom{n}{k} = \frac{n!}{k!(n-k)!}\ .$$

Thus, Λ^1 has dimension n, very much like Λ^{n-1}. Λ^n has dimension 1, while there is no space Λ^m whose dimension is greater than n.

The Exterior Derivative

The exterior derivative is the generalization of the total differential for functions, of the gradient of a function and of the curl and the divergence for vector fields in $\mathbb{R}^3$. It has the following properties:
It maps k-forms to $(k+1)$-forms (which may be zero),

$$\mathrm{d} : \Lambda^k(M) \to \Lambda^{k+1}(M) : \overset{k}{\omega} \mapsto \mathrm{d}\overset{k}{\omega} . \tag{2.15a}$$

When applied to a smooth function it yields the total differential

$$\mathrm{d} : f \mapsto \mathrm{d}f = \sum_i \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i . \tag{2.15b}$$

The exterior derivative fulfills a graded Leibniz rule with specific signs as follows: When applied to the exterior product of an r-form and an s-form ($r, s = 0, 1, \dots, n$) the result is

$$\mathrm{d}\big(\overset{r}{\omega} \wedge \overset{s}{\omega}\big) = \big(\mathrm{d}\overset{r}{\omega}\big) \wedge \overset{s}{\omega} + (-)^r\, \overset{r}{\omega} \wedge \big(\mathrm{d}\overset{s}{\omega}\big) . \tag{2.15c}$$

This resembles the familiar product rule of differential calculus except for the fact that the second term keeps its plus sign only if the first factor is an *even* form, but receives a minus sign if the degree r of the first factor is *odd*. As a rule of thumb one may remember that "shifting" the operator d past an r-form produces a sign $(-)^r$. Obviously, the exterior product $\overset{r}{\omega} \wedge \overset{s}{\omega}$ is an element of Λ^{r+s}, its exterior derivative is in Λ^{r+s+1}.

If d is applied twice the result is always zero

$$\mathrm{d} \circ \mathrm{d} = 0 . \tag{2.15d}$$

The following formula for the exterior derivative of a k-form in the representation (2.13) is useful in practice

$$\begin{aligned} \mathrm{d}\overset{k}{\omega} &= \sum_{i_1<\dots<i_k} \mathrm{d}\omega_{i_1\dots i_k}(x) \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \dots \wedge \mathrm{d}x^{i_k} \qquad (2.15\mathrm{e}) \\ &= \sum_{j=1}^{n} \sum_{i_1<\dots<i_k} \frac{\partial \omega_{i_1\dots i_k}(x)}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \dots \wedge \mathrm{d}x^{i_k} . \end{aligned}$$

It contains in the first step the total differential of the functions $\omega_{i_1,\dots,i_k}(x^1, \dots, x^n)$ which is to be calculated following the rule (2.15b) for functions. At the end of a calculation of this type one must reorder the base one-forms in order to arrange them in increasing order and keep track of the signs that this may produce.

► **Remarks**

1. For functions the property (2.15d) is nothing but the fact that the mixed second derivatives of a smooth function are equal. Indeed, taking the exterior derivative of $\mathrm{d}f$, one has

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \sum_i \left(\mathrm{d}\frac{\partial f}{\partial x^i}\right) \wedge \mathrm{d}x^i && \text{(according to rule (2.15e))} \\
&= \sum_{k \neq i} \frac{\partial^2 f}{\partial x^k \partial x^i}\, \mathrm{d}x^k \wedge \mathrm{d}x^i && \text{(with formula (2.15b)} \\
& && \text{for total differentials)} \\
&= \sum_{k<i} \left\{ \frac{\partial^2 f}{\partial x^k \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^k} \right\} \mathrm{d}x^k \wedge \mathrm{d}x^i = 0 \\
& && \text{(antisymmetry of base-forms)} .
\end{aligned}$$

   For forms of higher degree the rule (2.15d) follows from the Leibniz rule (2.15c).

2. Contemplating the series of spaces $\Lambda^1(M)$, ..., $\Lambda^k(M)$, ..., $\Lambda^n(M)$, one notices that their dimensions follow the binomial series $\binom{n}{1} = n, \ldots, \binom{n}{n} = 1$ but that the series of numbers in Pascal's triangle is incomplete. The number $\binom{n}{0} = 1$, i.e. the dimension of $\Lambda^0(M)$ is missing. Conversely, the exterior derivative $\mathrm{d}f$ of a function f is a one-form and, according to (2.15a), the operator d leads from $\Lambda^k(M)$ to $\Lambda^{k+1}(M)$. This suggests interpretation of the smooth functions in the framework of exterior forms as *zero-forms,*

$$f \in \mathfrak{F}(M) \quad \text{(smooth functions on } M\text{)}, \quad f \in \Lambda^0(M) .$$

3. If the application of d to an exterior k-form is zero this form is said to be *closed,*

$$\mathrm{d}\omega = 0 , \qquad \omega \in \Lambda^k(M) .$$

   For instance, the total differential of a function is a closed form because $\mathrm{d} \circ \mathrm{d}f = \mathrm{d}(\mathrm{d}f) = 0$.

   In turn, it may happen that a $(k+1)$-form η can be written as the exterior derivative of a k-form ω, i.e.

$$\eta = \mathrm{d}\omega , \quad \eta \in \Lambda^{k+1}(M) , \quad \omega \in \Lambda^k(M) .$$

   A form of this kind is said to be an *exact* form. Clearly, every exact form is also a closed form. Regarding the converse one has: On $M = \mathbb{R}^n$ every closed k-form can be written as the derivative of a $(k-1)$-form. Note that on more general manifolds this holds only locally. This is the content of Poincaré's lemma[1].

Hodge Dual Forms

The space $\mathbb{R}^n$ not only is a smooth manifold but is also orientable. In other terms, an ordered basis $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \dots, \hat{\boldsymbol{e}}_n)$ spans a generalized parallelepiped which can be assigned a sign. The spaces $\Lambda^k(M)$ and $\Lambda^{(n-k)}(M)$ have the same dimension because of the equality of binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$$

and they are isomorphic. One defines a bijective mapping, the so-called $\star$-operation, which associates to every k-form a $(n-k)$-form. The image of a k-form ω under Hodge dualism is denoted by $\star\omega$. Defining it by means of the action of forms onto unit vectors, the k-form ω is mapped to the $(n-k)$-form $\star\omega$ by

$$(\star\omega)\left(\hat{\boldsymbol{e}}_{i_{k+1}}, \dots, \hat{\boldsymbol{e}}_{i_n}\right) = \varepsilon_{i_1 \dots i_k i_{k+1} \dots i_n} \omega\left(\hat{\boldsymbol{e}}_{i_1}, \dots, \hat{\boldsymbol{e}}_{i_k}\right) . \tag{2.16}$$

For example, in $\mathbb{R}^3$ one has

$$\star \, \mathrm{d}x^i = \frac{1}{2} \sum_{j,k} \varepsilon_{ijk} \, \mathrm{d}x^j \wedge \mathrm{d}x^k \,, \tag{2.17a}$$

$$\star \left(\mathrm{d}x^i \wedge \mathrm{d}x^j\right) = \varepsilon_{ijk} \, \mathrm{d}x^k \,, \tag{2.17b}$$

$$\star \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) = 1 \,. \tag{2.17c}$$

In this example the spaces of one-forms and of two-forms are isomorphic because of the equality $\binom{3}{1} = \binom{3}{2} = 3$ and because $\mathbb{R}^3$ is orientable. Likewise, the three-forms and the functions are related one-to-one. (Note that in this case there is only one base three-form.)

In the general case of an $\mathbb{R}^n$ bijectivity is established by the relation

$$\star \left((\star\omega)\right) = (-)^{k(n-k)}\omega \,, \quad (\omega \in \Lambda^k) \,. \tag{2.18}$$

Applying the $\star$-operation to a k-form twice takes it back to the original form, up to a sign which depends on its degree.

The star operation and the exterior derivative can be combined to a new and very interesting operator. As anticipated in equations (1.49) and (1.50) define

$$\delta = (-)^{n(k+1)+1} \star \mathrm{d}\star \,, \tag{2.19a}$$

$$\boldsymbol{\Delta}_{\mathrm{LdR}} = \mathrm{d} \circ \delta + \delta \circ \mathrm{d} \,. \tag{2.19b}$$

[1] The Poincaré lemma applies to any open neighbourhood $U \subset M$ of the point $p \in M$ which can be contracted to p without leaving the manifold M.

The first of these operators, in some sense, is the counterpart of the exterior derivative. Indeed, one easily verifies that δ *lowers* the degree of the exterior form by one,

$$\delta : \Lambda^k(M) \longrightarrow \Lambda^{(k-1)}(M) .$$

The first mapping $\star$ leads from degree k to degree $(n-k)$, the application of d converts it to an $(n-k+1)$-form, and a second $\star$-mapping yields a $(n-(n-k+1)) = (k-1)$-form. By the same token one sees that the Laplace-de-Rham operator $\mathbf{\Delta}_{\mathrm{LdR}}$ does not change the degree of the form on which it acts.

Example 2.1

On the space $\mathbb{R}^3$ there exist the spaces Λ^0 and Λ^3 both of which have the dimension $\binom{3}{0} = 1 = \binom{3}{3}$, as well as the spaces Λ^1 and Λ^2 which both have dimension $\binom{3}{1} = 3 = \binom{3}{2}$. Regarding the base forms one has

$$\star\ \mathrm{d}x^i = \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\ \mathrm{d}x^j \wedge \mathrm{d}x^k , \tag{2.20a}$$

$$\star\left(\mathrm{d}x^i \wedge \mathrm{d}x^j\right) = \sum_k \varepsilon_{ijk}\ \mathrm{d}x^k , \tag{2.20b}$$

$$\star\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) = 1 . \tag{2.20c}$$

Example 2.2

This example is particularly important for electrodynamics, though it repeats an example whose details are worked out, for example in [ME], Sect. 5.4.5. Note that we made use of it in Sect. 1.6.1 above. Let $\boldsymbol{a}$ be a vector field on $M = \mathbb{R}^3$. Define the covariant components $a_i = a^i$ (with $\boldsymbol{a} \equiv \boldsymbol{a}(\boldsymbol{x})$), a one-form and a two-form, respectively, by

$$\overset{1}{\omega}_{\boldsymbol{a}} = \sum_{i=1}^{3} a_i(\boldsymbol{x})\ \mathrm{d}x^i , \tag{2.21a}$$

$$\overset{2}{\omega}_{\boldsymbol{a}} = \frac{1}{2}\sum_{i,j,k} \varepsilon_{ijk} a_i(\boldsymbol{x})\ \mathrm{d}x^j \wedge \mathrm{d}x^k . \tag{2.21b}$$

(The numerical factor in (2.21b) accounts for the antisymmetric permutations of (i, j, k).) On account of the relation (2.20b) and the formula (1.48b) one sees that

$$\star\ \overset{2}{\omega}_{\boldsymbol{a}} = \frac{1}{2}\sum_{i,j,k} \varepsilon_{ijk} a_i(\boldsymbol{x}) \varepsilon_{jkl}\ \mathrm{d}x^l = \overset{1}{\omega}_{\boldsymbol{a}} ,$$

i.e. the Hodge dual of the two-form (2.21b) coincides with the original one-form (2.21a).

The exterior derivative of the first form (2.21a) yields the two-form corresponding to the curl of $\boldsymbol{a}$, and then, after application of $\star$, the one-form constructed with the curl, viz.

$$\mathrm{d}\overset{1}{\omega}_{\boldsymbol{a}} = \frac{1}{2}\sum_{i,j,k} \varepsilon_{ijk}\big(\nabla \times \boldsymbol{a}\big)_i \, \mathrm{d}x^j \wedge \mathrm{d}x^k , \quad \text{or} \quad \star\, \mathrm{d}\overset{1}{\omega}_{\boldsymbol{a}} = \overset{1}{\omega}_{\nabla\times\boldsymbol{a}} . \tag{2.22}$$

The exterior derivative of the two-form (2.21b) yields the divergence of $\boldsymbol{a}$,

$$\mathrm{d}\overset{2}{\omega}_{\boldsymbol{a}} = \big(\nabla\cdot\boldsymbol{a}\big)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 , \quad \text{or} \quad \star\, \mathrm{d}\overset{2}{\omega}_{\boldsymbol{a}} = \nabla\cdot\boldsymbol{a} . \tag{2.23}$$

The action of the Laplace–de-Rham operator on a function or on a one-form of the type (2.21a) gives the results, respectively,

$$\boldsymbol{\Delta}_{\mathrm{LdR}} f = -\boldsymbol{\Delta} f(\boldsymbol{x}) , \tag{2.24a}$$

$$\boldsymbol{\Delta}_{\mathrm{LdR}}\, \overset{1}{\omega}_{\boldsymbol{a}} = -\sum_{i=1}^{3}\big(\boldsymbol{\Delta} a_i(\boldsymbol{x})\big)\, \mathrm{d}x^i , \tag{2.24b}$$

where $\boldsymbol{\Delta}$ denotes the customary Laplace(–Beltrami) operator, i.e. $\boldsymbol{\Delta} = \partial_1^2 + \partial_2^2 + \partial_3^2$, acting on smooth functions in either case.

Fields and Sources in Maxwell's Equations

We still remain in a fixed reference frame where spacetime has the structure $\mathbb{R}_t \times \mathbb{R}^3$ with a fixed division of spacetime into the factor to be called time and the remainder describing the well-known laboratory space of an experimenter. All quantities which are related by Maxwell's equations are defined as geometric objects over $\mathbb{R}^3$ but depend parametrically on time. Surely, this is a restricted perspective because it rests on a subjective perception of time and space. Nevertheless, the fundamental laws of electrodynamics in integral form give direct hints at the geometric role of fields and densities.

Faraday's law (1.12) contains, on the one side, the path integral of the tangential component of the electric field, and, on the other side, the integral of the magnetic flux over a surface whose boundary is that path. The path $\mathcal{C}$, by itself, is a smooth, closed manifold with dimension $\dim\mathcal{C} = 1$. The surface whose boundary is $\mathcal{C}$, is also a smooth manifold with dimension $\dim F(\mathcal{C}) = 2$. Quite generally, on an orientable manifold M with metric g and $\dim M = n$, the exterior n-form

$$\Omega^{(n)} = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \tag{2.25}$$

defines the so-called volume form. Here $|g|$ is the determinant, or more precisely, the absolute value of the determinant of the metric, $|g| = |\det\{g_{ik}\}|$. The exterior form which is proportional to the only base element that exists in $\Lambda^n(M)$, carries the orientation of the basis through the order of factors in the product $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. It is independent of the choice of the coordinate system. This is shown as follows: Let Φ be a diffeomorphism which relates the coordinates $(x^1, \ldots, x^n)$ to new coordinates $(y^1, \ldots, y^n)$. The metric tensor which in the original coordinates has the form $\{g_{ij}(x)\}$, is replaced by $\bar{g}_{kl}(y)$ in the new coordinates, and we have

$$(x^1, \ldots, x^n) \longleftrightarrow (y^1, \ldots, x^n) , \quad \bar{g}_{kl}(y) = \sum_{i,j=1}^{n} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} g_{ij}(x) .$$

Hence, the determinants of the metric tensors g and $\bar{g}$ are related by

$$|\bar{g}|\,(y) = \left(\det\left(\frac{\partial x^i}{\partial y^k}\right)\right)^2 |g|\,(x) .$$

If the map Φ preserves the orientation, i.e. if the two coordinate systems have the same orientation, one takes the square root and obtains

$$\sqrt{|\bar{g}|} = \det\left(\frac{\partial x^i}{\partial y^k}\right) \sqrt{|g|} .$$

Therefore, the expansions of an arbitrary smooth n-form in the first and in the second coordinate system, respectively, are related by

$$\begin{aligned} \overset{n}{\omega} &= a(x)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \bar{a}(y)\, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n \quad \text{with} \\ \bar{a}(y) &= a(x) \det\left(\frac{\partial x^i}{\partial y^k}\right) . \end{aligned}$$

This shows that, indeed, the volume form (2.25), $\Omega^{(n)}$, is invariant.

Example 2.3

This is an example in dimension 2 where calculations are particularly simple:

$$\begin{aligned} \Omega^{(2)}(x) &= \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &= \sqrt{|g|} \left(\frac{\partial x^1}{\partial y^1} \mathrm{d}y^1 + \frac{\partial x^1}{\partial y^2} \mathrm{d}y^2\right) \wedge \left(\frac{\partial x^2}{\partial y^1} \mathrm{d}y^1 + \frac{\partial x^2}{\partial y^2} \mathrm{d}y^2\right) \\ &= \sqrt{|g|} \left\{\frac{\partial x^1}{\partial y^1} \frac{\partial x^2}{\partial y^2} - \frac{\partial x^1}{\partial y^2} \frac{\partial x^2}{\partial y^1}\right\} \mathrm{d}y^1 \wedge \mathrm{d}y^2 \\ &= \sqrt{|g|} \det\left(\frac{\partial x^i}{\partial y^k}\right) \mathrm{d}y^1 \wedge \mathrm{d}y^2 \\ &= \sqrt{|\bar{g}|}\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \equiv \Omega^{(2)}(y) . \end{aligned}$$

One sees that not only the volume element but also the orientation of the coordinate system is conserved. The transition between x and y contains the Jacobi determinant of the transformation which carries a well-defined sign. Liouville's theorem on the conservation of a domain of initial conditions, of its volume and orientation, provides a good illustration.

These arguments and the example show that integration over an n-dimensional manifold M must have the form

$$\int_M (\text{integrand})\, \Omega^{(n)} \,, \quad \text{with } \Omega^{(n)} \text{ the volume form on } M \,.$$

Expressed differently this means that only integration of n-forms over the whole of M is meaningful.

A detailed discussion of integration on smooth manifolds would go beyond the scope of this section and also of this book. Therefore, I concentrate here on some plausibility arguments which emerge from Maxwell's equations in integral form, and refer to the literature on differential geometry for a more rigourous presentation. (For a concise, though short introduction and, in particular, a proof of Stokes' theorem in the general form of the equation (1.8b) see, e. g. [Arnol'd 1988, Sect. 36].)

Let us return for a moment to the original integral form of Faraday's law (1.12). The closed curve $\mathcal{C}$ over which one integrates, geometrically speaking, is a one-dimensional manifold embedded in $\mathbb{R}^3$. It inherits an induced metric from $g_{ik} = \operatorname{diag}(1, 1, 1)$. In the spirit of what was noticed above the path integral over the tangential component $\mathrm{d}\boldsymbol{s} \cdot \boldsymbol{E}(t, \boldsymbol{x})$ of the electric field must be the integral of a *one*-form on $\mathcal{C}$, and, therefore, on $\mathbb{R}^3$, by the embedding of the curve in space. Thus, it seems natural to associate to the electric field a one-form of the kind of (2.21a), viz.

$$\overset{1}{\omega}_{\boldsymbol{E}} := E_1(t, \boldsymbol{x})\, \mathrm{d}x^1 + E_2(t, \boldsymbol{x})\, \mathrm{d}x^2 + E_3(t, \boldsymbol{x})\, \mathrm{d}x^3 \,. \tag{2.26a}$$

On the right-hand side of Faraday's law (1.12) the normal component of the field $\boldsymbol{B}$ is integrated over a surface F which is also embedded in $\mathbb{R}^3$. If one compares this with the definition of the two-form (2.21b) including its characteristic ordering of indices, and takes account of the statement that only two-forms can be integrated consistently over surfaces ($\dim F = 2$), one realizes that, from a geometric point of view, $\boldsymbol{B}$ must be associated to a *two*-form of the type of (2.21b),

$$\begin{aligned} \overset{2}{\omega}_{\boldsymbol{B}} := {} & B_1(t, \boldsymbol{x})\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2(t, \boldsymbol{x})\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\ & + B_3(t, \boldsymbol{x})\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \,. \end{aligned} \tag{2.26b}$$

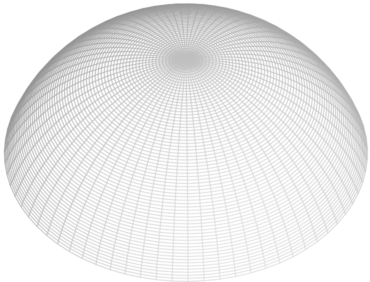


Fig. 2.1 Spherical calotte in 3-space. Faraday's law is formulated by exterior forms on this surface

Example 2.4

A simple, though physically unrealistic example may be helpful. Consider the rectangle in the $(1, 2)$-plane which is defined by the vectors $\boldsymbol{v} = v\hat{\boldsymbol{e}}_1$ and $\boldsymbol{w} = w\hat{\boldsymbol{e}}_2$. The integral of the one-form (2.26a) over the edges of the rectangle is seen to be the integral of the tangential component of $\boldsymbol{E}$ along this curve. Regarding the restriction of the two-form (2.26b) to the surface of the rectangle, conversely, only the third term survives whose coefficient, indeed, is B_3.

Example 2.5

The following example is closer to physics and it should be studied carefully because, on the one hand, it illustrates the assertion that only the integration of an n-form over an n-dimensional manifold is meaningful. On the other hand, it shows that the integral form of Faraday's law written in terms of exterior forms,

$$\int_{\partial F} \overset{1}{\omega}_{\boldsymbol{E}} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_{F} \overset{2}{\omega}_{\boldsymbol{B}} \ ,$$

is identical with the customary form (1.12) of that law.

Let the surface F be the calotte of a sphere with radius $r = R$ shown in Fig. 2.1 which is enclosed between the latitude defined by the angle θ_0 and the north pole ($\theta = 0$). Its boundary ∂F is the circle of latitude with fixed $\theta = \theta_0$ and azimuth in the interval $\phi \in [0, 2\pi]$. In this example it is useful to use spherical polar coordinates r, θ, ϕ instead of the cartesian coordinates x^1, x^2, x^2,

$$x^1 = r\sin\theta\cos\phi \ , \quad x^2 = r\sin\theta\sin\phi \ , \quad x^3 = r\cos\theta \ .$$

The first problem consists in determining the base one-forms $\mathrm{d}u^k$ in polar coordinates and to expand the two exterior forms in terms of these. Denoting by $\hat{\boldsymbol{e}}_i$ the cartesian unit vectors, by $\hat{\boldsymbol{a}}_1 \equiv \hat{\boldsymbol{e}}_r$ (pointing in radial direction), $\hat{\boldsymbol{a}}_2 \equiv \hat{\boldsymbol{e}}_\theta$ (tangential to the meridian), and $\hat{\boldsymbol{a}}_3 \equiv \hat{\boldsymbol{e}}_\phi$ (tangential to circle of latitude) the spherical unit vectors, the following relation holds

$$\begin{aligned}
\hat{\boldsymbol{a}}_1 &= \hat{\boldsymbol{e}}_1 \sin\theta\cos\phi + \hat{\boldsymbol{e}}_2 \sin\theta\sin\phi + \hat{\boldsymbol{e}}_3\cos\theta\ , \\
\hat{\boldsymbol{a}}_2 &= \hat{\boldsymbol{e}}_1 \cos\theta\cos\phi + \hat{\boldsymbol{e}}_2 \cos\theta\sin\phi - \hat{\boldsymbol{e}}_3\sin\theta\ , \\
\hat{\boldsymbol{a}}_3 &= -\hat{\boldsymbol{e}}_1 \sin\phi + \hat{\boldsymbol{e}}_2\cos\phi\ .
\end{aligned}$$

The base forms $\mathrm{d}u^k$ are dual to the base vectors $\hat{\boldsymbol{a}}_j$ and, therefore, must fulfill the relations $\mathrm{d}u^k(\hat{\boldsymbol{a}}_j) = \delta^k_j$. Both systems refer to real and orthogonal coordinates. Hence, the same transformation formulae hold for base one-forms and for base vectors,

$$\begin{aligned}
\mathrm{d}u^1 &= \mathrm{d}x^1 \sin\theta\cos\phi + \mathrm{d}x^2 \sin\theta\sin\phi + \mathrm{d}x^3\cos\theta\ , \\
\mathrm{d}u^2 &= \mathrm{d}x^1 \cos\theta\cos\phi + \mathrm{d}x^2 \cos\theta\sin\phi - \mathrm{d}x^3\sin\theta\ , \\
\mathrm{d}u^3 &= -\mathrm{d}x^1 \sin\phi + \mathrm{d}x^2\cos\phi\ .
\end{aligned}$$

This partial result may be interpreted in two different ways: One calculates the action of $\mathrm{d}u^k$ on the unit vectors $\hat{\boldsymbol{a}}_j$, makes use of the relation $\mathrm{d}x^p(\hat{\boldsymbol{e}}_q) = \delta^p_q$ and deduces the expected relation $\mathrm{d}u^k(\hat{\boldsymbol{a}}_j) = \delta^k_j$. Alternatively, one utilizes the differentials

$$\begin{aligned}
\mathrm{d}x^1 &= \sin\theta\cos\phi\ \mathrm{d}r + r\cos\theta\cos\phi\ \mathrm{d}\theta - r\sin\theta\sin\phi\ \mathrm{d}\phi\ , \\
\mathrm{d}x^2 &= \sin\theta\sin\phi\ \mathrm{d}r + r\cos\theta\sin\phi\ \mathrm{d}\theta + r\sin\theta\cos\phi\ \mathrm{d}\phi\ , \\
\mathrm{d}x^3 &= \cos\theta\ \mathrm{d}r - r\sin\theta\ \mathrm{d}\theta\ ,
\end{aligned}$$

to determine the line element

$$\begin{aligned}
(\mathrm{d}s)^2 &= \sum_{i=1}^{3}(\mathrm{d}x^i)^2 = \sum_{k=1}^{3}(\mathrm{d}u^k)^2 \\
&= (\mathrm{d}r)^2 + (r\,\mathrm{d}\theta)^2 + (r\sin\theta\,\mathrm{d}\phi)^2\ .
\end{aligned}$$

As a result one obtains the representation of the base one-forms in spherical polar coordinates:

$$\mathrm{d}u^1 = \mathrm{d}r\ , \quad \mathrm{d}u^2 = r\ \mathrm{d}\theta\ , \quad \mathrm{d}u^3 = r\sin\theta\ \mathrm{d}\phi\ .$$

One then applies these results to the exterior forms of electrodynamics to obtain on the circle of latitude $\theta = \theta_0$

$$\overset{1}{\omega}_{\boldsymbol{E}} = \sum_{i=1}^{3} E_i\,\mathrm{d}x^i = [-E_1\sin\phi + E_2\cos\phi]\,R\sin\theta\ \mathrm{d}\phi = E_\phi\,\mathrm{d}u^3\ .$$

As expected, the one-form appears as the tangential component oriented along the circle of latitude. Its integral over that circle is

$$\int_{\partial F} \overset{1}{\omega}_{\boldsymbol{E}} = \int_{0}^{2\pi} R \sin\theta \, \mathrm{d}\phi \; E_\phi \; .$$

Regarding the two-form of the magnetic induction, its restriction to the spherical calotte implies that $\mathrm{d}u^1 = 0$ so that only the base two-form $\mathrm{d}u^2 \wedge \mathrm{d}u^3$ contributes. Inserting the results given above one finds

$$\begin{aligned} & B_1 \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2 \, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + B_3 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ & = [B_1 \sin\theta \cos\phi + B_2 \sin\theta \sin\phi + B_3 \cos\theta] \; \mathrm{d}u^2 \wedge \mathrm{d}u^3 \, . \end{aligned}$$

The base form $\mathrm{d}u^2 \wedge \mathrm{d}u^3$ fixes the orientation of the normal to the surface and is equal to $R^2 \sin\theta \; \mathrm{d}\theta \, \mathrm{d}\phi$. The expression in square brackets is the normal component $B_n = \boldsymbol{B} \cdot \hat{\boldsymbol{n}}$ of the magnetic induction, the normal $\hat{\boldsymbol{n}}$ being directed *outwards* on the calotte. Thus, the integral of the two-form is

$$\int_F \overset{2}{\omega}_{\boldsymbol{B}} = R^2 \int_0^{\theta_0} \sin\theta \; \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \; \boldsymbol{B} \cdot \hat{\boldsymbol{n}}$$

and one recovers the integral form of Faraday's law.

Following similar arguments as for $\boldsymbol{B}$ one deduces from Gauss' law (1.14) that one must associate to the field $\boldsymbol{D}$ a *two*-form – in contrast to the electric field $\boldsymbol{E}$,

$$\begin{aligned} \overset{2}{\omega}_{\boldsymbol{D}} := {} & D_1(t, \boldsymbol{x}) \, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + D_2(t, \boldsymbol{x}) \, \mathrm{d}x^3 \wedge \mathrm{d}x^1 \\ & + D_3(t, \boldsymbol{x}) \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \, . \end{aligned} \tag{2.27a}$$

It should be clear that Maxwell's equations when written in terms of exterior forms, can only relate forms of equal degree. The second inhomogeneous Maxwell equation (2.1d), considered in vacuum, i.e. with $\boldsymbol{j} \equiv 0$, without loss of generality, indicates that the curl of the field $\boldsymbol{H}$ must be equivalent to a two-form. Therefore, the field $\boldsymbol{H}$ itself must be associated to a one-form of the type (2.21a),

$$\overset{1}{\omega}_{\boldsymbol{H}} := H_1(t, \boldsymbol{x}) \, \mathrm{d}x^1 + H_2(t, \boldsymbol{x}) \, \mathrm{d}x^2 + H_3(t, \boldsymbol{x}) \, \mathrm{d}x^3 \, . \tag{2.27b}$$

Regarding the source terms in the inhomogeneous equations (2.1c) and (2.1d) one sees that to the charge density one must associate a three-form, to the current density a two-form, respectively, as follows,

$$\overset{3}{\omega}_{\varrho} := \varrho(t,\boldsymbol{x})\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \tag{2.28a}$$

$$\overset{2}{\omega}_{\boldsymbol{j}} := j_1(t,\boldsymbol{x})\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + j_2(t,\boldsymbol{x})\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + j_3(t,\boldsymbol{x})\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \tag{2.28b}$$

These assignments were deduced from the inhomogeneous equations but they can also be made plausible from the integral fundamental laws. Indeed, the charge density always appears integrated over three-dimensional volume in order to yield physical charges, while the current density is integrated over cross sections of conductors such as to yield current strengths.

Maxwell's equations can now be formulated in terms of exterior forms such that their local form (2.1a–2.1d) follow by comparison of coefficients for forms of equal degree. Both the two homogeneous Maxwell equations (2.1a) and (2.1b), and the two inhomogeneous equations (2.1c) and (2.1d) take a very simple form. They read

$$\mathrm{d}\,\overset{2}{\omega}_{\boldsymbol{B}} = 0 \,, \tag{2.29a}$$

$$\mathrm{d}\,\overset{1}{\omega}_{\boldsymbol{E}} + \frac{1}{c}\frac{\partial}{\partial t}\,\overset{2}{\omega}_{\boldsymbol{B}} = 0 \,, \tag{2.29b}$$

$$\mathrm{d}\,\overset{2}{\omega}_{\boldsymbol{D}} = 4\pi\,\overset{3}{\omega}_{\varrho} \,, \tag{2.29c}$$

$$\mathrm{d}\,\overset{1}{\omega}_{\boldsymbol{H}} - \frac{1}{c}\frac{\partial}{\partial t}\,\overset{2}{\omega}_{\boldsymbol{D}} = \frac{4\pi}{c}\,\overset{2}{\omega}_{\boldsymbol{j}} \,. \tag{2.29d}$$

The first of these equations follows from Example 2.2, making use of (2.23), while the second is obtained using (2.22) of the same example. Similar arguments apply to the case of equations (2.29c) and (2.29d). The first and the third equations relate exterior three-forms, the second and the fourth relate exterior two-forms.

The continuity equation expressed in terms of exterior forms reads

$$\frac{\partial}{\partial t}\,\overset{3}{\omega}_{\varrho} + \mathrm{d}\,\overset{2}{\omega}_{\boldsymbol{j}} = 0 \,, \qquad \Longrightarrow \qquad \frac{\partial \varrho(t,\boldsymbol{x})}{\partial t} + \nabla\cdot\boldsymbol{j}\,(t,\boldsymbol{x}) = 0 \tag{2.30}$$

where again the equation (2.23) was used. The second equation on the right follows by comparison of coefficients.

► **Remarks**

1. Contemplating the definitions (2.26a), (2.26b), (2.27a), and (2.27b), an important observation comes to mind: All exterior forms are *invariant* under rotations $\mathrm{R} \in \mathrm{SO}(3)$ of the frame of reference. In contrast to the

components of the original fields they do not depend on the choice of coordinate system. The covariance of the Maxwell equations (2.29a–2.29d) with respect to rotations is obvious and needs not be checked separately.

2. Their behaviour under space reflection Π introduced in Sect. 2.2.1, is also of interest. The one-form (2.26a) of the electric field and the two-form (2.26b) of the induction field are invariant under Π. By the same token the peculiar difference in the behaviour of the original vector fields $\boldsymbol{E}$ and $\boldsymbol{B}$ under Π is understood: The electric field is a genuine vector field and corresponds to a *one*-form, the induction field which has the "wrong" transformation behaviour under space reflection corresponds to a *two*-form.
3. The spacetime on which Maxwell's equations are formulated, is $\mathbb{R}^4$, i.e. an orientable manifold. As long as we study only proper rotations $\mathsf{R} \in \mathrm{SO}(3)$ the orientation is not changed and all four exterior forms corresponding to the fields remain unchanged. Space reflection reverses the orientation. In contrast to the one-form (2.26a) of the electric field, the one-form (2.27b) of the magnetic field changes its sign. A similar conclusion follows from the comparison of (2.26b) with (2.27a). Exterior forms of this kind are defined on nonorientable manifolds and are called *twisted forms*.
4. Although this cannot be the last word on this topic it is instructive to summarize the behaviour of the exterior forms that are related by (2.29a–2.29d), and by (2.30), under the three discrete transformations Π, T, and C. Table 2.1 is based on the analysis of Sect. 2.2.1.
5. This analysis as a first attempt of a geometric interpretation of the Maxwell fields remains unsatisfactory because the fields depend not only on $\boldsymbol{x} \in \mathbb{R}^3$, but also on the time coordinate $t \in \mathbb{R}_t$, hence, are defined over a *four*-dimensional manifold. In the following section we will quit the fixed frame of reference that was assumed here, and will prove the covariance of Maxwell's equations under Lorentz transformations. This will lead quite naturally to generalizing these definitions in such a way that the fields and the source terms become exterior forms over Minkowski space.
6. If like in Sect. 1.6.3 one wishes to describe the electric field and the magnetic induction by means of potentials – still within a fixed division of spacetime into coordinate space and time – it is useful to define the one-form

$$\overset{1}{\omega}_{\boldsymbol{A}} := \sum_{i=1}^{3} A_i(t, \boldsymbol{x})\, \mathrm{d}x^i \tag{2.31}$$

whose coefficients are the components of the vector potential $\boldsymbol{A}(t, \boldsymbol{x})$. The scalar potential $\Phi(t, \boldsymbol{x})$ is a function and can be interpreted as a zero-form over the space $\mathbb{R}^3$. The representations (1.55a) and (1.55b) of the induction

Table 2.1 Behaviour of the electromagnetic exterior forms under the three discrete transformations. Note, however, that the behaviour under T and, hence, under the product ΠTC will be modified to some extent when these forms are defined over space *and* time

	Π	T	C	ΠTC
$\overset{1}{\omega}_{\boldsymbol{E}}$	$+$	$+$	$-$	$-$
$\overset{2}{\omega}_{\boldsymbol{B}}$	$+$	$-$	$-$	$+$
$\overset{1}{\omega}_{\boldsymbol{H}}$	$-$	$-$	$-$	$-$
$\overset{2}{\omega}_{\boldsymbol{D}}$	$-$	$+$	$-$	$+$
$\overset{3}{\omega}_{\varrho}$	$-$	$+$	$-$	$+$
$\overset{2}{\omega}_{\boldsymbol{j}}$	$-$	$-$	$-$	$-$

field and the electric field, respectively, written by means of exterior forms, become

$$\overset{2}{\omega}_{\boldsymbol{B}} = \mathrm{d}\,\overset{1}{\omega}_{\boldsymbol{A}}\,, \tag{2.32a}$$

$$\overset{1}{\omega}_{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial}{\partial t}\,\overset{1}{\omega}_{\boldsymbol{A}} - \mathrm{d}\Phi\,. \tag{2.32b}$$

Taking the exterior derivative of the first of these, one obtains

$$\mathrm{d}\,\overset{2}{\omega}_{\boldsymbol{B}} = \mathrm{d}\big(\mathrm{d}\,\overset{1}{\omega}_{\boldsymbol{A}}\big) = 0\,, \quad \text{or} \quad \nabla\cdot\boldsymbol{B}(t,\boldsymbol{x}) = 0\,.$$

Here use was made of the property (2.15d) of the exterior derivative. This repeats the well-known fact that an induction field that can be represented by a vector potential has divergence zero automatically. The exterior derivative of the second equation (2.32b), in turn, yields the conclusion that the curl of $\boldsymbol{E}$ is related to the time derivative of $\boldsymbol{A}$,

$$\mathrm{d}\,\overset{1}{\omega}_{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial}{\partial t}\mathrm{d}\,\overset{1}{\omega}_{\boldsymbol{A}}\,, \quad \text{or} \quad \nabla\times\boldsymbol{E}(t,\boldsymbol{x}) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\boldsymbol{A}(t,\boldsymbol{x})\,.$$

If the vector potential is independent of time the electric field is irrotational.

This representation has a further unsatisfactory feature: Although to the vector potential $\boldsymbol{A}$ it associates a *one*-form over $\mathbb{R}^3$, the scalar potential is described by a *zero*-form. Furthermore, no real use is made of the time dependence of these quantities.

Why is there this asymmetry between space and time?

2.3 Lorentz Covariance of Maxwell's Equations

Space and time translations,

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{a} , \quad t' = t + s ,$$

rotations ($t' = t$, $\boldsymbol{x}' = \mathsf{R}\boldsymbol{x}$) in $\mathbb{R}^3$, space reflection diag $(1, -1, -1, -1)$, and time reversal diag$(-1, 1, 1, 1)$ have the same effect in the Galilei group and in the Poincaré group. Only the Special Galilei transformations

$$\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix} \longmapsto \begin{pmatrix} t' = t \\ \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t \end{pmatrix} , \quad \text{or} \quad \begin{pmatrix} t' \\ |\boldsymbol{x}'\rangle \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{0} \\ |\boldsymbol{v}\rangle & \mathbb{1}_3 \end{pmatrix} \begin{pmatrix} t \\ |\boldsymbol{x}\rangle \end{pmatrix} \tag{2.33}$$

differ in an essential way from the Special Lorentz transformations (also called *boosts*)

$$\begin{pmatrix} x^0 \\ |\boldsymbol{x}\rangle \end{pmatrix} \longmapsto \begin{pmatrix} x'^0 \\ |\boldsymbol{x}'\rangle \end{pmatrix} = \begin{pmatrix} \gamma & \frac{1}{c}\gamma\langle\boldsymbol{v}| \\ \frac{1}{c}\gamma|\boldsymbol{v}\rangle & \mathbb{1}_3 + \frac{\gamma^2}{c^2(\gamma+1)}|\boldsymbol{v}\rangle\langle\boldsymbol{v}| \end{pmatrix} \begin{pmatrix} x^0 \\ |\boldsymbol{x}\rangle \end{pmatrix} , \tag{2.34}$$

where the time variable is replaced by the equivalent length $x^0 = ct$. A Galilei transformation of this class, taken along the 1-axis $|\boldsymbol{v}\rangle = v|\hat{\boldsymbol{e}}_1\rangle$, for example, reads

$$t' = t , \quad x'^1 = x^1 + vt , \quad x'^2 = x^2 , \quad x'^3 = x^3 ,$$

or, in a somewhat different notation, using $x^0 = ct$ and $\beta = v/c$,

$$\begin{aligned} x'^0 &= x^0 , & x'^1 &= \beta x^0 + x^1 , \\ x'^2 &= x^2 , & x'^3 &= x^3 , \end{aligned}$$

while in the case of the Lorentz group one has, instead,

$$\begin{aligned} x'^0 &= \gamma x^0 + \gamma\beta x^1 , & x'^1 &= \gamma\beta x^0 + \gamma x^1 , \\ x'^2 &= x^2 , & x'^3 &= x^3 , \end{aligned}$$

with $\beta = |\boldsymbol{v}|/c$ and $\gamma = (1 - \beta^2)^{-1/2}$.

Under a special Lorentz transformation which relates two observers moving with constant velocity $\boldsymbol{v}$ relative to each other, neither the electric field $\boldsymbol{E}$ by itself nor the induction field $\boldsymbol{B}$ by itself, can have a simple transformation behaviour. This can be seen in different ways. A first, intuitive argument is the following:

The argument relates to the Biot–Savart law (1.18) and starts from the model of a single point charge q assumed to be moving with constant velocity $\boldsymbol{v}$ relative to an inertial observer. In his frame of reference K the observer sees the particle moving along a straight line with constant velocity, first approaching and then flying off, so that the strength of the particle's Coulomb field increases and decreases in the course of time. In addition, he perceives the particle which passes by as an electric current density $\boldsymbol{j}(t, \boldsymbol{x}) = q\,\delta\big(\boldsymbol{x} - (\boldsymbol{v}t + \boldsymbol{x}_0)\big)$ which according to (1.18) creates an $\boldsymbol{H}$-field – and, as he is in vacuum, an induction field $\boldsymbol{B} \equiv \boldsymbol{H}$ – which is time and space dependent. Another inertial observer who travels along with the particle sees something radically different: In his frame of reference K' the particle is at rest and creates the spherically symmetric electric field $\boldsymbol{E}' = q\,\hat{\boldsymbol{r}}/r^2$. Since there is no electric current there is also no magnetic or induction field, $\boldsymbol{H}' = \boldsymbol{B}' = 0$. As only *relative* motion is relevant, the two observers are equivalent and the Maxwell equations should have the same physical interpretation in the two frames of reference. Independently of whether the relative velocity is small as compared to, or close to the speed of light, the special Lorentz transformation $\mathsf{L}(\boldsymbol{v}) : (\boldsymbol{E}(t, \boldsymbol{x}), \boldsymbol{B}(t, \boldsymbol{x})) \mapsto (\boldsymbol{E}'(t', \boldsymbol{x}'), \boldsymbol{B}'(t', \boldsymbol{x}'))$ must mix the two types of fields.

A more rigourous analytical argument starts from the Lorentz force (1.44e) and is worked out in more detail in Sect. 2.3.4 below. It leads straightforwardly to the correct transformation behaviour: Maxwell's equations are found to be covariant under the Lorentz group.

We start with a summary of the most important properties of the Poincaré and the Lorentz groups, but refer to [ME] for a more detailed exposition.

2.3.1 Poincaré and Lorentz Groups

A Poincaré transformation is a general affine transformation of the coordinates x of spacetime $M = \mathbb{R}^4$, as well as of tangent vectors[2] $v \in T_x M$,

$$(\Lambda, a) : x \longmapsto x' = \Lambda x + a\,, \quad y \longmapsto y' = \Lambda y + a\,, \tag{2.35a}$$

which leave the generalized (squared) distance

$$(x - y)^2 = \big(x^0 - y^0\big)^2 - \big(\boldsymbol{x} - \boldsymbol{y}\big)^2 \tag{2.35b}$$

invariant. Here $(x^0 = ct, \boldsymbol{x})$ is the decomposition of x into temporal and spatial parts, respectively, in a given system of reference K which may, but need not be an inertial system. By convention the components are labelled by *Greek* indices whenever one deals with all four of them, for time and for space, and by *Latin* indices if only the spatial components are concerned,

$$x = \{x^\mu \,|\, \mu = 0, 1, 2, 3\} = \big(x^0, \{x^i \,|\, i = 1, 2, 3\}\big)^T = \big(x^0, \boldsymbol{x}\big)^T .$$

When it is equal to zero the invariant (2.35b) describes the causal relationship between emission of a light quantum at x, i.e. at time $t_x = x^0/c$ and position $\boldsymbol{x}$, and

[2] As the base manifold $\boldsymbol{M}$ is the flat space $\mathbb{R}^4$ all tangent spaces $T_x\mathbb{R}^4$ can be identified with this space. As a consequence, the points $x \in \boldsymbol{M}$ and the vectors $v \in T_x\boldsymbol{M}$ have the same transformation behaviour.

its detection at the world point y, i.e. at time $t_y = y^0/c$ and position $\boldsymbol{y}$. It expresses the experimentally confirmed *constancy of the speed of light:* In all inertial frames the speed of light has the universal value

$$c = 2.99792458 \cdot 10^8 \text{ m s}^{-1} . \tag{2.36}$$

The squared distance (2.35b) is invariant under Poincaré transformations, independently of whether it is zero (i.e. *lightlike*), positive (i.e. *timelike*), or negative (i.e. *spacelike*). A notation which is equivalent to (2.35b) makes use of the *metric tensor* $\mathrm{g} = \{g_{\mu\nu}\} = \mathrm{diag}(1, -1, -1, -1)$. It reads

$$(x-y)^2 = \sum_{\mu,\nu=0}^{3} (x^\mu - y^\mu) g_{\mu\nu} (x^\nu - y^\nu) \equiv (x^\mu - y^\mu) g_{\mu\nu} (x^\nu - y^\nu) , \tag{2.37}$$

where in the second step Einstein's summation convention was introduced which says that two equal indices, one of which is covariant, while the other is contravariant, are to be summed over from 0 to 3. Conventionally covariant indices are written as *lower* indices, contravariant indices are noted as *upper* ones.

Inserting the transformation (2.35a) into the formula (2.37), and requesting the equality $(x' - y')^2 = (x-y)^2$ for all inertial systems, the translation term a cancels out in the difference of x and y. There remains a condition on the homogeneous part of the transformation (2.35a), viz.

$$\Lambda^T \mathrm{g} \Lambda = \mathrm{g} . \tag{2.38a}$$

This equation is the essential condition for the Lorentz group from which all characteristic properties of Lorentz transformations are deduced. One should note the analogy to the rotation group in $\mathbb{R}^3$: The defining property of the rotation group O(3) in three-dimensional space with the metric $\mathrm{g}|_{\mathbb{R}^3} = \mathrm{diag}(1, 1, 1)$ is

$$\mathsf{R}^T \mathbb{1}_3 \mathsf{R} = \mathbb{1}_3 ,$$

from which one concludes that $(\det \mathsf{R})^2 = 1$ and that $\mathsf{R}^{-1} = \mathsf{R}^T$, i.e. that R is orthogonal.

Equation (2.38a), written out in components, leads to a number of consequences that we describe schematically as follows. With the notation $\Lambda = \{\Lambda^\mu{}_\nu\}$ and using the summation convention the equation (2.38a) reads more explicitly

$$\Lambda^\mu{}_\sigma g_{\mu\nu} \Lambda^\nu{}_\tau = g_{\sigma\tau} . \tag{2.38b}$$

Note that μ and ν are summation indices, while σ and τ assume fixed values on both sides of the equation. The relative position of indices in the left-hand factor of (2.38b) seems in conflict with the rules of matrix multiplication but, in fact, is correct because it is the transpose of Λ which appears here.

Depending on the values of the fixed indices σ and τ, equation (2.38b) yields for

$$\sigma = 0\,, \tau = 0: \quad \left(\Lambda^0{}_0\right)^2 - \sum_{j=1}^{3} \left(\Lambda^j{}_0\right)^2 = 1\,, \tag{2.38c}$$

$$\sigma = i\,, \tau = k: \quad \Lambda^0{}_i \Lambda^0{}_k - \sum_{j=1}^{3} \Lambda^j{}_i \Lambda^j{}_k = -\delta_{ik}\,, \tag{2.38d}$$

$$\sigma = 0\,, \tau = k: \quad \Lambda^0{}_0 \Lambda^0{}_k - \sum_{j=1}^{3} \Lambda^j{}_0 \Lambda^j{}_k = 0\,. \tag{2.38e}$$

One concludes from the first of these the alternatives that

$$\text{either} \quad \text{(a):}\ \Lambda^0{}_0 \geqslant +1\,, \quad \text{or} \quad \text{(b):}\ \Lambda^0{}_0 \leqslant -1\,. \tag{2.39a}$$

Lorentz transformations which have the property (a) map the time coordinate forward, i.e. into the future. They are called *orthochronous*. Calculation of the determinant of the two sides of (2.38a), remembering that Λ is real, yields

$$\left(\det \Lambda\right)^2 = 1\,, \quad \text{hence, either (c):}\ \det \Lambda = +1 \quad \text{or (d):}\ \det \Lambda = -1\,. \tag{2.39b}$$

Thus, the four possible combinations of the properties (a) to (d) show that the Lorentz group has four disjoint branches. These are denoted by $\pm$ for the sign of the determinant, and by an arrow which points upward if $\Lambda^0{}_0$ is larger than or equal to $+1$, downward if $\Lambda^0{}_0$ is smaller than or equal to -1. The branch $L_+^\uparrow$, called the *proper, orthochronous Lorentz group*, contains all elements Λ with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geqslant +1$. As one easily verifies, this is a subgroup of the Lorentz group: It contains the identity $\mathbb{1}_4$; the product of two transformations $\Lambda_1, \Lambda_2 \in L_+^\uparrow$ is an element of $L_+^\uparrow$, and so is the inverse Λ^{-1} of every element $\Lambda \in L_+^\uparrow$.

Space reflection $\Lambda = \Pi = \mathrm{diag}(1, -1, -1, -1)$ has $\det \Lambda = -1$, and $\Lambda^0{}_0 = +1$, hence, is an element of the branch $L_-^\uparrow$. Time reversal $\mathsf{T} = \mathrm{diag}(-1, 1, 1, 1)$ belongs to $L_-^\downarrow$, while the product $\Pi\mathsf{T}$ of space reflection and time reversal has determinant $+1$, but $\Lambda^0{}_0 = -1$, hence, belongs to the branch $L_+^\downarrow$. As an essential lesson from this analysis we note that one knows the entire Lorentz group once one understands its subgroup $L_+^\uparrow$, the proper orthochronous Lorentz group. Indeed, every element of $L_-^\uparrow$ can be written as the product of an element of $L_+^\uparrow$ with Π, every element of $L_-^\downarrow$ as the product of an element of $L_+^\uparrow$ with T, and every element of $L_+^\downarrow$ as the product of an element of $L_+^\uparrow$ with $\Pi\mathsf{T}$.

The key to the proper orthochronous Lorentz group is provided by the *decomposition theorem,* which asserts that every element of $L_+^\uparrow$ can be written, in a unique manner, as the product of a rotation

$$\mathcal{R} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathsf{R} \end{pmatrix} , \qquad \text{with} \quad \mathsf{R} \in \mathrm{SO}(3) ,$$

and a special Lorentz transformation, cf. (2.34). Thus, one has

$$\Lambda = \mathsf{L}(\boldsymbol{v})\,\mathcal{R} , \qquad \Lambda \in L_+^\uparrow . \tag{2.40a}$$

The entries of the 4×4-matrix $\mathsf{L}(\boldsymbol{v})$ are determined by the velocity

$$\boldsymbol{v} = \frac{c}{\Lambda^0{}_0}\left(\Lambda^1{}_0, \Lambda^2{}_0, \Lambda^3{}_0\right)^T , \tag{2.40b}$$

hence, by the entries $\Lambda^\mu{}_0$ of the given transformation Λ. The entries of the orthogonal 3×3-matrix R are calculated from the formulae

$$R^{ik} = \Lambda^i{}_k - \frac{\Lambda^i{}_0 \Lambda^0{}_k}{1 + \Lambda^0{}_0} . \tag{2.40c}$$

For a proof of this important theorem see, for example [ME], Sect. 4.5.1.

2.3.2 Relativistic Kinematics and Dynamics

To the best of our knowledge, all *charged* particles of nature are massive particles. Physical trajectories on which such a particle moves at a velocity smaller than c, are described by *world lines* $x(\tau)$ whose tangent vector field is everywhere *time*like. The Lorentz invariant parameter τ denotes proper time, i.e. the time that an observer who travels with the particle reads on his clock. The trajectory is described by the function $x(\tau)$ in a way independent of any choice of specific coordinates. The corresponding velocity is characterized by a four-vector which is given by, in a coordinate-free way,

$$u(\tau) := \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau) . \tag{2.41}$$

Without loss of generality, the invariant square of u can be normalized such that $u^2 = c^2$.

In the rest system the proper time coincides with the coordinate time of K_0 and one has $\mathrm{d}\tau = \mathrm{d}t$. With respect to a moving system K the line element is $(\mathrm{d}s)^2 = c^2(\mathrm{d}\tau)^2 = c^2(\mathrm{d}t)^2 - (\mathrm{d}\boldsymbol{x})^2$ so that

$$(\mathrm{d}\tau)^2 = (\mathrm{d}t)^2 - \tfrac{1}{c^2}(\mathrm{d}\boldsymbol{x})^2 = (1 - \beta^2)(\mathrm{d}t)^2 = (\mathrm{d}t)^2/\gamma^2 .$$

In the momentary rest system K_0 of the particle (whose existence is guaranteed when $m \neq 0$) the velocity four-vector is

$$u(\tau)|_{\mathsf{K}_0} = \left(c, \mathbf{0}\right)^T , \tag{2.41a}$$

while in the "laboratory" system K of another observer relative to whom the particle is moving, it is

$$u(\tau)|_{\mathsf{K}} = \left(\gamma c, \gamma \boldsymbol{v}\right)^T \,. \tag{2.41b}$$

The relativistic variant of the momentum is the four-vector $p := mu$. It comprises the spatial momentum $\boldsymbol{p} = m\gamma\boldsymbol{v}$ and the corresponding energy (divided by c) $p^0 = mc\gamma = E_p/c$. In the frame of reference K of the observer, i.e. in the laboratory system, one has

$$p|_{\mathsf{K}} = \left(\tfrac{1}{c}E, \boldsymbol{p}\right)^T \,, \text{ with } E = \gamma mc^2 = \sqrt{(\boldsymbol{p}c)^2 + (mc^2)^2}\,, \quad \boldsymbol{p} = m\gamma\boldsymbol{v}\,, \tag{2.42a}$$

while in the momentary rest system of the particle one has, of course,

$$p|_{\mathsf{K}_0} = \left(mc, \boldsymbol{0}\right)^T \,, \quad E|_{\mathsf{K}_0} = mc^2\,, \quad \boldsymbol{p}|_{\mathsf{K}_0} = \boldsymbol{0}\,. \tag{2.42b}$$

One easily verifies that (2.41b) follows from (2.41a) by the action of the special Lorentz transformation $\mathsf{L}(\boldsymbol{v})$, cf. (2.34). In a similar way one verifies that

$$p|_{\mathsf{K}} = \mathsf{L}(\boldsymbol{p})\; p|_{\mathsf{K}_0}\,,$$

$$\text{with} \quad \mathsf{L}(\boldsymbol{p}) = \frac{1}{mc^2}\begin{pmatrix} E & c\langle\boldsymbol{p}| \\ c|\boldsymbol{p}\rangle & mc^2\,\mathbb{1}_3 + \frac{c^2}{(E+mc^2)}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \end{pmatrix}$$

where L was converted from a parametrization in terms of the velocity $\boldsymbol{v}$ to a parametrization in terms of the spatial momentum $\boldsymbol{p}$, using the relations (2.42a). Indeed, one has

$$\begin{pmatrix} E & c\langle\boldsymbol{p}| \\ c|\boldsymbol{p}\rangle & mc^2\,\mathbb{1}_3 + \frac{c^2}{(E+mc^2)}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \end{pmatrix}\begin{pmatrix} mc \\ |\boldsymbol{0}\rangle \end{pmatrix} = mc^2\begin{pmatrix} (\frac{E}{c}) \\ |\boldsymbol{p}\rangle \end{pmatrix}.$$

The relativistic, Lorentz covariant version of Newton's second law reads

$$m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x(\tau) = f(x)\,. \tag{2.43}$$

It is obtained from the usual nonrelativistic formula $m\ddot{\boldsymbol{x}} = \boldsymbol{F}_{\mathrm{N}}(\boldsymbol{x})$ in which $\boldsymbol{F}_{\mathrm{N}}$ denotes the force field of Newtonian mechanics, by a "boost" from the rest system K_0. Thus, in the rest system

$$m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x(\tau)\bigg|_{\mathsf{K}_0} = m(0, \ddot{\boldsymbol{x}})^T \quad \text{and} \quad f|_{\mathsf{K}_0} = (0, \boldsymbol{F}_{\mathrm{N}})^T$$

must hold. The action of the special Lorentz transformation $\mathsf{L}(\boldsymbol{v})$ on these two four-vectors,

$$\begin{pmatrix} f^0 \\ |\boldsymbol{f}\rangle \end{pmatrix} = \begin{pmatrix} \gamma & \frac{1}{c}\gamma\langle\boldsymbol{v}| \\ \frac{1}{c}\gamma|\boldsymbol{v}\rangle & \mathbb{1}_3 + \frac{\gamma^2}{c^2(\gamma+1)}|\boldsymbol{v}\rangle\langle\boldsymbol{v}| \end{pmatrix}\begin{pmatrix} 0 \\ |\boldsymbol{F}_{\mathrm{N}}\rangle \end{pmatrix}, \tag{2.44}$$

yields the individual components of f, using the notation $\langle \boldsymbol{a}|\boldsymbol{c}\rangle \equiv \boldsymbol{a}\cdot\boldsymbol{c}$, as follows

$$f^0 = \frac{1}{c}\gamma\left(\boldsymbol{v}\cdot\boldsymbol{F}_{\mathrm{N}}\right), \tag{2.44a}$$

$$|\boldsymbol{f}\rangle = \boldsymbol{F}_{\mathrm{N}} + \frac{\gamma^2}{c^2(\gamma+1)}\left(\boldsymbol{v}\cdot\boldsymbol{F}_{\mathrm{N}}\right)|\boldsymbol{v}\rangle \ . \tag{2.44b}$$

These expressions can be converted to another form which is instructive: Take the scalar product of (2.44b) with $\boldsymbol{v}$ and use the relation

$$\frac{\boldsymbol{v}^2}{c^2} = \beta^2 = \frac{\gamma^2-1}{\gamma^2} = (\gamma-1)\frac{\gamma+1}{\gamma^2} \ ,$$

to obtain

$$(\boldsymbol{v}\cdot\boldsymbol{f}) = \left\{1 + \frac{\gamma^2}{\gamma+1}\beta^2\right\}\left(\boldsymbol{v}\cdot\boldsymbol{F}_{\mathrm{N}}\right) = \gamma\left(\boldsymbol{v}\cdot\boldsymbol{F}_{\mathrm{N}}\right) .$$

Thus, the zero-component of (2.44a) can be written alternatively $f^0 = (1/c)(\boldsymbol{v}\cdot\boldsymbol{f})$. Regarding the space component (2.44b), make use once more of $\beta^2 = (\gamma^2-1)/\gamma^2$, and insert the identity

$$(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v} = \boldsymbol{v}^2\boldsymbol{a} + \boldsymbol{v}\times\left(\boldsymbol{v}\times\boldsymbol{a}\right)$$

to obtain

$$\begin{aligned}
\boldsymbol{f} &= \boldsymbol{F}_{\mathrm{N}} + \frac{\gamma^2}{\gamma+1}\left\{\frac{1}{c^2}\boldsymbol{v}\times\left(\boldsymbol{v}\times\boldsymbol{F}_{\mathrm{N}}\right) + \beta^2\boldsymbol{F}_{\mathrm{N}}\right\} \\
&= \boldsymbol{F}_{\mathrm{N}}(1+\gamma-1) + \frac{\gamma}{c}\boldsymbol{v}\times\left(\frac{\gamma}{c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{F}_{\mathrm{N}}\right)\right) \\
&= \gamma\left[\boldsymbol{F}_{\mathrm{N}} + \frac{1}{c}\,\boldsymbol{v}\times\left(\frac{\gamma}{c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{F}_{\mathrm{N}}\right)\right)\right] .
\end{aligned} \tag{2.44c}$$

The spatial part of the left-hand side of the equation of motion (2.43), when expressed in terms of the time derivative of the spatial momentum, is equal to $\gamma\,\mathrm{d}\boldsymbol{p}/\,\mathrm{d}t$, so that the equation of motion divided by γ reads

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}_{\mathrm{N}} + \frac{1}{c}\,\boldsymbol{v}\times\left(\frac{\gamma}{c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{F}_{\mathrm{N}}\right)\right). \tag{2.43a}$$

The time component satisfies the differential equation

$$mc\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{c}\left(\boldsymbol{F}_{\mathrm{N}}\cdot\boldsymbol{v}\right) . \tag{2.43b}$$

(It is easily verified that (2.43b) follows from (2.43a).) The first of these equations shows a striking similarity to the equation of motion of a charged particle under the action of the Lorentz force, with $\boldsymbol{F}_\mathrm{N}$ taking the role of $q\boldsymbol{E}$, and

$$\text{“}q\boldsymbol{B}\text{”} \equiv \frac{\gamma}{c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{F}_\mathrm{N}\right) \equiv \frac{q\gamma}{c(\gamma+1)}\left(\boldsymbol{v}\times\boldsymbol{E}\right)$$

appearing in the magnetic term of the force.

2.3.3 Lorentz Force and Field Strength

The Lorentz force with its characteristic dependence on the velocity

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\Big(\boldsymbol{E}(t,\boldsymbol{x}) + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}(t,\boldsymbol{x})\Big) \tag{2.45}$$

can be cast into the form of the equation of motion (2.43). The space part is obtained by multiplication of (2.45) with a factor $\gamma = 1/\sqrt{1-\boldsymbol{v}^2/c^2}$, the temporal part is obtained from the scalar product of (2.45) with the vector $(\gamma/c)\boldsymbol{v}$:

$$m\gamma\frac{\mathrm{d}}{\mathrm{d}t}(\gamma c) = \gamma\frac{1}{c}\,q\boldsymbol{E}\cdot\boldsymbol{v}\,, \tag{2.45a}$$

$$m\gamma\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\boldsymbol{v}) = \gamma\Big(q\boldsymbol{E}(t,\boldsymbol{x}) + \frac{q}{c}\,\boldsymbol{v}\times\boldsymbol{B}(t,\boldsymbol{x})\Big)\,. \tag{2.45b}$$

One sees again the analogy between the differential equations (2.45b) and (2.43a), as well as between the equations (2.45a) and (2.43b). The left-hand sides of (2.45a) and of (2.45b), written covariantly, are $m(\mathrm{d}u^\mu/\mathrm{d}\tau)$; their right-hand sides can be expressed in terms of the four-velocity u as follows. In the frame of reference K define

$$F^{\mu\nu}(x) := \begin{pmatrix} 0 & -E^1(x) & -E^2(x) & -E^3(x) \\ +E^1(x) & 0 & -B^3(x) & +B^2(x) \\ +E^2(x) & +B^3(x) & 0 & -B^1(x) \\ +E^3(x) & -B^2(x) & +B^1(x) & 0 \end{pmatrix}, \quad x = (t,\boldsymbol{x})^T\,, \tag{2.46}$$

and let this field act on $u_\nu = g_{\nu\sigma}u^\sigma = (\gamma c, -\gamma\boldsymbol{v})^T$. Using the summation convention one has

$$F^{\mu\nu}u_\nu = \begin{pmatrix} 0 & -E^1(x) & -E^2(x) & -E^3(x) \\ +E^1(x) & 0 & -B^3(x) & +B^2(x) \\ +E^2(x) & +B^3(x) & 0 & -B^1(x) \\ +E^3(x) & -B^2(x) & +B^1(x) & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ -\gamma v^1 \\ -\gamma v^2 \\ -\gamma v^3 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(\boldsymbol{E}(x)\cdot\boldsymbol{v}) \\ \gamma c E^1(x) + \gamma(v^2B^3(x) - v^3B^2(x)) \\ \gamma c E^2(x) + \gamma(v^3B^1(x) - v^1B^3(x)) \\ \gamma c E^3(x) + \gamma(v^1B^2(x) - v^2B^1(x)) \end{pmatrix}.$$

The equation of motion in its general form (2.43) appears here in the specific form

$$m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{q}{c}\,F^{\mu\nu}u_\nu\ . \tag{2.47}$$

In the frame K it is seen to be identical with the differential equations (2.45a) and (2.45b).

This observation raises an important question:

Does the combination of the electric field and the magnetic induction by the definition (2.46) *have a deeper and more general significance than just to reformulate the Lorentz force* (2.45) *in a compact form with respect to the special frame of reference* K*?*

In other terms the question we are asking is the following: In another frame K' which differs from K by a Lorentz transformation (which is to say that with K also K' is an inertial system) the Lorentz force can be expressed in the same compact form, i.e. as $F'^{\mu\nu}u'_\nu$. Are the fields $F'^{\mu\nu}$ and $F^{\mu\nu}$ also related by Lorentz transformations? More precisely, is it true that with

$$u' = \Lambda u \quad \text{also} \quad F' = \Lambda\, F\, \Lambda^T \text{ holds true?}$$

or, written in components,

$$u'^\sigma = \Lambda^\sigma{}_\mu u^\mu\ , \quad F'^{\sigma\tau}(x') = \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu F^{\mu\nu}(x)\ ?$$

If this were so then the equation of motion (2.47) would be Lorentz covariant. Its right-hand side $F^{\mu\nu}u_\nu$, summed over ν, is a Lorentz vector and thus transforms with Λ like its left-hand side. The equation of motion has the same form in every inertial system. The question that is raised then narrows down to the question:

Are the Maxwell equations covariant with regard to the transformations $\Lambda \in L_+^\uparrow$ *as suggested by the special form of the Lorentz force?*

The analysis of this question is the subject of the next Section. Before turning to it let us collect the inverse formulae that express the electric and induction fields in terms of $F^{\mu\nu}$. These are

$$E^i = F^{i0} = -F^{0i}\ , \qquad (i = 1, 2, 3)\ , \tag{2.48a}$$

$$B^i = -\frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}F^{jk}\ , \qquad (i = 1, 2, 3)\ . \tag{2.48b}$$

The object $F^{\mu\nu}(x)$ will turn out to be a tensor field, the *tensor field of electromagnetic field strengths.* It is called, somewhat shorter, *field strength tensor*[3]. Its definition (2.46) shows that this tensor is antisymmetric. In fact, one could have

[3] Its geometric role will be clarified in Chap. 5 below.

deduced this property directly from the equation of motion (2.47): Because of the property $u^2 = c^2 = \text{const.}$ one has

$$\frac{1}{2}\frac{\mathrm{d}u^2}{\mathrm{d}\tau} = u_\mu \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \,.$$

Therefore, by contraction of (2.47) with u_μ, one concludes that for all x

$$u_\mu F^{\mu\nu}(x) u_\nu = 0 \,.$$

This can only be correct if $F^{\nu\mu}(x) = -F^{\mu\nu}(x)$: The tensor $u_\mu u_\nu$ which is *symmetric* in μ and ν, when contracted with the *antisymmetric* tensor $F^{\mu\nu}$, gives zero. Conversely, if $F^{\mu\nu}$ had a symmetric term this would not give zero upon contraction with $u_\mu u_\nu$.

2.3.4 Covariance of Maxwell's Equations

The homogeneous Maxwell equations (2.1a) and (2.1b) are easily expressed in terms of the tensor field $F^{\mu\nu}(x)$. We show that they read as follows

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \,, \quad \text{with } \lambda \neq \mu \neq \nu \in (0,1,2,3) \,. \tag{2.49}$$

An alternative notation is obtained if one introduces the Levi-Civita symbol in dimension 4 whose properties are:

$$\begin{aligned} \varepsilon_{0123} &= +1 \\ \varepsilon_{\mu\nu\sigma\tau} &= +1 \quad \text{for } (\mu,\nu,\sigma,\tau) = \text{even permutation of } (0,1,2,3) \\ \varepsilon_{\mu\nu\sigma\tau} &= -1 \quad \text{for } (\mu,\nu,\sigma,\tau) = \text{odd permutation of } (0,1,2,3) \end{aligned} \tag{2.50}$$

while $\varepsilon_{\mu\nu\sigma\tau} = 0$ in all other cases, i.e. whenever two or more indices are equal. In terms of this totally antisymmetric symbol the equations (2.49) become

$$\varepsilon_{\mu\nu\sigma\tau}\partial^\nu F^{\sigma\tau}(x) = 0 \,, \quad (\mu = 0,1,2,3) \,. \tag{2.49a}$$

It is not difficult to verify that (2.49a) summarizes the four homogeneous Maxwell equations. In doing so one must recall that

$$\partial^0 = \partial_0 = \frac{\partial}{\partial x^0} \,, \quad \text{but} \quad \partial^i = -\partial_i = -\frac{\partial}{\partial x^i} = -(\nabla)_i \,.$$

Equation (2.49a) with $\mu = 0$ and with $\varepsilon_{0\nu\sigma\tau} \equiv \varepsilon_{0ijk} = \varepsilon_{ijk}$ (where in the last step one sees the usual ε-symbol in dimension 3) yields

$$\begin{aligned} 0 = \varepsilon_{0\nu\sigma\tau}\partial^\nu F^{\sigma\tau}(x) &= \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\partial^i F^{jk}(t,\boldsymbol{x}) \\ &= 2\left[\varepsilon_{123}\partial^1 F^{23} + \varepsilon_{231}\partial^2 F^{31} + \varepsilon_{312}\partial^3 F^{12}\right] = 2\nabla\cdot\boldsymbol{B}(t,\boldsymbol{x}) \,. \end{aligned}$$

This is the Maxwell equation (2.1a). If the first free index of (2.49a) is taken to be 1, one of the remaining indices ν, σ, and τ, must equal 0, while the other two must be 2 and 3, respectively. In this case (2.49a) yields

$$\begin{aligned} 0 &= \varepsilon_{1023}\partial^0 F^{23} + \varepsilon_{1230}\partial^2 F^{30} + \varepsilon_{1302}\partial^3 F^{02} \\ &= -1\Big(\partial_0(-B^1) + (-\partial_2)E^3 + (-\partial_3)(-E^2)\Big) \\ &= \frac{1}{c}\frac{\partial B^1}{\partial t} + \frac{\partial E^3}{\partial x^2} - \frac{\partial E^2}{\partial x^3} \,. \end{aligned}$$

Obviously, this is the 1-component of the homogeneous Maxwell equation (2.1b). The other two space components are obtained by cyclic permutation of the indices $(1,2,3)$.

The *inhomogeneous* equations (2.1c) and (2.1d) are more difficult to translate to a covariant form because they contain source terms which are not genuine elements of Maxwell theory but should follow from a theory of *matter*. We start with a heuristic remark:

The volume element in $\mathbb{R}^4$ is invariant under $\Lambda \in L_+^\uparrow$, $\mathrm{d}^4x' = \mathrm{d}^4x$, or, with respect to a given frame of reference K, $\mathrm{d}x'^0\,\mathrm{d}^3x' = \mathrm{d}x^0\,\mathrm{d}^3x$. If $\varrho(t,\boldsymbol{x})$ is the charge density in that frame then the charge element

$$\mathrm{d}q = \varrho(t,\boldsymbol{x})\,\mathrm{d}^3x = \varrho'(t',\boldsymbol{x}')\,\mathrm{d}^3x' \,,$$

by its very nature is a (physical) invariant. This suggests that under Lorentz transformations the charge density should transform like the time component of a four-vector. This, in turn, is so if the charge density and the current density together build up a Lorentz four-vector, i.e. if

$$j(x) = \big(c\varrho(x), \boldsymbol{j}(x)\big)^T \,, \quad x = \big(x^0, \boldsymbol{x}\big)^T \,, \tag{2.51}$$

is a vector field transforming like a Lorentz vector. As we anticipated in Sect. 1.4.5 the continuity equation then has the compact and Lorentz invariant form (1.24b), $\partial_\mu j^\mu(x) = 0$. Of course, this is a question whose answer must be found outside of Maxwell theory proper. Charged matter which provides the sources of Maxwell's equations, must be described as well by a Lorentz covariant theory and must allow for a four-vector current $j(x)$ which is conserved. This is not obvious!

In what follows let us assume that this is valid and that the charged matter particles, i.e. electrons, atomic nuclei, ions, which compose macroscopic matter, obey a Lorentz covariant theory.

It is suggestive to combine the dielectric displacement field $\boldsymbol{D}$ and the magnetic field $\boldsymbol{H}$ in a tensor field analogous to (2.46),

$$\mathcal{F}^{\mu\nu}(x) := \begin{pmatrix} 0 & -D^1(x) & -D^2(x) & -D^3(x) \\ +D^1(x) & 0 & -H^3(x) & +H^2(x) \\ +D^2(x) & +H^3(x) & 0 & -H^1(x) \\ +D^3(x) & -H^2(x) & +H^1(x) & 0 \end{pmatrix} , \tag{2.52a}$$

which is antisymmetric, too. The fields are expressed in terms of $\mathcal{F}^{\mu\nu}$ by formulae analogous to (2.48a) and to (2.48b),

$$D^i = \mathcal{F}^{i0} = -\mathcal{F}^{0i} , \qquad (i = 1, 2, 3) , \tag{2.52b}$$

$$H^i = -\frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}\mathcal{F}^{jk} , \qquad (i = 1, 2, 3) . \tag{2.52c}$$

The inhomogeneous Maxwell equations now take the compact form

$$\partial_\mu \mathcal{F}^{\mu\nu}(x) = \frac{4\pi}{c}\, j^\nu(x) , \quad (\nu = 0, 1, 2, 3) . \tag{2.53}$$

As in the case of the homogeneous equations let us verify this equation in more detail.

For $\nu = 0$ the first index of $\mathcal{F}$ can only take the values 1, 2, and 3, so that (2.53) reduces to

$$\sum_{i=1}^{3} \partial_i \mathcal{F}^{i0}(x) = \nabla \cdot \boldsymbol{D}(t, \boldsymbol{x}) = \frac{4\pi}{c}\, c\varrho(t, \boldsymbol{x}) = 4\pi\varrho(t, \boldsymbol{x}) .$$

Obviously, this is the same as (2.1c).
For a space index, for example $\nu = 1$, the equation (2.53) yields

$$\begin{aligned}
\frac{4\pi}{c} j^1(t, \boldsymbol{x}) &= \partial_0 \mathcal{F}^{01}(x) + \partial_2 \mathcal{F}^{21}(x) + \partial_3 \mathcal{F}^{31}(x) \\
&= -\partial_0 D^1(t, \boldsymbol{x}) + \partial_2 H^3(t, \boldsymbol{x}) - \partial_3 H^2(t, \boldsymbol{x}) \\
&= \left(-\frac{1}{c}\frac{\partial \boldsymbol{D}(t, \boldsymbol{x})}{\partial t} + \nabla \times \boldsymbol{H}(t, \boldsymbol{x})\right)^1 .
\end{aligned}$$

This is the 1-component of the differential equation (2.1d); the remaining two components follow by cyclic permutation of the space indices $(1, 2, 3)$.

Under the assumption discussed above, i.e. the current density $j(x)$ transforms like a Lorentz vector, the inhomogeneous equations (2.53) are manifestly covariant: Their left-hand sides as well as the right-hand sides transform like vectors under $\Lambda \in L_+^\uparrow$.

► **Remarks**

1. The contraction of the Levi-Civita symbol in dimension 4 with the field strength tensor that was used in the homogeneous equations (2.49a) rather naturally leads one to define another covariant tensor field of degree 2, the *dual field strength tensor field,*

$$\star\, F_{\alpha\beta}(x) := \tfrac{1}{2}\varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu}(x) . \tag{2.54a}$$

The corresponding contravariant tensor field is

$$\star\, F^{\sigma\tau}(x) = g^{\sigma\alpha}\big(\star F_{\alpha\beta}(x)\big)g^{\beta\tau}\ . \tag{2.54b}$$

It is not difficult to calculate $\star F_{\alpha\beta}$ and then $\star F^{\sigma\tau}$. The result (up to completion by virtue of its antisymmetry)

$$\star\, F^{\sigma\tau} = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ & 0 & -E^3 & E^2 \\ & & 0 & -E^1 \\ & & & 0 \end{pmatrix} \tag{2.54c}$$

is interesting because it shows that replacing $F^{\mu\nu}$ by $\star F^{\mu\nu}$ means an exchange of electric field and magnetic induction according to the rule

$$F^{\mu\nu} \longmapsto \star F^{\mu\nu}\ :\ (\boldsymbol{E}, \boldsymbol{B}) \longmapsto (-\boldsymbol{B}, \boldsymbol{E})\ . \tag{2.55}$$

In the vacuum where $\boldsymbol{D} = \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}$, with our choice of units, and in the absence of external sources, the replacement (2.55) is a symmetry of Maxwell's equations (2.1a–2.1d). This symmetry is called *electric-magnetic duality*. It interchanges (2.1a) with (2.1c), as well as (2.1b) with (2.1d). This duality is closely related to the Hodge duality that we studied in Sect. 2.2.2, (2.16). We will return to this in Sect. 2.5.1 below.

2. Of central importance for the covariance of Maxwell's equations was the postulate of the constancy of the speed of light, cf. Sect. 2.3.1. If this were not valid the Maxwell equations would single out a special class of frames of reference whose elements can differ only by translations and rotations but not by special Lorentz transformations. In the early era of electromagnetism this class of frames was called the "ether", its characteristic property being that Maxwell's equations hold in the form given above and that the speed of light has the value (2.36). As is well known, the experiments of A. A. Michelson and E. W. Morley disproved this hypothesis. No effects were found that would show any comotion of light with a frame moving uniformly relative to the hypothetical ether. The speed of light has the same universal value in all inertial systems.
3. The full system of Maxwell equations are Lorentz covariant if and only if the four-current density $j(x)$ is a four-vector field. As was emphasized previously this is a condition concerning the sources in (2.53). The reformulation of Maxwell's equations by means of the tensor fields $F^{\mu\nu}(x)$ and $\mathcal{F}^{\mu\nu}(x)$, and of the current density $j^{\mu}(x)$, in the differential equations (2.49a) and (2.53), renders the covariance explicit.This observation is termed *manifest Lorentz covariance*. In the equivalent differential equations (2.49a) and (2.53) covariance is not evident.

4. The conservation of the four-current density $j(x)$ now is manifest, too. Indeed, calculating the (four-)divergence of the inhomogeneous equations (2.53), one finds
$$\partial_\nu \partial_\mu \mathcal{F}^{\mu\nu}(x) = 0 \, .$$
The differential operator $\partial_\nu \partial_\mu$ being symmetric in μ and ν is contracted with the antisymmetric tensor field $\mathcal{F}^{\mu\nu}(x)$. This is compatible with (2.53) only if
$$\partial_\nu j^\nu(x) = \frac{\partial}{\partial t}\varrho(t, \boldsymbol{x}) + \nabla \cdot \boldsymbol{j}(t, \boldsymbol{x}) = 0 \tag{2.56}$$
holds. Thus, the continuity equation is essential for (2.53) to hold true. It guarantees the universal law of conservation of electric charge.

2.3.5 Gauge Invariance and Potentials

As anticipated in Sect. 1.6.3 the scalar potential $\Phi(t, \boldsymbol{x})$ and the vector potential $\boldsymbol{A}(t, \boldsymbol{x})$ can be combined in the definition
$$A(x) := \left(\Phi(t, \boldsymbol{x}), \boldsymbol{A}(t, \boldsymbol{x})\right)^T \, . \tag{2.57}$$
The representation of the electric field and the induction field by the potentials Φ and $\boldsymbol{A}$ is equivalent to the representation of the field strength tensor in terms of the four-potential (2.57)
$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \, . \tag{2.58}$$
It is not difficult to verify this assertion: For $\nu = 0$ we have
$$\begin{aligned} F^{i0}(x) &= E^i(t, \boldsymbol{x}) = \partial^i A^0(t, \boldsymbol{x}) - \partial^0 A^i(t, \boldsymbol{x}) \\ &= -\partial_i \Phi(t, \boldsymbol{x}) - \frac{1}{c}\partial_t A^i(t, \boldsymbol{x}) \, , \end{aligned}$$
in agreement with (1.55b). Considering a space-space component, say $\mu = 3$ and $\nu = 2$, one has
$$\begin{aligned} F^{32} &= B^1(t, \boldsymbol{x}) = \partial^3 A^2(t, \boldsymbol{x}) - \partial^2 A^3(t, \boldsymbol{x}) \\ &= -\partial_3 A^2(t, \boldsymbol{x}) + \partial_2 A^3(t, \boldsymbol{x}) = \left(\nabla \times \boldsymbol{A}(t, \boldsymbol{x})\right)^1 \, . \end{aligned}$$
The components $B^2(t, \boldsymbol{x})$ and $B^3(t, \boldsymbol{x})$ are obtained in an analogous manner, in agreement with the noncovariant representation (1.55a).

By definition, gauge transformations of potentials are space- and time-dependent transformations which do not change the observable fields. Solving for the scalar and vector potentials they have the somewhat complicated form (1.57a) and (1.57b). In a Lorentz covariant formulation their appearance is simpler, they read

$$A^{\mu}(x) \longmapsto A'^{\mu}(x) = A^{\mu}(x) - \partial^{\mu}\chi(x) \; . \tag{2.59}$$

The covariant derivative is related to the time derivative and the gradient in $\mathbb{R}^3$ by

$$\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \; .$$

One sees at once that (2.59) is identical to the equations (1.57a) and (1.57b). As $\chi(x)$ is a smooth function, its mixed second derivatives $\partial_{\mu}\partial_{\nu}\chi(x)$ and $\partial_{\nu}\partial_{\mu}\chi(x)$ are equal. Thus, they cancel in the difference on the right-hand side of (2.58) so that the tensor field $F^{\mu\nu}(x)$ stays invariant. For the same reason, the homogeneous equations (2.49a) are fulfilled automatically when using the definition (2.58). Thus, one has the choice: Either one works exclusively with *observables*, i.e. with the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and imposes the homogeneous equations (2.49a), or one expresses the field strength tensor in terms of potentials $A^{\mu}(x)$. In this case the homogenous equations are redundant.

The inhomogeneous equations in vacuum become

$$\Box A^{\nu}(x) - \partial^{\nu}\bigl(\partial_{\mu}A^{\mu}(x)\bigr) = \frac{4\pi}{c} j^{\nu}(x) \; . \tag{2.60}$$

As one would have expected, the first term of the left-hand side contains the differential operator $\Box = \partial_{\mu}\partial^{\mu} = (1/c^2)\partial_t^2 - \boldsymbol{\Delta}$ which is characteristic for the wave equation. The second term on the left-hand side depends on the choice of the gauge. The right-hand side, finally, is the source term. For example, if one uses the Lorenz gauge (2.61) below, then (2.60) is precisely the inhomogeneous wave equation.

► **Remarks**

1. As we just noted the covariance of Maxwell's equations is guaranteed only if $F^{\mu\nu}(x)$ is a contravariant tensor field of degree 2 and $j^{\mu}(x)$ is a contravariant vector field with respect to Lorentz transformations. The four-potential $A^{\mu}(x)$ *may be* a Lorentz vector field. However, it is always possible to hide the manifest Lorentz covariance without modifying the covariance of the Maxwell equations and of its physical content. For example, instead of the Lorentz invariant condition
$$\partial_{\mu}A^{\mu}(x) = 0 \tag{2.61}$$
(this is the Lorenz condition (1.58)), one may choose classes of noncovariant gauges. One may wish to impose, for example, the Coulomb gauge (1.63) and single out a special class of gauges by this choice. Lorentz covariance of

Maxwell's equations is then no longer manifestly visible, even though it is not lost. One has made use of the freedom in the choice of gauge, though hiding the covariance, for the purpose of emphasizing other properties of the theory. In the case of the Coulomb gauges, for instance, this was the transversality of electromagnetic waves. As no physical prediction of the theory is changed by gauge transformations, the theory is the same, no matter which formulation one has chosen and independently of the theory appearing in different disguises.

2. As will become clear in a later section the gauge freedom (2.59), in essence, means invariance of Maxwell theory under the group of local U(1)-transformations,

$$\mathcal{U}(1) :$$
$$\left\{g \in \mathcal{F}(\mathbb{R}^4)\ ,\ \text{smooth function} \,|\, g(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}, \alpha(x) \text{ smooth, real}\right\} .$$

Here the term *local* means that to every point $x \in \mathbb{R}^4$ of spacetime a copy of the group

$$\mathrm{U}(1) = \left\{g \in \mathbb{C} \,\big|\, |g|^2 = 1\ ,\ \text{i.e. } g = \mathrm{e}^{\mathrm{i}\alpha}, \alpha \in [0, 2\pi]\right\}$$

is attached. This gauge group (which is Abelian) acts on the potentials A^μ by

$$A'^{\mu}(x) = A^\mu(x) - \mathrm{i}g(x)\partial^\mu g^{-1}(x)\ , \tag{2.62}$$

but acts also on the source terms in Maxwell's equations. Equation (2.62) is a special case of a transformation for more general, non-Abelian groups that will be studied in Chap. 5. For the moment we just note that (2.62) with $\alpha(x) = \chi(x)$ reproduces the formula (2.59).

3. In case all charge and current densities are localized the continuity equation (2.56) implies that the time derivative of the integral of the charge density over the whole space vanishes,

$$\begin{aligned}\partial_0 \iiint \mathrm{d}^3x\ j^0(x) &= \iiint \mathrm{d}^3x\ \partial_0 j^0(t, \boldsymbol{x})\\ &= -\iiint \mathrm{d}^3x\ \partial_i j^i(t, \boldsymbol{x}) = 0\ .\end{aligned}$$

This follows because the space integral, by Gauss' theorem (1.6), can be converted to a surface integral over a surface at infinity where the current density vanishes by assumption. The total charge contained in space

$$Q := \iiint \mathrm{d}^3x\ j^0(t, \boldsymbol{x}) \tag{2.63}$$

is conserved. Although this is a beautiful and important result it seems to single out a certain class of Lorentz systems for which the division into space and time is fixed. Yet, the conservation law (2.63) of electric charge is Lorentz invariant. This becomes plausible by the following argument:

Let Σ_0 be the hypersurface $x^0 = \text{const.}$ in Minkowski space[4]. One says that this surface is *space*like and we understand by this that any two points on Σ_0 are spacelike relative to each other. This means that in each point $x \in \Sigma_0$ the positively oriented normal $n(x)$ (with respect to the time direction) to the surface is *time*like. In the special case of Σ_0 one has $n^\mu(x) = (1, \mathbf{0})^T$ for all x. In the general case, one has $n^2(x) \equiv n_\mu(x) n^\mu(x) = 1$ for every space-like hypersurface. To be spacelike is a property of a hypersurface which is invariant under all $\Lambda \in L_+^\uparrow$. Thus, the assertion that the normal n is timelike remains true for all inertial observers who differ from one another by proper, orthochronous Lorentz transformations.

It is not difficult to guess the four-dimensional variant of Gauss' theorem (1.6)

$$\iiiint\limits_V \mathrm{d}^4 x \; \partial^\mu F_\mu(x) = \iiint\limits_{\Delta(V)} \mathrm{d}\sigma^\mu \, F_\mu(x) \, . \tag{2.64}$$

In this formula $\Delta(V)$ is a piecewise smooth, closed hypersurface in Minkowski space, V denotes the (four-dimensional) volume enclosed by it, and $F^\mu(x)$ is a smooth vector field. The integration on the right-hand side contains $\mathrm{d}\sigma^\mu = n^\mu(x)\,\mathrm{d}\sigma$, where $n^\mu(x)$ is the positive normal to the surface, and $\mathrm{d}\sigma$ is the surface element on $\Delta(V)$.

The total charge (2.63) is given by the integral of the current density $j(x)$ over the space-like hypersurface Σ_0,

$$Q = \iiint \mathrm{d}^3 x \; j^0(x) = \iiint\limits_{\Sigma_0} \mathrm{d}\sigma^\mu \, j_\mu(x) \, , \tag{2.63a}$$

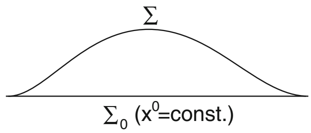
$$\left(\mathrm{d}\sigma^\mu = n^\mu \, \mathrm{d}\sigma \, , \; n^\mu = (1,0,0,0)^T \, , \; \mathrm{d}\sigma = \mathrm{d}^3 x\right) \, .$$

Consider now another space-like smooth hypersurface Σ which differs from Σ_0 only at finite distances in the way sketched in Fig. 2.2. Then the difference $\Sigma - \Sigma_0 =: \Delta(V)$ is a piecewise smooth, closed hypersurface which encloses a finite volume V. Gauss' theorem in the form of (2.64) applied to $\Delta(V)$ and V shows that the two integrals

$$Q' = \iiint\limits_{\Sigma} \mathrm{d}\sigma^\mu \, j_\mu(x) \quad \text{and} \quad Q = \iiint\limits_{\Sigma_0} \mathrm{d}\sigma^\mu \, j_\mu(x)$$

[4] Every smooth finite-dimensional surface that is embedded in a manifold with higher dimension is called a *hypersurface.*

Fig. 2.2 The three-dimensional hypersurface $x^0 = $ const. is deformed locally and continuously to the spacelike hypersurface Σ such that $\Sigma - \Sigma_0$ encloses a finite volume



differ only by the integral of $\partial^\mu j_\mu(x)$ over the volume V. This difference vanishes if and only if $j^\mu(x)$ is a conserved current. In this case the charge (2.63a) is independent of the choice of the space-like hypersurface Σ. Therefore, in spite of the apparent dependence on the splitting into space and time the definition (2.63) is Lorentz invariant.

2.4 Fields of a Uniformly Moving Point Charge

A special Lorentz transformation sort of "tumbles" electric and induction fields. While the transformation behaviour of the tensor field $F^{\mu\nu}(x)$ under $\mathsf{L}(\boldsymbol{v}) \in L_+^\uparrow$ is straightforward and transparent, this is not obviously so for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In order to clarify this matter and to work out its physical significance we study the example of a particle with charge q which moves at constant velocity $\boldsymbol{v}$ with respect to some inertial frame K (called a laboratory system, in short, in what follows). We calculate the fields $\boldsymbol{E}(t, \boldsymbol{x})$ and $\boldsymbol{B}(t, \boldsymbol{x})$, as measured by an observer who keeps a fixed position in the laboratory system.

Let the rest system of the particle be denoted by K'. It is chosen such that at $t = t' = 0$ it coincides with the laboratory system. The observer B is at rest relative to K, its spatial coordinates are $(0, b, 0)$; the charged particle sits at the origin of K', and, as seen from the laboratory system, moves with the constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{e}}_1$, i.e. along the 1-axis of K. This is sketched in Fig. 2.3. With $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$ the coordinates of B are

$$x'^0 = \gamma(x^0 - \beta x^1)\,, \qquad x'^2 = x^2\,,$$
$$x'^1 = \gamma(x^1 - \beta x^0)\,, \qquad x'^3 = x^3\,.$$

Inserting the laboratory coordinates $x^2 = b$ and $x^1 = 0 = x^3$ one has

$$B: \left(ct, 0, b, 0\right)\Big|_{(\text{rel. to }\mathsf{K})}\,,$$
$$B: \left(ct' = c\gamma t, x'^1 = -v\gamma t = -vt', x'^2 = b, x'^3 = 0\right)\Big|_{(\text{rel. to }\mathsf{K}')}\,.$$

The special Lorentz transformation which links K to K', and the formula which takes

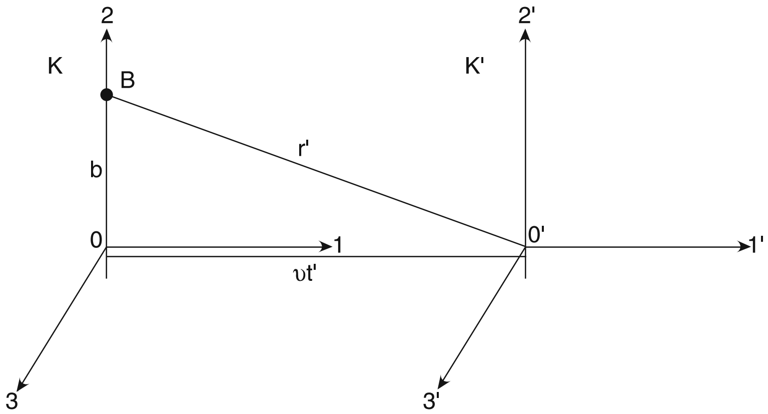


Fig. 2.3 An observer is assumed to be at rest relative to the inertial system K and has the position $(x^1 = 0, x^2 = b, x^3 = 0)$. He or she sees a charged particle moving at constant velocity which at the coordinate time $t = t' = 0$ passes through the origin of K

the field strength tensor from one system to the other are, respectively,

$$\mathsf{L}(-\boldsymbol{v}) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$F'^{\sigma\tau}(x') = \Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}\, F^{\mu\nu}(\Lambda^{-1}x') \,.$$

As we will see in a moment it suffices to calculate the time-space components only. With $\sigma = 0$ they are for the three values of τ, respectively,

$$\begin{aligned}
\tau = 1: \qquad & F'^{01} = -E'^{1} = \Lambda^{0}{}_{\mu}\Lambda^{1}{}_{\nu}F^{\mu\nu} \\
& = \Lambda^{0}{}_{0}\Lambda^{1}{}_{1}F^{01} + \Lambda^{0}{}_{1}\Lambda^{1}{}_{0}F^{10} \\
& = (\gamma)^2(-E^1) + (\gamma\beta)^2 E^1 = -E^1 \,, \\
\tau = 2: \qquad & F'^{02} = -E'^{2} = \Lambda^{0}{}_{\mu}\Lambda^{2}{}_{\nu}F^{\mu\nu} \\
& = \Lambda^{0}{}_{0}\Lambda^{2}{}_{2}F^{02} + \Lambda^{0}{}_{1}\Lambda^{2}{}_{2}F^{12} \\
& = \gamma(-E^2) + (-\gamma\beta)(-B^3) \,, \\
\tau = 3: \qquad & F'^{03} = -E'^{3} = \Lambda^{0}{}_{\mu}\Lambda^{3}{}_{\nu}F^{\mu\nu} \\
& = \Lambda^{0}{}_{0}\Lambda^{3}{}_{3}F^{03} + \Lambda^{0}{}_{1}\Lambda^{3}{}_{3}F^{13} \\
& = \gamma(-E^3) - \gamma\beta B^2 \,.
\end{aligned}$$

Here we made use of

$$\Lambda^{1}{}_{2} = 0 = \Lambda^{1}{}_{3} \,, \quad \Lambda^{2}{}_{1} = 0 = \Lambda^{2}{}_{3} \,, \quad \Lambda^{3}{}_{1} = 0 = \Lambda^{3}{}_{2} \,.$$

Thus, written more compactly, one obtains the formulae

$$E'^1 = E^1 \,, \tag{2.65a}$$
$$E'^2 = \gamma\big(E^2 - \beta B^3\big) \,, \tag{2.65b}$$
$$E'^3 = \gamma\big(E^3 + \beta B^2\big) \,. \tag{2.65c}$$

By applying a little gimmick the transformation behaviour of the $\boldsymbol{B}$ fields can be derived from the formulae (2.65a–2.65c). The tensor field $\star F^{\mu\nu}$, equation (2.54c), transforms in the same way as the tensor field $F^{\mu\nu}$, but, at the same time, it arises from the replacements (2.55). Therefore, one has

$$B'^1 = B^1 \,, \tag{2.65d}$$
$$B'^2 = \gamma\big(B^2 + \beta E^3\big) \,, \tag{2.65e}$$
$$B'^3 = \gamma\big(B^3 - \beta E^2\big) \,. \tag{2.65f}$$

Finally, it is not difficult to generalize these formulae to an arbitrary direction of the velocity $\boldsymbol{v}$: The results (2.65a–2.65f) show that the components which are parallel to $\boldsymbol{v}$ remain unchanged while the components perpendicular to $\boldsymbol{v}$ depend on the cross product of the velocity and the other field, respectively. Therefore, one has

$$E'_\parallel = E_\parallel \,, \quad B'_\parallel = B_\parallel \,, \tag{2.66a}$$
$$\boldsymbol{E}'_\perp = \gamma\left(\boldsymbol{E}_\perp + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) \tag{2.66b}$$
$$\boldsymbol{B}'_\perp = \gamma\left(\boldsymbol{B}_\perp - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{E}\right) \,. \tag{2.66c}$$

Returning to the concrete example $\boldsymbol{v} = v\hat{\boldsymbol{e}}_1$ and referring to Fig. 2.3 one has $r' = \sqrt{b^2 + (vt')^2}$. In its own rest system K' the particle creates the spherically symmetric electric field

$$\boldsymbol{E} = \frac{q}{r'^3}\boldsymbol{r}' \,.$$

There is no induction field. At the position of the observer one has, in particular,

$$E'^1 = -\frac{q}{r'^2}\frac{(vt')}{r'} \,, \quad E'^2 = \frac{q}{r'^2}\frac{b}{r'} \,, \quad E'^3 = 0 \,,$$
$$B'^1 = 0 = B'^2 = B'^3 \,.$$

As in the rest system $\boldsymbol{B}' = 0$, it follows from (2.65e) or (2.65f) that $B^2 = -\beta E^3$ and $B^3 = \beta E^2$. One inserts this into (2.65c) or (2.65b) to obtain $E'^2 = E^2/\gamma$ and $E'^3 = E^3/\gamma$, respectively. This is used to calculate the fields in the laboratory system K: At the position of B and with the given state of motion of the particle, the

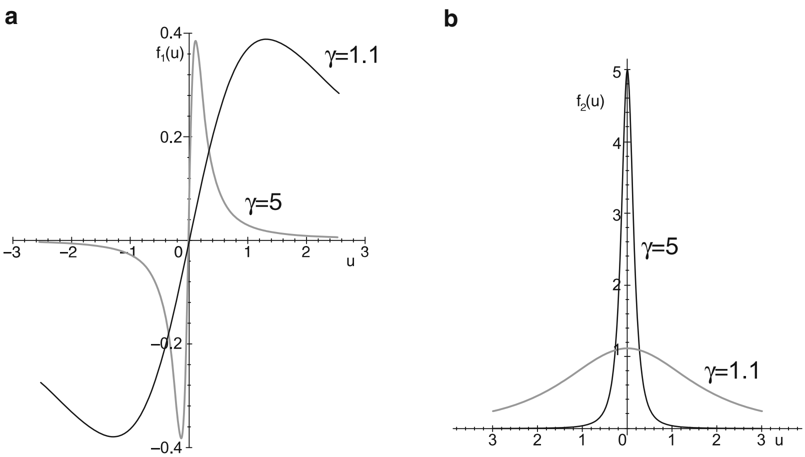


Fig. 2.4 **a** The component E^1 of the electric field in the direction of $\boldsymbol{v}$, and **b** the component E^2 perpendicular to $\boldsymbol{v}$ as a function of time, for two values of γ

field components are

$$E^1 = -q \frac{v\gamma t}{\left(b^2 + (v\gamma t)^2\right)^{\frac{3}{2}}} \,, \tag{2.67a}$$

$$E^2 = \gamma E'^2 = q \frac{\gamma b}{\left(b^2 + (v\gamma t)^2\right)^{\frac{3}{2}}} \,, \tag{2.67b}$$

$$\boldsymbol{B}_\perp = \frac{\gamma}{c} \boldsymbol{v} \times \boldsymbol{E} \,, \quad \text{i.e.} \quad B^3 = \frac{v}{c} \gamma E'^2 = \frac{v}{c} E^2 \,, \tag{2.67c}$$

and, evidently, $E^3 = E'^3 = 0$. In order to illustrate this result it is useful to introduce the dimensionless variable

$$u := \frac{ct}{b}$$

and to express the fields in units of q/b^2. Equations (2.67a) and (2.67b) then become, with $vt/b = \beta u$ and with $\beta^2\gamma^2 = \gamma^2 - 1$,

$$f_1(u) := -\frac{E^1}{(\frac{q}{b^2})} = \frac{\sqrt{\gamma^2 - 1}\, u}{\left(1 + (\gamma^2 - 1)u^2\right)^{\frac{3}{2}}} \,,$$

$$f_2(u) := \frac{E^2}{(\frac{q}{b^2})} = \frac{\gamma}{\left(1 + (\gamma^2 - 1)u^2\right)^{\frac{3}{2}}} \,.$$

Figure 2.4a shows the function f_1 as a function of u, i.e. as a function of coordinate time t. This function is odd, its maximum and its minimum are at $u_{\max/\min} =$

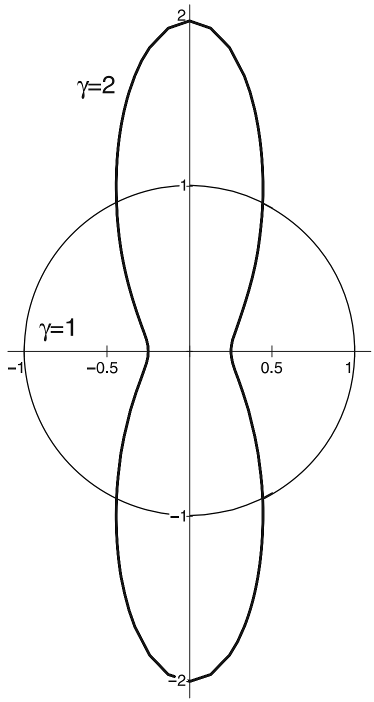


Fig. 2.5 The electric field at position $\boldsymbol{B}$, divided by $\boldsymbol{E}^{(0)}$, at a *fixed* time and as a function of φ. The direction of motion is the abscissa

$\mp 1/(\sqrt{2(\gamma^2-1)})$, respectively, its absolute value being $2/(3\sqrt{3})$ in both positions. Figure 2.4b shows the function $f_2(u)$. This function has its maximum at $u=0$. Its value at time zero is $f_2(0)=\gamma$, the width of this curve, i.e. the distance between the two points at which it has decreased to half its value at $u=0$, is found to be

$$\Delta u = \frac{2\sqrt{4^{\frac{1}{3}}-1}}{\sqrt{\gamma^2-1}} \simeq \frac{1.533}{\sqrt{\gamma^2-1}} \, .$$

The larger the value of γ, the more pronounced and narrow the "pulse" that the observer sees in the 2-direction. The phenomena seen *in* the direction of flight are the result of Lorentz contraction. This is seen most clearly if one calculates the electric field at the position of the observer B, in the laboratory system and at an arbitrary but fixed point in time. From the geometry of Fig. 2.5 one sees that

$$\frac{E^1}{E^2} = -\frac{vt}{b} \, ,$$

i.e. $\boldsymbol{E}$ has the same direction as the position vector $\boldsymbol{r}$. Regarding the denominator of the expressions (2.67a) and (2.67b) one calculates

$$\begin{aligned}
b^2 + (v\gamma t)^2 &= \gamma^2 \left(b^2 + (vt)^2\right) + b^2 \left(1 - \gamma^2\right) \\
&= \gamma^2 r^2 \left[1 + \frac{1-\gamma^2}{\gamma^2} \frac{b^2}{r^2}\right] = \gamma^2 r^2 \left(1 - \beta^2 \sin^2 \varphi\right) \,.
\end{aligned}$$

Thus, at the fixed time t the electric field at the position of B is

$$\boldsymbol{E}\left(t_{\text{fixed}}, \boldsymbol{r}\right) = \frac{q\boldsymbol{r}}{r^3 \gamma^2 \left(1 - \beta^2 \sin^2 \varphi\right)^{\frac{3}{2}}} \,. \tag{2.68}$$

The effect of Lorentz contraction can be read off this result: With $\boldsymbol{E}^{(0)} = q\boldsymbol{r}/r^3$ being the field of the particle at rest one has

$$\begin{aligned}
&\text{for } \varphi = \pm\frac{\pi}{2} : && \boldsymbol{E}\left(t_{\text{fixed}}, \boldsymbol{r}\right) = \gamma \boldsymbol{E}^{(0)}\left(t_{\text{fixed}}, \boldsymbol{r}\right) , \\
&\text{for } \varphi = 0 \text{ and } \pi : && \boldsymbol{E}\left(t_{\text{fixed}}, \boldsymbol{r}\right) = \frac{1}{\gamma^2} \boldsymbol{E}^{(0)}\left(t_{\text{fixed}}, \boldsymbol{r}\right) .
\end{aligned}$$

In the direction of motion ($\varphi = 0$ or π) the spherically symmetric field appears contracted when compared to the directions $\varphi = \pm\pi/2$ perpendicular to the motion.

2.5 Lorentz Invariant Exterior Forms and the Maxwell Equations

As was shown in Sect. 2.2.2 for the case of a fixed division of spacetime into time axis $\mathbb{R}_t$ and coordinate space $\mathbb{R}^3$, the association of simple exterior forms over $\mathbb{R}^3$ to the fields and potentials of Maxwell theory proved useful in reformulating Maxwell's equations in a concise and transparent manner. On the basis of this experience it is suggestive to interpret the observables and the potentials of Maxwell theory, written in covariant form, as geometric objects on Minkowski space $\mathbb{R}^4$. In this Section we show that the field strength tensor, the Lorentz force, and the external sources can be written as exterior forms which are even simpler than in the case of $\mathbb{R}^3$ and which satisfy simple and natural equations. By the same token we show that the apparent asymmetry between the electric field that was a *one*-form and the induction field that was a *two*-form on $\mathbb{R}^3$, disappears. Finally, we provide the basis for the generalization to non-Abelian gauge theories which are studied in Chap. 5.

2.5.1 Field Strength Tensor and Lorentz Force

The tensor field $F^{\mu\nu}(x)$ which in a given inertial system decomposes into observable $\boldsymbol{E}$-fields and $\boldsymbol{B}$-fields according to (2.46), is defined on Minkowski space $\big(\mathbb{R}^4, \mathrm{g} = \mathrm{diag}(1,-1,-1,-1)\big)$. Denoting the base one-forms over this space by $\mathrm{d}x^\mu$, the base two-forms by $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ $(\mu < \nu)$ we define

$$\omega_F := \sum_{\mu<\nu} F_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,. \tag{2.69}$$

The sums on μ and ν are written explicitly because of the condition $\mu < \nu$. If one prefers to also apply the summation convention here one must add the factor $1/2$. We have inserted the *co*variant tensor field in (2.69) which is obtained by calculating – now using the summation convention! –

$$F_{\mu\nu}(x) = g_{\mu\sigma} F^{\sigma\tau}(x) g_{\tau\nu} \,.$$

Like the coordinates the base one-forms

$$\mathrm{d}x^0 = c\,\mathrm{d}t\,,\ \mathrm{d}x^1\,,\ \mathrm{d}x^2\,,\ \mathrm{d}x^3$$

are ordered from 0 to 3, they refer to the chosen coordinate system. Note that we simplified the notation somewhat by omitting the degree of the form above the symbol. Obviously, the definition (2.69) indicates that one is dealing with a two-form.

Already at this point there is an important remark to be made: In contrast to the representation of the field strength tensor by $F^{\mu\nu}$, with its obvious recurrence to a given frame of reference, the definition of the two-form ω_F is Lorentz invariant. If, nevertheless, one sticks to the given frame in which, taking proper account of the signs from the two factors g,

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & 0 & -B_3 & +B_2 \\ -E_2 & +B_3 & 0 & -B_1 \\ -E_3 & -B_2 & +B_1 & 0 \end{pmatrix} \tag{2.69a}$$

there follows[5]

$$\begin{aligned} \omega_F &= \mathrm{d}x^0 \wedge \big[E_1\,\mathrm{d}x^1 + E_2\,\mathrm{d}x^2 + E_3\,\mathrm{d}x^3\big] \\ &\quad - \big[B_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + B_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2\,\mathrm{d}x^3 \wedge \mathrm{d}x^1\big] \\ &= \mathrm{d}x^0 \wedge \overset{1}{\omega}_{\boldsymbol{E}} - \overset{2}{\omega}_{\boldsymbol{B}}\,, \end{aligned} \tag{2.70}$$

[5] One should notice that on $\mathbb{R}^3$, the Euclidean space, one has $\boldsymbol{E}_i = \boldsymbol{E}^i$ and $\boldsymbol{B}_k = \boldsymbol{B}^k$. On $\mathbb{R}^3$ and with cartesian coordinates covariant and contravariant indices can be identified and need not be distinguished.

which contains the exterior forms defined in (2.26a) and (2.26b). This analysis explains at once why on $\mathbb{R}^3$ the field $\boldsymbol{E}$ is associated to a one-form, while the field $\boldsymbol{B}$ is associated to a two-form. On Minkowski space, in turn, both kinds of fields are represented by two-forms. Therefore, it is appropriate to replace the definitions (2.26a) and (2.26b) by the following:

$$\omega_E \equiv \overset{2}{\omega}_E := \sum_{i=1}^{3} E_i(t, \boldsymbol{x})\, \mathrm{d}x^0 \wedge \mathrm{d}x^i \,, \tag{2.71a}$$

$$\omega_B \equiv \overset{2}{\omega}_B := \frac{1}{2} \sum_{i,j,k} \varepsilon_{ijk} B_i(t, \boldsymbol{x})\, \mathrm{d}x^j \wedge \mathrm{d}x^k \,. \tag{2.71b}$$

In what follows we shall make use of either notation, the one defined in (2.69) but also the ones of (2.71a) and (2.71b).

Being a two-form ω_F can act on up to two vector fields. Thus, one has with

$$u = u^\alpha \partial_\alpha \quad \text{and} \quad v = v^\beta \partial_\beta$$

$$\begin{aligned}
\omega_F(u, v) &= \sum_{\mu<\nu} F_{\mu\nu}\left(v^\mu u^\nu - u^\mu v^\nu\right) = 2 \sum_{\mu<\nu} F_{\mu\nu} u^\nu v^\mu \\
&= 2 \frac{1}{2} \sum_{\mu,\nu} F_{\mu\nu} u^\nu v^\mu \,.
\end{aligned}$$

In turn, if one lets ω_F act on one vector field only, then one obtains a one-form

$$\omega_F(u, \bullet) = \sum_{\mu,\nu} F_{\mu\nu} u^\nu\, \mathrm{d}x^\mu \,.$$

Multiplication of this form with q/c yields the one-form which is to be associated to the Lorentz force (2.47) by way of the definition

$$\omega_{\text{Lor}} := \sum_{\mu=0}^{3} K_\mu(x)\, \mathrm{d}x^\mu \,, \quad \text{with} \quad K_\mu(x) = \frac{q}{c} \sum_{\nu=0}^{3} F_{\mu\nu}(x) u^\nu \,. \tag{2.72}$$

It is then not difficult to rewrite Maxwell's equations in their manifestly covariant form in terms of the two-form (2.69) and its Hodge dual. The Hodge star operation in Minkowski space is a bit more subtle to handle, as compared to the case of a Euclidean space and the definition (2.16), because of the characteristic signs of the metric. We list here the duals of all base forms over Minkowski space

$$\star\, \mathrm{d}x^\mu = \frac{1}{3!} g^{\mu\lambda} \varepsilon_{\lambda\nu\sigma\tau}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^\tau \,, \tag{2.73a}$$

$$\star \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) = \frac{1}{2!} g^{\mu\lambda} g^{\nu\varrho} \varepsilon_{\lambda\varrho\sigma\tau}\, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\tau \,, \tag{2.73b}$$

$$\star \left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma\right) = g^{\mu\lambda} g^{\nu\varrho} g^{\sigma\eta} \varepsilon_{\lambda\varrho\eta\tau}\, \mathrm{d}x^\tau \,, \tag{2.73c}$$

$$\star \left(\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) = \det \mathrm{g} = -1 \,. \tag{2.73d}$$

Remark
In these formulae $\varepsilon_{\alpha\beta\gamma\delta}$ denotes the totally antisymmetric Levi-Civita symbol in dimension four that was introduced in (2.50). Note the convention $\varepsilon_{0123} = +1$. The (inverse) metric which appears in the formulae (2.73a–2.73d) gives rise to signs because while the time-time element is $g^{00} = +1$, the space-space elements are $g^{ii} = -1$. This also implies a sign change in the relation (2.18) between the doubly dualized $\star\star\omega$ and the original ω which is different from the case of Euclidean spaces. Here it reads

$$\star\star\omega = (-)^{k(n-k)+1}\,\omega\ , \tag{2.74}$$

where the 1 in the exponent stems from the signature of the semi-Euclidian space $\mathbb{R}^{(p,q)}$ (with p space coordinates and q time coordinates), with $p + q = n$, and with the metric $\mathrm{g} = \big(1, 1, \dots\ (q\ \text{times}), -1, -1, \dots\ (p\ \text{times})\big)$. The signature s is the codimension of the biggest subspace on which the metric is *definite*. Consider the example of Minkowski space $\mathbb{R}^{(1,3)}$: The metric is $\mathrm{g} = \mathrm{diag}(1, -1, -1, -1)$. Its restriction to the space part $\mathrm{g}|_{\mathbb{R}^3}$ is negative definite, hence, $s = 4 - 3 = 1$.

Let us consider a few examples: The formula (2.73d) can be written, alternatively, as follows

$$\star\big(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^\tau\big) = g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} g^{\tau\delta} \varepsilon_{\alpha\beta\gamma\delta}\ .$$

Similarly, the dual of the constant function 1 is

$$\star 1 = \frac{1}{4!}\varepsilon_{\mu\nu\sigma\tau}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\sigma \wedge \mathrm{d}x^\tau\ .$$

From (2.73b) one obtains

$$\star\mathrm{d}x^0 \wedge \mathrm{d}x^1 = g^{00}g^{11}\varepsilon_{0123}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 = -\mathrm{d}x^2 \wedge \mathrm{d}x^3\ ,$$
$$\star\mathrm{d}x^2 \wedge \mathrm{d}x^3 = g^{22}g^{33}\varepsilon_{2301}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 = +\mathrm{d}x^0 \wedge \mathrm{d}x^1\ ,$$

and from there

$$\star\star\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 = -\mathrm{d}x^0 \wedge \mathrm{d}x^1\ ,$$

in agreement with (2.74) where $k = 2$ and $s = 1$. Starting from the formula (2.73a) one has

$$\begin{aligned}
\star\mathrm{d}x^0 &= g^{00}\varepsilon_{0123}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\ ,\\
\star\mathrm{d}x^1 &= g^{11}\varepsilon_{1023}\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = -\varepsilon_{1023}\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\\
&= \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\ ,\\
\star\mathrm{d}x^2 &= g^{22}\varepsilon_{2013}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 = -\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3\\
&= +\mathrm{d}x^0 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1\ ,\\
\star\mathrm{d}x^3 &= g^{33}\varepsilon_{3012}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\ .
\end{aligned}$$

(Note that the last three formulae show the cyclic symmetry in the space indices.) Upon comparison with the formula (2.73c) one sees that the base three-forms as well as the base one-forms do not change under the double star operation, in agreement with (2.74) for $n = 4$ and $s = 1$, $k = 1$ and $k = 3$, respectively.

The star operation applied to ω_F, (2.69), using the rules (2.73b) derived above, yields

$$\begin{aligned}\star\omega_F = &- E_1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 - E_2\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 - E_3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\ &- B_1\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 - B_2\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 - B_3\,\mathrm{d}x^0 \wedge \mathrm{d}x^3 \,.\end{aligned}$$

This is seen to be the two-form (2.70), with the replacements

$$\boldsymbol{E} \longmapsto -\boldsymbol{B}\,, \quad \boldsymbol{B} \longmapsto \boldsymbol{E}\,.$$

If one compares this with (2.55) it is clear that "dual" and "dual" are identical i.e. that

$$\star\,\omega_F = \omega_{(\star F)}\,. \tag{2.75}$$

The two-form (2.69), constructed from $\star F$, is the same as the Hodge dual of ω_F, (2.69).

Of course, the same construction can be applied to $\mathcal{F}^{\mu\nu}(x)$, the tensor field of the $\boldsymbol{D}$ and $\boldsymbol{H}$ fields, equation (2.52a). In analogy to (2.69) one defines

$$\omega_{\mathcal{F}} = \sum_{\mu<\nu} \mathcal{F}_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,. \tag{2.76}$$

Both types of two-forms, ω_F and $\omega_{\mathcal{F}}$, appear in the Maxwell equations to which we now turn.

2.5.2 Differential Equations for the Two-Forms ω_F and $\omega_{\mathcal{F}}$

The homogeneous Maxwell equations (2.49) or (2.49a) become very simple in the language of exterior forms. They just say that ω_F is a *closed* form,

$$\mathrm{d}\omega_F = 0\,. \tag{2.77}$$

This is verified by applying the formula (2.15e) for exterior derivatives:

$$\begin{aligned}\mathrm{d}\omega_F &= \big(\mu < \nu\big)\,\partial_\lambda F_{\mu\nu}(x)\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \frac{1}{2}\,\partial_\lambda F_{\mu\nu}(x)\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,.\end{aligned}$$

The three indices λ, μ, and ν must all be different. As the base three-forms are linearly independent, the coefficient

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \ ,$$

which multiplies the base form $\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ must vanish. This is the content of the equation (2.49). As an alternative, one may calculate the dual of $\mathrm{d}\omega_F$ by means of (2.73c):

$$\star\,\mathrm{d}\omega_F = \frac{1}{2}\,\partial^\nu F^{\sigma\tau}(x)\varepsilon_{\mu\nu\sigma\tau}\,\mathrm{d}x^\mu \ .$$

As the coefficient of every base one-form $\mathrm{d}x^\mu$ must vanish this yields the homogeneous Maxwell equations in the form (2.49a).

In order to obtain the inhomogeneous equations (2.53) one starts by calculating the exterior derivative of the dualized form $\star\omega_{\mathcal{F}}$,

$$\begin{aligned}\mathrm{d}\bigl(\star\omega_{\mathcal{F}}\bigr) &= \frac{1}{4}\,\partial_\alpha \mathcal{F}_{\mu\nu}(x)g^{\mu\lambda}g^{\nu\varrho}\varepsilon_{\lambda\varrho\beta\gamma}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \\ &= \frac{1}{4}\,\partial_\alpha \mathcal{F}^{\lambda\varrho}(x)\varepsilon_{\lambda\varrho\beta\gamma}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \ .\end{aligned}$$

The indices α, β, and γ of the base three-form must all be different from one another. In addition, as β and γ in the ε symbol must also differ from λ and from ϱ; one must have either $\lambda = \alpha$ or $\varrho = \alpha$. This becomes even more obvious if the last result is subject once more to the star operation. Making use of the tensor

$$\varepsilon^{\alpha\beta\gamma\delta} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\sigma}g^{\delta\tau}\varepsilon_{\mu\nu\sigma\tau}$$

and of the formula (2.73d), one has

$$\star\ \mathrm{d}\star\,\omega_{\mathcal{F}} = \frac{1}{4}\,\partial_\alpha \mathcal{F}^{\lambda\varrho}(x)\varepsilon_{\lambda\varrho\beta\gamma}\varepsilon^{\alpha\beta\gamma\delta}\,g_{\delta\eta}\,\mathrm{d}x^\eta \ .$$

Each coefficient multiplying a one-form $\mathrm{d}x^\eta$ must be considered separately. First one notes that the sums over β and γ can be evaluated by means of the formula

$$\varepsilon_{\lambda\varrho\beta\gamma}\varepsilon^{\alpha\beta\gamma\delta} = \varepsilon_{\beta\gamma\lambda\varrho}\varepsilon^{\beta\gamma\alpha\delta} = -2\left\{\delta^\alpha_\lambda\delta^\delta_\varrho - \delta^\alpha_\varrho\delta^\delta_\lambda\right\} \ . \tag{2.78}$$

Upon inserting this formula one obtains four times the same term so that

$$\star\ \mathrm{d}\star\,\omega_{\mathcal{F}} = -\partial_\lambda \mathcal{F}^{\lambda\varrho}(x)g_{\varrho\eta}\,\mathrm{d}x^\eta \ .$$

This equation contains the operator (2.19a) with $n = 4$, $k = 2$, $\delta = -\star\,\mathrm{d}\,\star$, so that

$$\delta\,\omega_{\mathcal{F}} = \partial_\lambda \mathcal{F}^{\lambda\varrho}(x)g_{\varrho\eta}\,\mathrm{d}x^\eta \ .$$

If, conversely, one compares the three-form (2.28a) of the charge density and the two-form (2.28b) of the current density one reckons that in the covariant formulation both will appear as three-forms over Minkowski space, viz.

$$\omega_j = \frac{1}{3!}\,\varepsilon_{\mu\alpha\beta\gamma}\, j^\mu(x)\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma\,. \tag{2.79}$$

Taking the dual of this expression and making use of the formula

$$\varepsilon_{\mu\alpha\beta\gamma}\varepsilon^{\alpha\beta\gamma\eta} = -\varepsilon_{\alpha\beta\gamma\mu}\varepsilon^{\alpha\beta\gamma\eta} = 3!\delta^\eta_\mu\,,$$

one has

$$\star\,\omega_j = j^\mu(x) g_{\mu\eta}\,\mathrm{d}x^\eta\,. \tag{2.79a}$$

From this and by comparison of coefficients of $\mathrm{d}x^\eta$ it is clear that the inhomogeneous Maxwell equations (2.53) in exterior forms must be

$$\delta\,\omega_{\mathcal{F}} = \frac{4\pi}{c}\star\omega_j\,. \tag{2.80}$$

Remarks

1. When one uses exclusively the geometric language in formulating electrodynamics in terms of exterior forms one simply writes F instead of ω_F, $\mathcal{F}$ instead of $\omega_{\mathcal{F}}$, etc. In this notation one has

$$F \equiv \frac{1}{2} F_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,.$$

   Covariance of Maxwell's equations in the form (2.77) and (2.80) is obvious because both are written in a coordinate-free way. Their independence of specific coordinates, in fact, means that Maxwell's equations hold in *all* inertial systems.
2. The homogeneous equations which are summarized in (2.77), do not make use yet of the metric on Minkowski space. In turn, the inhomogeneous equations which slumber in (2.80), depend on the Hodge star operation which assumes a metric. In their axiomatic approach to electrodynamics Hehl and Obukhov start from topological manifolds without presupposing the existence of a metric.
3. As one easily verifies the mapping δ, like the exterior derivative d, when applied twice, gives zero, $\delta\circ\delta = \star\mathrm{d}\star\circ\star\,\mathrm{d}\star = 0$. Therefore, by applying δ to the inhomogeneous equations (2.80), one concludes

$$\delta\big(\star\omega_j\big) = 0 \quad \text{or} \quad \partial_\mu j^\mu(x) = 0\,. \tag{2.81}$$

   This is the result found earlier: Current conservation follows from the inhomogeneous Maxwell equations. In other terms, only a *conserved* current can be a source of Maxwell's equations.

2.5.3 Potentials and Gauge Transformations

The four-potential (2.57) can be written as an exterior form as well, by using the covariant components $A_\nu(x) = g_{\nu\lambda} A^\lambda(x)$ in the definition of the following one-form

$$\omega_A := A_\nu(x)\,\mathrm{d}x^\nu \,. \tag{2.82}$$

Taking the exterior derivative, one finds

$$\begin{aligned}\mathrm{d}\omega_A &= \mathrm{d}A_\nu(x) \wedge \mathrm{d}x^\nu = \partial_\mu A_\nu(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \sum_{\mu<\nu} \Big(\partial_\mu A_\nu(x) - \partial_\mu A_\nu(x)\Big)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,.\end{aligned}$$

Comparison with the definition (2.69) shows that ω_F is the exterior derivative of ω_A, viz.

$$\omega_F = \mathrm{d}\omega_A \,. \tag{2.83}$$

We rediscover here a known fact in a particularly simple form: If one introduces potentials the homogeneous Maxwell equations are trivially fulfilled. Indeed, from the property (2.15d) of the exterior derivative

$$\omega_F = \mathrm{d}\omega_A \Longrightarrow \mathrm{d}\omega_F = \mathrm{d}^2\omega_A = 0 \,.$$

Likewise, the gauge transformations (2.59) fit well into the framework of exterior forms. Let $\Lambda(x)$ be a smooth function on Minkowski space. Its total differential $\mathrm{d}\Lambda$ is a one-form that may be added to ω_A without modifying the Maxwell equation (2.77):

$$\begin{aligned}\omega_A \mapsto \omega_{A'} = \omega_A + \mathrm{d}\Lambda \Longrightarrow \omega_{F'} &= \mathrm{d}\omega_{A'} = \mathrm{d}\omega_A + \mathrm{d}^2\Lambda \\ &= \mathrm{d}\omega_A \;= \omega_F \,.\end{aligned}$$

The exact form $\mathrm{d}\Lambda$ is closed. The gauge freedom in the choice of the potential $A_\mu(x)$ is equivalent to ω_A being determined only up to an arbitrary exact form. (Note, when comparing to (2.59), that $\Lambda(x)$ is the same function as $\chi(x)$, up to a sign.)

The action of the Laplace–de-Rham operator (2.19b) on a one-form of the kind of (2.82) is calculated by means of the relations (2.73a–2.73d) as follows:

$$\begin{aligned}\mathbf{\Delta}_{\mathrm{LdR}}\big(A_\mu\,\mathrm{d}x^\mu\big) &= \mathrm{d} \circ \delta\left(A_\mu\,\mathrm{d}x^\mu\right) + \delta \circ \mathrm{d}\left(A_\mu\,\mathrm{d}x^\mu\right) \\ &= -\big(\mathrm{d} \star \mathrm{d} \star + \star \mathrm{d} \star \mathrm{d}\big)\big(A_\mu\,\mathrm{d}x^\mu\big) \,.\end{aligned}$$

Using (2.73a) and (2.73d) the first term on the right-hand side leads to

$$-\frac{1}{3!}\partial_\varrho \partial_\lambda A_\mu g^{\mu\alpha} \varepsilon_{\alpha\nu\sigma\tau} \varepsilon^{\lambda\nu\sigma\tau}\,\mathrm{d}x^\varrho \,.$$

The contraction of the two ε-symbols is obtained from (2.78),

$$\varepsilon_{\alpha\nu\sigma\tau}\varepsilon^{\lambda\nu\sigma\tau} = \varepsilon_{\nu\sigma\tau\alpha}\varepsilon^{\nu\sigma\tau\lambda} = -2\big(\delta^\beta_\tau \delta^\lambda_\alpha - \delta^\beta_\alpha \delta^\lambda_\tau\big)\delta^\tau_\beta$$
$$= -2(4-1)\delta^\lambda_\alpha = -3!\,\delta^\lambda_\alpha \ ,$$

so that the first term gives

$$\partial_\varrho \partial_\lambda A^\lambda(x)\,\mathrm{d}x^\varrho \ .$$

For the second term one needs the formulae (2.73b) and (2.73c) for calculating

$$-\frac{1}{2}\partial_\eta \partial_\lambda A_\mu g^{\lambda\bar\lambda} g^{\mu\bar\mu} g_{\bar\gamma\gamma}\varepsilon_{\bar\lambda\bar\mu\alpha\beta}\varepsilon^{\eta\alpha\beta\bar\gamma}\,\mathrm{d}x^\gamma \ .$$

The contraction of the two ε-symbols is given in (2.78), the second term yields two contributions,

$$\partial^\lambda \partial_\lambda A_\mu \,\mathrm{d}x^\mu - \partial^\mu \partial_\lambda A_\mu \,\mathrm{d}x^\lambda \ .$$

Taking the sum of the two terms one obtains

$$\mathbf{\Delta}_{\mathrm{LdR}} A_\mu(x)\,\mathrm{d}x^\mu = \partial^\lambda\partial_\lambda A_\mu(x)\,\mathrm{d}x^\mu \equiv \big(\Box A_\mu(x)\big)\,\mathrm{d}x^\mu \ , \tag{2.84}$$

where $\Box = \partial_0^2 - \mathbf{\Delta}$ with $\mathbf{\Delta}$ the well-known Laplace(–Beltrami) operator in $\mathbb{R}^3$.

From this calculation and from the inhomogeneous equation (2.80) follows the equation of motion for ω_A:

$$\delta\omega_F = \delta \circ \mathrm{d}\omega_A = \mathbf{\Delta}_{\mathrm{LdR}}\,\omega_A - \mathrm{d}\circ\delta\,\omega_A = \frac{4\pi}{c} \star\omega_j \ .$$

Inserting here the expansions (2.82) and (2.79a), respectively, for the one-forms ω_A and $\star\omega_j$ in terms of base one-forms and comparing the coefficients multiplying $\mathrm{d}x^\mu$, one finds the differential equation

$$\Box A_\nu(x) - \partial_\nu\big(\partial^\mu A_\mu(x)\big) = \frac{4\pi}{c} j_\nu(x) \ . \tag{2.85}$$

Finally, by means of the inverse metric one raises the covariant index ν in the three terms of this equation and recovers the equation (2.60).

2.5.4 Behaviour Under the Discrete Transformations

In this section we study the behaviour of the exterior forms of Maxwell theory under space reflection Π, time reversal T, and charge conjugation C. Note that these exterior forms are now defined over four-dimensional Minkowski space! Comparing ω_E, (2.71a), with $\omega_{\boldsymbol{E}}$, (2.26a), one notices at once that these two forms differ in their transformation behaviour under time reversal. In contrast, the two-form $\omega_{\boldsymbol{B}}$,

Table 2.2 In the covariant formalism the electromagnetic exterior forms behave under the three discrete transformations as shown in the table

	Π	T	C	$\Pi\mathsf{TC}$
ω_E	$+$	$-$	$-$	$+$
ω_B	$+$	$-$	$-$	$+$
ω_F	$+$	$-$	$-$	$+$
$\omega_{\mathcal{F}}$	$+$	$-$	$-$	$+$
ω_A	$+$	$-$	$-$	$+$
ω_j	$-$	$+$	$-$	$+$

(2.71b), does not differ from the two-form (2.26b) over the space $\mathbb{R}^3$. The transformation behaviour of ω_E and of ω_B under Π, T, and C is now the same. This holds also for the two-form ω_F, (2.69), and, of course, also for $\omega_{\mathcal{F}}$, (2.76). These observations are listed in the first four rows of Table 2.2.

Considering ω_A as defined in (2.82) and taking into account that the scalar potential $\Phi(t, \boldsymbol{x})$ is a genuine scalar, the vector potential $\boldsymbol{A}$ is a genuine vector field over $\mathbb{R}^3$, and that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, it is clear that the one-form ω_A has the same transformation properties as the first four two-forms. This is noted in the fifth row of Table 2.2.

The three-form ω_j, (2.79), which for a given partition of Minkowski space into time and space contains the charge density $\varrho(t, \boldsymbol{x})$ and the current density $\boldsymbol{j}(t, \boldsymbol{x})$, has the same transformation behaviour as $\overset{3}{\omega}_{\varrho}$, (2.28a), cf. Table 2.1. The result obtained there can be taken over directly to Table 2.2. One easily verifies that the one-form $\star\omega_j$ which is its Hodge dual, is *even* under Π, but *odd* under T.

A common feature of all forms considered here is that they are invariant under the combined transformation $\Pi\mathsf{TC}$. It is instructive to compare the results with Table 2.1: The invariance under the combined transformation $\Pi\mathsf{TC}$ rests in an essential way on the fact that the exterior forms of Maxwell theory are defined on four-dimensional Minkowski space. Invariance of Maxwell theory, as well as of all other known theories of fundamental interactions, under the combined, so-called "PCT" symmetry touches upon a deeply significant result of quantum field theory.

2.5.5 * Covariant Derivative and Structure Equation

This section is merely a long remark which anticipates the more general framework of non-Abelian gauge theories. Therefore, it might not be fully understandable at this point and the reader might wish to come back to it at a later stage.

On the spaces $\Lambda^k(M)$ of exterior forms, $k = 1, \dots, n$, define the following differential operator:

$$\begin{aligned} D_A &:= \mathrm{d} + \mathrm{i}\,\frac{q}{\hbar c}\omega_A \\ &= \mathrm{i}\left(-\mathrm{i}\,\mathrm{d} + \frac{q}{\hbar c}\omega_A\right) . \end{aligned} \tag{2.86}$$

The simple rewriting in the second line of (2.86) serves the purpose of preparing a first intuitive understanding of this definition. Recall that in quantum mechanics the spatial momentum $\boldsymbol{p}$ is replaced by the operator $-\mathrm{i}\hbar\nabla$. Thus, by multiplying D_A with $\hbar$, one sees that (2.86) is the natural generalization of the term

$$\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}$$

whose square appears in the Hamiltonian function for a charged particle in external fields (see, e.g. [ME], Sect. 2.16), and which is prescribed by the principle of *minimal coupling*. In differential geometry as well as in quantum physics one calls this the *covariant derivative*.

When applied to an arbitrary exterior form ω the operator D_A shall act by

$$D_A\,\omega = \mathrm{d}\,\omega + \mathrm{i}\frac{q}{\hbar c}\omega_A \wedge \omega\ . \tag{2.86a}$$

Its action on functions (i.e. on zero-forms), in particular, reads

$$D_A\ f = \left(\partial_\mu f + \mathrm{i}\frac{q}{\hbar c}A_\mu f\right)\ \mathrm{d}x^\mu\ . \tag{2.86b}$$

The operator D_A is a linear combination of exterior derivative and exterior product with the one-form ω_A. In other terms, very much like d, D_A maps a k-form onto a $(k+1)$-form.

If one takes the square of the operator D_A, i.e. if one applies D_A twice successively on an arbitrary exterior form ω, one finds a remarkable result,

$$\begin{aligned} D_A \circ D_A\,\omega &= \left(\mathrm{d} + \mathrm{i}\frac{q}{\hbar c}\omega_A\right) \circ \left(\mathrm{d} + \mathrm{i}\frac{q}{\hbar c}\omega_A\right)\,\omega \\ &= \left\{\mathrm{d}\circ\mathrm{d} + \mathrm{i}\frac{q}{\hbar c}\big(\mathrm{d}\omega_A \wedge + \omega_A \wedge \mathrm{d}\big) + \left(\mathrm{i}\frac{q}{\hbar c}\right)^2 \omega_A \wedge \omega_A \wedge\right\}\,\omega\,. \end{aligned}$$

The first term in the curly brackets gives zero because of $\mathrm{d}^2 = 0$ (cf. (2.15d)). The third term vanishes as well by the asymmetry of the wedge product. As for the middle term, using the graded Leibniz rule (2.15c), one has

$$\begin{aligned} \mathrm{d}\omega_A \wedge \omega + \omega_A \wedge \mathrm{d}\omega &= \big(\mathrm{d}\omega_A\big) \wedge \omega - \omega_A \wedge \mathrm{d}\omega + \omega_A \wedge \mathrm{d}\omega \\ &= \big(\mathrm{d}\omega_A\big) \wedge \omega\ . \end{aligned}$$

From there and with (2.83) one obtains an important and most interesting result, viz.

$$D_A^2 = \mathrm{i}\frac{q}{\hbar c}(\mathrm{d}\omega_A) = \mathrm{i}\frac{q}{\hbar c}\omega_F \ . \tag{2.87}$$

It is particularly noteworthy that the operator $(\mathrm{d}\omega_A)$ saturates in itself, that is to say, the exterior derivative does not act further to the right – in contrast to the original operator D_A. The square of the covariant derivative D_A yields the two-form (2.69) of the field strengths, up to the factor $\mathrm{i}q/(\hbar c)$. In more mathematical terms (2.87) tells us that D_A^2 acts *linearly*. This does not hold for D_A.

These matters become a little more transparent if we replace the one-form ω_A and the two-form ω_F by a one-form A and a two-form F, respectively, which are defined as follows

$$A := \mathrm{i}\frac{q}{\hbar c}\omega_A \ , \quad F := \mathrm{i}\frac{q}{\hbar c}\omega_F \ . \tag{2.88a}$$

With these definitions the structure equations above simplify to

$$D_A = \mathrm{d} + A \ , \tag{2.88b}$$

$$D_A^2 = (\mathrm{d}A) + A \wedge A = (\mathrm{d}A) = F \ . \tag{2.88c}$$

Equations of this type are well known in differential geometry. The one-form A, defined here in a particularly simple case, is called the *connection form.* The operator $D_A = \mathrm{d} + A$ is the covariant derivative, while F is called the *curvature form* pertaining to the given connection. Indeed, one can show that $F = D_A^2$ may be interpreted as a "round trip" along a small closed loop, in analogy to what one would do in order to detect curvature in a hypersurface.

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