Chapter 1
Existence for Parabolic–Elliptic Degenerate Diffusion Problems

In this chapter we are concerned with the study of some boundary value problems with initial data formulated for parabolic–elliptic degenerate diffusion equations with advection, focusing especially on the fast diffusion case which involves a free boundary problem (case (a) in Introduction). After setting an adequate functional framework for each situation we transpose the boundary value problems into abstract formulations and study their well-posedness with specific methods of the theory of nonlinear evolution equations with m-accretive operators in Hilbert spaces. We investigate the conditions under which particular properties of the solutions, like uniqueness and time periodicity take place. We mention that the case without advection was studied in [58]. Numerical simulations applied to problems arisen in soil sciences complete the study and sustain the theoretical achievements.

Notation. We specify the functional spaces which will be further used.

Let Ω be a open bounded subset of $\mathbb{R}^N$ $(N \in \mathbb{N}^* = \{1, 2, \ldots\})$, with the boundary $\Gamma := \partial\Omega$ sufficiently smooth. The space variable is denoted by $x := (x_1, \ldots, x_N) \in \Omega$ and the time by $t \in (0, T)$, with T finite.

We shall work with the spaces $L^p(\Omega)$ (see [30], pp. 89), Sobolev spaces $W^{m,p}(\Omega)$ (see [30], pp. 263, 271) and the vectorial spaces $L^p(0, T; X)$, $W^{m,p}(0, T; X)$ where X is a Banach space (see [14], pp. 21), $m \geq 1$ and $p \in [1, \infty]$. Briefly, we recall that

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R};\ f \text{ measurable, } |f(x)|^p \text{ integrable}\},\ p \in [1, \infty),$$

$$L^\infty(\Omega) = \left\{ \begin{array}{l} f : \Omega \to \mathbb{R};\ f \text{ measurable and there is a constant } C \\ \text{such that } |f(x)| \leq C \text{ a.e. on } \Omega \end{array} \right\}$$

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are Banach spaces with the norms

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p},$$
$$\|f\|_{L^\infty(\Omega)} = \inf\{C; \ |f(x)| \le C \text{ a.e. on } \Omega\},$$

respectively. For $m \ge 1$ and $p \in [1, \infty]$ the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega); \ f \text{ measurable and } D^\alpha f \in L^p(\Omega), \text{ with } |\alpha| \le m\}$$

where α is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$, α_i is a positive integer and $D^\alpha = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$.

The norm is defined by

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{1 \le |\alpha| \le m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}, \text{ if } 1 \le p < \infty,$$
$$\|f\|_{W^{m,\infty}(\Omega)} = \max_{1 \le |\alpha| \le m} \|D^\alpha f\|^p_{L^\infty(\Omega)}, \text{ if } p = \infty.$$

We still denote $H^m(\Omega) = W^{m,2}(\Omega)$ which is a Hilbert space with the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{1 \le |\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.$$

Let X be a Banach space. We denote

$$L^p(0,T;X) = \left\{ \begin{array}{l} f : (0,T) \to X; \ f \text{ measurable and} \\ \|f(t)\|^p_X \text{ is Lebesgue integrable over } (0,T) \text{ for } p \in [1,\infty) \\ \text{and } ess \sup\limits_{t \in (0,T)} \|f(t)\|_X < \infty \text{ for } p = \infty \end{array} \right\},$$

$$W^{m,p}([0,T];X) = \{f \in \mathcal{D}'(0,T;X); \ \frac{d^j f}{dx_j} \in L^p(0,T;X), \ j = 1, \ldots, m\},$$

where $\mathcal{D}'(0,T;X)$ is the space of all continuous operators from $\mathcal{D}(0,T)$ to X. These spaces are endowed with the norms

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p\,dt\right)^{1/p},$$

$$\|f\|_{L^\infty(0,T;X)} = ess \sup_{t\in(0,T)} \|f(t)\|_X\,,$$

$$\|f\|_{W^{m,p}([0,T];X)} = \left(\sum_{j=1}^m \left\|\frac{d^j f}{dx^j}\right\|_{L^p(0,T;X)}^p\right)^{1/p},\ 1\le p<\infty,$$

$$\|f\|_{W^{m,\infty}([0,T];X)} = \max_{1\le j\le m}\left\|\frac{d^j f}{dx^j}\right\|_{L^\infty(0,T;X)}^p,\quad p=\infty.$$

By $C([0,T];X)$ we denote the space of continuous functions $f:[0,T]\to X$.

For simplicity, throughout the book we shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively.

For not overloading the notation, sometimes we do not indicate in the integrands the function arguments which are the integration variables.

1.1 Well-Posedness for the Cauchy Problem with Fast Diffusion

The first section is devoted to the study of a Cauchy problem for a fast diffusion equation with transport written for the unknown function $y(t,x)$, in which the degeneracy is induced by the vanishing of the time derivative coefficient $u(x)$, on a subset of nonzero measure of the space domain. The equation is accompanied by Dirichlet boundary conditions and an initial condition set for the function $u(x)y(t,x)$.

The problem to be studied is

$$\begin{aligned} \frac{\partial(u(x)y)}{\partial t} - \Delta\beta^*(y) + \nabla\cdot K_0(x,y) \ni f &\quad \text{in } Q:=(0,T)\times\Omega,\\ y(t,x)=0 &\quad \text{on } \Sigma:=(0,T)\times\Gamma, \\ (u(x)y(t,x))|_{t=0} = \theta_0(x) &\quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

1.1.1 Hypotheses for the Parabolic–Elliptic Case

Let ρ, y_s and β_s^* be given positive constants.

In this section $\beta^*:(-\infty,y_s]\to\mathbb{R}$ is a multivalued function defined as

$$\beta^*(r) := \begin{cases} \int_0^r \beta(\xi)d\xi, & r<y_s,\\ [\beta_s^*,+\infty), & r=y_s, \end{cases} \tag{1.2}$$

where $\beta : (-\infty, y_s) \to (\rho, +\infty)$ is assumed of class $C^1(-\infty, y_s)$ and monotonically increasing on $[0, y_s)$. We also make the hypothesis that it has the behavior

$$\beta(r) \geq \gamma_\beta \left|r\right|^m + \rho, \text{ for } r \leq 0, \tag{1.3}$$

and the blow up property

$$\lim_{r \nearrow y_s} \beta(r) = +\infty, \tag{1.4}$$

such that

$$\lim_{r \nearrow y_s} \int_0^r \beta(r) = \beta_s^*. \tag{1.5}$$

The blow up property (1.4) together with (1.5) account for the fast diffusion character of the first equation in (1.1). In (1.3) $\gamma_\beta \geq 0$ and $m \geq 0$. For the sake of simplicity we can take in the diffusion nondegenerate case $\gamma_\beta = 0$ and set

$$\beta(r) = \rho > 0, \text{ for any } r \leq 0, \tag{1.6}$$

without losing the generality. In fact in the nondegenerate diffusion case the requirement is $\beta(r) \geq \rho > 0$. The more general form (1.3) can be treated in the same way. Consequently, β^* gets the properties

$$\left(\zeta - \overline{\zeta}\right)(r - \overline{r}) \geq \rho(r - \overline{r})^2, \ \forall r, \overline{r} \in (-\infty, y_s], \ \zeta \in \beta^*(r), \ \zeta \in \beta^*(\overline{r}), \tag{1.7}$$

$$\lim_{r \to -\infty} \beta^*(r) = -\infty, \tag{1.8}$$

$$\lim_{r \nearrow y_s} \beta^*(r) = \beta_s^*. \tag{1.9}$$

The definition of the weak solution which we give a little later will specify the exact meaning of the boundary value problem (1.1).

The function u is considered smooth enough, nonnegative and bounded by the upper bound u_M, that can be taken any positive constant. Hence we assume

$$u \in W^{1,\infty}(\Omega), \ 0 \leq u(x) \leq u_M \text{ for any } x \in \Omega, \tag{1.10}$$

revealing the degeneration of the equation at the points where u is zero. To be more specific we assume that

$$u(x) = 0 \text{ on } \overline{\Omega_0}, \ u(x) > 0 \text{ on } \Omega_u = \Omega \backslash \overline{\Omega_0}, \tag{1.11}$$

where Ω_0 is a fixed open bounded subset of Ω with meas$(\Omega_0) > 0$ and $\overline{\Omega_0}$ is strictly contained in Ω, see Fig. 1.1. The common boundary of Ω_0 and Ω_u is denoted $\partial\Omega_0$ and is assumed to be regular enough.

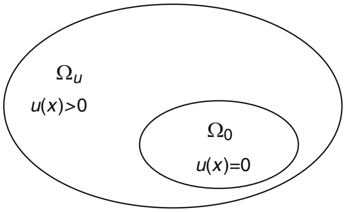

Fig. 1.1 Geometry of the problem

We also specify that the domain where u vanishes can be formed by a union of a finite number of subsets Ω_0 with the properties specified before, but we shall present the theory for only one subset.

Finally, the vector $K_0 : \Omega \times (-\infty, y_s]$ is assumed of the form

$$K_0(x,y) = \begin{cases} a(x)K(y), & x \in \Omega_u, \\ a(x), & x \in \Omega_0, \end{cases}$$

where $a(x) = (a_j(x))_{j=1,\dots,N}$,

$$a_j \in W^{1,\infty}(\Omega),\ a_j(x) = 0 \text{ in } \overline{\Omega_0},\ |a_j(x)| \le a_j^M, \text{ for } x \in \overline{\Omega}, \tag{1.12}$$

and $K : (-\infty, y_s] \to \mathbb{R}$ is Lipschitz continuous, i.e., there exists $M_K > 0$ such that

$$|K(r) - K(\overline{r})| \le M_K\, |r - \overline{r}|\,, \text{ for any } r, \overline{r} \in (-\infty, y_s]. \tag{1.13}$$

Moreover, we assume that K is bounded

$$|K(r)| \le K_s, \text{ for any } r \in \mathbb{R}. \tag{1.14}$$

The term $\nabla \cdot K_0(x,y)$ includes both a nonlinear advection term with the velocity $a(x)K'(y)$ and a nonlinear decay or source term with the rate $\nabla \cdot a$.

1.1.2 Functional Framework

We begin by establishing some notation and giving a few definitions.

Let us consider the Hilbert space $V = H_0^1(\Omega)$ with the usual Hilbertian norm

$$\|v\|_V = \left(\int_\Omega |\nabla v(x)|^2\, dx \right)^{1/2},$$

and its dual $V' = H^{-1}(\Omega)$.

The dual V' will be endowed with the scalar product

$$(y, \overline{y})_{V'} := \langle y, \psi \rangle_{V',V}, \tag{1.15}$$

where $\psi \in V$ is the solution to the elliptic problem

$$A_0 \psi = \overline{y}, \tag{1.16}$$

with $A_0 : V \to V'$ defined by

$$\langle A_0 v, \phi \rangle_{V',V} := \int_\Omega \nabla v \cdot \nabla \phi dx, \text{ for any } \phi \in V. \tag{1.17}$$

The notation $\langle y, \psi \rangle_{V',V}$ represents the pairing between V' and V and it reduces to the scalar product in $L^2(\Omega)$ if $y \in L^2(\Omega)$.

It is well known that $A_0 = -\Delta$ with Dirichlet boundary conditions is the canonical isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Moreover, it is isometric because

$$\|y\|_{V'} = \|\psi\|_V. \tag{1.18}$$

Indeed, by (1.15) and (1.16) we get

$$\|y\|_{V'}^2 = \langle y, \psi \rangle_{V',V} = \langle A_0 \psi, \psi \rangle_{V',V} = \|\psi\|_V^2,$$

where $\psi = A_0^{-1} y$.

We recall now the Poincaré inequality (see e.g., [30], pp. 290). Let Ω be a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary. For each $y \in H_0^1(\Omega)$ we have

$$\|y\| \leq c_P \|y\|_{H_0^1(\Omega)} \tag{1.19}$$

with c_P depending only on Ω and the dimension N.

We also recall that if $\theta \in L^2(\Omega)$ we have

$$\|\theta\|_{V'} \leq c_P \|\theta\|. \tag{1.20}$$

Indeed, by (1.15) and (1.18)

$$\|\theta\|_{V'}^2 = \langle \theta, \psi \rangle_{V',V} = \int_\Omega \theta \psi dx \leq \|\theta\| \|\psi\| \leq c_P \|\theta\| \|\psi\|_V = c_P \|\theta\| \|\theta\|_{V'}.$$

For $\theta(t) \in V'$, we denote by $\frac{d\theta}{dt}(t)$ the strong derivative of $\theta(t)$ in V', i.e.,

$$\frac{d\theta}{dt}(t) = \lim_{\varepsilon \to 0} \frac{\theta(t+\varepsilon) - \theta(t)}{\varepsilon} \text{ in } V'.$$

Finally, we specify that $u \in W^{1,\infty}(\Omega)$ is a multiplicator in V'. Let $\theta \in V'$. Noticing that $u\psi \in V$ for $\psi \in V$, we define

$$\langle u\theta, \psi \rangle_{V',V} := \langle \theta, u\psi \rangle_{V',V}\,, \text{ for any } \psi \in V,$$

and see by (1.15) that $u\theta$ is well defined since

$$\|u\theta\|_{V'}^2 = \langle u\theta, \psi \rangle_{V',V} = \langle \theta, u\psi \rangle_{V',V} \le \|\theta\|_{V'} \|u\psi\|_V \le C\,\|\psi\|_V = C\,\|u\theta\|_{V'}\,,$$

where $A_0\psi = u\theta$ and C includes the norm $\|u\|_{1,\infty} := \|u\|_{W^{1,\infty}(\Omega)}$.

Problem (1.1) will be approached under the following hypotheses for f and the initial datum:

$$f \in L^2(0,T;V'), \tag{1.21}$$

$$\begin{aligned} &\theta_0 \in L^2(\Omega),\ \theta_0 = 0 \text{ a.e. on } \Omega_0, \\ &\theta_0 \ge 0 \text{ a.e. on } \Omega_u,\ \frac{\theta_0}{u} \in L^2(\Omega_u),\ \frac{\theta_0}{u} \le y_s,\ \text{a.e. } x \in \Omega_u. \end{aligned} \tag{1.22}$$

We recall that $\Omega_u = \Omega \backslash \overline{\Omega_0}$ and it is an open subset of Ω. The non-negativeness assumed for θ_0 is in agreement with the physical interpretation of θ_0, that of a density (in general) or a temperature. From the mathematical point of view it does not diminish the generality.

We give now the definition of a *weak solution* to (1.1).

Definition 1.1. Let (1.21) and (1.22) hold. We call a *weak solution* to (1.1) a pair (y, ζ),

$$\begin{aligned} &y \in L^2(0,T;V), \\ &\zeta \in L^2(0,T;V),\ \zeta(t,x) \in \beta^*(y(t,x)) \text{ a.e. } (t,x) \in Q, \\ &uy \in C([0,T];L^2(\Omega)) \cap W^{1,2}([0,T];V'), \end{aligned} \tag{1.23}$$

which satisfies

$$\begin{aligned} &\left\langle \frac{d(uy)}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega (\nabla\zeta(t) - K_0(x,y(t))) \cdot \nabla\psi dx \\ &\quad = \langle f(t), \psi \rangle_{V',V}\,,\ \text{a.e. } t \in (0,T),\ \text{for any } \psi \in V, \end{aligned} \tag{1.24}$$

the initial condition $(uy(t))|_{t=0} = \theta_0$ and the boundedness condition

$$y(t,x) \le y_s \text{ a.e. } (t,x) \in Q. \tag{1.25}$$

It is easy to see that an equivalent form to (1.24), which will be used many times in this book is

$$\int_0^T \left\langle \frac{d(uy)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla\zeta - K_0(x,y)) \cdot \nabla\phi dxdt$$
$$= \int_0^T \langle f(t), \phi(t) \rangle_{V',V} dt, \text{ for any } \phi \in L^2(0,T;V). \tag{1.26}$$

A proof of the equivalence between (1.24) and (1.26) can be found in [84], pp. 81.

We also specify that a weak solution is a solution in the sense of distributions to (1.1). Indeed if we take $\phi \in C_0^\infty(Q)$ in (1.26) we get after some computations involving Green's and Ostrogradski's formulae (see [13], pp. 13) that

$$\int_Q \left(\frac{\partial(uy)}{\partial t} - \Delta\zeta + \nabla \cdot K_0(x,y) - f \right) \phi dxdt = 0, \ \forall \phi \in C_0^\infty(Q),$$

which means that

$$\frac{\partial(uy)}{\partial t} - \Delta\zeta + \nabla \cdot K_0(x,y) - f = 0 \text{ in } \mathcal{D}'(Q).$$

The boundary condition on Σ is immediately implied by the fact that the solution $y(t) \in V = H_0^1(\Omega)$ a.e. $t \in (0,T)$.

Now we pass to the abstract writing of our problem. We set

$$D(A) := \{y \in L^2(\Omega); \ \exists\zeta \in V, \ \zeta(x) \in \beta^*(y(x)) \text{ a.e. } x \in \Omega\}$$

and introduce the multivalued operator $A : D(A) \subset V' \to V'$ by

$$\langle Ay, \psi \rangle_{V',V} := \int_\Omega (\nabla\zeta - K_0(x,y)) \cdot \nabla\psi dx, \ \forall \psi \in V, \text{ for some } \zeta \in \beta^*(y).$$

With all these considerations we write the abstract evolution problem

$$\frac{d(uy)}{dt}(t) + Ay(t) \ni f(t), \text{ a.e. } t \in (0,T),$$
$$(uy(t))|_{t=0} = \theta_0. \tag{1.27}$$

We consider now the multiplication operator

$$M : D(A) \to L^2(\Omega), \ My := uy, \tag{1.28}$$

whose inverse M^{-1} is multivalued. Denoting

$$\theta(t,x) := u(x)y(t,x) \tag{1.29}$$

(and formally writing $y = M^{-1}\theta = \frac{\theta}{u}$) we can rewrite (1.27) in terms of θ as

$$\begin{aligned} \frac{d\theta}{dt}(t) + B\theta(t) \ni f(t), \text{ a.e. } t \in (0,T), \\ \theta(0) = \theta_0, \end{aligned} \tag{1.30}$$

where $B = AM^{-1}$ and

$$D(B) := \left\{ \theta \in L^2(\Omega);\ \frac{\theta}{u} \in L^2(\Omega),\ \exists \zeta \in V,\ \zeta(x) \in \beta^*\left(\frac{\theta}{u}(x)\right) \text{ a.e. } x \right\}.$$

We see that $\theta \in D(B)$ implies $\theta \in L^2(\Omega)$ and $y = \frac{\theta}{u} \in D(A)$. Conversely, if $y = \frac{\theta}{u} \in D(A)$ it follows that $\theta = uy \in D(B)$.

Besides the notion of weak solution previously given we recall the concepts of strong and mild solutions (see e.g., [11,29]). Let H be a Hilbert space and let us consider the problem

$$\begin{aligned} \frac{dz}{dt}(t) + Az(t) \ni f(t) \text{ a.e. } t \in (0,T), \\ z(0) = z_0, \end{aligned} \tag{1.31}$$

where $A : D(A) \subset H \to H$ is a nonlinear time-independent and possibly multivalued operator. Let $f \in L^1(0,T;H)$ be given, and $z_0 \in D(A)$.

A function $z \in C([0,T];H)$ is said to be a *strong solution* to the Cauchy problem (1.31) if z is absolutely continuous on any compact subinterval of $(0,T)$, satisfies (1.31) a.e. $t \in (0,T)$, $z(0) = z_0$ and $z(t) \in D(A)$ a.e. $t \in (0,T)$.

We remind that the absolute continuity on any compact subinterval of $(0,T)$ implies the a.e. differentiability on $(0,T)$, because H is a Hilbert space (generally this is true for a reflexive Banach space). Hence it is clear that a strong solution $z \in W^{1,1}([a,b];H)$, for all $0 < a < b < T$.

In literature by a *mild solution* to (1.31) it is meant a continuous function which is the uniform limit of solutions to a finite difference scheme corresponding to the problem (see [10,11]). We shall detail this definition in Chap. 2.

For a later use we still define $j : \mathbb{R} \to (-\infty,+\infty]$ by

$$j(r) := \begin{cases} \int_0^r \beta^*(\xi)d\xi, & r \le y_s, \\ +\infty, & r > y_s. \end{cases} \tag{1.32}$$

Next, we recall the concepts of lower semicontinuity (l.s.c.) and weakly lower semicontinuity and subdifferential.

Let X be a Banach space and let $\varphi : X \to [-\infty, \infty]$. The function φ is *proper* if $\varphi(x) \neq +\infty$. The function φ is *convex* if

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$$

for $\lambda \in [0,1]$ and any x_1, $x_2 \in X$.

The function φ is said *lower semicontinuous* at $x_0 \in X$ if

$$\liminf_{x \to x_0} \varphi(x) \geq \varphi(x_0).$$

If φ is l.s.c. at each point $x_0 \in X$ then it is l.s.c. on X.

A function φ is *sequentially weakly lower semicontinuous* on X if for any sequence $(x_n)_{n\geq 1}$, $x_n \in X$, such that $x_n \rightharpoonup x$ we have

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n), \ \forall x \in X.$$

Let φ be a proper convex lower semicontinuous function and let $x \in X$. The set

$$\partial\varphi(x) := \{x^* \in X'; \varphi(x) - \varphi(z) \leq \langle x^*, x - z\rangle_{X',X}\,, \ \forall z \in X\}$$

is called the *subdifferential* of φ at x.

Lemma 1.2. *The function j is proper, convex, lower semicontinuous and*

$$\partial j(r) = \begin{cases} \beta^*(r), & r < y_s \\ [\beta_s^*, +\infty), & r = y_s \\ \varnothing, & r > y_s. \end{cases} \tag{1.33}$$

Proof. First, we notice that

$$j(r) = \int_0^r \beta^*(\xi)d\xi \geq \frac{\rho}{2}r^2, \ \forall r \leq y_s. \tag{1.34}$$

Then, for $r \leq y_s$,

$$j(r) \leq j(y_s) = \lim_{r \nearrow y_s} \int_0^r \beta^*(\xi)d\xi \leq \lim_{r\nearrow y_s} \beta_s^* r = \beta_s^* y_s, \tag{1.35}$$

so j is proper. It is also obvious that j is convex.

We show now that j is lower semicontinuous. For $r < y_s$ the function j is continuous, so we have only to study what happens at y_s. Let us consider a sequence $(r_n)_{n\geq 1} \subset \mathbb{R}$, $r_n \leq y_s$, such that $r_n \to y_s$ and write

$$j(r_n) = \int_0^{r_n} \beta^*(\xi)d\xi = \int_0^{y_s} \chi_n(\xi)\beta^*(\xi)d\xi$$

where

$$\chi_n(\xi) = \begin{cases} 1 \text{ if } 0 \leq \xi \leq r_n, \\ 0 \text{ if } r_n < \xi \leq y_s. \end{cases}$$

We have $\chi_n(\xi)\beta^*(\xi) \geq 0$ and $\chi_n(\xi)\beta^*(\xi) \to \beta^*(\xi)$ a.e. on $(0, y_s)$ as $n \to \infty$. Using Fatou's lemma (see e.g., [13], pp. 3) we have

$$\liminf_{n\to\infty} j(r_n) = \liminf_{n\to\infty} \int_0^{y_s} \chi_n(\xi)\beta^*(\xi)d\xi \geq \int_0^{y_s} \beta^*(\xi)d\xi = j(y_s).$$

Finally we have to prove that $\beta^* = \partial j$. We begin with the inclusion $\beta^* \subset \partial j$. We have to prove that if $v \in \beta^*(r)$ then $v \in \partial j(r)$, for any $r \leq y_s$, i.e.,

$$j(r) - j(y) \leq v(r - y), \quad \text{for any } y \in \mathbb{R} \text{ and } r \leq y_s.$$

This inequality is obvious for $r < y_s$ and $y < y_s$ and for $r = y = y_s$.
Let $r = y_s$ and $y < y_s$. Then we have

$$j(y_s) - j(y) = \int_y^{y_s} \beta^*(\xi)d\xi = \lim_{r\nearrow y_s} \int_y^r \beta^*(\xi)d\xi \leq \beta_s^*(y_s - y) \leq v_s(y_s - y),$$

where $v_s \in [\beta_s^*, +\infty) = \beta^*(y_s)$. If $r < y_s$ and $y = y_s$, we have

$$j(r) - j(y_s) = -\int_r^{y_s} \beta^*(\xi)d\xi$$

and this comes back to the previous situation. If $r = y_s$ and $y > y_s$, then $j(y) = +\infty$ and the inequality is verified.

Now we notice that the function β^* is maximal monotone on $\mathbb{R}$. Indeed, the range $R(I + \beta^*) = \mathbb{R}$, this being implied by the observation that the equation $r + \beta^*(r) = g \in \mathbb{R}$ has a unique solution in $(-\infty, y_s]$. In conclusion, β^* is maximal and satisfies the inclusion $\beta^* \subset \partial j$, hence it should coincide with ∂j. So, we have proved (1.33) as claimed. □

1.1.3 Approximating Problem

The approach of the Cauchy problem (1.27), or equivalently (1.30) is based on some preliminary results. Since A is multivalued due to both M^{-1} and β^* we introduce an approximating problem by regularizing both of them. In this subsection we shall study the approximating problem while in the next subsection we shall prove that it converges in some sense to (1.27).

Thus, let ε be positive and replace u by

$$u_\varepsilon(x) := u(x) + \varepsilon,$$

and β^* by a regular single-valued function $\beta^*_\varepsilon : \mathbb{R} \to \mathbb{R}$. This can be defined as a regularization of β^* using mollifiers, or for convenience it can be taken of the form

$$\beta^*_\varepsilon(r) := \begin{cases} \beta^*(r), & r < y_s - \varepsilon \\ \beta^*(y_s - \varepsilon) + \frac{\beta^*_s - \beta^*(y_s-\varepsilon)}{\varepsilon}[r - (y_s - \varepsilon)], & r \geq y_s - \varepsilon. \end{cases} \tag{1.36}$$

The function β^*_ε is differentiable and has the derivative denoted β_ε bounded on $\mathbb{R}$, for each ε positive. Also, β^*_ε is monotonically increasing on $\mathbb{R}$,

$$(\beta^*_\varepsilon(r) - \beta^*_\varepsilon(\overline{r}))(r - \overline{r}) \geq \rho(r - \overline{r})^2, \ \text{ for } r, \overline{r} \in \mathbb{R}, \tag{1.37}$$

and

$$\lim_{r \to -\infty} \beta^*_\varepsilon(r) = -\infty, \quad \lim_{r \to +\infty} \beta^*_\varepsilon(r) = +\infty.$$

The function K is extended for $r \geq y_s$ by its value $K(y_s) \leq K_s$, but for the sake of simplicity we denote this extension still by K. Consequently, $K_0(x,r) = a(x)K(r)$ will extend K_0 by $a(x)K(y_s)$ for $r \geq y_s$.

Then we define the single-valued operator $A_\varepsilon : D(A_\varepsilon) \subset V' \to V'$, where

$$D(A_\varepsilon) := \{y \in L^2(\Omega);\ \beta^*_\varepsilon(y) \in V\},$$

$$\langle A_\varepsilon y, \psi \rangle_{V',V} := \int_\Omega (\nabla \beta^*_\varepsilon(y) - K_0(x,y)) \cdot \nabla \psi dx, \text{ for any } \psi \in V, \tag{1.38}$$

and we introduce the approximating Cauchy problem

$$\frac{d(u_\varepsilon y_\varepsilon)}{dt}(t) + A_\varepsilon y_\varepsilon(t) = f(t), \text{ a.e. } t \in (0,T),$$
$$u_\varepsilon y_\varepsilon(0) = \theta_0. \tag{1.39}$$

Denoting now $\theta_\varepsilon := u_\varepsilon y_\varepsilon$ we can write the equivalent approximating Cauchy problem in terms of θ_ε,

$$\frac{d\theta_\varepsilon}{dt}(t) + B_\varepsilon \theta_\varepsilon(t) = f(t), \text{ a.e. } t \in (0,T),$$
$$\theta_\varepsilon(0) = \theta_0. \tag{1.40}$$

The operator $B_\varepsilon : D(B_\varepsilon) \subset V' \to V'$ is single-valued, has the domain

$$D(B_\varepsilon) := \left\{ v \in L^2(\Omega);\ \beta_\varepsilon^*\left(\frac{v}{u_\varepsilon}\right) \in V \right\}$$

and is defined by

$$\langle B_\varepsilon v, \psi \rangle_{V',V} := \int_\Omega \left(\nabla \beta_\varepsilon^*\left(\frac{v}{u_\varepsilon}\right) - K_0\left(x, \frac{v}{u_\varepsilon}\right)\right) \cdot \nabla\psi dx, \text{ for any } \psi \in V. \tag{1.41}$$

In fact we note that $B_\varepsilon v = A_\varepsilon\left(\frac{v}{u_\varepsilon}\right)$ and $v \in D(B_\varepsilon)$ is equivalent to $\frac{v}{u_\varepsilon} \in D(A_\varepsilon)$.

Also, it is easily seen that $D(B_\varepsilon) = V$. Indeed, if $v \in D(B_\varepsilon)$ it follows that $\frac{v}{u_\varepsilon} \in V$ by the fact that the inverse of β_ε^* is Lipschitz, and from here we get that $v \in V$, since $u_\varepsilon \in W^{1,\infty}(\Omega)$. Conversely, $v \in V$ implies $\frac{v}{u_\varepsilon} \in V$ and taking into account that the derivative of β_ε^* is bounded for each $\varepsilon > 0$ we obtain that $\beta_\varepsilon^*\left(\frac{v}{u_\varepsilon}\right) \in V$. We recall that $u_\varepsilon = u + \varepsilon \in W^{1,\infty}(\Omega)$.

Definition 1.3. Let (1.21) and (1.22) hold. We call a *strong solution* to (1.40) a function

$$\theta_\varepsilon \in C([0,T]; L^2(\Omega)) \cap W^{1,2}([0,T]; V'),\ \beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \in L^2(0,T;V),$$

that satisfies (1.40), which can be still written

$$\left\langle \frac{d\theta_\varepsilon}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega \left(\nabla \beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) - K_0\left(x, \frac{\theta_\varepsilon}{u_\varepsilon}\right)\right) \cdot \nabla\psi dx$$
$$= \langle f(t), \psi \rangle_{V',V},\ \text{a.e. } t \in (0,T), \quad \text{for any } \psi \in V \tag{1.42}$$

and $\theta_\varepsilon(0) = \theta_0$.

Since by $\theta_\varepsilon := u_\varepsilon y_\varepsilon$, problems (1.40) and (1.39) are equivalent, it means that if θ_ε is a solution to (1.42) then y_ε is a solution to (1.39) and belongs to the same spaces as θ_ε.

An equivalent form to (1.42) can be written as

$$\int_0^T \left\langle \frac{d(u_\varepsilon y_\varepsilon)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla \beta_\varepsilon^*(y_\varepsilon) - K_0(x, y_\varepsilon)) \cdot \nabla\phi dxdt$$
$$= \int_0^T \langle f(t), \phi(t) \rangle_{V',V}, \quad \text{for any } \phi \in L^2(0,T;V). \tag{1.43}$$

1.1.4 Existence for the Approximating Problem

First we shall prove that, for each $\varepsilon > 0$, (1.40) has a unique solution θ_ε and consequently, (1.39) has a unique solution in their appropriate functional spaces. The proof is essentially based on the quasi m-accretivity of the operator B_ε on V'. Because we are working in Hilbert spaces, we recall the celebrated theorem of Minty (see [79], or [14], pp. 34), by which the notion of a maximal monotone operator is equivalent with that of m-accretive operator.

We say that B_ε is *quasi m-accretive* on V' if $\lambda I + B_\varepsilon$ is *monotone*,

$$((\lambda I + B_\varepsilon)\theta - (\lambda I + B_\varepsilon)\overline{\theta}, \theta - \overline{\theta})_{V'} \geq 0, \ \forall \theta, \overline{\theta} \in D(B_\varepsilon),$$

and surjective,

$$R(\lambda I + B_\varepsilon) = V',$$

for all $\lambda > \lambda_0$.

Lemma 1.4. *The operator B_ε is quasi m-accretive on V'.*

Proof. Let θ, $\overline{\theta} \in D(B_\varepsilon)$. We compute

$$\begin{aligned}\left(B_\varepsilon\theta - B_\varepsilon\overline{\theta}, \theta - \overline{\theta}\right)_{V'} &= \int_\Omega \nabla\left(\beta_\varepsilon^*\left(\frac{\theta}{u_\varepsilon}\right) - \beta_\varepsilon^*\left(\frac{\overline{\theta}}{u_\varepsilon}\right)\right)\cdot\nabla\psi dx \\ &\quad - \int_\Omega \left(K_0\left(x, \frac{\theta}{u_\varepsilon}\right) - K_0\left(x, \frac{\overline{\theta}}{u_\varepsilon}\right)\right)\cdot\nabla\psi dx\end{aligned}$$

where $\psi \in V$ is the solution to $A_0\psi = \theta - \overline{\theta}$. Recalling (1.12)–(1.13) and that $\varepsilon \leq u_\varepsilon(x) \leq u_M + \varepsilon$ we have

$$\begin{aligned}&\int_\Omega \left(K_0\left(x, \frac{\theta}{u_\varepsilon}\right) - K_0\left(x, \frac{\overline{\theta}}{u_\varepsilon}\right)\right)\cdot\nabla\psi dx \\ &\quad \leq \sum_{j=1}^N \int_{\Omega_u} M_K\left|a_j(x)\right|\left|\frac{\theta}{u_\varepsilon} - \frac{\overline{\theta}}{u_\varepsilon}\right|\left|\frac{\partial\psi}{\partial x_j}\right| dx \\ &\quad \leq \sum_{j=1}^N M_K a_j^M \left\|\frac{\theta - \overline{\theta}}{u_\varepsilon}\right\|_{L^2(\Omega_u)} \|\nabla\psi\|_{L^2(\Omega_u)} \\ &\quad \leq \frac{\overline{M}}{\varepsilon}\left\|\theta - \overline{\theta}\right\| \|\psi\|_V = \frac{\overline{M}}{\varepsilon}\left\|\theta - \overline{\theta}\right\| \left\|\theta - \overline{\theta}\right\|_{V'}, \qquad (1.44)\end{aligned}$$

where we have denoted $\overline{M} = M_K \sum\limits_{j=1}^{N} a_j^M$. Next, taking into account (1.37) we compute

$$\begin{aligned}
&((\lambda I + B_\varepsilon)\theta - (\lambda I + B_\varepsilon)\overline{\theta}, \theta - \overline{\theta})_{V'} \\
&\quad = \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \left(B_\varepsilon\theta - B_\varepsilon\overline{\theta}, \theta - \overline{\theta}\right)_{V'} \\
&\quad \geq \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \int_\Omega \left(\beta_\varepsilon^*\left(\frac{\theta}{u_\varepsilon}\right) - \beta_\varepsilon^*\left(\frac{\overline{\theta}}{u_\varepsilon}\right)\right)(\theta - \overline{\theta})dx \\
&\qquad - \frac{\overline{M}}{\varepsilon}\left\|\theta - \overline{\theta}\right\|\left\|\theta - \overline{\theta}\right\|_{V'} \\
&\quad \geq \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \frac{\rho}{2(u_M + \varepsilon)}\left\|\theta - \overline{\theta}\right\|^2 - \frac{\overline{M}^2}{2\varepsilon^2}\frac{u_M + \varepsilon}{\rho}\left\|\theta - \overline{\theta}\right\|_{V'}^2 \\
&\quad = \left(\lambda - \frac{\overline{M}^2}{2\varepsilon^2}\frac{u_M + \varepsilon}{\rho}\right)\left\|\theta - \overline{\theta}\right\|_{V'}^2 + \frac{\rho}{2(u_M + \varepsilon)}\left\|\theta - \overline{\theta}\right\|^2,
\end{aligned} \tag{1.45}$$

so that B_ε is quasi-monotone for $\lambda \geq \lambda_0 = \frac{\overline{M}^2(u_M+\varepsilon)}{2\rho\varepsilon^2}$. We recall that ε is positive fixed.

Next we have to prove that $R(\lambda I + B_\varepsilon) = V'$ for λ large, i.e., to show that the equation

$$\lambda\theta_\varepsilon + B_\varepsilon\theta_\varepsilon = g \tag{1.46}$$

has a solution $\theta_\varepsilon \in D(B_\varepsilon)$ for any $g \in V'$. If we denote $\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) = \zeta \in V$, due to the fact that β_ε^* is continuous and monotonically increasing on $\mathbb{R}$ and $R(\beta_\varepsilon^*) = (-\infty, \infty)$ it follows that its inverse

$$G\zeta := u_\varepsilon(\beta_\varepsilon^*)^{-1}(\zeta) \tag{1.47}$$

is continuous from V to $L^2(\Omega)$. Indeed, for ζ, $\overline{\zeta} \in V$

$$\begin{aligned}
\left\|G\zeta - G\overline{\zeta}\right\| &= \left\|u_\varepsilon\left((\beta_\varepsilon^*)^{-1}(\zeta) - (\beta_\varepsilon^*)^{-1}(\overline{\zeta})\right)\right\| \\
&\leq \frac{u_M + \varepsilon}{\rho}\left\|\zeta - \overline{\zeta}\right\| \leq \frac{(u_M + \varepsilon)c_P}{\rho}\left\|\zeta - \overline{\zeta}\right\|_V,
\end{aligned} \tag{1.48}$$

where we used (1.37) and Poincaré's inequality (with the constant c_P). So, (1.46) can be rewritten as

$$\lambda G\zeta + B_0\zeta = g \tag{1.49}$$

with $B_0 : V \to V'$ defined by

$$\langle B_0\zeta, \psi\rangle_{V',V} := \int_\Omega \left(\nabla\zeta - K_0\left(x, \frac{G\zeta}{u_\varepsilon}\right)\right) \cdot \nabla\psi dx, \ \forall\psi \in V. \tag{1.50}$$

We shall show that $\lambda G + B_0$ is surjective. First we have

$$\begin{aligned}
&\left\langle (\lambda G + B_0)\zeta - (\lambda G + B_0)\overline{\zeta}, \zeta - \overline{\zeta} \right\rangle_{V',V} \\
&= \lambda \int_\Omega (G\zeta - G\overline{\zeta})(\zeta - \overline{\zeta})dx + \int_\Omega \left|\nabla(\zeta - \overline{\zeta})\right|^2 dx \\
&\quad - \int_\Omega a(x)\left(K\left(\frac{G\zeta}{u_\varepsilon}\right) - K\left(\frac{G\overline{\zeta}}{u_\varepsilon}\right)\right)\cdot \nabla(\zeta - \overline{\zeta})dx \\
&\geq \int_\Omega \frac{\lambda\rho}{u_\varepsilon}(G\zeta - G\overline{\zeta})^2 dx + \int_\Omega \left|\nabla(\zeta - \overline{\zeta})\right|^2 dx \\
&\quad - \frac{\overline{M}}{\varepsilon}\left\|G\zeta - G\overline{\zeta}\right\| \left\|\zeta - \overline{\zeta}\right\|_V \\
&\geq \left(\frac{\lambda\rho}{u_M + \varepsilon} - \frac{\overline{M}^2}{2\varepsilon^2}\right)\left\|G\zeta - G\overline{\zeta}\right\|^2 + \frac{1}{2}\left\|\zeta - \overline{\zeta}\right\|_V^2,
\end{aligned}$$

so $\lambda G + B_0 : V \to V'$ is monotone and obviously coercive for $\lambda > \lambda_0$.

We recall that the operator $T : V \to V'$ is called *coercive* if

$$\lim_{n\to\infty} \frac{\langle Tz_n, z_n\rangle_{V',V}}{\|z_n\|_V} = +\infty$$

for any sequence $(z_n)_{n\geq 1}$ with $\lim\limits_{n\to\infty} \|z_n\|_V = +\infty$.

The inequality (1.48) implies also that the operator $\lambda G + B_0$ is continuous from V to $V^{'}$ and since it is monotone it follows that it is m-accretive. Being also coercive it is surjective (see [14], pp. 37). Therefore (1.49) has a solution meaning in fact that we have proved that (1.46) has a solution $\theta_\varepsilon \in D(B_\varepsilon)$, i.e., that B_ε is quasi m-accretive. □

Next we give an intermediate result that will be used in the existence proof of the solution to the approximating problem.

First we define

$$j_\varepsilon(r) := \int_0^r \beta_\varepsilon^*(\xi)d\xi, \ \forall r \in \mathbb{R}, \tag{1.51}$$

and notice that $\partial j_\varepsilon(r) = \beta_\varepsilon^*(r)$, for any $r \in \mathbb{R}$.

Let

$$\overline{K} = K_s(\mathrm{meas}(\Omega))^{1/2} \sum_{j=1}^N a_j^M.$$

Proposition 1.5. *Let $f \in L^2(0,T;V')$ and $\theta_0 \in L^2(\Omega)$. Then problem (1.40) has a unique strong solution satisfying*

$$\int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right)dx + \frac{1}{4}\int_0^t \left\|\frac{d\theta_\varepsilon}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \frac{1}{4}\int_0^t \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right)\right\|_V^2 d\tau$$
$$\leq \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right)dx + \int_0^T \|f(t)\|_{V'}^2\, dt + \overline{K}^2 T,\ t \in [0,T]. \tag{1.52}$$

Moreover,

$$\left\|\theta_\varepsilon(t) - \overline{\theta}_\varepsilon(t)\right\|_{V'}^2 + \frac{\rho}{u_M+\varepsilon}\int_0^t \left\|\left(\theta_\varepsilon - \overline{\theta}_\varepsilon\right)(\tau)\right\|^2 d\tau$$
$$\leq e^{\left(\frac{\overline{M}^2}{\varepsilon^2}\frac{u_M+\varepsilon}{\rho}+1\right)T}\left(\left\|\theta_0 - \overline{\theta}_0\right\|_{V'}^2 + \int_0^T \left\|f(t) - \overline{f}(t)\right\|_{V'}^2 dt\right) \tag{1.53}$$

where θ_ε and $\overline{\theta}_\varepsilon$ are two solutions to (1.40) corresponding to the pairs of data θ_0, f and $\overline{\theta}_0, \overline{f}$, respectively.

In addition, if $f \in W^{1,2}([0,T];L^2(\Omega))$ and $\theta_0 \in V$, then

$$\theta_\varepsilon, y_\varepsilon, \beta_\varepsilon^*(y_\varepsilon) \in L^2(0,T;H^2(\Omega)). \tag{1.54}$$

Proof. The proof is done in two steps. At the first step we take

$$\theta_0 \in D(B_\varepsilon),\ f \in W^{1,1}([0,T];V').$$

Hence the existence of a unique solution to (1.40)

$$\theta_\varepsilon \in C([0,T];V') \cap W^{1,\infty}([0,T];V') \cap L^\infty(0,T;D(B_\varepsilon)),$$
$$\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \in L^\infty(0,T;V)$$

follows from the general theorems for evolution equations with m-accretive operators (see [14], pp. 141).

By the properties assumed for β_ε^*, we deduce by (1.37) that its inverse is Lipschitz with the constant $\frac{1}{\rho}$, hence $\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right) \in D(B_\varepsilon) = H_0^1(\Omega)$ implies $\frac{\theta_\varepsilon}{u_\varepsilon}(t) \in H^1(\Omega)$, a.e. t. Since $(\beta_\varepsilon^*)^{-1}(0) = 0$ the trace of $\frac{\theta_\varepsilon}{u_\varepsilon}(t)$ (see [13], pp. 122) makes sense and vanishes on Γ. Therefore $\frac{\theta_\varepsilon}{u_\varepsilon} \in L^\infty(0,T;V)$. For proving the estimate (1.52) we test (1.40) for $\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \in V$ and integrate over $(0,t)\times\Omega$. Taking into account the relation

$$\begin{aligned}
&\int_0^t \left\langle \frac{d\theta_\varepsilon}{d\tau}(\tau), \beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right)\right\rangle_{V',V} d\tau \\
&\quad = \int_0^t \int_\Omega u_\varepsilon(x)\frac{d}{d\tau}\left(j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right)\right) dx d\tau \\
&\quad = \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right) dx - \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right) dx,
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right) dx + \int_0^t \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right)\right\|_V^2 d\tau \\
&\quad \le \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right) dx + \int_0^t \|f(\tau)\|_{V'} \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right)\right\|_V d\tau \\
&\qquad - \int_0^t \int_\Omega K_0\left(x, \frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right) \cdot \nabla\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right) dx d\tau.
\end{aligned}$$

From there, using (1.14) we get

$$\begin{aligned}
&\int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right) dx + \frac{1}{2}\int_0^t \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau)\right)\right\|_V^2 d\tau \\
&\quad \le \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right) dx + \int_0^T \|f(t)\|_{V'}^2 dt + \overline{K}^2 T, \text{ for } t \in [0,T].
\end{aligned} \tag{1.55}$$

Next, we multiply (1.40) scalarly in V' by $\frac{d\theta_\varepsilon}{dt}$ and integrate over $(0,t)$. By similar computations based on the definition of the scalar product in V', we get

$$\begin{aligned}
&\frac{1}{2}\int_0^t \left\|\frac{d\theta_\varepsilon}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right) dx \\
&\quad \le \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right) dx + \int_0^T \|f(t)\|_{V'}^2 dt + \overline{K}^2 T.
\end{aligned} \tag{1.56}$$

Adding the previous two inequalities we obtain (1.52).

In the second step we take

$$\theta_0 \in L^2(\Omega) = \overline{D(B_\varepsilon)},\ \ f \in L^2(0,T;V').$$

Since $W^{1,1}([0,T];V')$ is dense in $L^2(0,T;V')$ and $D(B_\varepsilon) = V$ is dense in $L^2(\Omega)$ we can take the sequences $(f_n)_{n\geq 1} \subset W^{1,1}([0,T];V')$ and $(\theta_0^n)_{n\geq 1} \subset D(B_\varepsilon)$ such that

$$f_n \to f \text{ strongly in } L^2(0,T;V'),$$
$$\theta_0^n \to \theta_0 \text{ strongly in } L^2(\Omega) \text{ as } n \to \infty.$$

Then, for each $\varepsilon > 0$, the problem

$$\frac{d\theta_\varepsilon^n}{dt}(t) + B_\varepsilon \theta_\varepsilon^n(t) = f_n(t),\ \text{ a.e. } t \in (0,T), \tag{1.57}$$
$$\theta_\varepsilon^n(0) = \theta_0^n$$

has, according to the first step, a unique solution θ_ε^n satisfying the estimate (1.52), namely,

$$\int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}(t)\right) dx + \frac{1}{4}\int_0^t \left\|\frac{d\theta_\varepsilon^n}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \frac{1}{4}\int_0^t \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}(\tau)\right)\right\|_V^2 d\tau$$
$$\leq \int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0^n}{u_\varepsilon}\right) dx + \int_0^T \|f_n(t)\|_{V'}^2\, dt + \overline{K}^2 T, \tag{1.58}$$

for any $t \in [0,T]$. We stress that ε is fixed.

We notice that j_ε is Lipschitz and by the definition of β_ε^* and j_ε we have

$$\int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_0^n}{u_\varepsilon}\right) dx \leq (u_M + \varepsilon)\frac{\beta_s^* - \beta^*(y_s - \varepsilon)}{2\varepsilon}\left\|\frac{\theta_0^n}{u_\varepsilon}\right\|^2, \tag{1.59}$$

whence

$$\int_\Omega u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}(t)\right) dx + \frac{1}{4}\int_0^t \left\|\frac{d\theta_\varepsilon^n}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \frac{1}{4}\int_0^t \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}(\tau)\right)\right\|_V^2 d\tau$$
$$\leq (u_M + \varepsilon)\frac{\beta_s^* - \beta^*(y_s - \varepsilon)}{2\varepsilon}\left\|\frac{\theta_0^n}{u_\varepsilon}\right\|^2 + \int_0^T \|f_n(t)\|_{V'}^2\, dt + \overline{K}^2 T \tag{1.60}$$
$$\leq (u_M + \varepsilon)\frac{\beta_s^* - \beta^*(y_s - \varepsilon)}{\varepsilon}\left\|\frac{\theta_0}{u_\varepsilon}\right\|^2 + \int_0^T \|f(t)\|_{V'}^2\, dt + \overline{K}^2 T + 2\varepsilon,$$

due to the strong convergence $\theta_0^n \to \theta_0$ and $f_n \to f$ as $n \to \infty$. Thus the right-hand side in (1.60) is independent of n, since ε is small, fixed, e.g. $\varepsilon \ll 1$.

Recalling (1.34), $j_\varepsilon(r) \geq \frac{\rho}{2} r^2$ for any $r \in \mathbb{R}$, we can write by (1.60) that

$$\frac{\rho}{(u_M+\varepsilon)} \|\theta_\varepsilon^n(t)\|^2 \leq (u_M+1) \frac{\beta_s^* - \beta^*(y_s - \varepsilon)}{\varepsilon} \left\| \frac{\theta_0}{u_\varepsilon} \right\|^2 + \int_0^T \|f(t)\|_{V'}^2 \, dt + \overline{K}^2 T + 2, \quad (1.61)$$

for any $t \in [0,T]$.

We deduce that $\left(\beta_\varepsilon^*\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}\right)\right)_n$ lies in a bounded subset of $L^2(0,T;V)$ and $\left(\frac{d\theta_\varepsilon^n}{dt}\right)_n$ is in a bounded subset of $L^2(0,T;V')$. Therefore we can select a subsequence, denoted still by the subscript n, such that

$$\frac{d\theta_\varepsilon^n}{dt} \rightharpoonup \frac{d\theta_\varepsilon}{dt} \text{ in } L^2(0,T;V') \text{ as } n \to \infty,$$

$$\beta_\varepsilon^*\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}\right) \rightharpoonup \zeta_\varepsilon \text{ in } L^2(0,T;V) \text{ as } n \to \infty.$$

The latter immediately implies that

$$\frac{\theta_\varepsilon^n}{u_\varepsilon} \rightharpoonup y_\varepsilon \text{ in } L^2(0,T;V) \text{ as } n \to \infty.$$

But $u_\varepsilon \in W^{1,\infty}(\Omega)$ and the sequence $(\theta_\varepsilon)_n = \left(u_\varepsilon \frac{\theta_\varepsilon^n}{u_\varepsilon}\right)_n$ is bounded in $L^2(0,T;V)$ so that we get

$$\theta_\varepsilon^n \rightharpoonup \theta_\varepsilon \text{ in } L^2(0,T;V) \text{ as } n \to \infty.$$

At this point we recall the following theorem (see [7, 77]).

Theorem (Aubin–Lions). *Let X_1, X_2, X_3 be three Banach spaces, X_1 and X_3 reflexive, $X_1 \subset X_2 \subset X_3$ with dense and continuous inclusions and the inclusion $X_1 \subset X_2$ is compact. Let $(z_n)_{n\geq 1}$ be a bounded sequence in $L^{p_1}(0,T;X_1)$ such that $(\frac{dz_n}{dt})_{n\geq 1}$ is bounded in $L^{p_3}(0,T;X_3)$. Then $(z_n)_{n\geq 1}$ is compact in $L^{p_2}(0,T;X_2)$, where $1 \leq p_1,\ p_2,\ p_3 < \infty$.*

On the basis of the previous convergencies and since V is compact in $L^2(\Omega)$ it follows by the above theorem that

$$\theta_\varepsilon^n \to \theta_\varepsilon \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } n \to \infty$$

and also (since $u_\varepsilon \geq \varepsilon$) that

$$\frac{\theta_\varepsilon^n}{u_\varepsilon} \to \frac{\theta_\varepsilon}{u_\varepsilon} \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } n \to \infty.$$

By (1.36) we have

$$\left\| \beta_\varepsilon^* \left(\frac{\theta_\varepsilon^n}{u_\varepsilon} \right) - \beta_\varepsilon^* \left(\frac{\theta_\varepsilon}{u_\varepsilon} \right) \right\|_{L^2(Q)} = \left\| \frac{\beta_s^* - \beta^*(y_s - \varepsilon)}{\varepsilon} \left(\frac{\theta_\varepsilon^n}{u_\varepsilon} - \frac{\theta_\varepsilon}{u_\varepsilon} \right) \right\|_{L^2(Q)}$$

and deduce that

$$\beta_\varepsilon^* \left(\frac{\theta_\varepsilon^n}{u_\varepsilon} \right) \to \beta_\varepsilon^* \left(\frac{\theta_\varepsilon}{u_\varepsilon} \right) \text{ in } L^2(Q), \text{ as } n \to \infty,$$

hence $\zeta_\varepsilon = \beta_\varepsilon^* \left(\frac{\theta_\varepsilon}{u_\varepsilon} \right)$ a.e. on Q.

Moreover, since K is Lipschitz it follows that

$$K \left(\frac{\theta_\varepsilon^n}{u_\varepsilon} \right) \to K \left(\frac{\theta_\varepsilon}{u_\varepsilon} \right) \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } n \to \infty.$$

Finally, the Ascoli–Arzelà theorem (see below) implies that

$$\theta_\varepsilon^n(t) \to \theta_\varepsilon(t) \text{ in } V', \text{ as } n \to \infty, \text{ uniformly in } t \in [0,T], \tag{1.62}$$

as we further prove. First we recall this theorem.

Theorem (Ascoli–Arzelà). *Let X be a Banach space and let $\mathcal{M} \subset C([0,T];X)$ be a family of functions such that*

(i) $\|u(t)\|_X \le C$, $\forall t \in [0,T]$, $u \in \mathcal{M}$,
(ii) *$\mathcal{M}$ is equi-uniformly continuous i.e., $\forall \varepsilon$, $\exists \delta(\varepsilon)$ such that*

$$\|u(t) - u(s)\|_X \le \varepsilon \textit{ if } |t-s| \le \delta(\varepsilon), \ \forall u \in \mathcal{M},$$

(iii) *For each $t \in [0,T]$ the set $\{u(t); u \in \mathcal{M}\}$ is compact in X.*

Then, $\mathcal{M}$ is compact in $C([0,T];X)$.

Indeed, the family $\mathcal{M} = (\theta_\varepsilon^n)_n \subset C([0,T];V')$ is bounded (this follows e.g., by (1.61)) and equi-uniformly continuous. To prove this, let $\varepsilon' > 0$ and consider that $\sigma(\varepsilon')$ exists such that $|t-s| \le \sigma(\varepsilon')$, for $0 \le s < t \le T$. We have

$$\begin{aligned} \left\|\theta_\varepsilon^n(t) - \theta_\varepsilon^n(s)\right\|_{V'} &= \left\| \int_s^t \frac{d\theta_\varepsilon^n}{dt}(\tau) d\tau \right\|_{V'} \le \int_s^t \left\| \frac{d\theta_\varepsilon^n}{dt}(\tau) \right\|_{V'} d\tau \\ &\le |t-s|^{1/2} \left\| \frac{d\theta_\varepsilon^n}{dt} \right\|_{L^2(0,T;V')} \le \varepsilon', \text{ for } \sigma(\varepsilon') \le \frac{\varepsilon'^2}{\gamma_0(\varepsilon)}, \ \forall \theta_\varepsilon^n \in \mathcal{M}, \end{aligned}$$

where $\gamma_0(\varepsilon)$ is the right-hand side in (1.60) which is independent of n. Still by (1.61) we get that the sequence $(\theta_\varepsilon^n(t))_n$ is bounded in $L^2(\Omega)$ for any $t \in [0,T]$ and since the injection of $L^2(\Omega)$ in V' is compact it follows that the sequence $(\theta_\varepsilon^n(t))_n$ is compact in V', for each $t \in [0,T]$. Hence the set $\mathcal{M}$ is compact in $C([0,T];V')$, i.e., we have (1.62).

From here we get that $\lim\limits_{n\to\infty} \theta_\varepsilon^n(0) = \theta_\varepsilon(0)$, whence $\theta_0 = \theta_\varepsilon(0)$.

By (1.57) we have that

$$B_\varepsilon \theta_\varepsilon^n = f_n - \frac{d\theta_\varepsilon^n}{dt} \rightharpoonup f - \frac{d\theta_\varepsilon}{dt} \text{ in } L^2(0,T;V'), \text{ as } n \to \infty.$$

Since B_ε is quasi m-accretive on V', its realization on $L^2(0,T;V')$ is quasi m-accretive too, hence it is demiclosed and the previous weak convergence together with the strong convergence $\theta_\varepsilon^n \to \theta_\varepsilon$ leads to

$$B_\varepsilon \theta_\varepsilon = f - \frac{d\theta_\varepsilon}{dt} \text{ in } L^2(0,T;V'),$$

(see [14], pp.100). We recall that a subset A of $X \times X$ is called *demiclosed* if it is strongly–weakly closed in $X \times X$, i.e., $z_n \to z$, $w_n \rightharpoonup w$ where $w_n \in Az_n$ imply $w \in Az$. Thus, we have got (1.40), and proved that this problem has the solution $\theta_\varepsilon \in C([0,T], L^2(\Omega)) \cap W^{1,2}([0,T];V') \cap L^2(0,T;V)$.

Finally, passing to limit in (1.58) as $n \to \infty$, and using the lower semicontinuity property we get (1.52), as claimed.

Consider now two problems (1.40) corresponding to the pairs of data θ_0, f and $\overline{\theta}_0, \overline{f}$. They have the solutions denoted θ_ε and $\overline{\theta}_\varepsilon$, respectively. We subtract the equations and multiply the difference by $(\theta_\varepsilon - \overline{\theta}_\varepsilon)(t)$, scalarly in V'. Then we integrate it over $(0,t)$. A few calculations on the basis of (1.45) lead us to

$$\begin{aligned}
&\left\|\theta_\varepsilon(t) - \overline{\theta}_\varepsilon(t)\right\|_{V'}^2 + \frac{\rho}{u_M+\varepsilon}\int_0^t \left\|\theta_\varepsilon(\tau) - \overline{\theta}_\varepsilon(\tau)\right\|^2 d\tau \le \left\|\theta_0 - \overline{\theta}_0\right\|_{V'}^2 \\
&\quad + \int_0^T \left\|f(t) - \overline{f}(t)\right\|_{V'}^2 dt + \left(\frac{\overline{M}^2(u_M+\varepsilon)}{\varepsilon^2\rho} + 1\right)\int_0^t \left\|\left(\theta_\varepsilon - \overline{\theta}_\varepsilon\right)(\tau)\right\|_{V'}^2 d\tau
\end{aligned}$$

which by the Gronwall's lemma implies (1.53). This also implies the uniqueness if the data are the same.

Finally, we give an idea for the proof of (1.54). Let $f \in W^{1,2}([0,T];L^2(\Omega))$ and $\theta_0 \in V$. A rigorous computation means to replace (1.40) by a time finite difference equation, to multiply it by $\frac{\beta_\varepsilon^*(y_\varepsilon(t+h)) - \beta_\varepsilon^*(y_\varepsilon(t))}{h}$ which is in V and to integrate with respect to t. For simplicity we present a more formal computation. We multiply (1.40) by $\frac{\partial \beta_\varepsilon^*(y_\varepsilon)}{\partial t}$ and integrate over $(0,t) \times \Omega$. We get

$$\begin{aligned}
&\int_0^t \int_\Omega u_\varepsilon \beta_\varepsilon(y_\varepsilon)\left(\frac{dy_\varepsilon}{d\tau}\right)^2 dx d\tau + \frac{1}{2}\int_0^t \frac{d}{d\tau}\left\|\nabla \beta_\varepsilon^*(y_\varepsilon(\tau))\right\|^2 d\tau \\
&\quad = \int_0^t \int_\Omega a(x) K(y_\varepsilon) \cdot \nabla\left(\frac{d\beta_\varepsilon^*(y_\varepsilon(\tau))}{d\tau}\right) dx d\tau + \int_0^t \int_\Omega f \frac{d\beta_\varepsilon^*(y_\varepsilon)}{d\tau} dx d\tau.
\end{aligned}$$

After the integration with respect to τ in the second term on the left-hand side, we obtain

$$\int_0^t \int_\Omega u_\varepsilon \beta_\varepsilon(y_\varepsilon) \left(\frac{dy_\varepsilon}{d\tau}\right)^2 dx d\tau + \frac{1}{2} \|\beta_\varepsilon^*(y_\varepsilon(t))\|_V^2 - \frac{1}{2} \|\beta_\varepsilon^*(y_\varepsilon(0))\|_V^2$$

$$= \int_\Omega a(x) K(y_\varepsilon(t)) \cdot \nabla \beta_\varepsilon^*(y_\varepsilon(t)) dx - \int_\Omega a(x) K(y_\varepsilon(0)) \cdot \nabla \beta_\varepsilon^*(y_\varepsilon(0)) dx$$

$$- \int_0^t \int_\Omega a(x) \frac{\partial K(y_\varepsilon)}{\partial \tau} \cdot \nabla \beta_\varepsilon^*(y_\varepsilon(\tau)) dx d\tau$$

$$+ \int_\Omega f(t) \beta_\varepsilon^*(y_\varepsilon(t)) dx - \int_\Omega f(0) \beta_\varepsilon^*(y_\varepsilon(0)) dx - \int_0^t \int_\Omega \frac{\partial f}{\partial \tau} \beta_\varepsilon^*(y_\varepsilon) dx d\tau.$$

Next we have

$$\int_0^t \int_\Omega u_\varepsilon \beta_\varepsilon(y_\varepsilon) \left(\frac{dy_\varepsilon}{d\tau}\right)^2 d\tau dx + \frac{1}{2} \|\beta_\varepsilon^*(y_\varepsilon(t))\|_V^2$$

$$\leq C_0(\varepsilon) + \overline{M} \|y_\varepsilon(t)\| \|\beta_\varepsilon^*(y_\varepsilon(t))\|_V + \overline{M} \int_0^t \left\| \frac{dy_\varepsilon}{d\tau}(\tau) \right\| \|\beta_\varepsilon^*(y_\varepsilon(\tau))\|_V d\tau$$

$$+ c_P \|f(t)\| \|\beta_\varepsilon^*(y_\varepsilon(t))\|_V + c_P \int_0^t \left\| \frac{\partial f}{\partial \tau}(\tau) \right\| \|\beta_\varepsilon^*(y_\varepsilon(\tau))\|_V d\tau,$$

where

$$C_0(\varepsilon) = \frac{1}{2} \left\| \beta_\varepsilon^* \left(\frac{\theta_0}{u_\varepsilon}\right) \right\|_V^2 + \overline{M} \left\| \frac{\theta_0}{u_\varepsilon} \right\| \left\| \beta_\varepsilon^* \left(\frac{\theta_0}{u_\varepsilon}\right) \right\|_V + c_P \|f(0)\| \left\| \beta_\varepsilon^* \left(\frac{\theta_0}{u_\varepsilon}\right) \right\|_V. \tag{1.63}$$

By $\beta_\varepsilon(y_\varepsilon) \geq \rho$ and (1.52) we deduce

$$\rho\varepsilon \int_0^t \int_\Omega \left(\frac{\partial y_\varepsilon}{\partial \tau}\right)^2 dx d\tau + \frac{1}{4} \|\beta_\varepsilon^*(y_\varepsilon)\|_V^2$$

$$\leq C_0(\varepsilon) + \frac{\rho}{2} \int_0^t \int_\Omega \varepsilon \left(\frac{\partial y_\varepsilon}{\partial \tau}\right)^2 dx d\tau + \frac{1}{2} \left(\frac{\overline{M}^2}{\rho\varepsilon} + 1\right) \int_0^t \|\beta_\varepsilon^*(y_\varepsilon(\tau))\|_V^2 d\tau$$

$$+ 2\overline{M}^2 \|y_\varepsilon(t)\|^2 + 2c_P^2 \|f(t)\|^2 + \frac{c_P^2}{2} \int_0^t \left\| \frac{\partial f}{\partial \tau}(\tau) \right\|^2 d\tau,$$

whence we get $\frac{dy_\varepsilon}{dt} \in L^2(Q)$, $\beta_\varepsilon^*(y_\varepsilon) \in L^\infty(0,T;V)$ for each $\varepsilon > 0$.

We continue with some other computations based on the arguments developed in [84], Theorem 2.6, pp. 156. These are very long and technical so we do no longer provide them. We obtain an estimate of the form

$$\begin{aligned}
&\|\beta_\varepsilon^*(y_\varepsilon)\|^2_{W^{1,2}([0,T];L^2(\Omega))} + \|\beta_\varepsilon^*(y_\varepsilon)\|^2_{L^\infty(0,T;V)} + \|\beta_\varepsilon^*(y_\varepsilon)\|^2_{L^2(0,T;H^2(\Omega))} \\
&\quad \leq \gamma_1 \frac{\beta_s^* - \beta^*(y_s-\varepsilon)}{\varepsilon} \\
&\qquad \times \left(\left\| \beta_\varepsilon^*\left(\frac{\theta_0}{u+\varepsilon}\right)\right\|_V^2 + \int_\Omega j_\varepsilon\left(\frac{\theta_0}{u+\varepsilon}\right) dx + \|f(t)\|^2_{W^{1,2}([0,T];L^2(\Omega))} + 1 \right),
\end{aligned} \tag{1.64}$$

where γ_1 is a constant depending on the problem data. Since $\theta_0 \in V$ it follows that $\frac{\theta_0}{u+\varepsilon} \in V$ and $j_\varepsilon\left(\frac{\theta_0}{u+\varepsilon}\right) \in L^1(\Omega)$, so that by (1.64) we get that $\beta_\varepsilon^*(y_\varepsilon) \in L^2(0,T;H^2(\Omega))$. By a direct computation we also get that $a_j K(y_\varepsilon) \in L^2(0,T;H^1(\Omega))$, $j=1,\dots,N$.

For a later use we specify that these imply the flux continuity across a surface, i.e.,

$$(K_0(x,y_\varepsilon(t)) - \nabla\beta_\varepsilon^*(y_\varepsilon(t)))\cdot\nu \text{ is continuous across } \Gamma_c, \text{ a.e. } t\in(0,T), \tag{1.65}$$

where Γ_c is any surface included in Ω and ν is the outer normal to Γ_c. Indeed, since each component $\eta_i(t)$ of the flux vector belongs to $H^1(\Omega)$, a.e. t it follows that its trace on any line crossing the surface Γ_c is continuous. Therefore the normal component of the gradient is continuous across any Γ_c and in particular across $\partial\Omega_0$. □

1.1.5 Convergence of the Approximating Problem

Theorem 1.6. *Let (1.21) and (1.22) hold. Then, the Cauchy problem (1.27) has at least a weak solution* (y^*,ζ).

Proof. Let us assume (1.21) and (1.22), i.e.,

$$\theta_0 \in L^2(\Omega),\ \theta_0 = 0 \text{ a.e. on } \Omega_0,$$

$$\theta_0 \geq 0 \text{ a.e. on } \Omega_u,\ \frac{\theta_0}{u} \in L^2(\Omega_u),\ \frac{\theta_0}{u} \leq y_s,\ \text{a.e. } x\in\Omega_u.$$

According to Proposition 1.5 there exists a unique solution to (1.40), with the properties (1.52), (1.53). Then, it follows that

$$\int_{\Omega} j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right)dx = \int_{\Omega_0} j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right)dx + \int_{\Omega_u} j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right)dx = \int_{\Omega_u} j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right)dx$$

since $\frac{\theta_0}{u_\varepsilon} = 0$ a.e. on $\overline{\Omega_0}$. Using (1.35) and the fact that $u_\varepsilon = u + \varepsilon > u$ on Ω_u, we still obtain

$$\begin{aligned}\int_{\Omega} j_\varepsilon\left(\frac{\theta_0}{u_\varepsilon}\right)dx &= \int_{\Omega_u}\int_0^{\theta_0/u_\varepsilon} \beta_\varepsilon^*(r)drdx \\ &\leq \int_{\Omega_u}\int_0^{\theta_0/u} \beta_\varepsilon^*(r)drdx \leq \beta_s^* y_s \mathrm{meas}(\Omega),\end{aligned}$$

and so the right-hand side in (1.52) becomes essentially independent of ε,

$$\begin{aligned}&\int_{\Omega} u_\varepsilon(x) j_\varepsilon\left(\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right)dx + \int_0^t \left\|\frac{d\theta_\varepsilon}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \int_0^t \left\|\beta_\varepsilon^*\left(\frac{\theta_\varepsilon(\tau)}{u_\varepsilon}\right)\right\|_V^2 d\tau \\ &\leq 4(u_M+\varepsilon)\left(\beta_s^* y_s \mathrm{meas}(\Omega) + \int_0^T \|f(t)\|_{V'}^2 dt + \overline{K}^2 T\right), \ t \in [0,T]. \quad (1.66)\end{aligned}$$

Then, using (1.34) we get

$$\begin{aligned}&\left\|\sqrt{u_\varepsilon}\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right\|^2 \\ &\leq \frac{8}{\rho}(u_M+\varepsilon)\left(\beta_s^* y_s \mathrm{meas}(\Omega) + \int_0^T \|f(t)\|_{V'}^2 dt + \overline{K}^2 T\right), \ t \in [0,T].\end{aligned} \quad (1.67)$$

Next, we write again

$$\theta_\varepsilon = \left(\sqrt{u_\varepsilon}\frac{\theta_\varepsilon}{u_\varepsilon}\right)\sqrt{u_\varepsilon}$$

and obtain

$$\|\theta_\varepsilon(t)\|^2 \leq \frac{8}{\rho}(u_M+\varepsilon)^2\left(\beta_s^* y_s \mathrm{meas}(\Omega) + \int_0^T \|f(t)\|_{V'}^2 dt + \overline{K}^2 T\right), \ t \in [0,T]. \quad (1.68)$$

Therefore, the right-hand side terms in the estimates (1.66)–(1.68) are bounded by constants (since ε is small, e.g., $\varepsilon << 1$).

1.1.5.1 Passing to the Limit as $\varepsilon \to 0$

On the basis of these estimates we can select a subsequence denoted still by the subscript ε, such that

$$\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \rightharpoonup \zeta \text{ in } L^2(0,T;V), \text{ as } \varepsilon \to 0, \tag{1.69}$$

$$y_\varepsilon = \frac{\theta_\varepsilon}{u_\varepsilon} \rightharpoonup y \text{ in } L^2(0,T;V), \text{ as } \varepsilon \to 0, \tag{1.70}$$

$$\sqrt{u_\varepsilon}\frac{\theta_\varepsilon}{u_\varepsilon} \overset{w*}{\to} \chi \text{ in } L^\infty(0,T;L^2(\Omega)), \text{ as } \varepsilon \to 0. \tag{1.71}$$

But

$$\theta_\varepsilon = u_\varepsilon \frac{\theta_\varepsilon}{u_\varepsilon} \tag{1.72}$$

and since $u_\varepsilon \to u$ uniformly on Ω and $u \in W^{1,\infty}(\Omega)$ we have that

$$\|\theta_\varepsilon\|_{L^2(0,T;V)} \leq \text{ constant independent of } \varepsilon, \tag{1.73}$$

and so

$$\theta_\varepsilon \rightharpoonup \theta \text{ in } L^2(0,T;V), \text{ as } \varepsilon \to 0. \tag{1.74}$$

By (1.66) we still deduce that

$$\frac{d\theta_\varepsilon}{dt} \rightharpoonup \frac{d\theta}{dt} \text{ in } L^2(0,T;V'), \text{ as } \varepsilon \to 0, \tag{1.75}$$

and by (1.40) we have

$$\Delta\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \rightharpoonup \frac{d\theta}{dt} - f \text{ in } L^2(0,T;V'), \text{ as } \varepsilon \to 0. \tag{1.76}$$

Also, by (1.70), (1.72), (1.74) and $u_\varepsilon \to u$ uniformly we deduce that

$$\theta = uy \text{ a.e. on } Q, \tag{1.77}$$

and obviously

$$\theta = 0 \text{ a.e. on } Q_0, \tag{1.78}$$

where $Q_0 := (0,T) \times \Omega_0$. Using (1.71) and (1.70) we still obtain that

$$\sqrt{u_\varepsilon}y_\varepsilon \overset{w*}{\to} \chi = \sqrt{u}y \text{ in } L^\infty(0,T;L^2(\Omega)). \tag{1.79}$$

Again, by the Ascoli–Arzelà theorem we deduce that

$$\theta_\varepsilon(t) \to \theta(t) \text{ in } V', \text{ as } \varepsilon \to 0, \text{ uniformly in } t \in [0,T]. \tag{1.80}$$

Thus,

$$\theta_0 = \lim_{\varepsilon\to 0} \theta_\varepsilon(0) = \theta(0) = (uy(t))|_{t=0}\,.$$

By the Aubin–Lions theorem $(\theta_\varepsilon)_\varepsilon$ is compact in $L^2(0,T;L^2(\Omega))$, i.e.,

$$\theta_\varepsilon \to \theta \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0. \tag{1.81}$$

We set now for $\delta > 0$ arbitrarily small

$$\Omega_\delta := \{x \in \Omega;\ u(x) > \delta\}, \quad Q_\delta := (0,T) \times \Omega_\delta. \tag{1.82}$$

We recall that

$$\Omega_u := \{x \in \Omega;\ u(x) > 0\}, \quad Q_u := (0,T) \times \Omega_u \tag{1.83}$$

and notice that Ω_δ and Ω_u are open. We have

$$\frac{1}{u_\varepsilon} = \frac{1}{u+\varepsilon} < \frac{1}{\delta} \text{ on } \Omega_\delta,$$

so that, by (1.81) and (1.70) we can conclude that

$$y_\varepsilon = \frac{1}{u_\varepsilon}\theta_\varepsilon \to \frac{\theta}{u} := y \text{ in } L^2(0,T;L^2(\Omega_\delta)), \tag{1.84}$$

and a.e. in Q_δ, $\forall\delta > 0$. Still by (1.70) we have that

$$y_\varepsilon = \frac{\theta_\varepsilon}{u_\varepsilon} \rightharpoonup y \text{ in } L^2(0,T;L^2(\Omega_u)). \tag{1.85}$$

1.1.5.2 Convergence of $\beta^*_\varepsilon(y_\varepsilon)$ on Q_u

Let $(t,x) \in Q_\delta$. First, we shall prove that

$$\zeta(t,x) \in \beta^*(y(t,x)) \text{ a.e. on } Q_\delta, \tag{1.86}$$

where ζ is given by (1.69). This will be proved using the fact that j is the potential of β^*, i.e., $\beta^* = \partial j$.

To this end we establish some relations. We note that

$$j_\varepsilon(z) \to j(z), \text{ as } \varepsilon \to 0, \quad \text{for any } z \in \mathbb{R}. \tag{1.87}$$

This assertion is clear for $z < y_s - \varepsilon$, where $j_\varepsilon(z) \equiv j(z)$.
For $y_s - \varepsilon \le z < y_s$ we compute

$$|j_\varepsilon(z) - j(z)| = \left| \int_{y_s-\varepsilon}^{z} (\beta_\varepsilon^*(\xi) - \beta^*(\xi))d\xi \right| \le 2\beta_s^* \varepsilon \to 0 \text{ as } \varepsilon \to 0,$$

where we recall that $\beta_s^* = \lim_{r \nearrow y_s} \beta^*(r)$ (see (1.5)).
For $z \ge y_s$ we have

$$j_\varepsilon(z) = \int_0^{y_s-\varepsilon} \beta_\varepsilon^*(\xi)d\xi + \int_{y_s-\varepsilon}^{z} \beta_\varepsilon^*(\xi)d\xi = \int_0^{y_s-\varepsilon} \beta^*(\xi)d\xi$$
$$+ \beta^*(y_s-\varepsilon)[z-(y_s-\varepsilon)] + \frac{\beta_s^* - \beta^*(y_s-\varepsilon)}{2\varepsilon}[z-(y_s-\varepsilon)]^2.$$

Therefore, we have $\lim_{\varepsilon\to 0} j_\varepsilon(z) = j(y_s)$ for $z = y_s$ and

$$\lim_{\varepsilon\to 0} j_\varepsilon(z) = +\infty = j(z) \text{ for } z > y_s.$$

Now, we are going to show that

$$\int_{Q_\delta} j(y)dxdt \le \liminf_{\varepsilon\to 0} \int_{Q_\delta} j_\varepsilon(y_\varepsilon)dxdt. \tag{1.88}$$

Let ε be small, e.g., $\varepsilon < \frac{y_s}{2}$. We can write

$$\int_{Q_\delta} j_\varepsilon(y_\varepsilon(t,x))dxdt \tag{1.89}$$
$$= \int_{Q_1^\varepsilon} j_\varepsilon(y_\varepsilon(t,x))dxdt + \int_{Q_2^\varepsilon} j_\varepsilon(y_\varepsilon(t,x))dxdt + \int_{Q_3^\varepsilon} j_\varepsilon(y_\varepsilon(t,x))dxdt,$$

where

$$Q_1^\varepsilon = \{(t,x) \in Q_\delta;\ y_\varepsilon(t,x) < y_s - \varepsilon\},$$
$$Q_2^\varepsilon = \{(t,x) \in Q_\delta;\ y_s - \varepsilon \le y_\varepsilon(t,x) \le y_s\},$$
$$Q_3^\varepsilon = \{(t,x) \in Q_\delta;\ y_s < y_\varepsilon(t,x)\}.$$

We compute each term apart. For $(t,x) \in Q_1^\varepsilon$ we have

$$j_\varepsilon(y_\varepsilon(t,x)) = \int_0^{y_\varepsilon(t,x)} \beta_\varepsilon^*(\xi)d\xi = \int_0^{y_\varepsilon(t,x)} \beta^*(\xi)d\xi = j(y_\varepsilon(t,x)).$$

For $(t,x) \in Q_2^\varepsilon$ we write

$$\begin{aligned} j_\varepsilon(y_\varepsilon(t,x)) &= \int_0^{y_s-\varepsilon} \beta_\varepsilon^*(\xi)d\xi + \int_{y_s-\varepsilon}^{y_\varepsilon(t,x)} \beta_\varepsilon^*(\xi)d\xi \\ &= j(y_s-\varepsilon) + \beta^*(y_s-\varepsilon)[y_\varepsilon(t,x)-(y_s-\varepsilon)] \\ &\quad + \frac{\beta_s^*-\beta^*(y_s-\varepsilon)}{2\varepsilon}[y_\varepsilon(t,x)-(y_s-\varepsilon)]^2 \\ &\geq j(y_s-\varepsilon) \end{aligned}$$

because the last two terms in the sum are positive on Q_2^ε (β^* is positive for a positive argument and so $\beta^*(y_s-\varepsilon)>0$).

Next, if $(t,x) \in Q_3^\varepsilon$, taking into account that $\beta_\varepsilon^*(r) \geq \beta^*(r)$ for $r < y_s$ and $\beta_\varepsilon^*(y_s) = \beta_s^*$ we have

$$\begin{aligned} j_\varepsilon(y_\varepsilon(t,x)) &= \int_0^{y_s} \beta_\varepsilon^*(\xi)d\xi + \int_{y_s}^{y_\varepsilon(t,x)} \beta_\varepsilon^*(\xi)d\xi \\ &\geq \int_0^{y_s} \beta^*(\xi)d\xi + \beta^*(y_s-\varepsilon)(y_\varepsilon(t,x)-y_s) \\ &\quad + \frac{\beta_s^*-\beta^*(y_s-\varepsilon)}{2\varepsilon}(y_\varepsilon(t,x)-y_s)^2 \\ &\geq j(y_s). \end{aligned}$$

We resume (1.89), writing

$$\begin{aligned} &\int_{Q_\delta} j_\varepsilon(y_\varepsilon(t,x))dxdt \geq \int_{Q_1^\varepsilon} j(y_\varepsilon(t,x))dxdt + \int_{Q_2^\varepsilon} j(y_s-\varepsilon)dxdt + \int_{Q_3^\varepsilon} j(y_s)dxdt \\ &\quad = \int_{Q_\delta} j(y(t,x))dxdt + \int_{Q_1^\varepsilon} (j(y_\varepsilon(t,x)) - j(y(t,x)))dxdt \\ &\quad + \int_{Q_2^\varepsilon} (j(y_s-\varepsilon) - j(y(t,x)))dxdt + \int_{Q_3^\varepsilon} (j(y_s) - j(y(t,x)))dxdt \end{aligned} \tag{1.90}$$

and we treat again each term apart.

Since $y_\varepsilon \to y$ in $L^2(Q_\delta)$ it follows that on a subsequence $y_\varepsilon \to y$ a.e. on Q_δ, and in particular this is true on Q_1^ε and Q_2^ε. Moreover, $y \to j(y)$ is continuous if $y \leq y_s$ and so we have

$$j(y_\varepsilon(t,x)) - j(y(t,x)) \to 0 \text{ a.e. on } Q_1^\varepsilon, \text{ as } \varepsilon \to 0.$$

Then $j(y_\varepsilon(t,x)) \le j(y_s-\varepsilon) \le j(y_s)$ if $(t,x) \in Q_1^\varepsilon$ and so $|j(y_\varepsilon(t,x)) - j(y(t,x))| \le 2j(y_s)$. In conclusion by the Lebesgue dominated convergence theorem we deduce that

$$\left| \int_{Q_1^\varepsilon} (j(y_\varepsilon(t,x)) - j(y(t,x)))dxdt \right|$$
$$= \int_{Q_\delta} \left| (j(y_\varepsilon(t,x)) - j(y(t,x)))\chi_{Q_1^\varepsilon}(t,x) \right| dxdt \to 0,$$

where $\chi_{Q_1^\varepsilon}$ is the characteristic function of the set Q_1^ε. For the second term in the sum (1.90) we write

$$\int_{Q_2^\varepsilon} (j(y_s-\varepsilon) - j(y(t,x)))dxdt$$
$$= \int_{Q_2^\varepsilon} (j(y_s-\varepsilon) - j(y_\varepsilon(t,x)))dxdt + \int_{Q_2^\varepsilon} (j(y_\varepsilon(t,x) - j(y(t,x)))dxdt.$$

The last term on the right-hand side converges to 0 by a similar argument as before, using the Lebesgue dominated convergence theorem. For the first term we recall that $y \to j(y)$ is Lipschitz if $y \le y_s$ and we have

$$\left| \int_{Q_2^\varepsilon} (j(y_s-\varepsilon) - j(y_\varepsilon(t,x)))dxdt \right| \le \left| \int_{Q_\delta} (j(y_s-\varepsilon) - j(y_\varepsilon(t,x)))\chi_{Q_2^\varepsilon}(t,x)dxdt \right|$$
$$\le \beta_s^* \int_{Q_\delta} |y_s - \varepsilon - y_\varepsilon(t,x)|\, dxdt$$
$$\le \beta_s^* \mathrm{meas}(Q_\delta)\varepsilon \to 0 \text{ as } \varepsilon \to 0,$$

where $\chi_{Q_2^\varepsilon}$ is the characteristic function of the set Q_2^ε.

For the third term in (1.90) we write

$$\int_{Q_3^\varepsilon} (j(y_s) - j(y(t,x)))dxdt = \int_{Q_\delta} (j(y_s) - j(y(t,x)))\chi_{Q_3^\varepsilon}(t,x)dxdt$$

where $\chi_{Q_3^\varepsilon}$ is the characteristic function of the set Q_3^ε.

We are going to show that

$$y(t,x) \le y_s \text{ a.e. on } Q_\delta$$

which will imply that the integral on Q_3^ε is nonnegative.

Thus, on the basis of these results coming back to (1.90) we deduce

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_{Q_\delta} j_\varepsilon(y_\varepsilon(t,x))dxdt &\geq \int_{Q_\delta} j(y(t,x))dxdt \\ &+\liminf_{\varepsilon\to 0}\left(\int_{Q_1^\varepsilon}(j(y_\varepsilon(t,x))-j(y(t,x)))\,dxdt\right. \\ &\qquad\left.+\int_{Q_2^\varepsilon}(j(y_s-\varepsilon)-j(y(t,x)))dxdt\right) \\ &=\int_{Q_\delta} j(y(t,x))dxdt\end{aligned}$$

and so (1.88) is proved.

It remains to prove the assertion that $y(t,x)\leq y_s$ a.e. on Q_δ. We recall (1.66) which implies in particular

$$\int_0^t \|\beta_\varepsilon^*\,(y_\varepsilon(\tau))\|^2_{L^2(Q_\delta)}\,d\tau\leq C$$

that can be still written

$$\int_0^t \|\beta_\varepsilon^*\,(y_\varepsilon(\tau))\|^2_{L^2(Q_3^\varepsilon)}\,d\tau+\int_0^t \|\beta_\varepsilon^*\,(y_\varepsilon(\tau))\|^2_{L^2(Q_\delta\setminus Q_3^\varepsilon)}\,d\tau\leq C.$$

The second term is positive and bounded, $\beta_\varepsilon^*\,(y_\varepsilon(\tau,x))\leq\beta_s^*$ on $Q_\delta\setminus Q_3^\varepsilon=\{(t,x);\,y_\varepsilon(t,x)\leq y_s\}$, and replacing the expression of β_ε^* we obtain

$$\int_{Q_\delta}\left\{\beta^*(y_s-\varepsilon)+\frac{\beta_s^*-\beta^*(y_s-\varepsilon)}{\varepsilon}[y_\varepsilon-(y_s-\varepsilon)]\right\}^2\chi_{Q_3^\varepsilon}(t,x)dxdt\leq C.$$

Further we have

$$\int_{Q_\delta}\left(\frac{\beta_s^*-\beta^*(y_s-\varepsilon)}{\varepsilon}\right)^2(y_\varepsilon-y_s)^2\chi_{Q_3^\varepsilon}(t,x)dxdt\leq C$$

because $\beta^*(y_s-\varepsilon)>0$. We recall that β^* is convex, which implies that

$$\frac{\beta_s^*-\beta^*(y_s-\varepsilon)}{\varepsilon}>\beta(y_s-\varepsilon)$$

and so we get

$$\int_{Q_\delta}(y_\varepsilon-y_s)^2\chi_{Q_3^\varepsilon}(t,x)dxdt=\int_{Q_\delta}\{(y_\varepsilon-y_s)^+\}^2dxdt\leq\frac{C}{\beta^2(y_s-\varepsilon)}$$

where $(y_\varepsilon - y_s)^+$ represents the positive part of $(y_\varepsilon - y_s)$. Now we pass to the limit (recalling that $y_\varepsilon \to y$ in $L^2(Q_\delta)$ by (1.84)) and take into account that β blows up at y_s, getting

$$\int_{Q_\delta} \{(y - y_s)^+\}^2 dxdt \leq 0$$

whence we deduce that $y(t,x) \leq y_s$ a.e. on Q_δ.

Now we resume the proof of the convergence of β_ε^* on Q_δ. Since

$$j_\varepsilon(r) \leq j_\varepsilon(z) + \beta_\varepsilon^*(r)(r - z), \text{ for any } r, z \in \mathbb{R},$$

we can write the inequality in particular for $z : (0,T) \times \Omega_\delta \to \mathbb{R}$, $z \in L^2(Q_\delta)$ and $r = y_\varepsilon$. We have

$$\int_{Q_\delta} j_\varepsilon(y_\varepsilon) dxdt \leq \int_{Q_\delta} j_\varepsilon(z) dxdt + \int_{Q_\delta} \beta_\varepsilon^*(y_\varepsilon)(y_\varepsilon - z) dxdt. \tag{1.91}$$

Assume $z \leq y_s$. Then $j_\varepsilon(z) \leq \beta_s^* y_s$ and using (1.87) we deduce by the Lebesgue dominated convergence theorem (see [13], pp. 3) that

$$\lim_{\varepsilon \to 0} \int_{Q_\delta} j_\varepsilon(z) dxdt = \int_{Q_\delta} j(z) dxdt.$$

Next, we remind that $\beta_\varepsilon^*(y_\varepsilon) \rightharpoonup \zeta$ in $L^2(0,T;V)$ and $y_\varepsilon \to y$ in $L^2(0,T;L^2(\Omega_\delta))$. By passing to limit as $\varepsilon \to 0$ in (1.91) and taking into account (1.88) we obtain that

$$\int_{Q_\delta} j(y) dxdt \leq \int_{Q_\delta} j(z) dxdt + \int_{Q_\delta} \zeta(y - z) dxdt, \ \forall z \in L^2(Q_\delta), \ z \leq y_s. \tag{1.92}$$

This implies that $\partial j = \zeta$. Here is the argument. Let us fix $(t_0, x_0) \in Q_\delta$, choose w arbitrary in $\mathbb{R}$, $w \leq y_s$, and define

$$z(t,x) := \begin{cases} y(t,x), & (t,x) \notin B_r(t_0,x_0) \\ w, & (t,x) \in B_r(t_0,x_0), \end{cases}$$

where $B_r(t_0, x_0)$ is the ball of centre (t_0, x_0) and radius $r > 0$. We denote $\overline{B}_r(t_0,x_0) = Q_\delta \backslash B_r(t_0,x_0)$. Then, (1.92) yields

$$\begin{aligned} &\int_{B_r(t_0,x_0)} j(y) dxdt + \int_{\overline{B}_r(t_0,x_0)} j(y) dxdt \\ &\leq \int_{B_r(t_0,x_0)} j(z) dxdt + \int_{\overline{B}_r(t_0,x_0)} j(z) dxdt \\ &\quad + \int_{B_r(t_0,x_0)} \zeta(y - z) dxdt + \int_{\overline{B}_r(t_0,x_0)} \zeta(y - z) dxdt. \end{aligned}$$

Taking into account the choice of $z(t,x)$ we have

$$\int_{B_r(t_0,x_0)} j(y)dxdt + \int_{\overline{B}_r(t_0,x_0)} j(y)dxdt$$
$$\leq \int_{B_r(t_0,x_0)} j(w)dxdt + \int_{\overline{B}_r(t_0,x_0)} j(y)dxdt$$
$$+ \int_{B_r(t_0,x_0)} \zeta(y-w)dxdt + \int_{\overline{B}_r(t_0,x_0)} \zeta(y-y)dxdt$$

from where it remains

$$\int_{B_r(t_0,x_0)} j(y)dxdt \leq \int_{B_r(t_0,x_0)} j(w)dxdt + \int_{B_r(t_0,x_0)} \zeta(y-w)dxdt.$$

We recall the following definition. Let l be a Lebesgue measurable function on a set S and let $z_0 \in S$. The point z_0 is called *a Lebesgue point* for l if

$$\lim_{r\to 0} \frac{1}{\text{meas}(B_r(z_0))} \int_{B_r(z_0)} l(x)dx = l(z_0).$$

The set of the points at which the previous relation holds is called the set of Lebesgue points. We also recall that the set of Lebesgue points for an integrable function l on a set S has the Lebesgue measure equal to that of S, namely almost all points in S are Lebesgue for l.

Thus, let us assume now that (t_0, x_0) considered before is a Lebesgue point for j. Dividing the inequality by meas$(B_r(x_0,t_0))$ and letting $r \to 0$ we get

$$j(y(t_0,x_0)) \leq j(w) + \zeta(t_0,x_0)\left(y(t_0,x_0) - w\right),\ \forall w \in \mathbb{R},\ w \leq y_s.$$

By the definition of j we get $\zeta(t,x) \in \beta^*(y(t,x))$ a.e. $(t,x) \in Q_\delta$. Then, since δ is arbitrary and $Q_u = \bigcup_{\delta>0} Q_\delta$, we infer that

$$\zeta(t,x) \in \beta^*(y(t,x)) \text{ a.e. on } Q_u,$$

and we deduce that

$$y(t,x) \leq y_s \text{ a.e. on } Q_u.$$

Finally, since $\left(K\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right)\right)_\varepsilon$ is bounded in $L^2(Q)$ we have

$$K\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \rightharpoonup \kappa \text{ in } L^2(Q),\ \text{ as } \varepsilon \to 0$$

and we assert that $\kappa = K(y)$. Indeed,

$$K\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \rightharpoonup \kappa \text{ in } L^2(Q_u),\ \text{ as } \varepsilon \to 0,$$

too. On the other hand, K being Lipschitz it follows by (1.84) that $\left(K\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right)\right)_{\varepsilon>0}$ is strongly convergent on each subset Q_δ,

$$K\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \to K(y) \text{ in } L^2(Q_\delta), \text{ as } \varepsilon \to 0.$$

By the uniqueness of the limit the restriction of the weak limit function κ to Q_δ must coincide with $K(y)$ and this also implies that

$$\kappa(t,x) = K(y(t,x)) \text{ a.e. on } Q_u.$$

On the subset Q_0 the function $a(x)K\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) = 0$, so by the definition of K_0 we get

$$K_0\left(x, \frac{\theta_\varepsilon}{u_\varepsilon}\right) \rightharpoonup K_0(x,y) \text{ in } L^2(Q), \text{ as } \varepsilon \to 0.$$

Finally, we derive a relation which will serve a little later. Assume first that $f \in W^{1,2}([0,T]; L^2(\Omega))$ and $\theta_0 \in V$.

We recall that $\zeta(t) \in V$ a.e. $t \in (0,T)$. Since this regularity is not sufficient to define its normal derivative to a surface $\Gamma_c \subset \Omega$, we define a generalized normal derivative of it $\frac{\partial \zeta(t)}{\partial \nu}$, as an element of a distribution space on Γ_c. As a matter of fact $\frac{\partial \zeta(t)}{\partial \nu} \in H^{-1/2}(\Gamma_c)$ which is the dual of $H^{1/2}(\Gamma_c)$ (see the definitions of these spaces in [78]).

Assume that Γ_c is a smooth surface surrounding the domain $\Omega_c \subset \Omega$, i.e., $\Gamma_c = \partial\Omega_c$. If $\eta \in H^1(\Omega_c)$ and $\Delta\eta \in (H^1(\Omega_c))'$ then we define $\frac{\partial \eta}{\partial \nu} \in H^{-1/2}(\Gamma_c)$ by

$$\begin{aligned}&\left\langle \frac{\partial \eta}{\partial \nu}, tr(\psi) \right\rangle_{H^{-1/2}(\Gamma_c), H^{1/2}(\Gamma_c)} \\ &\quad = \langle \Delta\eta, \psi \rangle_{(H^1(\Omega_c))', H^1(\Omega_c)} + \int_{\Omega_c} \nabla\eta \cdot \nabla\psi dx, \ \forall \psi \in H^1(\Omega_c). \qquad (1.93)\end{aligned}$$

In particular, for $\eta = \zeta(t)$, $\Omega_c = \Omega_0$ with the boundary $\Gamma_0 = \partial\Omega_0$ we define the outward normal derivative $\frac{\partial^+}{\partial \nu}\zeta(t)$ a.e. $t \in (0,T)$, by

$$\begin{aligned}&\left\langle \frac{\partial^+ \zeta(t)}{\partial \nu}, tr(\psi) \right\rangle_{H^{-1/2}(\Gamma_0), H^{1/2}(\Gamma_0)} \\ &\quad = \langle \Delta\zeta(t), \psi \rangle_{(H^1(\Omega_0))', H^1(\Omega_0)} \\ &\qquad + \int_{\Omega_0} \nabla\zeta(t) \cdot \nabla\psi dx, \ \forall \psi \in H^1(\Omega_0), \text{ a.e. } t \in (0,T), \qquad (1.94)\end{aligned}$$

where $tr(\psi)$ is the trace of $\psi \in H^1(\Omega_0)$ on Γ_0.

In a similar way, considering $\Omega_u = \Omega \backslash \Omega_0$ which has the common boundary $\Gamma_0 = \partial\Omega_0$ with Ω_0, we define $\frac{\partial^-}{\partial\nu}\zeta(t)$ on Γ_0 by the relation

$$\left\langle \frac{\partial^-\zeta(t)}{\partial\nu}, tr(\psi) \right\rangle_{H^{-1/2}(\Gamma_0),H^{1/2}(\Gamma_0)}$$
$$= \langle \Delta\zeta(t), \psi \rangle_{(H^1(\Omega_u))',H^1(\Omega_u)} + \int_{\Omega_u} \nabla\zeta(t) \cdot \nabla\psi dx, \ \forall\psi \in V, \ \text{a.e. } t \in (0,T), \tag{1.95}$$

where $tr(\psi)$ is the trace of $\psi \in V$ on Γ_0.

Thus we can obtain the continuity of the generalized normal derivative across the surface Γ_c, in particular across Γ_0. Indeed by (1.65) we have

$$\int_{\Gamma_0} (K_0^+(x, y_\varepsilon(t)) - \nabla\zeta_\varepsilon^+(t)) \cdot \nu^+ \psi d\sigma$$
$$= \int_{\Gamma_0} (K_0^-(x, y_\varepsilon(t)) - \nabla\zeta_\varepsilon^-(t)) \cdot \nu^- \psi d\sigma, \ \forall\psi \in V, \ \text{a.e. } t \in (0,T),$$

where $\zeta_\varepsilon = \beta_\varepsilon^*(y_\varepsilon)$ and the superscripts $+$ and $-$ denote the restrictions of the functions on Ω_0 and Ω_u, respectively. Also, ν^+ and ν^- are the outer normal derivatives to Γ_0 from Ω_0 and Ω_u, respectively. Since $\nabla\zeta_\varepsilon(t)$ is bounded in $L^2(\Omega)$ independently on ε, a.e. t (see (1.52)) we can pass to the limit and get

$$\left\langle (K_0^+(\cdot, y(t)) - \nabla\zeta^+(t)) \cdot \nu^+, \psi \right\rangle_{H^{-1/2}(\Gamma_0),H^{1/2}(\Gamma_0)}$$
$$= \left\langle K_0^-(\cdot, y(t)) - \nabla\zeta^-(t) \cdot \nu^-, \psi \right\rangle_{H^{-1/2}(\Gamma_0),H^{1/2}(\Gamma_0)}, \ \forall\psi \in V, \ \text{a.e. } t \in (0,T) \tag{1.96}$$

where the normal derivatives $\frac{\partial^+\zeta(t)}{\partial\nu} = \nabla\zeta^+(t) \cdot \nu^+$ and $\frac{\partial^-\zeta(t)}{\partial\nu} = \nabla\zeta^-(t) \cdot \nu^-$ are considered in the generalized sense (1.94) and (1.95). For simplicity, here we denoted $tr(\psi)$ still by ψ.

Now we can pass to limit as $\varepsilon \to 0$ in (1.43) and obtain

$$\int_0^T \left\langle \frac{d(uy)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla\zeta - K_0(x,y)) \cdot \nabla\phi dx dt$$
$$= \int_0^T \int_\Omega f\phi dx dt, \ \ \text{for any } \phi \in L^2(0,T;V), \tag{1.97}$$

where ζ is given by (1.69), $\zeta = \lim\limits_{\varepsilon\to 0} \beta_\varepsilon^*(y_\varepsilon)$.

In particular if $\phi \in C_0^\infty(Q_u)$ we get

$$\int_0^T \left\langle \frac{d(uy)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_{Q_u} (\nabla\zeta - K_0(x,y)) \cdot \nabla\phi dxdt$$
$$= \int_0^T \int_{\Omega_u} f\phi dxdt, \tag{1.98}$$

where $\zeta \in \beta^*(y)$ a.e. on Q_u. We have taken into account that

$$\frac{d(uy)}{dt} = \begin{cases} \frac{\partial(uy)}{\partial t}, & \text{if } uy > 0 \\ 0, & \text{if } uy = 0 \end{cases} \tag{1.99}$$

where $\frac{\partial(uy)}{\partial t}$ is the derivative in the sense of distributions.

If we take $\phi \in C_0^\infty(Q_0)$ we obtain

$$\int_{Q_u} (\nabla\zeta - K_0(x,y)) \cdot \nabla\phi dxdt = \int_0^T \int_{\Omega_0} f\phi dxdt, \tag{1.100}$$

where ζ is given by (1.69).

1.1.6 Construction of the Solution

Now we consider the following equations in the sense of distributions

$$\frac{\partial(uy)}{\partial t} - \Delta\zeta + \nabla \cdot K_0(x,y) \ni f \ \text{ in } Q,$$
$$\zeta = 0 \ \text{ on } \Sigma, \tag{1.101}$$

obtained from (1.97) for $\phi \in C_0^\infty(Q)$, where ζ is given by (1.69),

$$\frac{\partial(uy)}{\partial t} - \Delta\zeta + \nabla \cdot K_0(x,y) \ni f \ \text{ in } Q_u = (0,T) \times \Omega_u,$$
$$\zeta = 0 \ \text{ on } \Sigma, \tag{1.102}$$

with $\zeta(t,x) \in \beta^*(y(t,x))$ a.e. $(t,x) \in Q_u$ and

$$-\Delta\zeta \ni f \ \text{ in } Q_0 = (0,T) \times \Omega_0 \tag{1.103}$$

with ζ given again by (1.69).

The common boundary $\partial\Omega_0$ of the domains Ω_u and Ω_0 is regular. Since $\zeta \in L^2(0,T;V)$ we deduce that for a.e. $t \in (0,T)$ the trace of the function

$\zeta(t)$ on any line $\mathcal{L}_0 \subset \Omega$ crossing the boundary $\partial\Omega_0$ belongs to V, so that it is continuous across $\mathcal{L}_0$. Thus if we take $x_0 \in \partial\Omega_0$ then

$$\zeta^-(t) := \lim_{\substack{x\to x_0 \\ x\in\mathcal{L}_0\cap\Omega_u}} \zeta(t) = \lim_{\substack{x\to x_0 \\ x\in\mathcal{L}_0\cap\Omega_0}} \zeta(t) = \zeta^+(t) \text{ a.e. } t \in (0,T).$$

We take into account that $\zeta^-(t) \in \beta^*(y(t))$ a.e. $t \in (0,T)$, hence ζ turns out to be the solution to the elliptic problem

$$\begin{aligned} -\Delta\zeta(t) &= f(t) && \text{in } \Omega_0, \text{ a.e. } t \in (0,T) \\ \zeta(t) &= \zeta^-(t) \in \beta^*(y(t)) && \text{on } \partial\Omega_0, \text{ a.e. } t \in (0,T), \end{aligned} \tag{1.104}$$

where y is the solution to (1.102) in Q_u.

Now we can construct the function

$$y^*(t,x) := \begin{cases} y(t,x), & \text{if } (t,x) \in Q_u \\ (\beta^*)^{-1}(\zeta(t,x)), & \text{if } (t,x) \in Q_0 \end{cases} \tag{1.105}$$

and show that it is the solution to (1.27). Since $\zeta \in L^2(0,T;V)$ it follows that $y^* \in L^2(0,T;D(A))$, whence $y^* \le y_s$ a.e. on Q. This function belongs also to the spaces specified in (1.23) (for the derivative we take into account (1.99)).

We have to check that y^* satisfies (1.26). If we plug y^* given by (1.105) in (1.26) and we take into account (1.99), (1.96) we obtain

$$\begin{aligned} &\int_0^T \left\langle \frac{d(uy^*)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla\zeta - K_0(x,y^*)) \cdot \nabla\phi dxdt \\ &= \int_0^T \left\langle \frac{d(uy^*)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_{Q_u} (\nabla\zeta - K_0(x,y)) \cdot \nabla\phi dxdt \\ &\quad + \int_{Q_0} (\nabla\zeta - K_0(x,y^*)) \cdot \nabla\phi dxdt \\ &= \int_0^T \int_{\Omega_u} f\phi dxdt + \int_0^T \int_{\Omega_0} f\phi dxdt = \int_0^T \int_\Omega f\phi dxdt, \end{aligned}$$

for any $\phi \in L^2(0,T;V)$, $\zeta \in \beta^*(y^*)$ a.e. on Q. Here we used (1.98) and (1.100).

Now, let $f \in L^2(0,T;V')$ and $\theta_0 \in L^2(\Omega)$. The previous relation remains true, by density, but we do not provide all arguments because they are similar with those given up to now. So, we obtain (1.26) as claimed and this ends the existence proof. □

Now we are going to specify a physical interpretation of the solution, stating that the previous proof also implies

Corollary 1.7. *The solution y^* to problem (1.1) given by Theorem 1.6 is the solution to the transmission problem*

$$\begin{aligned} \frac{\partial(u(x)y^*)}{\partial t} - \Delta\beta^*(y^*) + \nabla\cdot K_0(x,y^*) &\ni f \qquad \text{in } Q_u,\\ -\Delta\beta^*(y^*) &\ni f \qquad \text{in } Q_0,\\ \zeta^+ &= \zeta^- \qquad \text{on } \Sigma_0=(0,T)\times\partial\Omega_0,\\ (K_0^+(x,y^*)-\nabla\zeta^+)\cdot\nu^+ &= (K_0^-(x,y^*)-\nabla\zeta^-)\cdot\nu^+ \quad \text{on } \Sigma_0,\\ y^*(t,x) &= 0 \qquad \text{on } \Sigma:=(0,T)\times\Gamma,\\ (u(x)y^*(t,x))|_{t=0} &= \theta_0(x) \quad \text{in } \Omega. \end{aligned} \tag{1.106}$$

Proof. Let $f\in W^{1,2}([0,T];L^2(\Omega))$. Let us write that y^* is a solution to (1.1)

$$\int_0^T \left\langle \frac{d(uy^*)}{dt}(t),\phi(t)\right\rangle_{V',V} dt + \int_{Q_u}(\nabla\zeta - K_0(x,y))\cdot\nabla\phi dxdt$$
$$+\int_{Q_0}(\nabla\zeta-K_0(x,y^*))\cdot\nabla\phi dxdt$$
$$=\int_0^T\int_\Omega f\phi dxdt,$$

whence, expressing the integrals on Q_u and Q_0 in another way, we get

$$\int_0^T\left\langle\frac{d(uy^*)}{dt}(t)-\Delta\zeta(t)+\nabla\cdot a(x)K_0(x,y^*(t))-f(t),\phi(t)\right\rangle_{(H^1(\Omega_u))',H^1(\Omega_u)} dt$$
$$-\int_0^T\left\langle (K_0^-(\cdot,y^*(t))-\nabla\zeta^-(t))\cdot\nu^-,\phi(t)\right\rangle_{H^{-1/2}(\partial\Omega_0),H^{1/2}(\partial\Omega_0)} dt$$
$$+\int_0^T\langle-\Delta\zeta(t)-f(t),\phi(t)\rangle_{(H^1(\Omega_0))',H^1(\Omega_0)}dt$$
$$-\int_0^T\left\langle K_0^+(\cdot,y^*(t))-\nabla\zeta^+(t)\cdot\nu^+,\phi(t)\right\rangle_{H^{-1/2}(\partial\Omega_0),H^{1/2}(\partial\Omega_0)} dt$$
$$=0,$$

for any $\phi\in C_0^\infty(Q)$. Using (1.102) and (1.103) we get

$$(K_0^-(\cdot,y^*(t))-\nabla\zeta^-(t)))\cdot\nu^- + (K_0^+(\cdot,y^*(t))-\nabla\zeta^+(t))\cdot\nu^+ = 0 \text{ a.e. } t, \text{ on } \partial\Omega_0$$

where $\nu^-=-\nu^+$. The result remains true for $f\in L^2(0,T;V')$, by density. □

This means that the flux is conserved across the boundary Σ_0, which from the physical point of view is natural. As a matter of fact (1.106) is an equivalent form of (1.27).

Finally, we mention that the presence of the advection term in nonlinear degenerate diffusion problems, as well as in periodic problems as we shall see, may induce difficulties in proving the solution uniqueness, especially when using energetic relations. This is not a singular situation, because as it is well known there are many nonlinear problems in which uniqueness has remained an open problem (e.g. Navier–Stokes equation in 3D, nonlinear wave equation). In general uniqueness follows under restrictive assumptions and in diffusion with transport problems one can observe that it is ensured when the diffusion dominates the advection. In media with low porosity it can also be shown that a small enough velocity of the fluid is a condition guaranteeing the flow uniqueness. So, we give next a uniqueness result, establishing in fact a sufficient condition in (1.107) below. Its interpretation is that the advection vector in absolute value is of the same order of magnitude as the square root of the porosity. For the case when (1.107) is not obeyed one can accept that the approximating solution (which is unique) is an appropriate candidate for the solution to the physical model (1.1).

Proposition 1.8. *Under the hypotheses of Theorem 1.6 assume in addition that there exists* $k_u > 0$ *such that*

$$|a(x)| \leq k_u \sqrt{u(x)} \text{ for any } x \in \Omega. \tag{1.107}$$

Then the solution to (1.1) is unique a.e. on Q.

Proof. Assume that we have two solutions (y^*, ζ) and $(\overline{y}^*, \overline{\zeta})$ to (1.27) corresponding to the same data f and θ_0. We subtract (1.27) written for y^* and $\overline{y}^*$, multiply the difference scalarly in V' by $u(y^* - \overline{y}^*)(t)$, and integrate over $(0, t)$. We get

$$\int_0^t \left(\frac{d(u(y^* - \overline{y}^*))}{d\tau}(\tau), u(y^* - \overline{y}^*)(\tau) \right)_{V'} d\tau + \int_0^t \int_\Omega \nabla(\zeta - \overline{\zeta}) \cdot \nabla\psi dx d\tau$$
$$= \int_0^t \int_\Omega (K(y^*) - K(\overline{y}^*)) a(x) \cdot \nabla\psi dx d\tau, \tag{1.108}$$

where $A_0\psi = u(y^* - \overline{y}^*)$. Next we have

$$\frac{1}{2} \left\| u(y^* - \overline{y}^*)(t) \right\|_{V'}^2 + \int_0^t \int_\Omega (\zeta - \overline{\zeta}) u(y^* - \overline{y}^*) dx d\tau$$
$$\leq N M_K k_u \int_0^t \int_\Omega \left| \sqrt{u}(y^* - \overline{y}^*) \right| |\nabla\psi| \, dx d\tau$$
$$\leq N M_K k_u \int_0^t \left\| \sqrt{u}(y^* - \overline{y}^*)(\tau) \right\| \left\| u(y^* - \overline{y}^*)(\tau) \right\|_{V'} d\tau$$

whence, recalling (1.7) we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|u(y^*-\overline{y}^*)(t)\right\|_{V'}^2+\rho\int_0^t\int_{\Omega}u(y^*-\overline{y}^*)^2dxd\tau \\
&\quad\leq\frac{\rho}{2}\int_0^t\int_{\Omega}u(y^*-\overline{y}^*)^2dxd\tau+\frac{1}{2\rho}(NM_Kk_u)^2\int_0^t\left\|u(y^*-\overline{y}^*)(\tau)\right\|_{V'}^2d\tau.
\end{aligned}$$

Therefore, by Gronwall lemma (see [29]), $\left\|u(y^*-\overline{y}^*)(t)\right\|_{V'}^2\leq 0$ and we deduce that $uy^*(t)=u\overline{y}^*(t)$ for any $t\in[0,T]$. It follows that the solution is unique a.e. on the set Q_u where $u(x)>0$. Therefore, using (1.104) which is satisfied by $\zeta(t)\in\beta^*(y^*(t))$ and $\overline{\zeta}(t)\in\beta^*(y^*(t))$ we write the problem satisfied by their difference

$$\begin{aligned}
\Delta(\zeta-\overline{\zeta})(t)&=0 \text{ in } \Omega_0,\ \text{ a.e. } t\in(0,T),\\
(\zeta-\overline{\zeta})(t)&=0 \text{ on } \partial\Omega_0,\ \text{ a.e. } t\in(0,T).
\end{aligned}$$

This implies that $\zeta(t)=\overline{\zeta}(t)$ a.e. t and since $(\beta^*)^{-1}$ is single valued we get that $y^*(t)=\overline{y}^*(t)$ a.e. on Ω_0.Then the solution uniqueness follows a.e. on Q. □

Finally we would like to make a short comment about the continuity of the solution with respect to the nonlinear functions, without entering into details. We recall that such a property has been studied in [25] in the case of Richards' equation.

First we focus on the approximating problem (1.40). Let $(K_j)_j$ be such that $K_j(r)\to K(r)$ as $j\to\infty$, and $(\beta_j^*)_j$ be a family of graphs such that $(\beta_j^*)_j$ converges to β^* in the sense of the resolvent, that is

$$(1+\lambda\beta_j^*)^{-1}z\to(1+\lambda\beta^*)^{-1}z,\ \text{as } j\to\infty,\ \forall\lambda>0,\ \forall z\in\mathbb{R}.$$

Then

$$(I+\lambda B_\varepsilon^j)^{-1}g\to(I+\lambda B_\varepsilon)^{-1}g \text{ as } j\to\infty,\ \text{ for } g\in V',$$

where B_ε^j are the quasi m-accretive operators in (1.40) corresponding to $(\beta_j^*)_\varepsilon$. Then by Trotter–Kato theorem for nonlinear semigroups (see [14], pp. 168) it follows that the corresponding sequence of solutions $(\theta_j)_\varepsilon$ is convergent to θ_ε as $j\to\infty$ in $C([0,T];V')$. This continuity result can be further used to get the continuity for the solution to the limit equation when $\varepsilon\to 0$.

1.1.7 Numerical Results

We end this chapter with numerical simulations for the solution to (1.1). We imagine some scenarios for a real-world model of water infiltration into a nonhomogeneous porous medium (soil) in which a solid intrusion with zero porosity (a rock) is present. Assuming that the model (1.1) is already written in a dimensionless form, let us consider the expressions

$$\beta(r) = \frac{c(c-1)}{(c-r)^2}, \; K(r) = \frac{(c-1)r^2}{c-r} \text{ for } r \in [0,1), \quad c > 1, \tag{1.109}$$

given by the parametric model of Broadbridge and White (see [33]). These functions characterize the water infiltration into a soil whose properties are strongly nonlinear when c is in a neighborhood of 1 and weakly nonlinear for larger values of c (e.g., $c \geq 1.2$).

We see that here $\lim_{r \to y_s = 1} \beta(r)$ is finite. This may be obtained by a jump of the function C (defined in Introduction) at $r = r_s = 0$ from a positive value at the left to 0 at the right (see case (a) in Introduction), such that the function β^* is multivalued at $r = 1$. All the results proved in this section apply to this case as well.

The computations are done in the 2D case in the domain

$$\Omega = \{(x_1, x_2); x_1 \in (0,5), \; x_2 \in (0,5)\},$$

with Ω_0 the circle with center in $(2,3)$ and radius $\delta = 0.1$,

$$\Omega_0 = \{(x_1, x_2); (x_1 - 2)^2 + (x_2 - 3)^2 \leq 0.1^2\}$$

and the function u (expressing the porosity of the soil) is chosen of the form

$$u(x_1, x_2) := \begin{cases} 0, & \text{in } \Omega_0 \\ \frac{(x_1-2)^2 + (x_2-3)^2 - 0.1^2}{100}, & \text{in } \Omega_u. \end{cases} \tag{1.110}$$

In the computations we take $u_\varepsilon(x_1, x_2) = u(x_1, x_2) + 10^{-9}$.

The functions β^* and K_0 with the properties considered in this section are

$$\beta^*(r) = \begin{cases} \frac{(c-1)r}{c-r}, & r \in [0,1) \\ [1, \infty), & r = 1, \end{cases} \quad K_0(x, r) = \begin{cases} a(x) \begin{cases} \frac{(c-1)r^2}{c-r}, & r \in [0,1) \\ 1, & r \geq 1 \end{cases} & \text{in } \Omega_0 \\ 0, & \text{in } \Omega_u. \end{cases}$$

The other data are: $\theta_0(x_1, x_2) = 0, \;\; a(x_1, x_2) = (1,1)$, meaning that the initial soil is dry and the advection is along both directions, and

$$f(t, x_1, x_2) = \begin{cases} t^2, & \text{in } \Omega_u \\ 0, & \text{in } \Omega_0. \end{cases}$$

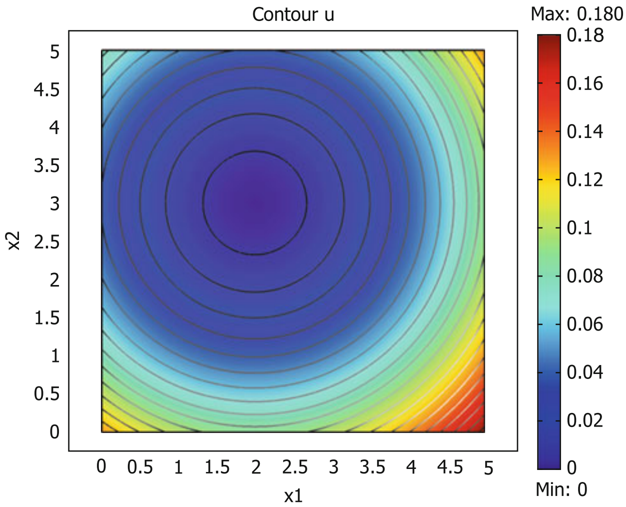

Fig. 1.2 Contour plot of the function u given by (1.110)

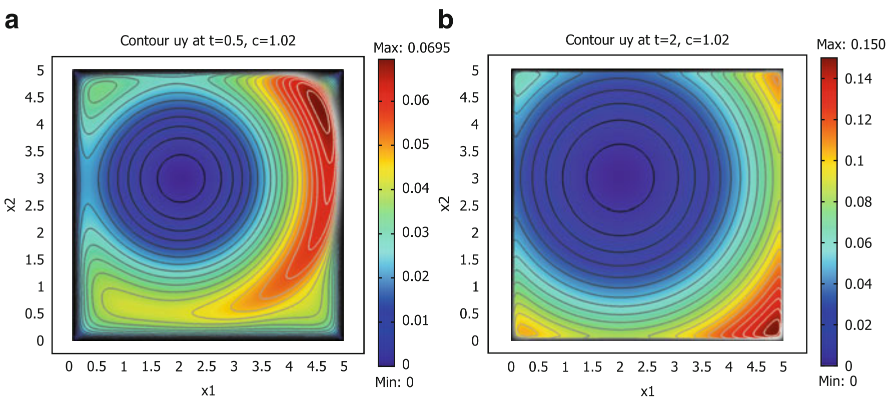

Fig. 1.3 Solution $\theta = uy$ in the parabolic–elliptic degenerate case for u given by (1.110) and $c = 1.02$

The algorithm is adapted from [39] for this degenerate case and the computations are done by using the software package Comsol Multiphysics (see [40]).

In Fig. 1.2 it is represented the contour plot of the function $x_3 = u(x_1, x_2)$, i.e., the projection of this surface on the plane x_1Ox_2.

We are interested in some comparisons. In Fig. 1.3a, b we see the evolution of $\theta = uy$ (representing the volumetric water content or soil moisture) computed for $c = 1.02$ (a strongly nonlinear soil) at two moments of time

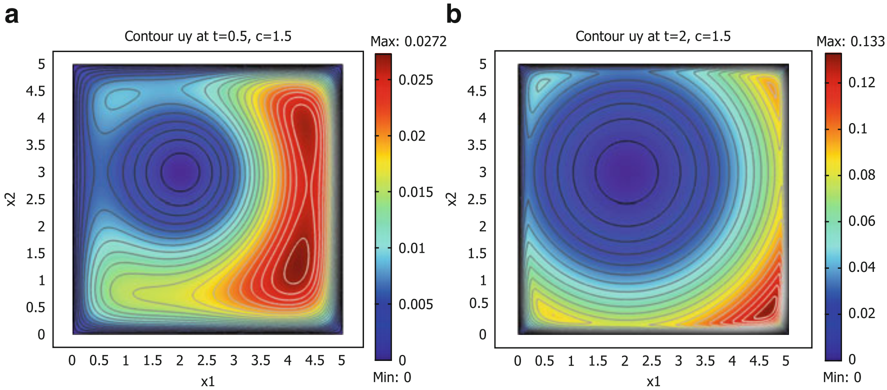

Fig. 1.4 Solution $\theta = uy$ in the parabolic–elliptic degenerate case for u given by (1.110) and $c = 1.5$

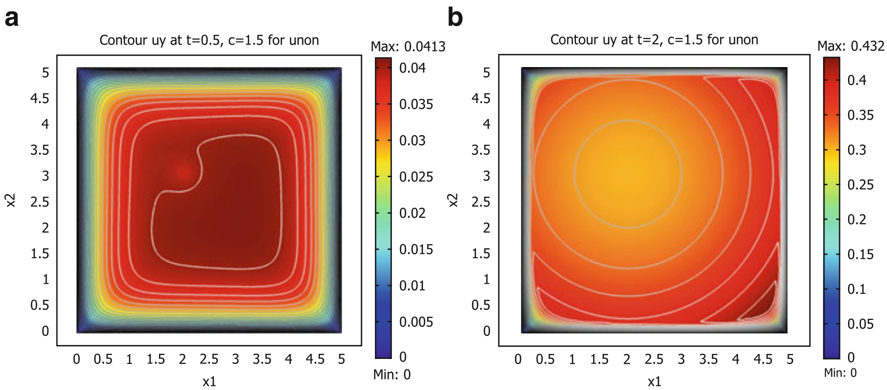

Fig. 1.5 Solution θ in the parabolic–elliptic nondegenerate case for u given by (1.111) and $c = 1.5$

$t = 0.5$ (Fig. 1.3a) and $t = 2$ (Fig. 1.3b), while in Fig. 1.4a, b we see the evolution of θ computed for $c = 1.5$ (a weakly nonlinear soil).

Then we compare the graphics in Fig. 1.4a, b with those drawn in Fig. 1.5a, b corresponding to the nondegenerate case with u positive given by the relation

$$unon(x_1, x_2) = u(x_1, x_2) + 0.3 \tag{1.111}$$

and $c = 1.5$. This describes a porous medium with a higher porosity which does not vanish, in which we see that the volumetric water content θ can reach higher values than in porosity vanishing case.

1.2 Well-Posedness for the Cauchy Problem with Very Fast Diffusion

Let us consider the problem

$$\begin{aligned} \frac{\partial(u(x)y)}{\partial t} - \Delta\beta^*(y) + \nabla\cdot K_0(x,y) &= f(t,x) \quad \text{in } Q, \\ y(t,x) &= 0 \qquad\quad \text{on } \Sigma, \\ (u(x)y(t,x))|_{t=0} &= \theta_0(x) \quad\ \text{in } \Omega, \end{aligned} \tag{1.112}$$

in which β^* is a single valued function, β and β^* blow-up at $r = y_s$,

$$\lim_{r\nearrow y_s} \beta(r) = +\infty, \quad \lim_{r\nearrow y_s} \beta^*(r) = \lim_{r\nearrow y_s} \int_0^r \beta(s)ds = +\infty \tag{1.113}$$

(see case (b) in Introduction) and

$$\beta(r) = \rho > 0, \text{ for any } r \leq 0.$$

The functions u, a_i and K are assumed to be as in the fast diffusion case, i.e., obeying (1.10)–(1.14).

In this case we introduce the function $j : \mathbb{R} \to (-\infty, +\infty]$ by

$$j(r) := \begin{cases} \int_0^r \beta^*(\xi)d\xi, & r < y_s, \\ +\infty, & r \geq y_s, \end{cases}$$

and specify that j is proper, convex, l.s.c. and

$$\partial j(r) = \begin{cases} \beta^*(r), & r < y_s, \\ +\infty, & r \geq y_s, \end{cases}$$

(see the proof in [84], pp. 74).

Let us assume that

$$f \in L^2(0,T;V'), \tag{1.114}$$

$$\theta_0 \in L^2(\Omega),\ \theta_0 = 0 \text{ a.e. on } \Omega_0, \tag{1.115}$$

$$\theta_0 \geq 0 \text{ a.e. on } \Omega_u,\ \frac{\theta_0}{u} \in L^2(\Omega_u),\ j\left(\frac{\theta_0}{u}\right) \in L^1(\Omega).$$

Definition 1.9. Let (1.114) and (1.115) hold. We call a *weak solution* to (1.112) a function y such that

$$y \in L^2(0,T;V),\ \beta^*(y) \in L^2(0,T;V),$$
$$uy \in C([0,T];L^2(\Omega)) \cap W^{1,2}([0,T];V'),$$

which satisfies

$$\left\langle \frac{d(uy)}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega (\nabla\beta^*(y)(t) - K_0(x,y(t))) \cdot \nabla\psi dx$$
$$= \langle f(t), \psi\rangle_{V',V},\ \text{a.e. } t \in (0,T),\ \text{for any } \psi \in V,$$

the initial condition $(uy(t))|_{t=0} = \theta_0$ and the boundedness condition

$$y(t,x) < y_s \text{ a.e. } (t,x) \in Q.$$

In the same way as in the previous section we can write the abstract Cauchy problem

$$\frac{d(uy)}{dt}(t) + Ay(t) = f(t),\ \text{a.e. } t \in (0,T), \tag{1.116}$$
$$(uy(t))|_{t=0} = \theta_0,$$

where

$$D(A) := \left\{ y \in L^2(\Omega);\ \beta^*(y) \in V \right\}$$

and $V = H_0^1(\Omega)$, with the dual $V' = H^{-1}(\Omega)$.

Then we pass to (1.30) by denoting $\theta(t,x) = u(x)y(t,x)$.

Next we shall prove that (1.116) has a weak solution.

Theorem 1.10. *Let us assume (1.114) and (1.115). Then, the Cauchy problem (1.116) has at least a weak solution y^*. In addition, if (1.107) holds, then the solution is unique.*

Proof. The proof is led as in the case of fast diffusion, with some modifications imposed by the blowing-up of β^*. First, we introduce the approximating functions β_ε and β_ε^* by

$$\beta_\varepsilon(r) := \begin{cases} \beta(r), & r < y_s - \varepsilon \\ \beta(y_s - \varepsilon), & r \geq y_s - \varepsilon, \end{cases} \tag{1.117}$$

$$\beta_\varepsilon^*(r) := \begin{cases} \beta^*(r), & r < y_s - \varepsilon \\ \beta^*(y_s - \varepsilon) + \beta(y_s - \varepsilon)[r - (y_s - \varepsilon)], & r \geq y_s - \varepsilon \end{cases} \tag{1.118}$$

and the approximating problem (1.40). It has a unique strong solution satisfying estimate (1.52), by using the same arguments as in Proposition 1.5. Then, if $j\left(\frac{\theta_0}{u}\right) \in L^1(\Omega)$ one can see that the upper bound of this estimate does not depend on ε and the proof can be continued as in Theorem 1.6.

The delicate point is to show the convergence of $\beta_\varepsilon^*(y_\varepsilon)$ to $\beta^*(y)$ in $L^2(0,T;L^2(\Omega_u))$. This is implied by the convergencies (1.84), (1.85)

$$y_\varepsilon \to y \text{ in } L^2(0,T;L^2(\Omega_\delta)) \text{ as } \varepsilon \to 0,$$
$$y_\varepsilon \rightharpoonup y \text{ in } L^2(0,T;L^2(\Omega_u)) \text{ as } \varepsilon \to 0$$

and (1.69)

$$\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \rightharpoonup \zeta \text{ in } L^2(0,T;V), \text{ as } \varepsilon \to 0. \tag{1.119}$$

We claim that $\zeta = \beta^*(y)$ a.e. on Q_u. For this we set

$$Q_{\delta s} := \{(t,x) \in Q_\delta;\ y(t,x) = y_s\},\ Q_{\delta n} := \{(t,x) \in Q_\delta;\ y(t,x) < y_s\}.$$

Then, if $(t,x) \in Q_{\delta n}$ we have $\beta_\varepsilon(r) = \beta(r)$ (for ε small enough) and we can write

$$\begin{aligned}\beta_\varepsilon^*(y_\varepsilon(t,x)) &= \int_0^{y_\varepsilon(t,x)} \beta_\varepsilon(r)dr = \int_0^{y_\varepsilon(t,x)} \beta(r)dr \\ &\to \int_0^{y(t,x)} \beta(r)dr = \beta^*(y(t,x)) \text{ a.e. on } Q_{\delta n},\ \text{as } \varepsilon \to 0.\end{aligned}$$

If $(t,x) \in Q_{\delta s}$, then two situations may arise:

(p$_1$) there is a sequence $\varepsilon_k \to 0$ such that $y_{\varepsilon_k}(t,x) \geq y_s - \varepsilon_k$.
(p$_2$) for all $\varepsilon < \varepsilon_0$ we have $y_\varepsilon(t,x) < y_s - \varepsilon$.

In the case (p$_2$) the previous argument for $(t,x) \in Q_{\delta n}$ applies and $\beta_\varepsilon^*(y_\varepsilon) \to \beta^*(y)$ a.e. for $(t,x) \in Q_{\delta s}$.

In the case (p$_1$) we have

$$\begin{aligned}\beta_{\varepsilon_k}^*(y_{\varepsilon_k}(t,x)) &= \int_0^{y_{\varepsilon_k}(t,x)-\varepsilon_k} \beta(r)dr + \int_{y_{\varepsilon_k}(t,x)-\varepsilon_k}^{y_{\varepsilon_k}(t,x)} \beta\left(y_s - \varepsilon_k\right)dr \\ &= \int_0^{y_{\varepsilon_k}(t,x)-\varepsilon_k} \beta(r)dr + \varepsilon_k\beta(y_s - \varepsilon_k) \to +\infty = \beta^*(y_s), \\ &\quad \text{as } \varepsilon_k \to 0,\end{aligned}$$

because $\int_0^{y_s} \beta(r)dr = +\infty$, pursuant to (1.113). Hence, selecting a subsequence (denoted still by the subscript ε), we have that

$$\beta_\varepsilon^*(y_\varepsilon) \to \beta^*(y) \text{ a.e. on } Q_\delta \text{ as } \varepsilon \to 0.$$

But $(\beta_\varepsilon^*(y_\varepsilon))_{\varepsilon>0}$ is bounded in $L^2(Q_\delta)$ by (1.66) and since it converges a.e. on Q_δ, it follows that $\beta_\varepsilon^*(y_\varepsilon) \rightharpoonup \beta^*(y)$ in $L^2(Q_\delta)$. Then we get that $\zeta = \beta^*(y)$ a.e. on Q_δ and since δ is arbitrarily small we obtain $\zeta = \beta^*(y)$ a.e. on Q_u.

Here we have applied a consequence of Mazur theorem saying that if O is a bounded open set of finite measure and $(f_n)_{n\geq 1}$ is a sequence bounded in $L^2(O)$ such that $f_n \to f$ a.e. on O, then $f_n \rightharpoonup f$ in $L^2(O)$ as $n \to \infty$.

The proof is continued as in Theorem 1.6 and Proposition 1.8. □

1.3 Existence of Periodic Solutions in the Parabolic–Elliptic Degenerate Case

In this section we deal with the study of periodic solutions to the degenerate fast diffusion equation introduced in Sect. 1.1, under the hypothesis of a T-periodic function f. To this end, we first investigate the existence of a periodic solution to an intermediate problem restraint to a period T and extend then the result by periodicity to the time space $\mathbb{R}_+ = (0, \infty)$. The proof involves an appropriate approximating periodic problem and the existence of a solution is shown via a fixed point theorem on the basis of the results for the approximating problem (1.40). This result will also allow to characterize the behavior at large time of the solution to a Cauchy problem with periodic data.

We recall some previous papers dealing with periodic problems for degenerate linear equations. In [16] a problem of the type

$$\frac{d}{dt}(My(t)) + Ay(t) = f(t),\ 0 \leq t \leq 1,$$

with the periodic condition $(My)(0) = (My)(1)$ has been studied. Here M and A are two closed linear operators from a complex Banach space into itself, under the assumptions that the domain $D(A)$ of A is continuously embedded in $D(M)$ and A has a bounded inverse. Assuming suitable hypotheses on the modified resolvent $(\lambda M + A)^{-1}$, it has been proved that problem admits one 1-periodic solution. Some examples of applications to partial differential equations and ordinary differential equations have been given. The latter case has been studied in the paper [17].

The nondegenerate fast diffusion case with a nonlinear transport term has been approached in the paper [87], while the degenerate case without advection has been studied in [59].

As in Sect. 1.1, Ω is an open bounded subset of $\mathbb{R}^N$ and T is finite. We consider the problem

$$\begin{aligned} \frac{\partial(u(x)y)}{\partial t} - \Delta\beta^*(y) + \nabla\cdot K_0(x,y) &\ni f \text{ in } \mathbb{R}_+\times\Omega,\\ y(t,x) &= 0 \text{ on } \mathbb{R}_+\times\Gamma, \\ (u(x)y(\tau,x))|_{\tau=t} - (u(x)y(\tau,x))|_{\tau=t+T} &= 0 \text{ in } \mathbb{R}_+\times\Omega, \end{aligned} \tag{1.120}$$

under the assumption of the T-periodicity of the function f,

$$f(t,x) = f(t+T,x) \text{ a.e. } (t,x)\in\mathbb{R}_+\times\Omega. \tag{1.121}$$

The hypotheses made for β^*, K_0 and u are preserved as they were presented in Sect. 1.1 and we assume that $f\in L^\infty_{loc}(\mathbb{R}_+;V')$.

We begin with the study of the existence for the solution to the problem on a time period

$$\begin{aligned} \frac{\partial(u(x)y)}{\partial t} - \Delta\beta^*(y) + \nabla\cdot K_0(x,y) &\ni f \text{ in } Q=(0,T)\times\Omega,\\ y(t,x) &= 0 \text{ on } \Sigma=(0,T)\times\Gamma, \\ (u(x)y(t,x))|_{t=0} - (u(x)y(t,x))|_{t=T} &= 0 \text{ in } \Omega. \end{aligned} \tag{1.122}$$

Then, this solution will be extended by periodicity to all $t\in\mathbb{R}_+$.

1.3.1 Solution Existence on the Time Period $(0,T)$

The functional framework for this problem is the same as in Sect. 1.1.

Definition 1.11. Let $f\in L^\infty(0,T;V')$. We call a *weak solution* to (1.122) a pair (y,ζ) such that

$$\begin{aligned} &y\in L^2(0,T;V),\ y(t,x)\le y_s \text{ a.e. } (t,x)\in Q,\\ &uy\in C([0,T];L^2(\Omega))\cap W^{1,2}([0,T];V'),\\ &\zeta\in L^2(0,T;V),\ \zeta(t,x)\in\beta^*(y(t,x)) \text{ a.e. } (t,x)\in Q, \end{aligned}$$

satisfying the equation

$$\int_0^T \left\langle \frac{d(uy)}{dt}(t), \phi(t) \right\rangle_{V',V} dt + \int_Q (\nabla\zeta - K_0(x,y)) \cdot \nabla\phi dxdt$$
$$= \int_0^T \langle f(t), \phi(t) \rangle_{V',V} dt, \text{ for any } \phi \in L^2(0,T;V)$$

and the condition $(u(x)y(t,x))|_{t=0} - (u(x)y(t,x))|_{t=T} = 0$ in Ω.

With the same notation and definitions as in Sect. 1.1. we consider the periodic approximating problem

$$\frac{d(u_\varepsilon y_\varepsilon)}{dt}(t) + A_\varepsilon u_\varepsilon(t) = f(t) \text{ a.e. } t \in (0,T), \quad (1.123)$$
$$u_\varepsilon(y_\varepsilon(0) - y_\varepsilon(T)) = 0$$

which is equivalent with

$$\frac{d\theta_\varepsilon}{dt}(t) + B_\varepsilon\theta_\varepsilon(t) = f(t) \text{ a.e. } t \in (0,T), \quad (1.124)$$
$$\theta_\varepsilon(0) = \theta_\varepsilon(T),$$

by the function replacement $\theta_\varepsilon = u_\varepsilon y_\varepsilon$, with A_ε and B_ε given by (1.38) and (1.41), respectively.

Let us denote

$$C_f = \frac{2}{\rho}\left(\|f\|^2_{L^\infty(0,T;V')} + \overline{K}^2 \right), \quad (1.125)$$

where $\overline{K} = K_s(\text{meas}(\Omega))^{1/2} \sum_{j=1}^{N} a_j^M$ was defined in Proposition 1.5. We also recall that ρ was specified in (1.6), $\overline{M} = M_K \sum_{j=1}^{N} a_j^M$ and by c_P we have denoted the constant in the Poincaré inequality.

We are going to prove the following existence result.

Theorem 1.12. *Let $f \in L^\infty(0,T;V')$. Then, the periodic approximating problem (1.124) has a unique solution*

$$\theta_\varepsilon \in C([0,T];L^2(\Omega)) \cap W^{1,2}([0,T];V') \cap L^2(0,T;V), \quad (1.126)$$

$$\beta_\varepsilon^*\left(\frac{\theta_\varepsilon}{u_\varepsilon}\right) \in L^2(0,T;V). \quad (1.127)$$

Moreover, the solution satisfies the estimate

$$\int_0^T \left\| \frac{d\theta_\varepsilon}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^T \left\| \beta_\varepsilon^* \left(\frac{\theta_\varepsilon}{u_\varepsilon}(\tau) \right) \right\|_V^2 d\tau \tag{1.128}$$
$$\leq 4 \left(\int_0^T \|f(t)\|_{V'}^2 \, dt + \overline{K}^2 T \right).$$

Proof. We apply a fixed point result and start this by fixing in (1.124) $\theta_\varepsilon(0)$ in $L^2(\Omega)$ and denoting it by v, i.e.,

$$\theta_\varepsilon(0) := v \in L^2(\Omega).$$

Hence we have to deal with the Cauchy problem

$$\frac{d\theta_\varepsilon}{dt}(t) + B_\varepsilon \theta_\varepsilon(t) = f(t) \text{ a.e. } t \in (0,T), \tag{1.129}$$
$$\theta_\varepsilon(0) = v,$$

whose well-posedness for $v \in L^2(\Omega)$ has already been studied in Sect. 1.1, Proposition 1.5. Thus, (1.129) has a unique solution (1.126)–(1.127).

Let us consider the set

$$\mathcal{S}_\varepsilon := \left\{ z \in L^2(\Omega);\ \left\| \frac{z}{\sqrt{u_\varepsilon}} \right\| \leq R_\varepsilon \text{ a.e. } x \in \Omega \right\} \tag{1.130}$$

where R_ε is a positive constant for each $\varepsilon > 0$. We define the mapping

$$\Psi_\varepsilon : \mathcal{S}_\varepsilon \to \mathcal{S}_\varepsilon,\ \Psi_\varepsilon(v) = \theta_\varepsilon(T), \text{ for any } v \in \mathcal{S}_\varepsilon$$

where $\theta_\varepsilon(t)$ is the solution to (1.129).

Since (1.129) has a unique solution for $v \in \mathcal{S}_\varepsilon$, the mapping Ψ_ε is single-valued and we are going to show that it has a fixed point by the Schauder–Tychonoff theorem (see e.g., [67], pp. 148), working in the weak topology. We begin by checking the conditions of this theorem.

(i1) It is obvious that $\mathcal{S}_\varepsilon$ is a convex, bounded and strongly closed subset of $L^2(\Omega)$. Hence it is weakly compact in $L^2(\Omega)$.

(i2) Next, we have to show the inclusion $\Psi_\varepsilon(\mathcal{S}_\varepsilon) \subset \mathcal{S}_\varepsilon$.

The solution $\theta_\varepsilon \in C([0,T]; L^2(\Omega))$ and so $\theta_\varepsilon(T) = u_\varepsilon y_\varepsilon(T) \in L^2(\Omega)$. We test (1.129) for $\frac{\theta_\varepsilon}{u_\varepsilon} \in V$ and recalling (1.37) and (1.14) we get

$$\frac{1}{2}\frac{d}{dt}\left\| \frac{\theta_\varepsilon}{\sqrt{u_\varepsilon}}(t) \right\|^2 + \rho \left\| \frac{\theta_\varepsilon}{u_\varepsilon}(t) \right\|_V^2 \leq \|f(t)\|_{V'} \left\| \frac{\theta_\varepsilon}{u_\varepsilon}(t) \right\|_V + \overline{K} \left\| \frac{\theta_\varepsilon}{u_\varepsilon}(t) \right\|_V$$
$$\leq \frac{\rho}{2} \left\| \frac{\theta_\varepsilon}{u_\varepsilon}(t) \right\|_V^2 + \frac{1}{\rho}\left(\|f\|_{L^\infty(0,T;V')}^2 + \overline{K}^2 \right).$$

Next, applying the Poincaré inequality we have

$$\frac{d}{dt}\left\|\frac{\theta_\varepsilon}{\sqrt{u_\varepsilon}}(t)\right\|^2+\frac{\rho}{c_P^2}\left\|\frac{\theta_\varepsilon}{u_\varepsilon}(t)\right\|^2\le C_f$$

and using the relation $u_\varepsilon(x)\le u_M+\varepsilon<u_M+1$ (since ε is arbitrarily small) we obtain

$$\frac{d}{dt}\left\|\frac{\theta_\varepsilon}{\sqrt{u_\varepsilon}}(t)\right\|^2+\rho_0\left\|\frac{\theta_\varepsilon}{\sqrt{u_\varepsilon}}(t)\right\|^2\le C_f$$

with $\rho_0=\frac{\rho}{(u_M+1)c_P^2}$. Integrating on $(0,t)$ with $t\in[0,T]$ we get

$$\left\|\frac{\theta_\varepsilon}{\sqrt{u_\varepsilon}}(t)\right\|^2\le\left\|\frac{v}{\sqrt{u_\varepsilon}}\right\|^2\exp(-\rho_0 t)+\frac{C_f}{\rho_0}(1-\exp(-\rho_0 t)).$$

Now if $R_\varepsilon^2\ge\frac{C_f}{\rho_0}$ (and this is true since R_ε is large enough) and $v\in\mathcal{S}_\varepsilon$ it follows that

$$\left\|\frac{\theta_\varepsilon}{\sqrt{u_\varepsilon}}(t)\right\|\le R_\varepsilon,\ \text{ for any } t\in[0,T].$$

Thus, we have obtained that $\theta_\varepsilon(T)=\Psi_\varepsilon(v)\in\mathcal{S}_\varepsilon$ and therefore, it follows that $\Psi_\varepsilon(\mathcal{S}_\varepsilon)$ is weakly compact, too.

(i3) Finally, we have to prove that the mapping Ψ_ε is weakly continuous.

For that we consider a sequence

$$\{v^n\}_{n\ge1}\subset\mathcal{S}_\varepsilon,\ v^n\rightharpoonup v \text{ in } L^2(\Omega) \text{ as } n\to\infty,$$

and will show that

$$\Psi_\varepsilon(v^n)\rightharpoonup\Psi_\varepsilon(v) \text{ in } L^2(\Omega) \text{ as } n\to\infty.$$

We introduce the approximating problem

$$\frac{d\theta_\varepsilon^n}{dt}(t)+B_\varepsilon\theta_\varepsilon^n(t)=f(t),\ \text{a.e. } t\in(0,T),$$
$$\theta_\varepsilon^n(0)=v^n.$$

This has a unique solution

$$\theta_\varepsilon^n\in C([0,T];V')\cap W^{1,2}([0,T];V')\cap L^2(0,T;V),\ \beta_\varepsilon^*\left(\frac{\theta_\varepsilon^n}{u_\varepsilon}\right)\in L^2(0,T;V)$$

satisfying the estimate (1.52). Now, by (1.59)

$$\int_{\Omega} u_\varepsilon j_\varepsilon \left(\frac{\theta_\varepsilon^n}{u_\varepsilon}\right) dx \leq (u_M + \varepsilon)\frac{\beta_s^* - \beta^*(y_s - \varepsilon)}{2\varepsilon} \left\| \frac{v^n}{u_\varepsilon} \right\|^2$$

and

$$\left\| \frac{v^n}{u_\varepsilon} \right\| \leq \frac{1}{\sqrt{\varepsilon}} \left\| \frac{v^n}{\sqrt{u_\varepsilon}} \right\| \leq \frac{1}{\sqrt{\varepsilon}} R_\varepsilon$$

due to the fact that $v^n \in \mathcal{S}_\varepsilon$. Therefore (1.52) written for θ_ε^n is bounded independently of n, and we can proceed like in Proposition 1.5 to show that θ_ε^n tends in some appropriate space to θ_ε which turns out to be the solution to (1.129). This implies also the convergence

$$\theta_\varepsilon^n(T) \to \theta_\varepsilon(T) \text{ in } V', \text{ as } n \to \infty$$

due to the Ascoli–Arzelà theorem (see (1.62)). Hence

$$\Psi_\varepsilon(v^n) = \theta_\varepsilon^n(T) \rightharpoonup \theta_\varepsilon(T) = \Psi_\varepsilon(v) \text{ in } L^2(\Omega),$$

and because $\mathcal{S}_\varepsilon$ is weakly closed it follows that $\theta_\varepsilon(T) \in \mathcal{S}_\varepsilon$.

Now the Schauder–Tychonoff theorem ensures that Ψ_ε has a fixed point, implying that

$$\theta_\varepsilon(0) = \theta_\varepsilon(T) \text{ or } u_\varepsilon\theta_\varepsilon(0) = u_\varepsilon\theta_\varepsilon(T).$$

Consequently, (1.124) has at least a solution.

The estimate (1.128) follows immediately by (1.52) in Proposition 1.5, for $t = T$.

Uniqueness is proved as in Proposition 1.5, taking the same data in (1.53). This ends the proof of Theorem 1.12. □

Theorem 1.13. *Let $f \in L^\infty(0, T; V')$. Then, the periodic problem (1.122) has at least a solution (y^*, ζ) such that*

$$\begin{aligned} y^* &\in L^2(0,T;V), \\ uy^* &\in C([0,T];L^2(\Omega)) \cap W^{1,2}([0,T];V'), \\ \zeta &\in L^2(0,T;V),\ \zeta(t,x) \in \beta^*(y^*(t,x)) \text{ a.e. } (t,x) \in Q, \\ y^*(t,x) &\leq y_s \text{ a.e. } (t,x) \in Q. \end{aligned}$$

If (1.107) and

$$\rho > N M_K k_u c_P \sqrt{u_M} \tag{1.131}$$

are satisfied the solution is unique a.e. on Q.

Proof. The proof of the existence is based on the same arguments and is led in the same way as in Theorem 1.6, including the construction of y^*, with the corresponding modifications due to the periodicity condition. Thus in the approximating problem in Theorem 1.6, $\theta_\varepsilon(0) = u_\varepsilon y_\varepsilon(0) = u_\varepsilon y_\varepsilon(T) = \theta_\varepsilon(T)$ and by (1.80) we get $(uy)|_{t=0} = (uy)|_{t=T}$ property which is inherited by uy^*. Obviously, $uy^* = 0$ in Q_0.

Assume now (1.107) and that there exist two solutions (y^*, ζ) and $(\overline{y}^*, \overline{\zeta})$ to (1.122) corresponding to the same periodic data f. We subtract (1.122) written for y^* and $\overline{y}^*$ and multiply the difference scalarly in V' by $u(y^* - \overline{y}^*)(t)$,

$$\left(\frac{d(u(y^* - \overline{y}^*))}{dt}(t), u(y^* - \overline{y}^*)(t)\right)_{V'} + \int_\Omega \nabla(\zeta(t) - \overline{\zeta}(t)) \cdot \nabla\psi(t)dx$$
$$= \int_\Omega (K(y^*(t)) - K(\overline{y}^*(t))a(x) \cdot \nabla\psi(t)dx$$

where $A_0\psi(t) = u(y^* - \overline{y}^*)(t)$, a.e. t, (where we recall that $A_0 = -\Delta$ with Dirichlet boundary conditions (see (1.17)). Integrating over $(0, T)$ and proceeding as in Proposition 1.8 we get

$$\frac{1}{2}\left\|u(y^* - \overline{y}^*)(T)\right\|_{V'}^2 - \frac{1}{2}\left\|u(y^* - \overline{y}^*)(0)\right\|_{V'}^2 + \frac{\rho}{2}\int_0^T \int_\Omega u(y^* - \overline{y}^*)^2 dxdt$$
$$\leq \frac{1}{2\rho}(NM_K k_u)^2 \int_0^T \left\|u(y^* - \overline{y}^*)(\tau)\right\|_{V'}^2 d\tau$$
$$\leq \frac{1}{2\rho}(NM_K k_u c_P \sqrt{u_M})^2 \int_0^T \left\|\sqrt{u}(y^* - \overline{y}^*)(\tau)\right\|^2 d\tau,$$

where c_P is the constant in the Poincaré inequality. We apply the solution periodicity and it remains that $\|\sqrt{u}(y^* - \overline{y}^*)\|_{L^2(Q)}^2 = 0$. This implies that $uy^* = u\overline{y}^*$ a.e. on Q and then we continue as in Proposition 1.8. □

1.3.2 Solution Existence on $\mathbb{R}_+$

Now we can extend the previous result to $t \in \mathbb{R}_+$. We resume problem (1.120) and prove

Theorem 1.14. *Let us assume*

$$f \in L^\infty_{loc}(\mathbb{R}_+; V'),\ f(t,x) = f(t+T, x) \text{ a.e. } (t,x) \in \mathbb{R}_+ \times \Omega.$$

Then problem (1.120) has at least a solution $y \in L^2_{loc}(\mathbb{R}_+; V)$ satisfying

$$\theta = uy \in C(\mathbb{R}_+; L^2(\Omega)) \cap W^{1,2}_{loc}(\mathbb{R}_+; V'),$$
$$y(t,x) \leq y_s \text{ a.e. } (t,x) \in \mathbb{R}_+ \times \Omega,$$
$$\zeta \in L^2_{loc}(\mathbb{R}_+; V), \text{ where } \zeta(t,x) \in \beta^*(y(t,x)) \text{ a.e. } (t,x) \in \mathbb{R}_+ \times \Omega.$$

If (1.107) and (1.131) are satisfied then the solution is unique.

Proof. We consider first (1.120) on $(0,T)$ with $f|_{(0,T)}$. We obtain (1.122) which has a periodic solution with $(u(x)y(t,x))|_{t=0} = (u(x)y(t,x))|_{t=T}$ in Ω. Then we consider (1.120) on $(T, 2T)$ with the periodicity condition $(u(x)y(t,x))|_{t=T} = (u(x)y(t,x))|_{t=2T}$ in Ω. We make the transformation $t' = t - T$ and denote $\widetilde{y}(t',x) = y(t'+T,x)$ with $t' \in [0,T]$. Using now the periodicity of the function f we find again problem (1.122) which has a periodic solution $\widetilde{y}(t')$ with $\widetilde{\theta} = u\widetilde{y} \in C([0,T]; L^2(\Omega))$, such that $(u(x)\widetilde{y}(t',x))|_{t'=0} = (u(x)\widetilde{y}(t',x))|_{t'=T}$. Coming back to the variable t we obtain that (1.120) has a periodic solution such that $u(x)y(t,x)$ is continuous on $[T, 2T]$ and this extends by continuity the solution obtained on $[0,T]$. The procedure is continued in this way on each time period. Moreover, if a satisfies (1.107) and (1.131) the solution is unique on each period. □

1.3.3 Longtime Behavior of the Solution to a Cauchy Problem with Periodic Data

Finally we are going to characterize the longtime behavior of the solution y to problem (1.1) with a T-periodic function f. The domain Q is taken $\mathbb{R}_+ \times \Omega$, and we assume that the solution starts from the initial condition θ_0. Let

$$f \in L^\infty_{loc}(\mathbb{R}_+; V'),\ f(t,x) = f(t+T,x) \text{ a.e. } (t,x) \in \mathbb{R}_+ \times \Omega, \tag{1.132}$$
$$\theta_0 \in L^2(\Omega),\ \theta_0 = 0 \text{ a.e. on } \Omega_0,$$
$$\theta_0 \geq 0 \text{ a.e. on } \Omega_u,\ \frac{\theta_0}{u} \in L^2(\Omega_u),\ \frac{\theta_0}{u}(x) \leq y_s \text{ a.e. } x \in \Omega_u$$

and we recall that u_M is the maximum of u and c_P is the constant in Poincaré inequality (1.19).

Proposition 1.15. *Let us assume (1.107) and (1.131). Then, the solution to the Cauchy problem (1.1) with f periodic of period T satisfies*

$$\lim_{t\to\infty} \|(uy - u\omega)(t)\|_{V'} = 0 \tag{1.133}$$

exponentially, where ω is the unique periodic solution to (1.120) and y is the unique solution to (1.1).

Proof. By Theorem 1.14 the solution to (1.120) is unique and let us denote it by ω. We multiply the difference of (1.1) and (1.120) by $u(y(t) - \omega(t))$ scalarly in V', and we get

$$\frac{1}{2}\frac{d}{dt}\|u(y-\omega)(t)\|_{V'}^2 + \rho\int_\Omega u(y-\omega)^2(t)dx$$
$$\leq NM_K k_u c_P \sqrt{u_M}\left\|\sqrt{u}(y-\omega)(t)\right\|^2.$$

Therefore, applying (1.131) we obtain

$$\frac{d}{dt}\|u(y-\omega)(t)\|_{V'}^2 + \delta\left\|\sqrt{u}(y-\omega)(t)\right\|^2 \leq 0$$

with $\delta = \rho - NM_K k_u c_P \sqrt{u_M}$.

We have that

$$\int_\Omega u(y(t)-\omega(t))^2 dx \geq \frac{1}{u_M}\int_\Omega u^2(y(t)-\omega(t))^2 dx \geq \frac{1}{u_M c_P^2}\|u(y-\omega)(t)\|_{V'}^2$$

hence

$$\frac{d}{dt}\|u(y-\omega)(t)\|_{V'}^2 + \delta_0\|u(y-\omega)(t)\|_{V'}^2 \leq 0$$

with $\delta_0 = \frac{\delta}{u_M c_P^2}$. We deduce that

$$\|u(y-\omega)(t)\|_{V'}^2 \leq e^{-\delta_0 t}\|\theta_0 - (u\omega)(0)\|_{V'}^2$$

and this implies (1.133). □

Referring to applications in real-world problems we remark that the behavior (1.133) of the solution to the Cauchy problem (1.1) with a periodic f is possible only if the advection is done with a velocity in absolute value lower than the porosity u and the diffusion processes has a sufficient high diffusion coefficient. This means that the velocity must be sufficient small in comparison with the pore dimension and that the diffusivity should dominate the advection.

1.3.4 Numerical Results

We shall provide some simulations intended to show the behavior at large time of the solution to (1.1) with a periodic f.

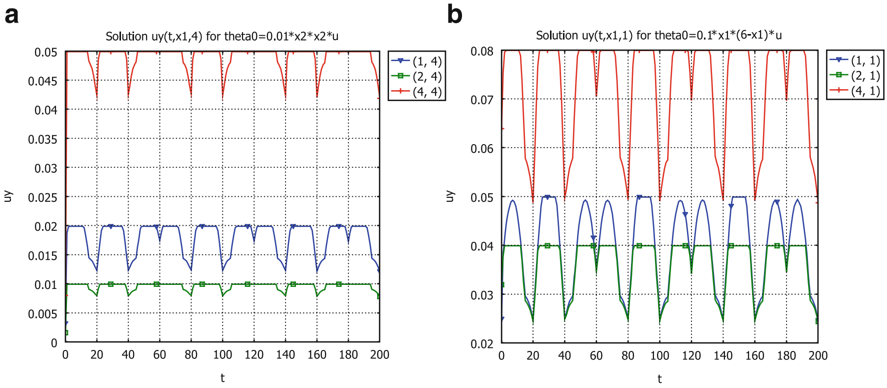

Fig. 1.6 Asymptotic behavior of $\theta = uy$ solution to (1.1) in the periodic parabolic–elliptic degenerate case

The computations are done in 2D with $\Omega = (0,5) \times (0,5)$, with the same data for Ω_0, u, β^* and K as in Sect. 1.1, (1.110), (1.109), $a = (1,1)$, $c = 1.5$ (a weakly nonlinear porous medium),

$$f(t, x_1, x_2) = \begin{cases} \left(\left|\sin \frac{\pi}{20} t\right| + \left|\cos \frac{\pi}{30} t\right|\right), & x \in \Omega_u \\ 0, & x \in \Omega_0 \end{cases}$$

and two different initial data. In Fig. 1.6a the values $\theta(t,x) = u(x)y(t,x)$ are computed for

$$\theta_0(x_1, x_2) = 0.01 x_2^2 u(x_1, x_2)$$

and represented at $x = (x_1, 4)$, $x_1 = 1, 2, 4$.

In Fig. 1.6b there are the graphics $\theta(t,x) = u(x)y(t,x)$ at $x = (x_1, 1)$, $x_1 = 1, 2, 4$, computed for

$$\theta_0(x_1, x_2) = 0.1 x_1 (6 - x_1) u(x_1, x_2). \tag{1.134}$$

We can see that after some time the solutions to (1.1) become periodic.

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