Chapter 2
Auxiliary Properties of Evolution Inclusions Solutions for Earth Data Processing

A great number of collectives of mathematicians, mechanicians, geophysicists (mainly theorists), engineers goes in for qualitative investigation of nonlinear mathematical models of evolution processes and fields of different nature, in particular, problems deal with the dynamics of solutions of non-stationary problems. Far from complete list of results concern the given direction is in works [4, 5, 7, 9–17, 19]. The last results deal with the studying of multivalued, in general case, dynamics of solutions of mathematical models with nonlinear, nonsmooth, nonmonotonic interaction functions as a rule are based on the theory of global and trajectory attractors for m-semiflows of solutions [4,21,24,37]. At that, properties for solutions of considered evolution problem concern with dissipativity of system and closedness (in certain sense) of resolving operator [4, 10, 11, 21, 24, 32, 36, 37]. Note that such properties of solutions for each equation are usually checked separately. At that we succeed to consider problems with linear main part of differential operator appeared in problem [4, 10, 11, 32, 37]. On the other hand, energetic extensions and Nemytskii operators for differential operators appeared in generalized settings of different problems of mathematical physics, problems on a manifold with boundary and without boundary, problems with delay, stochastic partial differential equations, problems with degenerates, as a rule have (as corresponding choice of the phase space) common properties concern growth conditions (usually no more that polynomial), sign conditions, pseudomonotony [14, 16, 19, 22, 25, 41, 42]. In general case as such restrictions for determinative parameters of a problem we succeed to prove only the existence of weak solutions of differential-operator inclusion, but not always this proof is constructive [14, 16, 19, 22, 25, 41, 42]. Hence, the problem of the existence of trajectory and global attractors and investigation of their structure for weak solutions of differential-operator equations in infinite-dimensional spaces with interaction functions of pseudomonotone type is actual one. Here we consider some additional properties of solutions for the first and second order autonomous evolution inclusions with pointwice pseudomonotone multivalued maps. This properties are connected with dissipation and closuredness of graph for resolving operator. The results of this chapter are borrowed from [6, 8, 13, 15, 18, 21, 23, 24, 28, 29, 40, 43].

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2.1 Preliminaries

At first let us consider constructions, presented in [41, 42].

At an analysis and control of different geophysical and socio-economical processes it is often appears such problem: at a mathematical modelling of effects related to friction and viscosity, quantum effects, a description of different nature waves the existing "gap" between rather high degree of the mathematical theory of analysis and control for non-linear processes and fields and practice of its using in applied scientific investigations make us require rather stringent conditions for interaction functions. These conditions related to linearity, monotony, smoothness, continuity and can substantially have an influence on the adequacy of mathematical model. Let us consider for example some diffusion process. Its mathematical model has the next form:

$$\begin{cases} y_t - \Delta y + f(y) = g(t,x) \quad in \quad \Omega \times (\tau; T), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{\tau=T} = y_0, \end{cases} \tag{2.1}$$

here $n \geq 2$, $\Omega \subset \mathbf{R}^n$ is a bounded domain with a rather smooth boundary, $-\infty < \tau < T < +\infty$, $g : \Omega \times (\tau; T) \to \mathbf{R}$, $y_0 : \Omega \to \mathbf{R}$ are rather regular functions, $f : \mathbf{R} \to \mathbf{R}$ is an interaction function, $y : \Omega \times (\tau; T) \to \mathbf{R}$ is an unknown function. It is well known that if f is a rather smooth function and satisfies for example the next condition of no more than polynomial growth:

$$\exists p > 1, \quad \exists c > 0 : \quad |f(s)| \leq c(1 + |s|^{p-1}) \quad \forall s \in \mathbf{R}, \tag{2.2}$$

then problem (2.1) has a unique rather regular solution. Let us consider the case when f is continuous and initial data and external forces are nonregular (for example $y_0 \in L_2(\Omega)$, $g \in L_2(\Omega \times (\tau; T))$). Then, as a rule, we consider the generalized setting of problem (2.1):

$$\begin{cases} y'(t) + A(y(t)) + B(y(t)) = g(t) \quad \text{for a.e.} \quad t \in (\tau; T), \\ y(\tau) = y_0, \end{cases} \tag{2.3}$$

here $A : V_1 \to V_1^*$ is an energetic extension of operator "$-\Delta$", $B : V_2 \to V_2^*$ is the Nemytskii operator for F, $V_1 = H_0^1(\Omega)$ is a real Sobolev space, $V_2 = L_p(\Omega)$, $V_1^* = H^{-1}(\Omega)$, $V_2^* = L_q(\Omega)$, q is the conjugated index, y' is a derivative of an element $y \in L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2)$ and it is considered in the sense of the space $\mathscr{D}^*([\tau; T], V_1^* + V_2^*)$.

A solution of problem (2.3) in the class $W = \{y \in L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2) \big| y' \in L_2(\tau, T; V_1^*) + L_q(\tau, T; V_2^*)\}$ refers to be the generalized solution of problem (2.1).

To prove the existence of solutions for problem (2.1) as a rule we need to add supplementary "signed condition" for an interaction function f, for example,

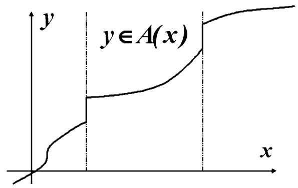


Fig. 2.1 The monotone multivalued map

$$\exists \alpha, \beta > 0 : \quad f(s)s \ge \alpha |s|^p - \beta \quad \forall s \in \mathbf{R}. \tag{2.4}$$

But we do not succeed in proving the uniqueness of the solution of such problem in the general case. Note that technical condition (2.4) provides a dissipation too. We remark also that different conditions for parameters of problem (2.1) provide corresponding conditions for generated mappings A and B.

Problem (2.3) is usually investigated in more general case:

$$\begin{cases} y' + \mathscr{A}(y) = g, \\ y(\tau) = y_0, \end{cases} \tag{2.5}$$

here $\mathscr{A} : X \to X^*$ is the Nemytskii operator for $A + B$,

$$\mathscr{A}(y)(t) = A(y(t)) + B(y(t)) \quad \text{for a.e.} \quad t \in (\tau; T), \ y \in X,$$
$$X = L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2), \quad X^* = L_2(\tau, T; V_1^*) + L_q(\tau, T; V_2^*).$$

Solutions of problem (2.5) are also searched in the class $W = \{y \in X \big| y' \in X^*\}$.

In cases when the continuity of the interaction function f have an influence on the adequacy of mathematical model fundamentally then problem (2.1) is reduced to such problem (Fig. 2.1):

$$\begin{cases} y_t - \Delta y + F(y) \ni g(x,t) \quad \text{in } Q = \Omega \times (\tau; T), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{\tau=T} = y_0, \end{cases} \tag{2.6}$$

here

$$F(s) = [\underline{f}(s), \overline{f}(s)], \quad \underline{f}(s) = \varliminf_{t \to s} f(t), \quad \overline{f}(s) = \varlimsup_{t \to s} f(t), \quad s \in \mathbf{R},$$
$$[a, b] = \{\alpha a + (1-\alpha) b \big| \alpha \in [0, 1]\},$$

$-\infty < a < b < +\infty$.

A solution of such differential-operator inclusion

$$\begin{cases} y' + \mathscr{A}(y) \ni g, \\ y(\tau) = y_0, \end{cases} \tag{2.7}$$

is usually thought to be the generalized solution of problem (2.6). Here $\mathscr{A}$: $X \rightrightarrows X^*$,

$$\mathscr{A}(u) = \{p \in X^* \big| p(t) \in A(u(t)) + B(u(t)), \text{ for a.e. } t \in (\tau; T)\}, \ u \in X,$$

$A : V_1 \to V_1^*$ is the energetic extension of "$-\Delta$" in $H_0^1(\Omega)$, $B : V_2 \to C_v(V_2^*)$ is the Nemytskii operator for F:

$$B(v) = \{z \in V_2^* \big| z(x) \in F(v(x)) \text{ for a.e. } x \in \Omega\}, \quad v \in V_2.$$

Taking into account all variety of classes of mathematical models for different nature geophysical processes and fields we propose rather general approach to investigation of them in this book. Further we will study classes of mathematical models in terms of general properties of generated mappings like $\mathscr{A}$.

Let us consider some denotations and results, that we will use in this book. Let X be a Banach space, X^* be its topologically adjoint,

$$\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbf{R}$$

be the canonical duality between X and X^*, 2^{X^*} be a family of all subsets of the space X^*, let $A : X \to 2^{X^*}$ be the multivalued map,

$$\text{graph}A = \{(\xi; y) \in X^* \times X \,|\xi \in A(y)\}\,,$$
$$\text{Dom}A = \{y \in X \,|A(y) \neq \emptyset\}\,.$$

The multivalued map A is called strict if $\text{Dom}A = X$. Together with every multivalued map A we consider its upper

$$[A(y), \xi]_+ = \sup_{d \in A(y)} \langle d, \xi \rangle_X$$

and lower

$$[A(y), \xi]_- = \inf_{d \in A(y)} \langle d, \xi \rangle_X$$

support functions, where $y, \xi \in X$. Let also

$$\|A(y)\|_+ = \sup_{d \in A(y)} \|d\|_{X^*}, \quad \|A(y)\|_- = \inf_{d \in A(y)} \|d\|_{X^*}, \quad \|\emptyset\|_+ = \|\emptyset\|_- = 0.$$

For arbitrary sets $C, D \in 2^{X^*}$ we set

$$\text{dist}(C, D) = \sup_{e \in C} \inf_{d \in D} \|e - d\|_{X^*}, \quad d_H(C, D) = \max\{\text{dist}(C, D), \text{dist}(D, C)\}\,.$$

Then, obviously,

$$\|A(y)\|_+ = d_H(A(y), 0) = \text{dist}(A(y), 0), \quad \|A(y)\|_- = \text{dist}(0, A(y)).$$

Together with the operator $A : X \to 2^{X^*}$ let us consider the following maps

$$\mathrm{co}A : X \to 2^{X^*} \quad \text{and} \quad \overset{*}{\overline{\mathrm{co}}}\, A : X \to 2^{X^*},$$

defined by relations

$$(\mathrm{co}A)(y) = \mathrm{co}(A(y)) \quad \text{and} \quad (\overset{*}{\overline{\mathrm{co}}}\, A(y)) = \overset{*}{\overline{\mathrm{co}}}\, (A(y))$$

respectively, where $\overset{*}{\overline{\mathrm{co}}}\, (A(y))$ is the weak star closure of the convex hull $\mathrm{co}(A(y))$ for the set $A(y)$ in the space X^*. Besides for every $G \subset X$

$$(\mathrm{co}A)(G) = \bigcup_{y \in G} (\mathrm{co}A)(y), \quad (\overset{*}{\overline{\mathrm{co}}}\, A)(G) = \bigcup_{y \in G} (\overset{*}{\overline{\mathrm{co}}}\, A)(y).$$

Further we will denote the strong, weak and weak star convergence by $\to$, $\xrightarrow{w}$, $\xrightarrow{*}$ or $\to$, $\rightharpoonup$, $\rightharpoonup$ respectively. As $C_v(X^*)$ we consider the family of all nonempty convex closed bounded subsets from X^*.

Proposition 2.1. *[41, Proposition 1.2.1] Let $A, B, C : X \rightrightarrows X^*$. Then for all $y, v, v_1, v_2 \in X$ the following statements take place:*

1. *The functional $X \ni u \to [A(y), u]_+$ is convex, positively homogeneous and lower semicontinuous;*
2. $[A(y), v_1 + v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+$,
   $[A(y), v_1 + v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_-$,
   $[A(y), v_1 + v_2]_+ \ge [A(y), v_1]_+ + [A(y), v_2]_-$,
   $[A(y), v_1 + v_2]_- \le [A(y), v_1]_+ + [A(y), v_2]_-$;
3. $[A(y) + B(y), v]_+ = [A(y), v]_+ + [B(y), v]_+$,
   $[A(y) + B(y), v]_- = [A(y), v]_- + [B(y), v]_-$;
4. $[A(y), v]_+ \le \|A(y)\|_+ \|v\|_X$,
   $[A(y), v]_- \le \|A(y)\|_- \|v\|_X$;
5. $\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_+ = \|A(y)\|_+$, $\| \overset{*}{\overline{\mathrm{co}}}\, A(y) \|_- = \|A(y)\|_-$,
   $[A(y), v]_+ = \left[\overset{*}{\overline{\mathrm{co}}}\, A(y), v \right]_+$, $[A(y), v]_- = \left[\overset{*}{\overline{\mathrm{co}}}\, A(y), v \right]_-$;
6. $\|A(y) - B(y)\|_+ \ge | \, \|A(y)\|_+ - \|B(y)\|_- \, |$,
   $\|A(y) - B(y)\|_- \ge \|A(y)\|_- - \|B(y)\|_+$;
7. $d \in \overset{*}{\overline{\mathrm{co}}}\, A(y) \quad \Leftrightarrow \quad \forall \omega \in X \; [A(y), \omega]_+ \ge \langle d, w \rangle_X$;
8. $d_H(A(y), B(y)) \ge |\|A(y)\|_+ - \|B(y)\|_+|$,
   $d_H(A(y), B(y)) \ge |\|A(y)\|_+ - \|B(y)\|_+|$,
   where d_H is Hausdorff metric;
9. $\mathrm{dist}(A(y) + B(y), C(y)) \le \mathrm{dist}(A(y), C(y)) + \mathrm{dist}(B(y), 0)$,
   $\mathrm{dist}(C(y), A(y) + B(y)) \le \mathrm{dist}(C(y), A(y)) + \mathrm{dist}(0, B(y))$,
   $d_H(A(y) + B(y), C(y)) \le d_H(A(y), C(y)) + d_H(B(y), 0)$;

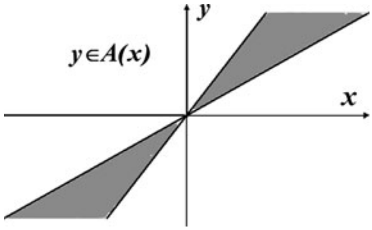


Fig. 2.2 The "−"-coercive multivalued map

10. For any $D \subset X^*$ and bounded $E \in C_v(X^*)$

$$\operatorname{dist}(D, E) = \operatorname{dist}(\overset{*}{\overline{\text{co}}}\, D, E).$$

Proposition 2.2. *[41, Proposition 1.2.2] The inclusion $d \in \overset{*}{\overline{\text{co}}}\, A(y)$ holds true if and only if one of the following relations takes place (Fig. 2.2):*

$$\begin{aligned} \textit{either} \quad & [A(y), v]_+ \ge \langle d, v\rangle_X \quad \forall v \in X, \\ \textit{or} \quad & [A(y), v]_- \le \langle d, v\rangle_X \quad \forall v \in X. \end{aligned}$$

Proposition 2.3. *[41, Proposition 1.2.3] Let $D \subset X$ and $a(\cdot, \cdot) : D \times X \to \mathbf{R}$. For each $y \in D$ the functional $X \ni w \mapsto a(y, w)$ is positively homogeneous, convex and lower semicontinuous if and only if there exists the multivalued map $A : X \to 2^{X^*}$ such that $D(A) = D$ and*

$$a(y, w) = [A(y), w]_+ \qquad \forall y \in D(A),\ w \in X.$$

Proposition 2.4. *[41, Proposition 1.2.4] The functional $\|\cdot\|_+ : C_v(X^*) \to \mathbf{R}_+$ satisfies the following properties:*

1. $\{\bar{0}\} = A \quad \Leftrightarrow \quad \|A\|_+ = 0$,
2. $\|\alpha A\|_+ = |\alpha|\|A\|_+, \quad \forall \alpha \in \mathbf{R},\ A \in C_v(X^*)$,
3. $\|A + B\|_+ \le \|A\|_+ + \|B\|_+ \quad \forall A, B \in C_v(X^*)$.

Proposition 2.5. *[41, Proposition 1.2.5] The functional $\|\cdot\|_- : C_v(X^*) \to \mathbf{R}_+$ satisfies the following properties:*

1. $\bar{0} \in A \quad \Leftrightarrow \quad \|A\|_- = 0$,
2. $\|\alpha A\|_- = |\alpha|\|A\|_-, \quad \forall \alpha \in \mathbf{R},\ A \in C_v(X^*)$,
3. $\|A + B\|_- \le \|A\|_- + \|B\|_- \quad \forall A, B \in C_v(X^*)$.

Let us remark that any multivalued map $A : X \to 2^{X^*}$, naturally generates *upper* and, accordingly, *lower form*:

$$[A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, w\rangle_X, \qquad [A(y), \omega]_- = \inf_{d \in A(y)} \langle d, w\rangle_X, \quad y, \omega \in X.$$

Thus, together with the classical coercivity condition for operator A:

$$\frac{\langle A(y), y\rangle_X}{\|y\|_X} \to +\infty, \qquad \text{as} \qquad \|y\|_X \to +\infty,$$

which ensures the important a priori estimations, arises $+$-coercivity (and, accordingly, $-$-coercivity):

$$\frac{[A(y), y]_{+(-)}}{\|y\|_X} \to +\infty, \qquad \text{as} \qquad \|y\|_X \to +\infty.$$

$+$-coercivity is much weaker condition than $-$-coercivity.

2.2 Pointwise Pseudomonotone Maps

In this section we consider Nemytskii operator properties for classes of pseudomonotone multivalued maps, considered in [20] (see paper and references therein). This properties we obtain, analyzing Theorem proofs from [20]. At that we consider weaker properties for operators connected with measurability and obtain stronger results, that we use in further sections.

For evolution triple (V, H, V^*),[1] $p > 1$ we consider a multivalued (in the general case) map $A : V \rightrightarrows V^*$. We suppose

$(A1)$ $v \to A(v)$ is a pseudomonotone map such that

- (a) $A(u) \in C_v(V^*)$ $\forall u \in V$, i.e. the set $A(u)$ is a nonempty, closed and convex one for all $u \in V$;
- (b) If $u_j \to u$ weakly in V and $d_j \in A(u_j)$ is such that

$$\varlimsup_{j\to+\infty} \langle d_j, u_j - u\rangle_V \le 0,$$

then

$$\varliminf_{j\to+\infty} \langle d_j, u_j - \omega\rangle_V \ge [A(u), u-\omega]_- \;\forall \omega \in V.$$

$(A2)$ $\exists c_1 > 0$:

$$\|A(u)\|_+ \le c_1(1 + \|u\|_V^{p-1}) \;\forall u \in V;$$

[1]That is, V is a real reflexive separable Banach space embedded into a real Hilbert space H continuously and densely, H is identified with its conjugated space H^*, V^* is a dual space to V. So, we have such chain of continuous and dense embeddings: $V \subset H \equiv H^* \subset V^*$ (see, for example, [42]).

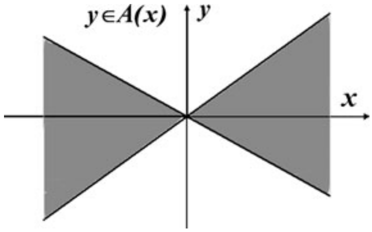


Fig. 2.3 The "+"-coercive multivalued map, but not "−"-coercive

$(A3)$ $\exists c_2, c_3 > 0:$

$$[A(u), u]_- \geq c_2 \|u\|_V^p - c_3 \;\; \forall u \in V.$$

We consider a reflexive separable Banach space V_σ such that $V_\sigma \subset V$ with dense and continuous embedding. Therefore, we have the chain of continuous and dense embeddings (Fig. 2.3):

$$V_\sigma \subset V \subset H \equiv H^* \subset V^* \subset V_\sigma^*,$$

where V_σ^* is dual space to V_σ. Let us set: $S = [\tau, T], -\infty < \tau < T < +\infty, q > 1 :$ $\frac{1}{p} + \frac{1}{q} = 1,$

$$X = L_p(S; V),\;\; X^* = L_q(S; V^*),\;\; X_\sigma = L_p(S; V_\sigma),\;\; X_\sigma^* = L_q(S; V_\sigma^*),$$
$$W = \{y \in X \mid y' \in X^*\},\;\; W_\sigma = \{y \in X \mid y' \in X_\sigma^*\}.$$

Lemma 2.1. *Under above conditions for any* $y \in X$

$$\hat{A}(y) = \{g \in X^* \mid g(t) \in A(y(t)) \text{ for a.e. } t \in S\} \neq \emptyset.$$

Moreover,

$$\exists C_1 > 0: \qquad \|\hat{A}(y)\|_+ \leq c_1(1 + \|y\|_X^{p-1}) \;\; \forall y \in X; \tag{2.8}$$

$$\exists C_2, C_3 > 0: \;\; [\hat{A}(y), y]_- \geq C_2 \|y\|_X^p - C_3 \;\; \forall y \in X. \tag{2.9}$$

Proof. Let $y \in X$. Then there exists a sequence of "step functions" [16, Chap. IV] $\{y_n\}_{n \geq 1} \subset X$ such that

$$y_n \to y \text{ in } X, \tag{2.10}$$

$$\text{for a.e. } t \in S \;\; y_n(t) \to y(t) \text{ in } V, \;\; n \to +\infty. \tag{2.11}$$

We remark that

$$y_n(t) = a_{k,n} \text{ for a.e. } t \in A_{k,n},$$

where $n \geq 1$, $k = 1, \dots, m_n$, $m_n \in \mathbf{N}$, $A_{k,n}$ is measurable set, $A_{k,n} \cap A_{j,n} = \emptyset$, $k \neq j$, $\bigcup_{k=1}^{m_n} A_{k,n} = S$, $a_{k,n} \in V$.

Let for $n \geq 1$, $k = 1, \dots, m_n$ $d_{k,n} \in A(a_{k,n})$ be an arbitrary. For any $n \geq 1$ we consider a "step function" $d_n \in X^*$ such that $d_n(t) = d_{k,n}$, for a.e. $t \in A_{k,n}$, $k = \overline{1, m_n}$.

Thus $\forall n \geq 1$ for a.e. $t \in S$ $d_n(t) \in A(y_n(t))$. In virtue of Condition $(A2)$ and (2.10) we obtain that up to a subsequence $\{d_{n_k}\}_{k\geq 1} \subset \{d_n\}_{n\geq 1}$ for some $d \in X^*$ the next convergence is fulfilled:

$$d_{n_k} \to d \text{ weakly in } X^*, \ k \to +\infty. \tag{2.12}$$

To finish the proof of Lemma 2.1 it is sufficiently to show that $d \in \hat{A}(y)$. From (2.11) and Condition $(A2)$ it follows that for a.e. $t \in S$

$$\langle d_n(t), y_n(t) - y(t)\rangle_V \to 0, \ n \to +\infty. \tag{2.13}$$

As V is separable Banach space then there exists a countable dense system of vectors $\{v_j\}_{j\geq 1} \subset V$.

We finish the proof into several steps.

Step 1. In virtue of the pseudomonotony of A, from (2.11), (2.13) it follows that

$$\text{for a.e. } t \in S \ \forall j \geq 1 \ \varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - \omega_j\rangle_V$$

$$= \varliminf_{n\to+\infty} \langle d_n(t), y(t) - \omega_j\rangle_V \geq [A(y(t)), y(t) - \omega_j]_-. \tag{2.14}$$

Step 2. Due to Conditions $(A2)$ and $(A3)$ it follows that $\forall n, j \geq 1$, for a.e. $s \in S$

$$\langle d_n(s), y_n(s) - \omega_j\rangle_V \geq c_2\|y_n(s)\|_V^p - c_3 - c_1(1 + \|y_n(s)\|_V^{p-1})\|\omega_j\|_V.$$

Now using Young's inequality, we can obtain

$$c_1\|y_n(s)\|_V^{p-1}\|\omega_j\|_V \leq c_2\|y_n(s)\|_V^p + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}}.$$

Letting

$$c_{4,j} = c_1\|\omega_j\|_V + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}} + c_3 > 0,$$

we finally get

$$\forall n, j \geq 1, \ \text{for a.e. } t \in S \ \langle d_n(s), y_n(s) - \omega_j\rangle_V \geq -c_{4,j}. \tag{2.15}$$

Step 3. From (2.10) and (2.12) we have that $\forall t_1, t_2 \in S$, $t_1 < t_2$,

$$\int_{t_1}^{t_2} \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V ds \to \int_{t_1}^{t_2} \langle d(s), y(s) - \omega_j \rangle_V ds. \tag{2.16}$$

Step 4. In virtue of (2.10), (2.12), (2.15), (2.16) and Fatou's lemma $\forall j \geq 1$, $\forall t \in S$, $\forall h > 0 : t + h \in S$, we obtain

$$\begin{aligned}
\int_t^{t+h} \langle d(s), y(s) - \omega_j \rangle_V ds &= \lim_{k \to +\infty} \int_t^{t+h} \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V ds \\
&\geq \varliminf_{k \to +\infty} \int_t^{t+h} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h \\
&\geq \int_t^{t+h} \varliminf_{k \to +\infty} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h \\
&= \int_t^{t+h} \varliminf_{k \to +\infty} \langle d_{n_k}(s), y(s) - \omega_j \rangle_V ds.
\end{aligned}$$

Because of $\forall \varphi \in L_1(S)$

$$\frac{1}{h} \int_0^h \varphi(s + \cdot) ds \to \varphi(\cdot) \text{ in } L_1(S), \ h \searrow 0,$$

we have:

$$\begin{aligned}
\text{for a.e. } t \in S, \ \forall j \geq 1, \ \langle d(t), y(t) - \omega_j \rangle_V &\geq \varliminf_{k \to +\infty} \langle d_{n_k}(t), y(t) - \omega_j \rangle_V \\
&\geq [A(y(t)), y(t) - \omega_j]_-.
\end{aligned}$$

Step 5. In virtue of $\{y(t) - \omega_j\}_{j \geq 1}$ is dense in V for a.e. $t \in S$ we finally obtain that $d(t) \in A(y(t))$ for a.e. $t \in S$, i.e. $d \in \hat{A}(y)$.
The proof of (2.8), (2.9) is trivial [20].

Lemma 2.2. *Under the above listed conditions, if* $y_n \to y$ *weakly in* W_σ, $\{d_n\}_{n \geq 1} \subset X^* : d_n(t) \in A(y_n(t))$ *for a.e.* $t \in S$, $\forall n \geq 1$, *and*

$$\varlimsup_{n \to +\infty} \langle d_n, y_n - y \rangle_X \leq 0,$$

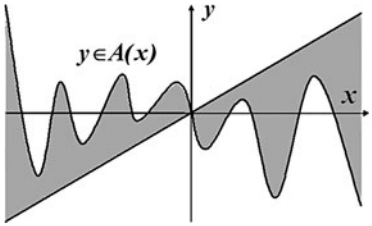


Fig. 2.4 The weakly "$-$"-coercive multivalued map

we have

$$\lim_{n\to+\infty}\int\limits_S |\langle d_n(t), y_n(t)-y(t)\rangle_V|\,dt = 0. \tag{2.17}$$

Proof. We define $\hat{A}(y)=\{g\in X^*\,|\,g(t)\in A(y(t))$ for a.e. $t\in S\}$, $y\in X$. From Lemma 2.1 the set $\hat{A}(y)$ is nonempty. It is clear that $\hat{A}(y)$ is a closed and convex set, i.e. $\hat{A}(y): X\to C_v(X^*)$ (Fig. 2.4).

Let $y_n\to y$ in W_σ, $d_n\in\hat{A}(y_n)$ $\forall n\geq 1$, and we suppose that

$$\overline{\lim_{n\to+\infty}}\langle d_n, y_n-y\rangle_X\leq 0. \tag{2.18}$$

First we prove (2.17). We note that there is a set of measure zero, $\Sigma_1\subset S$ such that for $t\notin\Sigma_1$, we have that

$$d_n(t)\in A(y_n(t))\text{ for all } n\geq 1.$$

Similarly to [20, p. 7] we verify the following claim.
Claim: Let $y_n\to y$ weakly in W_σ and let $t\notin\Sigma_1$. Then

$$\underline{\lim}_{n\to+\infty}\langle d_n(t), y_n(t)-y(t)\rangle_V\geq 0.$$

Proof of the claim. Fix $t\notin\Sigma_1$ and suppose to the contrary that

$$\underline{\lim}_{n\to+\infty}\langle d_n(t), y_n(t)-y(t)\rangle_V<0. \tag{2.19}$$

Then up to a subsequence $\{d_{n_k}, y_{n_k}\}_{k\geq1}\subset\{d_n, y_n\}_{n\geq1}$ we have

$$\lim_{k\to+\infty}\langle d_{n_k}(t), y_{n_k}(t)-y(t)\rangle_V=\underline{\lim}_{n\to+\infty}\langle d_n(t), y_n(t)-y(t)\rangle_V<0. \tag{2.20}$$

Therefore, for all rather large k, Conditions $(A2)$ and $(A3)$ implies

$$c_2\|y_{n_k}(t)\|_V^p-c_3\leq\|A(y_{n_k}(t))\|_+\|y(t)\|_V\leq c_1(1+\|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V.$$

which implies $\{\|y_{n_k}(t)\|_V\}_{k\geq 1}$ and consequently $\{\|d_{n_k}(t)\|_{V^*}\}_{k\geq 1}$ are bounded sequences. $\{\|d_{n_k}(t)\|_{V^*}\}_{k\geq 1}$ is bounded one independently on n_k in virtue of the assumption that $A : V \to C_v(V^*)$ is bounded map and we just showed that $\{\|y_{n_k}(t)\|_V\}_{k\geq 1}$ is bounded sequence. In virtue of the continuity of embedding $W_\sigma \subset C(S; V_\sigma^*)$ we obtain that $y_{n_k}(t) \to y(t)$ weakly in V_σ^* and in virtue of boundedness of $\{y_{n_k}(t)\}_{k\geq 1}$ in V we finally have

$$\forall t \in S\setminus\Sigma_1 \;\; y_{n_k}(t) \to y(t) \text{ weakly in } V, \;\; k \to +\infty. \tag{2.21}$$

The pseudomonotony condition for A, (2.19)–(2.21) implies that

$$\begin{aligned}\varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V &\geq [A(y(t)), y(t) - y(t)]_- \\ &= 0 > \varlimsup_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V.\end{aligned}$$

We obtain a contradiction.

The claim is proved.

Now we continue the proof of the lemma. It follows from the claim that for a.e. $t \in S$, in fact for any $t \notin \Sigma_1$, we have

$$\varliminf_{n\to+\infty} \langle d_n(t), y_n(t) - y(t)\rangle_V \geq 0. \tag{2.22}$$

Now also from the "–"-coercivity condition, $(A3)$, if $\omega \in X$

$$\langle d_n(t), y_n(t) - \omega(t)\rangle_V \geq c_2\|y_n(t)\|_V^p - c_3 - c_1(1 + \|y_n(t)\|_V^{p-1})\|\omega(t)\|_V$$
for a.e. $t \in S\setminus\Sigma_1$.

Using $p - 1 = \frac{p}{q}$, the right side of the above inequality equals to

$$c_2\|y_n(t)\|_V^p - c_3 - c_1\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V - c_1\|\omega(t)\|_V.$$

Now using Young's inequality, we can obtain a constant $c(c_1, c_2)$ depending on c_1, c_2 such that

$$c_1\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V \leq \frac{c_2}{2}\|y_n(t)\|_V^p + \|\omega(t)\|_V^p \cdot c(c_1, c_2).$$

Letting $c_4 = \max\{c_3 + \frac{c_1}{q}; c(c_1, c_2) + \frac{c_1}{p}\}$ it follows that

$$\langle d_n(t), y_n(t) - \omega(t)\rangle_V \geq -c_4(1 + \|\omega(t)\|_V^p) \text{ for a.e. } t \in S. \tag{2.23}$$

Letting $\omega = y$, we can use Fatou's lemma and we obtain

$$\varliminf_{n\to+\infty}\int_0^T [\langle d_n(t), y_n(t)-y(t)\rangle_V + c_4(1+\|y(t)\|_V^p)]dt$$

$$\geq \int_0^T \varliminf_{n\to+\infty} [\langle d_n(t), y_n(t)-y(t)\rangle_V + c_4(1+\|y(t)\|_V^p)]dt \geq c_4\int_0^T (1+\|y(t)\|_V^p)dt.$$

Therefore,

$$0 \geq \varlimsup_{n\to+\infty}\langle d_n, y_n-y\rangle_X \geq \varliminf_{n\to+\infty}\int_S \langle d_n(t), y_n(t)-y(t)\rangle_V dt$$

$$= \varliminf_{n\to+\infty}\langle d_n, y_n-y\rangle_X \geq \int_S \varliminf_{n\to+\infty}\langle d_n(t), y_n(t)-y(t)\rangle_V dt = 0,$$

showing that

$$\lim_{n\to+\infty}\langle d_n, y_n-y\rangle_X = 0. \tag{2.24}$$

From (2.23),

$$\forall n\geq 1\ \forall t\notin \Sigma_1\ 0\leq \langle d_n(t), y_n(t)-y(t)\rangle_V^- \leq c_4(1+\|y(t)\|_V^p),$$

where $a^- = \max\{0,-a\}$, for $a\in\mathbf{R}$. Thanks to (2.22) we know that for a.e. t, $\langle d_n(t), y_n(t)-y(t)\rangle_V \geq -\varepsilon$ for all rather large n. Therefore, for such n, $\langle d_n(t), y_n(t)-y(t)\rangle_V^- \leq \varepsilon$, if $\langle d_n(t), y_n(t)-y(t)\rangle_V < 0$ and $\langle d_n(t), y_n(t)-y(t)\rangle_V^- = 0$, if $\langle d_n(t), y_n(t)-y(t)\rangle_V \geq 0$. Therefore, $\lim_{n\to+\infty}\langle d_n(t), y_n(t)-y(t)\rangle_V^- = 0$ and we can apply the dominated convergence theorem and conclude that

$$\lim_{n\to+\infty}\int_S \langle d_n(t), y_n(t)-y(t)\rangle_V^- = \int_S \lim_{n\to+\infty}\langle d_n(t), y_n(t)-y(t)\rangle_V^- dt = 0$$

from (2.22). Now by (2.24) and the above equation we have

$$\lim_{n\to+\infty}\int_S \langle d_n(t), y_n(t)-y(t)\rangle_V^+ dt$$

$$= \lim_{n\to+\infty}\int_0^T [\langle d_n(t), y_n(t)-y(t)\rangle_V + \langle d_n(t), y_n(t)-y(t)\rangle_V^-]dt$$

$$= \lim_{n\to+\infty}\langle d_n, y_n-y\rangle_X = 0.$$

Therefore,

$$\lim_{n\to+\infty}\int_S |\langle d_n(t), y_n(t)-y(t)\rangle_V|\,dt = 0.$$

The lemma is proved.

Lemma 2.3. *Under the conditions of Lemma 2.2 we additionally have that up to a subsequence* $\{y_{n_k}, d_{n_k}\}_{k\ge1} \subset \{y_n, d_n\}_{n\ge1}$ *for a.e.* $t \in S$ $y_{n_k}(t) \to y(t)$ *weakly in* V*, and* $\langle d_{n_k}(t), y_{n_k}(t)-y(t)\rangle_V \to 0$, $k \to +\infty$.

Proof. Let $y_n \to y$ weakly in W_σ, $\{d_n\}_{n\ge1} \subset X_\sigma^* : d_n(t) \in A(y_n(t))$ for a.e. $t \in S$ $\forall n \ge 1$ and

$$\overline{\lim_{n\to+\infty}} \langle d_n), y_n - y\rangle_X \le 0.$$

In virtue of Lemma 2.2 we obtain

$$\lim_{n\to+\infty}\int_S |\langle d_n(t), y_n(t)-y(t)\rangle_V|\,dt = 0. \tag{2.25}$$

Due to the continuous embedding $W_\sigma \subset C(S; V_\sigma^*)$ we have

$$\forall t \in S \;\; y_n(t) \to y(t) \text{ weakly in } V_\sigma^*, \;\; n \to +\infty. \tag{2.26}$$

From (2.25) it follows that $\exists\{d_{n_k}, y_{n_k}\}_{k\ge1} \subset \{d_n, y_n\}_{n\ge1}$ such that

$$\text{for a.e. } t \in S \;\; \langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0, \;\; k \to +\infty.$$

Let $\Sigma_1 \subset S$ be a set of measure zero such that for $t \notin \Sigma_1$ d_{n_k}, $y_{n_k}(t)$, $y(t)$ are well-defined $\forall k \ge 1$ $d_{n_k}(t) \in A(y_{n_k}(t))$ $\forall k \ge 1$ and

$$\langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0, \;\; k \to +\infty.$$

In virtue of Conditions $(A1)$ and $(A3)$ we obtain

$$\forall t \notin \Sigma_1 \;\; \forall k \ge 1 \;\; \overline{\lim_{k\to+\infty}} \left(c_2\|y_{n_k}(t)\|_V^p - c_3 - c_1(1 + \|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V \right) \le 0.$$

Thus $\forall t \notin \Sigma_1$

$$\overline{\lim_{k\to+\infty}} \|y_{n_k}(t)\|_V^p \le c(c_1, c_2, c_3, p)(1 + \|y(t)\|_V^p).$$

Therefore, due to (2.26) we obtain that for a.e. $t \in S$ $y_{n_k}(t) \to y(t)$ weakly in V, $k \to +\infty$.

Lemma 2.4. *Let* $p > 1$*,* $A : V \to C_v(V^*)$ *satisfies Conditions* $(A1)$*,* $(A2)$ *and* $(A3)$*. Then* $\hat{A} : X \to C_v(X^*)$*,* $\hat{A}(y) = \{g \in X^* \mid g(t) \in A(y(t))$ *for a.e.* $t \in S\}$*,* $y \in X$*, is pseudomonotone on* W_σ*.*

Proof. Let $y_n \to y$ weakly in W_σ, $d_n \in \hat{A}(y_n)$ $\forall n \geq 1$ and $\overline{\lim\limits_{n\to+\infty}} \langle d_n, y_n - y\rangle_X \leq 0$. We need to show that for all $\omega \in X$ there exists $g(\omega) \in \hat{A}(y)$ such that

$$\underline{\lim\limits_{n\to+\infty}} \langle d_n, y_n - \omega\rangle_X \geq \langle g(\omega), y - \omega\rangle_X.$$

Suppose on the contrary that for some $\omega \in X$

$$\underline{\lim\limits_{n\to+\infty}} \langle d_n, y_n - \omega\rangle_X < [\hat{A}(y), y - \omega]_-. \tag{2.27}$$

On the other hand in virtue of Lemmas 2.2 and 2.3 we have that $\exists\{d_{n_k}, y_{n_k}\}_{k\geq 1} \subset \{d_n, y_n\}_{n\geq 1}$ such that

$$\underline{\lim\limits_{n\to+\infty}} \langle d_n, y_n - \omega\rangle_X = \lim_{k\to+\infty} \langle d_{n_k}, y_{n_k} - \omega\rangle_X \tag{2.28}$$

$$\text{for a.e. } t \in S \ \langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V \to 0,\ k \to +\infty, \tag{2.29}$$

$$\text{for a.e. } t \in S \ y_{n_k}(t) \to y(t) \text{ weakly in } V,\ k \to +\infty, \tag{2.30}$$

$$\int_S |\langle d_{n_k}(t), y_{n_k}(t) - y(t)\rangle_V| dt \to 0,\ k \to +\infty, \tag{2.31}$$

$$d_{n_k} \to d \text{ weakly in } X^*,\ k \to +\infty. \tag{2.32}$$

As V is separable Banach space then there exists a countable dense system of vectors $\{v_j\}_{j\geq 1} \subset V$.

We finish the proof into several steps.

Step 1. In virtue of the pseudomonotony of A, from (2.29), (2.30) it follows that

$$\text{for a.e. } t \in S\ \forall j\geq 1\ \underline{\lim\limits_{k\to+\infty}} \langle d_{n_k}(t), y_{n_k}(t) - \omega_j\rangle_V \geq [A(y(t)), y(t) - \omega_j]_-, \tag{2.33}$$

where $\underline{\lim\limits_{k\to+\infty}} \langle d_{n_k}(t), y_{n_k}(t) - \omega_j\rangle_V = \underline{\lim\limits_{k\to+\infty}} \langle d_{n_k}(t), y(t) - \omega_j\rangle_V$.

Step 2. Due to Conditions $(A2)$ and $(A3)$ it follows that $\forall k, j \geq 1$, for a.e. $s \in S$

$$\langle d_{n_k}(s), y_{n_k}(s) - \omega_j\rangle_V \geq c_2\|y_{n_k}(s)\|_V^p - c_3 - c_1(1 + \|y_{n_k}(s)\|_V^{p-1})\|\omega_j\|_V.$$

Now using Young's inequality, we can obtain

$$c_1\|y_n(s)\|_V^{p-1}\|\omega_j\|_V \leq c_2\|y_n(s)\|_V^p + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}}.$$

Letting

$$c_{4,j} = c_1\|\omega_j\|_V + c_2^{\frac{-p}{q}}\|\omega_j\|_V^p c_1^p p^{-1} q^{\frac{-p}{q}} + c_3 > 0,$$

we finally get

$$\forall k, j \geq 1, \text{ for a.e. } t \in S \ \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V \geq -c_{4,j}. \tag{2.34}$$

Step 3. From (2.31) and (2.32) we have that $\forall t_1, t_2 \in S$, $t_1 < t_2$,

$$\int_{t_1}^{t_2} \langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V ds \to \int_{t_1}^{t_2} \langle d(s), y(s) - \omega_j \rangle_V ds. \tag{2.35}$$

Step 4. In virtue of (2.31), (2.29), (2.35) and Fatou's lemma $\forall j \geq 1$, $\forall t \in S$, $\forall h > 0 : t + h \in S$, we obtain

$$\begin{aligned} &\int_t^{t+h} \langle d(s), y(s) - \omega_j \rangle_V ds \\ &= \varliminf_{k\to+\infty} \int_t^{t+h} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h \\ &\geq \int_t^{t+h} \varliminf_{k\to+\infty} [\langle d_{n_k}(s), y_{n_k}(s) - \omega_j \rangle_V + c_{4,j}] ds - c_{4,j} h \\ &= \int_t^{t+h} \varliminf_{k\to+\infty} \langle d_{n_k}(s), y(s) - \omega_j \rangle_V ds. \end{aligned} \tag{2.36}$$

Because of $\forall \varphi \in L_1(S)$

$$\frac{1}{h} \int_0^h \varphi(s + \cdot) ds \to \varphi(\cdot) \text{ in } L_1(S), \ h \searrow 0,$$

we have:

$$\text{for a.e. } t \in S, \ \forall j \geq 1, \ \langle d(t), y(t) - \omega_j \rangle_V \geq \varliminf_{k\to+\infty} \langle d_{n_k}(t), y(t) - \omega_j \rangle_V \geq$$

$$\geq [A(y(t)), y(t) - \omega_j]_-.$$

Step 5. In virtue of $\{y(t) - \omega_j\}_{j\geq 1}$ is dense in V for a.e. $t \in S$ we finally obtain that $d(t) \in A(y(t))$ for a.e. $t \in S$, i.e. $d \in \hat{A}(y)$ and due to (2.31), (2.32), (2.28) we have

$$\underline{\lim}_{n\to+\infty} \langle d_n, y_n - \omega\rangle_X = \langle d, y - \omega\rangle_X \geq [\hat{A}(y), y - \omega]_-,$$

that contradicts to (2.27).
Lemma 2.4 is proved.

2.3 Auxiliary Properties of Solutions for the First Order Evolution Inclusions with Uniformly Coercive Mappings

Let us consider the additional properties for the first order autonomous evolution inclusions.

2.3.1 *The Setting of the Problem*

For evolution triple $(V; H; V^*)$, multi-valued (in the general case) map $A : V \overset{\to}{\to} V^*$ and exciting force $f \in V^*$ we consider a problem of investigation of dynamics for all weak solutions defined for $t \geq 0$ of non-linear autonomous differential-operator inclusion

$$y'(t) + A(y(t)) \ni f, \tag{2.37}$$

as $t \to +\infty$ in the phase space H. Parameters of this problem satisfy the next properties:

(H_1) $p \geq 2$, $f \in V^*$;
(H_2) The embedding V into H is compact one;
(A_1) $\exists c > 0$: $\forall u \in V$, $\forall d \in A(u)$ $\quad \|d\|_{V^*} \leq c(1 + \|u\|_V^{p-1})$;
(A_2) $\exists \alpha, \beta > 0$: $\forall u \in V$, $\forall d \in A(u)$ $\quad \langle d, u\rangle_V \geq \alpha\|u\|_V^p - \beta$;
(A_3) $A : V \overset{\to}{\to} V^*$ is (generalized) pseudomonotone, i.e.

(a) For every $u \in V$ the set $A(u)$ is a nonempty, convex and weakly compact one in V^*;

(b) If $u_n \to u$ weakly in V, $d_n \in A(u_n)$ $\forall n \geq 1$ and $\overline{\lim}_{n\to+\infty} \langle d_n, u_n - u\rangle_V \leq 0$ then $\forall \omega \in V$ $\exists d(\omega) \in A(u)$:

$$\underline{\lim}_{n\to+\infty} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V.$$

Here $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbf{R}$ is a pairing in $V^* \times V$ coincident on $H \times V$ with the inner product $(\cdot,\cdot)$ in the Hilbert space H.

Remark 2.1. From properties (A_1)–(A_3) it follows that the map A is u.s.c. from every finite-dimensional subspace V into V^* equipped with the weak topology.

As a *weak solution* of evolution inclusion (2.37) on the interval $[\tau, T]$ we consider an element u of the space $L_p(\tau, T; V)$ such that for some $d \in L_q(\tau, T; V^*)$

$$d(t) \in A(y(t)) \quad \text{for almost each (a.e.) } t \in (\tau, T), \tag{2.38}$$

$$-\int_\tau^T (\xi'(t), u(t))dt + \int_\tau^T \langle d(t), \xi(t)\rangle_V dt = \int_\tau^T (f, \xi(t))dt \quad \forall \xi \in C_0^\infty([\tau, T]; V), \tag{2.39}$$

where $q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$.

2.3.2 Preliminaries

For fixed $\tau < T$ let us consider

$$X_{\tau,T} = L_p(\tau, T; V), \quad X^*_{\tau,T} = L_q(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} \mid u' \in X^*_{\tau,T}\},$$
$$\mathscr{A}_{\tau,T} : X_{\tau,T} \overset{\rightarrow}{\rightarrow} X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} \mid d(t) \in A(y(t)) \text{ for a.e. } t \in (\tau, T)\},$$
$$f_{\tau,T} \in X^*_{\tau,T}, \quad f_{\tau,T}(t) = f \text{ for a.e. } t \in (\tau, T),$$

where u' is a derivative of an element $u \in X_{\tau,T}$ in the sense of the space of distributions $\mathscr{D}([\tau, T]; V^*)$ (see, for example, [42]). Note that the space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative (see, for example [42]):

$$\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{2.40}$$

From Sect. 2.2, (A_1)–(A_3) it follows that $\mathscr{A}_{\tau,T} : X_{\tau,T} \overset{\rightarrow}{\rightarrow} X^*_{\tau,T}$ satisfies properties:

(B_1) $\exists C_1 > 0$: $\|d\|_{X^*_{\tau,T}} \le C_1(1 + \|y\|^{p-1}_{X_{\tau,T}})$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(B_2) $\exists C_2, C_3 > 0$: $\langle d, y\rangle_{X_{\tau,T}} \ge C_2\|y\|^p_{X_{\tau,T}} - C_3$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(B_3) $\mathscr{A}_{\tau,T} : X_{\tau,T} \overset{\rightarrow}{\rightarrow} X^*_{\tau,T}$ is (generalized) pseudomonotone on $W_{\tau,T}$ operator, i.e.

(a) For every $y \in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex and weakly compact one in $X^*_{\tau,T}$;

(b) $\mathscr{A}_{\tau,T}$ is u.s.c. from every finite dimensional subspace $X_{\tau,T}$ into $X^*_{\tau,T}$ endowed with the weak topology;

(c) If $y_n \to y$ weakly in $W_{\tau,T}$, $d_n \in \mathscr{A}_{\tau,T}(y_n)$ $\forall n \ge 1$, $d_n \to d$ weakly in $X^*_{\tau,T}$ and
$$\overline{\lim_{n\to+\infty}} \langle d_n, y_n - y\rangle_{X_{\tau,T}} \le 0$$
then $d \in \mathscr{A}_{\tau,T}(y)$ $\lim_{n\to+\infty} \langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$.

Here $\langle \cdot, \cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbf{R}$ is a pairing in $X^*_{\tau,T} \times X_{\tau,T}$ coincident on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, i.e.

$$\forall u \in L_2(\tau, T; H), \ \forall v \in X_{\tau,T} \quad \langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also (see [16, Theorem IV.1.17, c. 177]) that the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense, moreover,

$$\forall u, v \in W_{\tau,T} \quad (u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V\Big]dt. \tag{2.41}$$

From the definition of a derivative in the sense of $\mathscr{D}([\tau, T]; V^*)$ and equality (2.39) it directly follows such statement:

Lemma 2.5. *Each weak solution $u \in X_{\tau,T}$ of differential-operator inclusion (2.37) on the interval $[\tau, T]$ belongs to the space $W_{\tau,T}$. Moreover,*

$$u' + \mathscr{A}_{\tau,T}(u) \ni f_{\tau,T}. \tag{2.42}$$

Vice versa, if $u \in W_{\tau,T}$ satisfies (2.42) then u is a weak solution of (2.37) on $[\tau, T]$.

Properties (B_1)–(B_3), (H_1), [42] provide the existence of a weak solution of Cauchy problem (2.37) with initial data

$$y(\tau) = y_\tau \tag{2.43}$$

on the interval $[\tau, T]$ for an arbitrary $y_\tau \in H$. Therefore, the next result takes place:

Lemma 2.6. *$\forall \tau < T, y_\tau \in H$ Cauchy problem (2.37), (2.43) has a weak solution on the interval $[\tau, T]$. Moreover, each weak solution $u \in X_{\tau,T}$ of Cauchy problem (2.37), (2.43) on the interval $[\tau, T]$ belongs to $W_{\tau,T} \subset C([\tau, T]; H)$ and satisfies (2.42).*

Remark 2.2. Since $W_{\tau,T} \subset C([\tau, T]; H)$, for each weak solution of problem (2.37), in view of Lemma 2.5, initial data (2.43) has sense.

For fixed $\tau < T$ we denote

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \mid u \text{ is a weak solution of (2.37) on } [\tau, T],\ u(\tau) = u_\tau\}, \quad u_\tau \in H.$$

From Lemma 2.6 it follows that $\mathscr{D}_{\tau,T}(u_\tau) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(u_\tau) \subset W_{\tau,T}$ $\forall \tau < T$, $u_\tau \in H$.

We complete this section checking that the translation and concatenation of weak solutions is a weak solution too.

Lemma 2.7. *If $\tau < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$, then $v(\cdot) = u(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(u_\tau)$ $\forall s$. If $\tau < t < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,t}(u_\tau)$ and $v(\cdot) \in \mathscr{D}_{t,T}(u(t))$, then*

$$z(s) = \begin{cases} u(s), \ s \in [\tau, t], \\ v(s), \ s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(u_\tau)$.

Proof. The proof follows from the definition of solution (2.39), Lemma 2.5 and from $z \in W_{\tau,T}$ as soon as $v \in W_{\tau,t}$, $u \in W_{t,T}$ and $v(t) = u(t)$. Proving the last statement we can use the definition of a derivative in the sense $\mathscr{D}([\tau, T]; V^*)$, (2.41) and [16, Chap. IV] on the density of $C^1([t_1, t_2]; V)$ in W_{t_1,t_2} for $t_1 < t_2$.

2.3.3 Supplementary Properties of Solutions

The proof of the existence of compact global and trajectory attractors for evolutional inclusions and, in particular, equations of type (2.37) as a rule is based on properties of a family of weak solutions (2.37), related to the absorbing of the generated m-semiflow of solutions and its asymptotic compactness (see, for example, works [21, 24, 36, 37] and references there). The next lemma on a priory estimates for solutions and Theorem 2.1 on dependence of solutions on initial data are "key players" when investigating the dynamics for solutions of problem (2.37) as $t \to +\infty$.

Lemma 2.8. *There exist* $c_4, c_5, c_6, c_7 > 0$ *such that for any finite interval of time* $[\tau, T]$ *every weak solution* u *of problem (2.37) on* $[\tau, T]$ *satisfies estimates:* $\forall t \geq s$, $t, s \in [\tau, T]$

$$\|u(t)\|_H^2 + c_4 \int_s^t \|u(\xi)\|_V^p d\xi \leq \|u(s)\|_H^2 + c_5 \left(1 + \|f\|_{V^*}^2\right)(t - s), \tag{2.44}$$

$$\|u(t)\|_H^2 \leq \|u(s)\|_H^2 e^{-c_6(t-s)} + c_7 \left(1 + \|f\|_{V^*}^2\right). \tag{2.45}$$

Proof. The proof naturally follows from conditions for the parameters of problem (2.37) and Gronwall lemma.

Theorem 2.1. *Let* $\tau < T$, $\{u_n\}_{n\geq 1}$ *be an arbitrary sequence of weak solutions of (2.37) on* $[\tau, T]$ *such that* $u_n(\tau) \to \eta$ *weakly in* H. *Then there exist* $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ *and* $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ *such that*

$$\forall \varepsilon \in (0, T - \tau) \qquad \max_{t\in[\tau+\varepsilon,T]} \|u_{n_k}(t) - u(t)\|_H \to 0, \quad k \to +\infty. \tag{2.46}$$

Proof. Suppose that conditions of Theorem 2.1 are satisfied. Then in view of Lemma 2.5 for any $n \geq 1$ $u_n(\cdot) \in W_{\tau,T} \subset C([\tau, T]; H)$. Moreover, from Lemma 2.8, property (A_2) and relation (2.42) we have that

$$\forall n \geq 1 \quad \exists d_n \in \mathscr{A}_{\tau,T}(u_n): \quad u_n'(t) + d_n(t) = f \text{ for a.e. } t \in (\tau, T), \tag{2.47}$$

$$\exists C > 0: \quad \forall n \geq 1 \quad \|u_n\|_{X_{\tau,T}} + \|u_n'\|_{X^*_{\tau,T}} + \|u_n\|_{C([\tau,T];H)} + \|d_n\|_{X^*_{\tau,T}} \leq C. \tag{2.48}$$

Hence, due to the continuous embedding $W_{\tau,T} \subset C([\tau, T]; H)$ [16, Chap. IV], properties (H_2) (B_1), the compactness of the embedding $W_{\tau,T} \subset L_2(\tau, T; H)$ (see [22, Chap. 1]), and the reflexivity of the space $W_{\tau,T}$ with the graph norm of a derivative (2.40), we obtain that up to a subsequence $\{u_{n_k}, d_{n_k}\}_{k\geq 1} \subset \{u_n, d_n\}_{n\geq 1}$ for some $u \in W_{\tau,T}$, $d \in X^*_{\tau,T}$ the next convergence take place:

$$\begin{aligned} u_{n_k} &\to u \text{weakly in } X_{\tau,T}, \\ u'_{n_k} &\to u' \text{weakly in } X^*_{\tau,T}, \\ d_{n_k} &\to d \text{weakly in } X^*_{\tau,T}, \\ u_{n_k} &\to u \text{weakly in } C([\tau, T]; H), \\ u_{n_k} &\to u \text{in } L_2(\tau, T; H), \\ u_{n_k}(t) \to u(t) &\text{in } H \text{ for a.e. } t \in (\tau, T), \quad k \to +\infty. \end{aligned} \tag{2.49}$$

Let us complete the proof of this theorem in a few "steps".

Step 1. Prove that

$$\forall t \in (\tau, T] \quad u_{n_k}(t) \to u(t) \text{ in } H,\ k \to +\infty. \tag{2.50}$$

From Lemma 2.8 it follows that $\forall k \geq 1$ $\forall t \geq s$, $t, s \in [\tau, T]$

$$\|u_{n_k}(t)\|_H^2 - c_5(1 + \|f\|_H^2)t \leq \|u_{n_k}(s)\|_H^2 - c_5(1 + \|f\|_H^2)s. \tag{2.51}$$

Moreover, from (2.49) we have that for a.e. $s \in (\tau, T)$ for a.e. $t \in (s, T)$

$$\|u(t)\|_H^2 - c_5(1 + \|f\|_H^2)t \leq \|u(s)\|_H^2 - c_5(1 + \|f\|_H^2)s.$$

Since $u \in W_{\tau,T} \subset C([\tau, T]; H)$, then $\forall t \geq s$ $t, s \in [\tau, T]$

$$\|u(t)\|_H^2 - c_5(1 + \|f\|_H^2)t \leq \|u(s)\|_H^2 - c_5(1 + \|f\|_H^2)s. \tag{2.52}$$

Therefore, functions

$$J_k(t) = \|u_{n_k}(t)\|_H^2 - c_5(1 + \|f\|_H^2)t, \tag{2.53}$$

$$J(t) = \|u(t)\|_H^2 - c_5(1 + \|f\|_H^2)t, \tag{2.54}$$

are continuous and monotone nonincreasing one on $[\tau, T]$.
Moreover, since $u_{n_k}(t) \to u(t)$ in H for a.e. $t \in (\tau, T)$, then

$$J_k(t) \to J(t),\ k \to +\infty \text{ for a.e. } t \in (\tau, T). \tag{2.55}$$

Show that

$$\varlimsup_{k\to+\infty} J_k(t) \le J(t) \quad \forall t \in (\tau, T]. \tag{2.56}$$

From (2.55) it follows that

$$\forall t \in (\tau, T], \forall \varepsilon > 0\, \exists \bar{t} \in (\tau, t): \ |J(\bar{t}) - J(t)| < \varepsilon \text{ and } \lim_{k\to+\infty} J_k(\bar{t}) = J(\bar{t}).$$

Hence, $\forall k \ge 1$

$$J_k(t) - J(t) \le J_k(\bar{t}) - J(t) \le |J_k(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t)| < \varepsilon + |J_k(\bar{t}) - J(\bar{t})|.$$

Therefore,

$$\forall t \in (\tau, T], \ \forall \varepsilon > 0 \ \varlimsup_{k\to+\infty} J_k(t) \le J(t) + \varepsilon.$$

Hence (2.56) and, in particular, the inequality

$$\varlimsup_{k\to+\infty} \|u_{n_k}(t)\|_H^2 \le \|u(t)\|_H^2 \quad \forall t \in (\tau, T]$$

are true. From weak convergence $u_{n_k}(t)$ to $u(t)$ in H as $k \to +\infty$ $\forall t \in [\tau, T]$, inequality (2.56) and [16, Chap. I] we obtain (2.50).

Step 2. Show that

$$u' = f_{\tau,T} - d. \tag{2.57}$$

In view of Lemma 2.5 for any $k \ge 1$, $\xi \in C_0^\infty([\tau, T]; V)$ we have

$$-\langle \xi', u_{n_k} \rangle_{X_{\tau,T}} + \langle d_{n_k}, \xi \rangle_{X_{\tau,T}} = \langle f_{\tau,T}, \xi \rangle. \tag{2.58}$$

Passing to the limit as $k \to +\infty$ in the last relation we obtain

$$\forall \xi \in C_0^\infty([\tau, T]; V) \ -\langle \xi', u \rangle_{X_{\tau,T}} + \langle d, \xi \rangle_{X_{\tau,T}} = \langle f_{\tau,T}, \xi \rangle.$$

Therefore, using properties of the Bochner integral, we obtain $\forall \varphi \in C_0^\infty([\tau, T])$ $\forall h \in V$

$$-\left(\int_\tau^T u(s)\varphi'(s)ds, h \right) = -\int_\tau^T (h, u(s))_H \varphi'(s)ds$$

$$= \int_\tau^T \langle f - d(s), h \rangle_V \varphi(s)ds = \left\langle \int_\tau^T [f_{\tau,T}(s) - d(s)]\varphi(s)ds, h \right\rangle_V .$$

From the definition of a derivative of an element $u \in X_{\tau,T}$ in the sense of $\mathscr{D}^*([\tau, T]; V^*)$ it directly follows relation (2.57).

Step 3. Fix an arbitrary $\varepsilon \in (0, T-\tau)$ and show that

$$d(t) \in A(u(t)) \text{ for a.e. } t \in (\tau+\varepsilon, T), \tag{2.59}$$

using the pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$.
Consider restrictions $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, $d(\cdot)$ to the interval $[\tau+\varepsilon, T]$. To simplify the consideration we denote them by the same symbols: $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$ and $d(\cdot)$ respectively. From convergence (2.49), (2.50) we have that

$$\begin{aligned} &u_{n_k} \to u \text{ weakly in } W_{\tau+\varepsilon,T}, \\ &d_{n_k} \to d \text{ weakly in } X^*_{\tau+\varepsilon,T}, \\ &\forall t \in [\tau+\varepsilon, T] \quad u_{n_k}(t) \to u(t) \text{ in } H,\ k \to +\infty. \end{aligned} \tag{2.60}$$

Show that

$$\lim_{k\to+\infty} \langle d_{n_k}, u_{n_k} - u\rangle_{X_{\tau+\varepsilon,T}} = 0. \tag{2.61}$$

Indeed,

$$\begin{aligned} &\forall k \geq 1 \int_{\tau+\varepsilon}^{T} \langle d_{n_k}(s), u_{n_k}(s) - u(s)\rangle_V \, ds \\ &= \int_{\tau+\varepsilon}^{T} \big(f, u_{n_k}(s) - u(s)\big)\, ds - \int_{\tau+\varepsilon}^{T} \big\langle u'_{n_k}(s), u_{n_k}(s) - u(s)\big\rangle_V \, ds. \end{aligned} \tag{2.62}$$

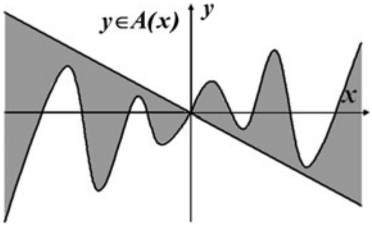
From (2.60) it follows that

$$\int_{\tau+\varepsilon}^{T} \big(f, u_{n_k}(s) - u(s)\big)\, ds \to 0,\ k \to +\infty. \tag{2.63}$$

From (2.41) and (2.60) we obtain that

$$\begin{aligned} &\int_{\tau+\varepsilon}^{T} \langle u'_{n_k}(s), u(s) - u_{n_k}(s)\rangle_V \, ds \\ &= \int_{\tau+\varepsilon}^{T} \langle u'_{n_k}(s), u(s)\rangle_V - \frac{1}{2}\left(\|u_{n_k}(T)\|_H^2 - \|u_{n_k}(\tau+\varepsilon)\|_H^2\right) \\ &\to \int_{\tau+\varepsilon}^{T} \langle u'(s), u(s)\rangle_V - \frac{1}{2}\left(\|u(\tau)\|_H^2 - \|u(\tau+\varepsilon)\|_H^2\right) = 0,\ k \to +\infty. \end{aligned} \tag{2.64}$$

Fig. 2.5 The weakly "+"-coercive, but not weakly "−"-coercive multivalued map



Pass to the limit as $k \to +\infty$ in (2.62). From (2.63) and (2.64) we obtain (2.61). So, due to (2.47), (2.60), (2.61) and in view of the pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$ we obtain (2.59).

Step 4. From the arbitrariness of $\varepsilon \in (0, T-\tau)$, convergence (2.49), relation (2.59) and the definition of $\mathscr{A}_{\tau,T}$ it follows that $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$.

Step 5. Let us prove (2.46). By contradiction suppose the existence of $\varepsilon > 0$, $L > 0$ and subsequence $\{u_{k_j}\}_{j\geq 1} \subset \{u_{n_k}\}_{k\geq 1}$ such that

$$\forall j \geq 1 \qquad \max_{t\in[\tau+\varepsilon,T]} \|u_{k_j}(t) - u(t)\|_H = \|u_{k_j}(t_j) - u(t_j)\|_H \geq L.$$

Without loss of generality we suggest that $t_j \to t_0 \in [\tau+\varepsilon, T]$, $j \to +\infty$. Therefore, by virtue of the continuity of $u : [\tau, T] \to H$, we have

$$\varliminf_{j\to+\infty} \|u_{k_j}(t_j) - u(t_0)\|_H \geq L. \tag{2.65}$$

On the other hand we prove that

$$u_{k_j}(t_j) \to u(t_0) \text{ in } H,\ \ j \to +\infty. \tag{2.66}$$

Step 5.1. Firstly let us show that

$$u_{k_j}(t_j) \to u(t_0) \text{ weakly in } H, \quad j \to +\infty. \tag{2.67}$$

For a fixed $h \in V$ from (2.49) it follows that the sequence of real functions $(u_{n_k}(\cdot), h) : [\tau, T] \to \mathbf{R}$ is uniformly bounded and equicontinuous. Taking into account inequality (2.48) and the density of embedding $V \subset H$ we obtain that $u_{n_k}(t) \to u(t)$ weakly in H uniformly on $[\tau, T]$, $k \to +\infty$. So, we obtain (2.67) (Fig. 2.5).

Step 5.2. Let us prove that

$$\varlimsup_{j\to+\infty} \|u_{k_j}(t_j)\|_H \leq \|u(t_0)\|_H. \tag{2.68}$$

We consider continuous nonincreasing functions J_{k_j}, J, $j \geq 1$, defined in (2.53), (2.54). Let us fix an arbitrary $\varepsilon_1 > 0$. From (2.55) and from the continuity of J it follows that

$$\exists \bar{t} \in (\tau, t_0): \quad \lim_{j\to+\infty} J_{k_j}(\bar{t}) = J(\bar{t}) \text{ and } |J(\bar{t}) - J(t_0)| < \varepsilon_1.$$

Then for rather large $j \geq 1$

$$J_{k_j}(t_j) - J(t_0) \leq |J_{k_j}(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t_0)| \leq |J_{k_j}(\bar{t}) - J(\bar{t})| + \varepsilon_1.$$

Therefore, $\overline{\lim\limits_{j\to+\infty}} J_{k_j}(t_j) \leq J(t_0) + \varepsilon_1$. From the arbitrariness of $\varepsilon_1 > 0$ and from $t_j \to t_0$, $j \to +\infty$, we obtain (2.68).

Step 5.3. Convergence (2.66) directly follows from (2.67), (2.68) and [16, Chap. I].

Step 5.4. To finish the proof of the theorem we remark that (2.66) contradicts (2.65). Therefore, (2.46) is true.

Corollary 2.1. *Let $\tau < T$, $\{u_n\}_{n\geq 1}$ be an arbitrary sequence of weak solutions of (2.37) on $[\tau, T]$ such that $u_n(\tau) \to \eta$ in H, $n \to +\infty$. Then there exists $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ and $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ such that $u_{n_k} \to u$ in $C([\tau, T]; H)$, $k \to +\infty$.*

Proof. The proof is similar to the proof of Theorem 2.1. The main difference is in the checking of the inequality $\overline{\lim\limits_{j\to+\infty}} J_{k_j}(t_j) \leq J(t_0)$, when $t_0 = \tau$, $t_j \to t_0$, $j \to +\infty$, $\{t_j\}_{j\geq 1} \subset [\tau, T]$ (see Step 5.2 from the proof of Theorem 2.1). In this case $\forall j \geq 1$ $J_{k_j}(t_j) - J(\tau) \leq J_{k_j}(\tau) - J(\tau)$. Since $u_n(\tau) \to u(\tau)$ in H, $n \to +\infty$, then $J_{k_j}(\tau) \to J(\tau)$, $j \to +\infty$. Therefore, $\overline{\lim\limits_{j\to+\infty}} J_{k_j}(t_j) \leq J(t_0)$.

2.4 Asymptotic Behavior of the First Order Evolution Inclusions

We note that problem (2.37) arises in many important models for distributed parameter control problems and that large class of identification problems enter our formulation. Let us indicate a problem which is one of motivations for the study of the autonomous evolution inclusion (2.37) [26]. In a subset Ω of $\mathbf{R}^3$, we consider the nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \triangle y = f \text{ in } \Omega \times (0, +\infty)$$

with initial conditions and suitable boundary ones. Here $y = y(x,t)$ represents the temperature at the point $x \in \Omega$ and time $t > 0$. It is supposed that $f = \bar{f} + \bar{\bar{f}}$, where $\bar{\bar{f}}$ is given and $\bar{f}$ is a known function of the temperature of the form

$$-\bar{f}(x,t) \in \partial j(x, y(x,t)) \text{ a.e. } (x,t) \in \Omega \times (0, +\infty).$$

Here $\partial j(x,\xi)$ denotes generalized gradient of Clarke with respect to the last variable of a function $j : \Omega \times \mathbf{R} \to \mathbf{R}$ which is assumed to be locally Lipschitz in ξ. The multivalued function $\partial j(x,\cdot) : \mathbf{R} \to 2^{\mathbf{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language it means that the law is characterized by the generalized gradient of a nonsmooth potential j.

The variational formulation of the above problem leads to the inclusion (6.5) with, for example, $H = L_2(\Omega)$, $V = H^1(\Omega)$, $A = -\triangle + \partial j$ and it is met, for example, in nonmonotone nonconvex interior semipermeability problems.

We remark that monotone semipermeability problems, leading to variation inequalities, have been studied in [15] under the assumption that $j(x,t)$ is a proper, lower semicontinuous, convex function which means that $\partial j(x,\cdot)$ is maximal monotone in $\mathbf{R}^2$.

Following the upper presented results under similar conditions to [15, 26, 30] we can state not only the existence of solutions for autonomous evolution objects but also investigate the dynamic of all weak solutions as $t \to +\infty$. We can also consider other examples from [15, 26, 30].

2.4.1 Existence of the Global Attractor

First we consider constructions presented in [24]. Denote the set of all nonempty (nonempty bounded) subsets of H by $P(H)$ ($\mathscr{B}(H)$). We recall that the multivalued map $G : \mathbf{R} \times H \to P(H)$ is said to be a *m-semiflow* if:

(a) $G(0,\cdot) = \mathrm{Id}$ (the identity map),

(b) $G(t+s,x) \subset G(t,G(s,x))\ \forall x \in H, t,s \in \mathbf{R}_+$;
m-semiflow *is a strict* one if $G(t+s,x) = G(t,G(s,x))\ \forall x \in H, t,s \in \mathbf{R}_+$.

From Lemmas 2.7 and 2.8 it follows that any weak solution can be extended to a global one defined on $[0,+\infty)$. For an arbitrary $y_0 \in H$ let $\mathscr{D}(y_0)$ be the set of all weak solutions (defined on $[0,+\infty)$) of problem (6.5) with initial data $y(0) = y_0$.

We define the m-semiflow G as $G(t,y_0) = \{y(t) \mid y(\cdot) \in \mathscr{D}(y_0)\}$.

Lemma 2.9. *G is the strict m-semiflow.*

Proof. Let $y \in G(t+s,y_0)$. Then $y = u(t+s)$, where $u(\cdot) \in \mathscr{D}(y_0)$. From Lemma 2.7 it follows that $v(\cdot) = u(s+\cdot) \in \mathscr{D}(u(s))$. Hence $y = v(t) \in G(t,u(s)) \subset G(t,G(s,y_0))$.

Vice versa, if $y \in G(t,G(s,y_0))$, then $\exists u(\cdot) \in \mathscr{D}(y_0)\ v(\cdot) \in \mathscr{D}(u(s))$: $y = v(t)$. Define the map

$$z(\xi) = \begin{cases} u(\xi), & \xi \in [0,s], \\ v(\xi - s), & \xi \in [s,t+s]. \end{cases}$$

From Lemma 2.7 it follows that $z(\cdot) \in \mathscr{D}(y_0)$. Hence $y = z(t+s) \in G(t+s,y_0)$.

We recall that the set $\mathscr{A}$ is said to be *a global attractor* G, if:

1. $\mathscr{A}$ is negatively semiinvariant (i.e. $\mathscr{A} \subset G(t, \mathscr{A})\ \forall t \geq 0$);
2. $\mathscr{A}$ is attracting, that is,

$$\operatorname{dist}(G(t, B), \mathscr{A}) \to 0, \quad t \to +\infty \quad \forall B \in \mathscr{B}(H), \tag{2.69}$$

where $\operatorname{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_H$ is the Hausdorff semidistance;

3. For any closed set $Y \subset H$ satisfying (2.69), we have $\mathscr{A} \subset Y$ (minimality). The global attractor is said to be *invariant* if $\mathscr{A} = G(t, \mathscr{A})\ \forall t \geq 0$. We prove the existence of the global attractor.

Theorem 2.2. *The m-semiflow G has the invariant compact in the phase space H global attractor $\mathscr{A}$.*

Proof. From Lemma 2.8 it follows that

$$\exists R, \tilde{\alpha} > 0: \quad \forall y_0 \in H,\ y(\cdot) \in \mathscr{D}(y_0),\ t \geq 0 \quad \|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-\tilde{\alpha} t} + R. \tag{2.70}$$

Therefore the ball $B_0 = \{u \in H \mid \|u\|_H \leq \sqrt{R+1}\}$ is the absorbing set, i.e. $\forall B \in \mathscr{B}(H)\ \exists T(B) > 0$: $\forall t \geq T(B)\ G(t, B) \subset B_0$. In particular, from (2.70) it follows that the set $\cup_{t \geq 0} G(t, B)$ is bounded one in $H\ \forall B \in \mathscr{B}(H)$.

Note also that from Theorem 2.1 it follows that the map $G(t, \cdot) : H \to \mathscr{B}(H)$ takes compact values and it is compact for $t > 0$ in that sense that it maps bounded sets into precompact one.

Show that the map $u_0 \to G(t, u_0)$ is upper semicontinuous [2, Definition 1.4.1, p. 38]. In order to do that it is sufficient to show [3, p. 45], that $\forall u_0 \in H$, $\forall \varepsilon > 0$ $\exists \delta(u_0, \varepsilon) > 0$: $\forall u \in B_\delta(u_0)\ G(t, u) \subset B_\varepsilon(G(t, u_0)) = \{z \in H \mid \operatorname{dist}(z, G(t, u_0)) < \varepsilon\}$. If it is not true then there exist $u_0 \in H$, $\varepsilon > 0$, $\{\delta_n\}_{n \geq 1} \subset (0, +\infty)$, $\{u_n\}_{n \geq 1} \subset H$ such that $\forall n \geq 1\ u_n \in B_{\delta_n}(u_0)$, $G(t, u_n) \not\subset B_\varepsilon(G(t, u_0))$ and $\delta_n \to 0$, $n \to +\infty$. Then $\forall n \geq 1\ \exists v_n(\cdot) \in \mathscr{D}(u_n)$: $v_n(t) \notin B_\varepsilon(G(t, u_0))$. Since $u_n \to u_0$ in H, $n \to +\infty$, then from Theorem 2.1 it follows that $v_n(t) \to v(t) \in G(t, u_0)$ in H, $n \to +\infty$, for some $v(\cdot) \in \mathscr{D}(u_0)$. We obtain contradiction with $\forall n \geq 1$ $\|v_n(t) - v(t)\|_H \geq \varepsilon$.

Thus the existence of the global attractor with required properties directly follows from results from Chap. 1.

2.4.2 Existence of the Trajectory Attractor

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in H} \mathscr{D}(y_0)$ of all weak solutions of inclusion (6.5) defined on the semi-infinite interval $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant* one, i.e. $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0\ u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h + s)$,

$s \geq 0$. We set the *translation semigroup* $\{T(h)\}_{h\geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$ on $\mathscr{K}_+$.

We shall construct the attractor of the translation semigroup $\{T(h)\}_{h\geq 0}$ acting on $\mathscr{K}_+$. On $\mathscr{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbf{R}_+; H)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbf{R}_+; H) \iff \forall M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; H),$$

where Π_M is the restriction operator to the interval $[0, M]$ [37, p. 18]. We denote the restriction operator to the semi-infinite interval $[0, +\infty)$ by Π_+.

We recall that the a $\mathscr{P} \subset C^{loc}(\mathbf{R}_+; H) \cap L_\infty(\mathbf{R}_+; H)$ is said to be *attracting* for the trajectory space $\mathscr{K}_+$ of inclusion (6.5) in the topology of $C^{loc}(\mathbf{R}_+; H)$ if for any bounded in $L_\infty(\mathbf{R}_+; H)$ set $\mathscr{B} \subset \mathscr{K}_+$ and any number $M \geq 0$ the following relation holds:

$$\mathrm{dist}_{C([0,M];H)}(\Pi_M T(t)\mathscr{B}, \Pi_M \mathscr{P}) \to 0, \quad t \to +\infty. \tag{2.71}$$

A set $\mathscr{U} \subset \mathscr{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; H)$ (see, for example, [37, Definition 1.2, p. 197]) if

(i) $\mathscr{U}$ is a compact set in $C^{loc}(\mathbf{R}_+; H)$ and bounded in $L_\infty(\mathbf{R}_+; H)$;
(ii) $\mathscr{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, i.e. $T(h)\mathscr{U} = \mathscr{U}\ \forall h \geq 0$;
(iii) $\mathscr{U}$ is an attracting set in the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbf{R}_+; H)$.

Let us consider inclusions (6.5) on the entire time axis. Similarly to the space $C^{loc}(\mathbf{R}_+; H)$ the space $C^{loc}(\mathbf{R}; H)$ is equipped with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbf{R}$ (see, for example, [37, p. 198]). A function $u \in C^{loc}(\mathbf{R}; H) \cap L_\infty(\mathbf{R}; H)$ is called a *complete trajectory* of inclusion (6.5) if $\forall h \in \mathbf{R}\ \Pi_+ u_h(\cdot) \in \mathscr{K}_+$ [37, p. 198]. Let $\mathscr{K}$ be a family all complete trajectories of inclusion (6.5). Note that

$$\forall h \in \mathbf{R},\ \forall u(\cdot) \in \mathscr{K} \quad u_h(\cdot) \in \mathscr{K}. \tag{2.72}$$

Lemma 2.10. *The set $\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}; H)$ and bounded in $L_\infty(\mathbf{R}; H)$. Moreover,*

$$\forall y(\cdot) \in \mathscr{K},\ \forall t \in \mathbf{R} \quad y(t) \in \mathscr{A}, \tag{2.73}$$

where $\mathscr{A}$ is the global attractor from Theorem 2.2.

Proof.

Step 1. Let us show that $\mathscr{K} \neq \emptyset$. Note that in view of [22, Theorem 3.1.1, p. 329] and conditions (A_1)–(A_3), (H_1), it follows that $\exists v \in V$: $A(v) = f$. We set $u(t) = v\ \forall t \in \mathbf{R}$. Then, $u \in \mathscr{K} \neq \emptyset$.

Step 2. Let us prove (2.73). For any $y \in \mathscr{K}$ $\exists d > 0$: $\|y(t)\|_H \le d$ $\forall t \in \mathbf{R}$. We set $B = \cup_{t\in\mathbf{R}}\{y(t)\} \in \mathscr{B}(H)$. Note that $\forall \tau \in \mathbf{R}$, $\forall t \in \mathbf{R}_+$ $y(\tau) = y_{\tau-t}(t) \in G(t, y_{\tau-t}(0)) \subset G(t, B)$. From Theorem 2.2 and from (2.69) it follows that $\forall \varepsilon > 0$ $\exists T > 0$: $\forall \tau \in \mathbf{R}$ $\mathrm{dist}(y(\tau), \mathscr{A}) \le \mathrm{dist}(G(T, B), \mathscr{A}) < \varepsilon$. Hence taking into account the compactness of $\mathscr{A}$ in H, for any $u(\cdot) \in \mathscr{K}$, $\tau \in \mathbf{R}$ it follows that $u(\tau) \in \mathscr{A}$.

Step 3. The boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}_+; H)$ it follows from (2.73) and from the boundedness of $\mathscr{A}$ in H.

Step 4. Let us check the compactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; H)$. In order to do that it is sufficient to check the precompactness and completeness.

Step 4.1. Let us check the precompactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; H)$. If it is not true then in view of (2.72), $\exists M > 0$: $\Pi_M\mathscr{K}$ is not precompact in $C([0, M]; H)$. Hence there exists a sequence $\{v_n\}_{n\ge1} \subset \Pi_M\mathscr{K}$, that has not a convergent in $C([0, M]; H)$ subsequence. On the other hand $v_n = \Pi_M u_n$, where $u_n \in \mathscr{K}$, $v_n(0) = u_n(0) \in \mathscr{A}$, $n \ge 1$. Since $\mathscr{A}$ is compact in H (see Theorem 2.2), then in view of Corollary 2.1, $\exists\{v_{n_k}\}_{k\ge1} \subset \{v_n\}_{n\ge1}$, $\exists\eta \in H$, $\exists v(\cdot) \in \mathscr{D}_{0,M}(\eta)$: $v_{n_k}(0) \to \eta$ in H, $v_{n_k} \to v$ in $C([0, T]; H)$, $k \to +\infty$. We obtained contradiction.

Step 4.2. Let us check the completeness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; H)$. Let $\{v_n\}_{n\ge1} \subset \mathscr{K}$, $v \in C^{loc}(\mathbf{R}; H)$: $v_n \to v$ in $C^{loc}(\mathbf{R}; H)$, $n \to +\infty$. From the boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}; H)$ it follows that $v \in L_\infty(\mathbf{R}; H)$. From Corollary 2.1 we have that $\forall M > 0$ the restriction $v(\cdot)$ to the interval $[-M, M]$ belongs to $\mathscr{D}_{-M,M}(v(-T))$. Therefore, $v(\cdot)$ is complete trajectory of inclusion (6.5). Thus, $v \in \mathscr{K}$.

Lemma 2.11. *Let $\mathscr{A}$ be a global attractor from Theorem 2.2. Then*

$$\forall y_0 \in \mathscr{A} \quad \exists y(\cdot) \in \mathscr{K}: \quad y(0) = y_0. \tag{2.74}$$

Proof. Let $y_0 \in \mathscr{A}$, $u(\cdot) \in \mathscr{D}(y_0)$. From (6.17), (2.69) we have that $\forall t \in \mathbf{R}_+$ $y(t) \in \mathscr{A}$. From Theorem 2.2 it follows that $G(1, \mathscr{A}) = \mathscr{A}$. Therefore,

$$\forall \eta \in \mathscr{A} \quad \exists \xi \in \mathscr{A},\ \exists \varphi_\eta(\cdot) \in \mathscr{D}_{0,1}(\xi): \quad \varphi_\eta(1) = \eta.$$

For any $t \in \mathbf{R}$ we set

$$y(t) = \begin{cases} u(t), & t \in \mathbf{R}_+, \\ \varphi_{y(-k+1)}(t+k), & t \in [-k, -k+1),\ k \in \mathbf{N}. \end{cases}$$

Note that $y \in C^{loc}(\mathbf{R}; H)$, $y(t) \in \mathscr{A}$ $\forall t \in \mathbf{R}$ (consequently $y \in L_\infty(\mathbf{R}; H)$) and in view of Lemma 2.7, $y \in \mathscr{K}$. At that $y(0) = y_0$.

Theorem 2.3. *Let $\mathscr{A}$ be a global attractor from Theorem 2.2. Then there exists the trajectory attractor $\mathscr{P} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$. At that the next formula holds:*

$$\mathscr{P} = \Pi_+\mathscr{K} = \Pi_+\{y \in \mathscr{K} \mid y(t) \in \mathscr{A}\ \forall t \in \mathbf{R}\}, \tag{2.75}$$

Proof. From Lemma 2.10 and the continuity of operator $\Pi_+ : C^{loc}(\mathbf{R}; H) \to C^{loc}(\mathbf{R}_+; H)$ it follows that the set $\Pi_+\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}_+; H)$ and bounded one in $L_\infty(\mathbf{R}_+; H)$. Moreover, the second equality in (2.75) holds. The strict invariance of $\Pi_+\mathscr{K}$ follows from the autonomy of inclusion (6.5).

Let us prove that $\Pi_+\mathscr{K}$ is the attracting set for the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbf{R}_+; H)$. Let $B \subset \mathscr{K}_+$ be a bounded set in $L_\infty(\mathbf{R}_+; H)$, $M \geq 0$. Let us check (2.71). If it is not true then there exist sequences $t_n \to +\infty$, $v_n(\cdot) \in B$ such that

$$\forall n \geq 1 \quad \mathrm{dist}_{C([0,T];H)}(\Pi_M v_n(t_n + \cdot), \Pi_M \mathscr{K}) \geq \varepsilon. \tag{2.76}$$

On the other hand, from the boundedness B in $L_\infty(\mathbf{R}_+; H)$ it follows that $\exists R > 0$: $\forall v(\cdot) \in B$, $\forall t \in \mathbf{R}_+$ $\|v(t)\|_H \leq R$. Hence, $\exists N \geq 1$: $\forall n \geq N$ $v_n(t_n) \in G(t_n, v_n(0)) \subset G(1, G(t_n - 1, v_n(0))) \subset G(1, \overline{B_R})$, where $\overline{B_R} = \{u \in H \mid \|u\|_H \leq R\}$. Therefore, taking into account (2.69) and the compactness of the map $G(1, \cdot) : H \to \mathscr{B}(H)$ (see the proof of Theorem 2.2) we have that $\exists \{v_{n_k}(t_{n_k})\}_{k\geq 1} \subset \{v_n(t_n)\}_{n\geq 1}$, $\exists z \in \mathscr{A}$: $v_{n_k}(t_{n_k}) \to z$ in H, $k \to +\infty$. Further, $\forall k \geq 1$ we set $\varphi_k(t) = v_{n_k}(t_{n_k} + t)$, $t \in [0, M]$. Note that $\forall k \geq 1$ $\varphi_k(\cdot) \in \mathscr{D}_{0,M}(v_{n_k}(t_{n_k}))$. Then from Corollary 2.1 there exists a subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_k\}_{k\geq 1}$ and an element $\varphi(\cdot) \in \mathscr{D}_{0,M}(z)$:

$$\varphi_{k_j} \to \varphi \text{ in } C([0, M]; H), \quad j \to +\infty. \tag{2.77}$$

At that, taking into account the invariance of $\mathscr{A}$ (see Theorem 2.2), $\forall t \in [0, M]$ $\varphi(t) \in \mathscr{A}$. In consequence of Lemma 2.11 there exist $y(\cdot), v(\cdot) \in \mathscr{K}$: $y(0) = z$, $v(0) = \varphi(M)$. For any $t \in \mathbf{R}$ we set

$$\psi(t) = \begin{cases} y(t), & t \leq 0, \\ \varphi(t), & t \in [0, M], \\ v(t - M), & t \geq M. \end{cases}$$

In view of Lemma 2.7 $\psi(\cdot) \in \mathscr{K}$. Therefore, from (2.76) we have:

$$\forall k \geq 1 \quad \|\Pi_M v_{n_k}(t_{n_k} + \cdot) - \Pi_M \psi(\cdot)\|_{C([0,M];H)} = \|\varphi_k - \varphi\|_{C([0,M];H)} \geq \varepsilon,$$

and we obtain the contradiction with (2.77).

Thus, the set $\mathscr{P}$ constructed in (2.75) is the trajectory attractor in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; H)$.

2.4.3 Comments

From results of Sects. 2.4.1 and 2.4.2 it follows that the m-semiflow G, constructed on all weak solutions of (6.5), has the compact invariant global attractor $\mathscr{A}$. For all weak solutions of (6.5), defined on semi-infinite interval $[0, +\infty)$, there exists the trajectory attractor $\mathscr{P}$. At that

$$\mathscr{A} = \mathscr{P}(0) = \{y(0) \mid y \in \mathscr{K}\}, \quad \mathscr{P} = \Pi_+\mathscr{K},$$

where $\mathscr{K}$ is the family of all complete trajectories of differential-operator inclusion (6.5) in $C^{loc}(\mathbf{R}; H) \cap L_\infty(\mathbf{R}; H)$. Therefore, the equality of global attractors in the sense of [24, Definition 6, p. 88] as well as [37, Definition 2.2, c. 201] is proved. Questions concerned with connectedness of constructed attractors in the general case are opened. Note that approaches proposed in works [24, 37] are based on properties of solutions for evolutional objects. The approach considered in this work is based on properties of an interaction function A from (6.5) and properties of phase spaces.

2.4.4 Conclusion

We investigated the dynamics as $t \to +\infty$ of all global weak solutions defined on $[0, +\infty)$ for a class of autonomous differential-operator inclusions with pseudomonotone nonlinear dependence between determinative parameters of a problem. We proved the existence of the global compact and compact trajectory attractors, investigated their structure and checked the equality of global attractors in the sense of Definition 6 from [24] as well as in the sense of Definition 2.2 from [37]. Obtained results allows us to study the dynamics of solutions of new classes of evolution equations of nonlinear mathematical models of geophysical and socioeconomical processes and fields with interaction function of pseudomonotone type satisfying the condition of "no more than polynomial growth" and standard sign condition.

2.5 Auxiliary Properties of Solutions for the Second Order Evolution Inclusions and Hemivariational Inequalities for Viscoelastic Processes

Let the next conditions are fulfilled (see Example 1):

(H_1) V, Z, H are Hilbert spaces; $H^* \equiv H$ and we have such chain of dense and compact embeddings:

$$V \subset Z \subset H \equiv H^* \subset Z^* \subset V^*;$$

(H_2) $f_0 \in V^*$;

(A_1) $\exists c > 0 : \forall u \in V, \forall d \in A_0(u)\ \|d\|_{V^*} \le c(1 + \|u\|_V)$;

(A_2) $\exists \alpha, \beta > 0 : \forall u \in V, \forall d \in A_0(u)\ \langle d, u\rangle_V \ge \alpha\|u\|_V^2 - \beta$;

(A_3) $A_0 = A_1 + A_2$, where $A_1 : V \to V^*$ is linear, selfconjugated, positive operator, $A_2 : V \overset{\to}{\to} V^*$ satisfies such conditions:

(a) There exists such Hilbert space Z, that the embedding $V \subset Z$ is dense and compact one and the embedding $Z \subset H$ is dense and continuous one;

(b) For any $u \in Z$ the set $A_2(u)$ is nonempty, convex and weakly compact one in Z^*;
(c) $A_2 : Z \rightrightarrows Z^*$ is a bounded map, i.e. A_2 converts bounded sets from Z into bounded sets in the space Z^*;
(d) $A_2 : Z \rightrightarrows Z^*$ is a demiclosed map, i.e. if $u_n \to u$ in Z, $d_n \to d$ weakly in Z^*, $n \to +\infty$, and $d_n \in A_2(u_n)$ $\forall n \geq 1$ then $d \in A_2(u)$;

(B_1) $B_0 : V \to V^*$ is a linear selfconjugated operator;

(B_2) $\exists \gamma > 0 : \langle B_0 u, u\rangle_V \geq \gamma \|u\|_V^2$.
Here $\langle \cdot, \cdot\rangle_V : V^* \times V \to \mathbf{R}$ is the duality in $V^* \times V$, coinciding on $H \times V$ with the inner product $(\cdot, \cdot)$ in Hilbert space H.

Note that from (A_1)–(A_3), [25, 42] it follows that the map A_0 satisfies such condition:
$(A_3)'$ $A_0 : V \rightrightarrows V^*$ is (generalized) λ_0-pseudomonotone, i.e.

(a) For any $u \in V$ the set $A_0(u)$ is nonempty, convex and weakly compact one in V^*;
(b) If $u_n \to u$ weakly in V, $n \to +\infty$, $d_n \in A_0(u_n)$ $\forall n \geq 1$ and $\overline{\lim\limits_{n\to\infty}} \langle d_n, u_n - u\rangle_V \leq 0$ then $\forall \omega \in V$ $\exists d(\omega) \in A_0(u)$:

$$\underline{\lim_{n\to+\infty}} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V;$$

(c) The map A_0 is upper semicontinuous one that acts from an arbitrary finite-dimensional subspace of V into V^*, endowed with weak topology.

Thus, we investigate the dynamic of all weak solutions of the second order nonlinear autonomous differential-operator inclusion

$$y''(t) + A_0(y'(t)) + B_0(y(t)) \ni f_0, \quad \text{for a.e. } t > 0 \tag{2.78}$$

as $t \to +\infty$, which are defined as $t \geq 0$, where parameters of the problem satisfy conditions (H_1), (H_2), (A_1)–(A_3), (B_1)–(B_2).

As a *weak solution* of the evolution inclusion (2.78) on the interval $[\tau, T]$ we consider such pair of elements $(u(\cdot), u'(\cdot))^T \in L_2(\tau, T; V \times V)$, that for some $d(\cdot) \in L_2(\tau, T; V^*)$

$$d(t) \in A_0(u'(t)) \quad \text{for almost every (a.e.)} \quad t \in (\tau, T),$$

$$-\int_\tau^T (\zeta'(t), u'(t))dt + \int_\tau^T \langle d(t), \zeta(t)\rangle_V dt$$

$$+\int_\tau^T \langle B_0 u(t), \zeta(t)\rangle_V dt = \int_\tau^T \langle f_0, \zeta(t)\rangle_V \quad \forall \zeta \in C_0^\infty([\tau, T]; V), \tag{2.79}$$

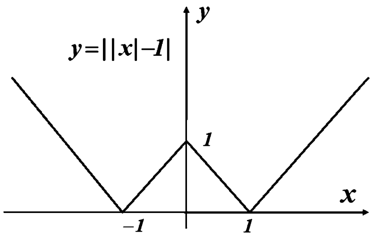


Fig. 2.6 Clarkes subdifferentiable nonconvex functional

where u' is the derivative of the element $u(\cdot)$ in the sense of the space of distributions $\mathscr{D}^*([\tau, T]; V^*)$.

As a *generalized solution* of the problem (3)–(7) we consider the weak solution of the corresponding problem (2.78). This definition is coordinated with Definition 3 from [25].

We have to note that abstract theorems on existence of solutions for such problems as the problem (2.78) and the optimal control problems for weaker conditions for parameters of problems are considered in works [25, 38, 39, 41, 42]. Here we consider Problem 2 from [25], for which we can (as follows from results of the given paper) have not only the abstract result on existence of weak solution but we can investigate the behaviour of all weak solutions as $t \to +\infty$ in the phase space $V \times H$ and study the structure of the global and trajectory attractors. Underline that results concerning multivalued dynamic of displacements and velocities can be applied to hemivariational inequalities with multidimensional superpotential laws (Fig. 2.6).

2.5.1 Preliminary Results

Further, without loss the generality, on the space V we consider the equivalent norm $\|u\|_V = \sqrt{\langle B_0 u, u\rangle_V}$, $u \in V$. The given norm is generated by the inner product $(u, v)_V = \langle B_0 u, v\rangle_V$, $u, v \in V$. For fixed $\tau < T$ let us consider

$$X_{\tau,T} = L_2(\tau, T; V), \quad X^*_{\tau,T} = L_2(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} | u' \in X^*_{\tau,T}\},$$

$$A_{\tau,T} : X_{\tau,T} \overset{\rightarrow}{\rightarrow} X^*_{\tau,T}, \quad \mathscr{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} | d(t) \in A_0(y(t)) \text{ for a.e. } t \in (\tau, T)\},$$

$$B_{\tau,T} : X_{\tau,T} \to X^*_{\tau,T}, \quad B_{\tau,T}(y)(t) = B_0(y(t)) \text{ for a.e. } t \in (\tau, T),$$

$$f_{\tau,T} \in X^*_{\tau,T}, \quad f_{\tau,T}(t) = f_0 \text{ for a.e. } t \in (\tau, T).$$

Note, that the space $W_{\tau,T}$ is the Hilbert space with the graph norm of the derivative (cf. [41, 42]):

$$\|u\|^2_{W_{\tau,T}} = \|u\|^2_{X_{\tau,T}} + \|u'\|^2_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{2.80}$$

From [25, Lemma 7, p. 516], (A$_1$), (A$_2$), (A$_3$)$'$ it follows that $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ satisfies the next conditions:

(N_1) $\exists C_1 > 0$: $\|d\|_{X^*_{\tau,T}} \le C_1(1 + \|y\|_{X_{\tau,T}})$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(N_2) $\exists C_2, C_3 > 0$: $\langle d, y\rangle_{X_{\tau,T}} \ge C_2\|y\|^2_{X_{\tau,T}} - C_3$ $\forall y \in X_{\tau,T}$, $\forall d \in \mathscr{A}_{\tau,T}(y)$;
(N_3) $\mathscr{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ is (generalized) w_λ-pseudomonotone on $W_{\tau,T}$, i.e.

(a) For any $y \in X_{\tau,T}$ the set $\mathscr{A}_{\tau,T}(y)$ is a nonempty, convex and weakly compact one in $X^*_{\tau,T}$;
(b) $\mathscr{A}_{\tau,T}$ is the upper semicontinuous map as the map that acts from an arbitrary finite dimensional subspace from $X_{\tau,T}$ into $X^*_{\tau,T}$, endowed by the weak topology;
(c) If $y_n \to y$ weakly in $W_{\tau,T}$, $d_n \in \mathscr{A}_{\tau,T}(y_n)$ $\forall n \ge 1$, $d_n \to d$ weakly in $X^*_{\tau,T}$ and

$$\overline{\lim_{n\to+\infty}} \langle d_n, y_n - y\rangle_{X_{\tau,T}} \le 0$$

then $d \in \mathscr{A}_{\tau,T}(y)$ and $\lim\limits_{n\to+\infty} \langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$.

Here $\langle\cdot,\cdot\rangle_{X_{\tau,T}} : X^*_{\tau,T} \times X_{\tau,T} \to \mathbf{R}$ is the pairing in $X^*_{\tau,T} \times X_{\tau,T}$ coinciding on $L_2(\tau, T; H) \times X_{\tau,T}$ with the inner product in $L_2(\tau, T; H)$, i.e.

$$\forall u \in L_2(\tau, T; H),\ \forall v \in X_{\tau,T} \quad \langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also (cf. [16, Theorem IV.1.17, P. 177]), that the embedding $W_{\tau,T} \subset C([\tau, T]; H)$ is continuous and dense one, moreover

$$\forall u, v \in W_{\tau,T} \quad (u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T \Big[\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V\Big]dt. \tag{2.81}$$

From the definition of derivative in the sense of $\mathscr{D}([\tau, T]; V^*)$ and the equality (2.79) it directly follows such statement:

Lemma 2.12. *Each weak solution $(y(\cdot), y'(\cdot))^T$ of the problem (2.78) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; V) \times W_{\tau,T}$. Moreover*

$$y'' + \mathscr{A}_{\tau,T}(y') + B_{\tau,T}(y) \ni f_{\tau,T}. \tag{2.82}$$

Vice versa, if $y(\cdot) \in C([\tau, T]; V)$, $y'(\cdot) \in W_{\tau,T}$ and $y(\cdot)$ satisfies (2.82), then $(y(\cdot), y'(\cdot))^T$ is a weak solution of (2.78) on $[\tau, T]$.

A weak solution of the problem (2.78) with initial data

$$y(\tau) = a, \quad y'(\tau) = b \tag{2.83}$$

on the interval $[\tau, T]$ exists for any $a \in V$, $b \in H$. It follows from [25, Theorem 11, p. 523]. Thus, the next lemma holds true.

Lemma 2.13. *For any $\tau < T$, $a \in V$, $b \in H$ the Cauchy problem (2.78), (2.83) has a weak solution $(y, y')^T \in X_{\tau,T} \times X_{\tau,T}$. Moreover, each weak solution $(y, y')^T$ of the Cauchy problem (2.78), (2.83) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; V) \times W_{\tau,T}$ and y satisfies (2.82).*

Remark 2.3. Since $W_{\tau,T} \subset C([\tau, T]; H)$, initial data (2.83) have sense.

Let us consider the next denotations: $E = V \times H$, $\forall \varphi_\tau = (a, b)^T \in E$

$$\mathscr{D}_{\tau,T}(\varphi_\tau) = \left\{ \begin{pmatrix} y(\cdot) \\ y'(\cdot) \end{pmatrix} \middle| \begin{array}{l} (y, y')^T \text{ is a weak solution of (2.78) on } [\tau, T], \\ y(\tau) = a, \ y'(\tau) = b \end{array} \right\}.$$

From Lemma 2.13 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; V) \times W_{\tau,T} \subset C([\tau, T]; E)$.

Let us complete the given subsection by checking that translation and concatenation of weak solutions is a weak solution too.

Lemma 2.14. *If $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, then $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$ $\forall s$. If $\tau < t < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi_\tau)$, then*

$$\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(\varphi_\tau)$.

Proof. The first part of the statement of this lemma follows from the autonomy of the inclusion (2.78). The proof of the second part follows from the definition of the solution of (2.79), Lemma 2.12 and from that fact that $z \in W_{\tau,T}$ as soon as $v \in W_{\tau,t}$, $u \in W_{t,T}$ and $v(t) = u(t)$, where

$$z(s) = \begin{cases} v(s), & s \in [\tau, t], \\ u(s), & s \in [t, T] \end{cases}$$

For the proof of the last we can use the definition of the derivative in the sense $\mathscr{D}([\tau, T]; V^*)$ and the formula (2.81).

2.5.2 Auxiliary Properties of the Resolving Operator

As a rule the proof of the existence of the compact global and trajectory attractors for evolution inclusions and, in particular, inclusions like (2.78) is based on properties of family of weak solutions of the problem (2.78) connected to absorbtion of the generated m-semiflow, closedness of its graph and its asymptotic compactness (cf. [21, 24, 36, 37] and references therein). The next lemma on a priory estimates and Theorem 2.4 on dependence of solutions on initial data play the key part when investigating the dynamic of solutions of the problem (2.78) as $t \to +\infty$.

Lemma 2.15. *There exists constants $c_1, c_2, c_3, c_4 > 0$ such that for any finite interval $[\tau, T]$ and for each weak solution $(y, y')^T$ of the problem (2.78) on $[\tau, T]$ the next estimates holds true: $\forall t \geq s$, $t, s \in [\tau, T]$*

$$\begin{aligned} &\|y'(t)\|_H^2 + \|y(t)\|_V^2 + \alpha \int_s^t \|y'(\xi)\|_V^2 d\xi \\ &\quad \leq \|y'(s)\|_H^2 + \|y(s)\|_V^2 + c_4(t-s)(\|f\|_{V^*}^2 + 1), \end{aligned} \tag{2.84}$$

$$\|y'(t)\|_H^2 + \|y(t)\|_V^2 \leq c_1(\|y'(s)\|_H^2 + \|y(s)\|_V^2)e^{-c_2(t-s)} + c_3(1 + \|f\|_{V^*}^2). \tag{2.85}$$

Proof. The inequality (2.84) obviously follows from Lemma 2.12 and Condition (A_2).

Let us prove now (2.85). We fix an arbitrary finite interval $[\tau, T]$ and an arbitrary weak solution $(y, y')^T$ of the problem (2.78) on $[\tau, T]$. Note that $y \in C([\tau, T]; V)$, $y' \in W_{\tau,T}$. For any $t \in [\tau, T]$ let us set

$$Y(t) = \frac{1}{2}\|y'(t)\|_H^2 + \frac{1}{2}\|y(t)\|_V^2 + \varepsilon(y'(t), y(t)),$$

where $\varepsilon = \frac{2\lambda_1 \alpha}{5 + 2\lambda_1 c^2} > 0$, $\lambda_1 > 0$ such that the next inequality takes place:

$$\lambda_1 \|u\|_H^2 \leq \|u\|_V^2 \quad \forall u \in V. \tag{2.86}$$

Firstly we check the next inequality

$$\frac{dY(t)}{dt} \leq -\alpha_1 Y(t) + \alpha_2 \quad \text{for a.e. } t \in (\tau, T), \tag{2.87}$$

where $\alpha_1 = \frac{\varepsilon\sqrt{\lambda_1}}{2(\varepsilon + 2\sqrt{\lambda_1})} > 0$, $\alpha_2 = \beta + 2\varepsilon c^2 + \|f\|_{V^*}^2(\frac{1}{2\alpha} + 2\varepsilon) > 0$.

From Conditions (A_1), (A_2) and the definition of a weak solution of the problem (2.78) on $[\tau, T]$ we have:

$$\begin{aligned} \frac{dY(t)}{dt} &= (y''(t), y'(t)) + \langle B_0 y(t), y'(t)\rangle_V + \varepsilon(y''(t), y(t)) + \varepsilon\|y'(t)\|_H^2 \\ &\leq -\alpha\|y'(t)\|_V^2 - \varepsilon\|y(t)\|_V^2 + \varepsilon\|y'(t)\|_H^2 + \|f\|_{V^*}\|y'(t)\|_V \\ &\quad + \varepsilon\|y(t)\|_V(c + \|f\|_{V^*}) + \varepsilon c\|y'(t)\|_V\|y(t)\|_V + \beta. \end{aligned} \tag{2.88}$$

Note that

$$c\|y'(t)\|_V\|y(t)\|_V \le \frac{c^2}{2}\|y'(t)\|_V^2 + \frac{1}{2}\|y(t)\|_V^2,$$

$$\|y(t)\|_V(c+\|f\|_{V^*}) \le \frac{\|y(t)\|_V^2}{4} + (c+\|f\|_{V^*})^2 \le \frac{\|y(t)\|_V^2}{4} + 2c^2 + 2\|f\|_{V^*}^2,$$

$$\|f\|_{V^*}\|y'(t)\|_V \le \frac{\alpha\|y'(t)\|_V^2}{2} + \frac{\|f\|_{V^*}^2}{2\alpha}.$$

Applying considered inequalities to the right part of (2.88), by the help of (2.86) we obtain:

$$\frac{dY(t)}{dt} \le -\frac{\varepsilon}{4}(\|y'(t)\|_H^2 + \|y(t)\|_V^2) + \beta + 2\varepsilon c^2 + \|f\|_{V^*}^2\left(\frac{1}{2\alpha} + 2\varepsilon\right). \quad (2.89)$$

Note that

$$|(y'(t), y(t))| \le \frac{1}{2\sqrt{\lambda_1}}(\|y'(t)\|_H^2 + \|y(t)\|_V^2). \quad (2.90)$$

Therefore, from inequalities (2.89), (2.90) we have (2.87).

From (2.87) and Gronwall-Bellman lemma we obtain

$$\forall \tau \le s \le t \le T \quad Y(t) \le Y(s)e^{-\alpha_1(t-s)} + \frac{\alpha_2}{\alpha_1}(1+\|f\|_{V^*}^2).$$

Thus, in view of (2.90), the next inequality takes place:

$$\forall t \in [\tau, T] \quad \|y'(t)\|_H^2 + \|y(t)\|_V^2$$

$$\le \frac{\sqrt{\lambda_1}+\varepsilon}{\sqrt{\lambda_1}-\varepsilon}\left((\|y'(s)\|_H^2 + \|y(s)\|_V^2)e^{-\alpha_1(t-s)} + \frac{\alpha_2}{\alpha_1}(1+\|f\|_{V^*}^2)\right).$$

Note that $\sqrt{\lambda_1} > \varepsilon$, in view of $\alpha \le c$ and $2\lambda^2 - 2\lambda + 5 \ge 0 \ \forall \lambda \in \mathbf{R}$, in particular for $\lambda = \sqrt{\lambda_1}c$.

If we set $c_1 = \frac{\sqrt{\lambda_1}+\varepsilon}{\sqrt{\lambda_1}-\varepsilon} > 0$, $c_2 = \alpha_1$, $c_3 = \frac{\alpha_2}{\alpha_1}\cdot c_1 > 0$, we obtain the necessary inequality.

Theorem 2.4. *Let $\tau < T$, $\{(u_n, u_n')^T\}_{n\ge1}$ be an arbitrary sequence of weak solutions of (2.78) on $[\tau, T]$ such that $u_n(\tau) \to u_\tau$ weakly in V, $u_n'(\tau) \to u_\tau'$ weakly in H. Then there exist $\{(u_{n_k}, u_{n_k}')^T\}_{k\ge1} \subset \{(u_n, u_n')^T\}_{n\ge1}$ and $(u, u')^T \in \mathscr{D}_{\tau,T}((u_\tau, u_\tau')^T)$ such that*

$$\forall \varepsilon \in (0, T-\tau) \ \max_{t\in[\tau+\varepsilon,T]}\|u_{n_k}'(t) - u'(t)\|_H \to 0, \ k \to +\infty, \quad (2.91)$$

$$u_{n_k}(t) \to u(t) \textit{ weakly in } V, \textit{ uniformly on } [\tau, T], \ k \to +\infty. \quad (2.92)$$

If supplementary $(u_n(\tau), u_n'(\tau))^T \to (u_\tau, u_\tau')^T$ *in* E, $n \to +\infty$, *then* $(u_{n_k}(\cdot), u_{n_k}'(\cdot))^T \to (u(\cdot), u'(\cdot))^T$ *in* $C([\tau, T]; E)$, $k \to +\infty$.

Proof. Under conditions of the theorem in view of Lemma 2.12, for any $n \geq 1$ $\varphi_n = (u_n, u_n')^T \in C([\tau, T]; E)$. Moreover, from Lemma 2.13, 2.15 we obtain that

$$\begin{aligned} &\forall n \geq 1 \ \exists d_n \in \mathscr{A}_{\tau,T}(u_n') : \\ &u_n''(t) + d_n(t) + B_0 u_n(t) = f \text{ for a.e. } t \in (\tau, T); \end{aligned} \tag{2.93}$$

$$\begin{aligned} &\exists C > 0 : \ \forall n \geq 1 \ \|u_n'\|_{X_{\tau,T}} + \|u_n''\|_{X_{\tau,T}^*} \\ &+ \|u_n'\|_{C([\tau,T];H)} + \|d_n\|_{X_{\tau,T}^*} + \|u_n\|_{C([\tau,T];V)} \leq C. \end{aligned} \tag{2.94}$$

Note that $\forall n \geq 1$ $\forall t \in [\tau, T]$ $u_n(t) = v_n(t) + u_{\tau,n}$, where $v_n(t) = \int_\tau^t u_n'(s)ds$, $(u_{\tau,n}, u_{\tau,n}')^T = \varphi_{\tau,n}$. At that

$$\forall n \geq 1, \ \forall t, s \in [\tau, T] \ \|v_n(t) - v_n(s)\|_V \leq C|t - s|^{\frac{1}{2}}, \ v_n(0) = \bar{0}. \tag{2.95}$$

Therefore, from (2.93)–(2.95), continuity of the embedding $W_{\tau,T} \subset C([\tau, T]; H)$, compactness of the embedding $W_{\tau,T} \subset L_2(\tau, T; H)$, reflexivity of spaces $W_{\tau,T}$, $X_{\tau,T}$, $X_{\tau,T}^*$ we have that up to a subsequence $\{u_{n_k}, d_{n_k}\}_{k\geq 1} \subset \{u_n, d_n\}_{n\geq 1}$ for some $u \in C([\tau, T]; V)$, $u' \in W_{\tau,T}$, $d \in X_{\tau,T}^*$ the next convergences take place:

$$\begin{aligned} &v_{n_k} \to v \text{ in } C([\tau, T]; V), \ \ u_{n_k}(t) \to u(t) \text{ weakly in } V \ \forall t \in [\tau, T], \\ &u_{n_k}' \to u' \text{ weakly in } X_{\tau,T}, \ u_{n_k}'' \to u'' \text{ weakly in } X_{\tau,T}^*, \\ &d_{n_k} \to d \text{ weakly in } X_{\tau,T}^*, \ u_{n_k}' \to u' \text{ weakly in } C([\tau, T]; H), \\ &u_{n_k}' \to u' \text{ in } L_2(\tau, T; H), \ u_{n_k}'(t) \to u'(t) \text{ in } H \text{ for a.e. } t \in (\tau, T), \ \ k \to +\infty, \end{aligned} \tag{2.96}$$

where $v(\cdot) = u(\cdot) - u_\tau$. Let us complete the proof of the theorem in several steps.

Step 1. Show that

$$u'' = f_{\tau,T} - d - B_{\tau,T}(u). \tag{2.97}$$

Indeed, $\forall k \geq 1$, $\forall \zeta \in C_0^\infty([\tau, T]; V)$

$$\begin{aligned} &-\langle \zeta', u_{n_k}' \rangle_{X_{\tau,T}} + \langle d_{n_k}, \zeta \rangle_{X_{\tau,T}} + \langle B_{\tau,T}(v_{n_k}), \zeta \rangle_{X_{\tau,T}} \\ &+ \int_\tau^T \langle B_0 u_{n_k,\tau}, \zeta(t) \rangle_V dt = \langle f_{\tau,T}, \zeta \rangle_{X_{\tau,T}}. \end{aligned} \tag{2.98}$$

Further, let us pass in (2.98) to the limit as $k \to +\infty$. We obtain:

$$\forall \zeta \in C_0^\infty([\tau, T]; V) \quad -\langle \zeta', u' \rangle_{X_{\tau,T}} + \langle d, \zeta \rangle_{X_{\tau,T}}$$
$$+ \langle B_{\tau,T}(v), \zeta \rangle_{X_{\tau,T}} + \int_\tau^T \langle B_0 u_\tau, \zeta(t) \rangle_V dt = \langle f_{\tau,T}, \zeta \rangle_{X_{\tau,T}}.$$

Thus, using properties of Bochner's integral, $\forall \varphi \in C_0^\infty([\tau, T])\ \forall h \in V$

$$-\left(\int_\tau^T u'(s)\varphi'(s)ds, h \right)_H = -\int_\tau^T (h, u'(s))_H \varphi'(s)ds$$
$$= \int_\tau^T \langle f - d(s) - B_0 v(s) - B_0 u_\tau, h \rangle_H \varphi(s)ds$$
$$= \left\langle \int_\tau^T [f_{\tau,T}(s) - d(s) - B_{\tau,T}(u)(s)]\varphi(s)ds, h \right\rangle_V.$$

Finally, the relation (2.97) follows from the definition of derivative of an element u' in the sense $\mathscr{D}^*([\tau, T]; V^*)$.

Step 2. From (2.96) it follows that $\exists \{\varepsilon_j\}_{j \ge 1} \subset (\tau, T)$:

$$\varepsilon_j \searrow 0+,\ j \to +\infty, \ \ \forall j \ge 1\ u'_{n_k}(\tau + \varepsilon_j) \to u'(\tau + \varepsilon_j) \text{ in } H,\ k \to +\infty. \tag{2.99}$$

Let us fix an arbitrary $\varepsilon \in \{\varepsilon_j\}_{j \ge 1}$ and show that

$$d(t) \in A_0(u'(t)) \text{ for a.e. } t \in (\tau + \varepsilon, T), \tag{2.100}$$

using pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$.
Let us consider restrictions $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, $d(\cdot)$ on the interval $[\tau + \varepsilon, T]$. For the simplicity of suggestions we denote these restrictions by the same symbols: $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, $d(\cdot)$ correspondingly. From (2.96) it follows that

$$u'_{n_k} \to u' \text{ weakly in } W_{\tau+\varepsilon,T},\ d_{n_k} \to d \text{ weakly in } X^*_{\tau+\varepsilon,T},\ k \to +\infty. \tag{2.101}$$

Show that

$$\overline{\lim_{k\to+\infty}} \langle d_{n_k}, u'_{n_k} - u' \rangle_{X_{\tau+\varepsilon,T}} \le 0 \tag{2.102}$$

Indeed,

$$
\begin{aligned}
\forall k \geq 1 \quad & \int_{\tau+\varepsilon}^{T} \langle d_{n_k}(s), u'_{n_k}(s) - u'(s)\rangle_V ds \\
&= \int_{\tau+\varepsilon}^{T} \langle f, u'_{n_k}(s) - u'(s)\rangle_V ds + \int_{\tau+\varepsilon}^{T} \langle u''_{n_k}(s), u'(s) - u'_{n_k}(s)\rangle_V ds \\
&\quad + \int_{\tau+\varepsilon}^{T} \langle B_0 u_{\tau,n_k}, u'(s) - u'_{n_k}(s)\rangle_V ds \\
&\quad + \int_{\tau+\varepsilon}^{T} \langle B_0 v_{n_k}, u'(s) - u'_{n_k}(s)\rangle_V ds \\
&:= I_{1,k} + I_{2,k} + I_{3,k} + I_{4,k}.
\end{aligned} \tag{2.103}
$$

From (2.101) it follows that

$$
I_{1,k} \to 0, \quad k \to +\infty. \tag{2.104}
$$

In consequence of (2.81), (2.96) and (2.99) we obtain that

$$
\begin{aligned}
\forall k \geq 1 \;\; I_{2,k} &= \int_{\tau+\varepsilon}^{T} \langle u''_{n_k}(s), u'(s)\rangle_V ds - \frac{1}{2}(\|u'_{n_k}(T)\|_H^2 - \|u'_{n_k}(\tau+\varepsilon)\|_H^2), \\
\overline{\lim_{k\to+\infty}} I_{2,k} &\leq \int_{\tau+\varepsilon}^{T} \langle u''(s), u'(s)\rangle_V ds - \frac{1}{2}(\|u'(T)\|_H^2 - \|u'(\tau+\varepsilon)\|_H^2) = 0.
\end{aligned} \tag{2.105}
$$

In view of (2.95), (2.96) and properties of Bochner's integral we have that $\forall k \geq 1$

$$
I_{3,k} = \langle B_0 u_{\tau,n_k}, v(T) - v(\tau+\varepsilon) - v_{n_k}(T) + v_{n_k}(\tau+\varepsilon)\rangle_V \to 0, \; k \to +\infty. \tag{2.106}
$$

$$
\begin{aligned}
|I_{4,k}| &\leq \left| \int_{\tau+\varepsilon}^{T} \langle B_0 v(s), u'(s) - u'_{n_k}(s)\rangle_V ds \right| \\
&+ \|B_0\|_{\mathscr{L}(V;V^*)} \|v_{n_k} - v\|_{C([\tau,T];V)} \cdot 2C \cdot (T-\tau-\varepsilon)^{\frac{1}{2}} \to 0, \; k \to +\infty.
\end{aligned} \tag{2.107}
$$

Thus, if we pass in (2.103) to the upper limit as $k \to +\infty$, in view of (2.104)–(2.107), we obtain (2.102).

Further, due to (2.93), (2.101), (2.102) and pseudomonotony of $\mathscr{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$ we obtain (2.100).

Step 3. From (2.99), an arbitrariness of $\varepsilon \in \{\varepsilon_j\}_{j\geq 1}$, the relation (2.100) and definition of $\mathscr{A}_{\tau,T}(u')$ we obtain that $(u,u')^T \in \mathscr{D}_{\tau,T}((u_\tau, u'_\tau)^T)$.

Step 4. From (2.96) it directly follows (2.92).

Step 5. Let us check (2.91) using the method by contradiction. We suggest that $\exists \varepsilon > 0, \exists L > 0, \exists \{u_{k_j}\}_{j\geq 1} \subset \{u_{n_k}\}_{k\geq 1}$:

$$\forall j \geq 1 \ \max_{[\tau,T]} \|u'_{k_j}(t) - u'(t)\|_H = \|u'_{k_j}(t_j) - u'(t_j)\|_H \geq L.$$

Without loss of the generality we can guess that $t_j \to t_0 \in [\tau, T]$, $j \to +\infty$. Therefore, in view of continuity of $u' : [\tau, T] \to H$,

$$\underline{\lim}_{j\to+\infty} \|u'_{k_j}(t_j) - u'(t_0)\|_H \geq L. \tag{2.108}$$

On the other hand we show that

$$u'_{k_j}(t_j) \to u'(t_0) \text{ in } H,\ j \to +\infty. \tag{2.109}$$

Step 5.1. Firstly we prove that

$$u'_{k_j}(t_j) \to u'(t_0) \text{ weakly in } H,\ j \to +\infty. \tag{2.110}$$

For a fixed $h \in V$ from (2.96) it follows that the sequence of real functions $(u'_{n_k}(\cdot), h) : [\tau, T] \to \mathbf{R}$ is uniformly bounded and equipotentionally continuous one. Taking into account (2.96) and density of the embedding $V \subset H$ we obtain that $u'_{n_k}(t) \to u'(t)$ weakly in H uniformly on $[\tau, T]$, $k \to +\infty$, whence it follows (2.110).

Step 5.2. Let us prove that

$$\overline{\lim}_{j\to+\infty} \|u'_{k_j}(t_j)\|_H \leq \|u'(t_0)\|_H. \tag{2.111}$$

Note that in view of (2.93) and Condition (A_2) we obtain that

$$\begin{aligned}\forall j \geq 1 \text{ for a.e. } t \in (\tau, T)\ &\frac{d}{dt}(\|u'_{k_j}(t)\|_H^2 + 2\langle B_0 u_{\tau,k_j}, v_{k_j}(t)\rangle_V \\ &+\|v_{k_j}(t)\|_V^2) \\ &= \frac{d}{dt}(\|u'_{k_j}(t)\|_H^2 + \|u_{k_j}(t)\|_V^2) \leq \beta + \frac{\|f\|_{V^*}^2}{4\alpha} =: \bar{\beta}.\end{aligned}$$

Similarly,

$$\text{for a.e. } t \in (\tau, T) \ \frac{d}{dt}(\|u'(t)\|_H^2 + 2\langle B_0 u_\tau, v(t)\rangle_V + \|v(t)\|_V^2) \le \bar{\beta}.$$

Thus, real functions $\{J_j : [\tau, T] \to \mathbf{R} \mid j \ge 0\}$,

$$J_j(t) = \|u'_{k_j}(t)\|_H^2 + \|v_{k_j}(t)\|_V^2 + 2\langle B_0 u_{\tau,k_j}, v_{k_j}(t)\rangle_V - \bar{\beta}t, \tag{2.112}$$

$$J_0(t) = \|u'(t)\|_H^2 + \|v(t)\|_V^2 + 2\langle B_0 u_\tau, v(t)\rangle_V - \bar{\beta}t, \ t \in [\tau, T], \tag{2.113}$$

are steadily nonincreasing, continuous and in view of (2.96),

$$\text{for a.e. } t \in (\tau, T) \ J_j(t) \to J_0(t), \ j \to +\infty. \tag{2.114}$$

Let us fix an arbitrary $\varepsilon_1 > 0$. From (2.114) and continuity of J_0 it follows that

$$\exists \bar{t} \in (\tau, t_0) : \ J_j(\bar{t}) \to J_0(\bar{t}), \ j \to +\infty \text{ and } |J_0(\bar{t}) - J_0(t_0)| < \varepsilon_1.$$

Then for rather big $j \ge 1$ $J_j(t_j) - J_0(t_0) \le J_j(\bar{t}) - J_0(\bar{t}) + |J_0(\bar{t}) - J_0(t_0)| < |J_j(\bar{t}) - J_0(\bar{t})| + \varepsilon_1$. From the arbitrariness of $\varepsilon_1 > 0$ we have $\overline{\lim_{j\to+\infty}} J_j(t_j) \le J_0(t_0)$. Hence, taking into account (2.96), we obtain (2.111).

Step 5.3. Equation (2.109) directly follows from (2.110), (2.111) and [16].

Step 5.4. For the completeness of the proof of (2.91) note that (2.109) contradicts with (2.108). Therefore, the validity of (2.91) is checked.

Step 6. Supplementary we suggest that

$$(u_n(\tau), u'_n(\tau))^T \to (u_\tau, u'_\tau)^T \text{ in } E, \ k \to +\infty. \tag{2.115}$$

Step 6.1. From (2.115) and (2.96) it directly follows that

$$u_{n_k} \to u \text{ in } C([\tau, T], V), \ k \to +\infty. \tag{2.116}$$

Step 6.2. For the completeness of this proof it remains to check that

$$u'_{n_k} \to u' \text{ in } C([\tau, T]; H). \tag{2.117}$$

Let us check (2.117) using the method by contradiction. We suggest that $\exists L_1 > 0$, $\exists \{u_{k_j}\}_{j\ge1} \subset \{u_{n_k}\}_{k\ge1}$

$$\forall j \ge 1 \ \|u'_{k_j} - u'\|_{C([\tau,T];H)} = \|u'_{k_j}(t_j) - u'(t_j)\|_H \ge L_1. \tag{2.118}$$

Repeating upper considered suggestions from Step 5 of the proof, taking into account (2.91), without loss of the generality, we can guess that

$$t_j \to \tau,\ u'_{k_j}(t_j) \to u'(\tau) \text{ weakly in } H,\ j \to +\infty;$$
$$\varliminf_{j\to+\infty} \|u'_{k_j}(t_j) - u'(\tau)\|_H \geq L_1 \tag{2.119}$$

Let us consider a sequence of steadily non-decreasing continuous functions $\{J_j\}_{j\geq 0}$, defined in (2.112), (2.113). Since $\forall j \geq 1$ $J_j(t_j) - J_0(\tau) \leq J_j(\tau) - J_0(\tau)$, then in view of (2.96) we obtain that $\varlimsup_{j\to+\infty} J_j(t_j) \leq J_0(\tau)$ and, therefore, $\varlimsup_{j\to+\infty} \|u'_{k_j}(t_j)\|_H \leq \|u'(\tau)\|_H$. The last inequality together with (2.119) contradicts with (2.118). The theorem is proved.

2.6 Auxiliary Properties of Solutions for the Second Order Evolution Inclusions and Hemivariational Inequalities for Piezoelectric Fields

Now we consider a mathematical model which describes the contact between a piezoelectric body and a foundation (see Example 2). For evolution triple $(V; H; V^*)$, linear operators $R : H \to H$, $G : V \to V^*$ and locally Lipschitz functional $J : H \to \mathbf{R}$ we consider a problem of investigation of dynamics for all weak solutions defined for $t \geq 0$ of non-linear second order autonomous differential-operator inclusion:

$$u''(t) + Ru'(t) + Gu(t) + \partial J(u(t)) \ni \bar{0} \quad \text{a.e.} \quad t > 0. \tag{2.120}$$

We need the following hypotheses:.

$\underline{H(R)}$ $R : H \to H$ is a linear symmetric such that $\exists \gamma > 0 : (Rv, v)_H = \gamma\|v\|_H^2$ $\forall v \in H$;
$\underline{H(G)}$ $G : V \to V^*$ is linear, symmetric and $\exists c_G > 0 : \langle Gv, v\rangle_V \geq c_G\|v\|_V^2$ $\forall v \in V$;
$\underline{H(J)}$ $J : H \to \mathbf{R}$ is a function such that

(i) $J(\cdot)$ is locally Lipschitz and regular [12], i.e.

- For any $x, v \in H$, the usual one-sided directional derivative $J'(x; v) = \lim_{t\searrow 0} \frac{J(x+tv)-J(x)}{t}$ exists,
- For all $x, v \in H$, $J'(x; v) = J^\circ(x; v)$, where $J^\circ(x; v) = \varlimsup_{y\to x, t\searrow 0} \frac{J(y+tv)-J(y)}{t}$;

(ii) $\exists c_1 > 0$: $\|\partial J(v)\|_+ \leq c_1(1 + \|v\|_H)$ $\forall v \in H$;
(iii) $\exists c_2 > 0$:
$$[\partial J(v), v]_- \geq -\lambda\|v\|_H^2 - c_2 \quad \forall v \in H,$$

where $\partial J(v) = \{p \in H \,|\, (p,w)_H \le J^\circ(v;w)\ \forall w \in H\}$ denotes the Clarke subdifferential of $J(\cdot)$ at a point $v \in H$ (see [12] for details), $\lambda \in (0,\lambda_1)$, $\lambda_1 > 0$: $c_G\|v\|_V^2 \ge \lambda_1\|v\|_H^2\ \forall v \in V$;

($\underline{H_0}$) V is a Hilbert space.

The phase space for Problem (2.120) we define Hilbert space $E = V \times H$. Let $-\infty < \tau < T < +\infty$.

Definition 2.1. The function $(u(\cdot),u'(\cdot))^T \in L_\infty(\tau,T;E)$ is called a *weak solution* for (2.120) on (τ,T), if there exists $d \in L_2(\tau,T;H)$, $d(t) \in \partial J(u(t))$ for a.e. $t \in (\tau,T)$, such that $\forall \psi \in V$, $\forall \eta \in C_0^\infty(\tau,T)$

$$-\int_\tau^T (u'(t),\psi)_H\eta'(t)dt+\int_\tau^T \left[(u'(t),\psi)_H + (u(t),\psi)_H + (d(t),\psi)_H\right]\eta(t)dt = 0,$$

We consider a class of functions $W_\tau^T = C([\tau,T];E)$. Further $\gamma, c_1, c_2, \lambda, \lambda_1$ we recall parameters of Problem (2.120). The main purpose of this work is to investigate the long-time behavior (as $t \to +\infty$) of all weak solutions for the problem (2.120).

To simplify our conclusions from Conditions $H(G)$, $H(R)$ we suppose that

$$\begin{aligned}(u,v)_V &= \langle Gu,v\rangle_V,\ \|v\|_V^2 = \langle Gu,v\rangle_V,\ c_G = 1,\ \gamma(u,v)_H = (Ru,v)_H,\ \gamma\|v\|_H^2 \\ &= (Rv,v)_H\ \forall u,v \in V.\end{aligned} \tag{2.121}$$

Lebourgues mean value theorem [12, Chap. 2] provides the existence of constants $c_3, c_4 > 0$ and $\mu \in (0,\lambda_1)$:

$$|J(u)| \le c_3(1+\|u\|_H^2),\quad J(u) \ge -\frac{\mu}{2}\|u\|_H^2 - c_4 \quad \forall u \in H. \tag{2.122}$$

Lemma 2.16. *Let $J : H \to \mathbf{R}$ be a locally Lipschitz and regular functional, $y \in C^1([\tau,T];H)$. Then for a.e. $t \in (\tau,T)$ $\exists \frac{d}{dt}(J\circ y)(t) = (p, y'(t))\ \forall p \in \partial J(y(t))$. Moreover, $\frac{d}{dt}(J\circ y)(\cdot) \in L_1(\tau,T)$.*

Proof. Since $y \in C^1([\tau,T];H)$ then y is strictly differentiable at the point t_0 for any $t_0 \in (\tau,T)$. Hence, taking into account the regularity of J and [12, Theorem 2.3.10], we obtain that the functional $J\circ y$ is regular one at $t_0 \in (\tau,T)$ and $\partial(J\circ y)(t_0) = \{(p,y'(t_0))|\ p \in \partial J(y(t_0))\}$. On the other hand, since $y \in C([\tau,T];H)$, $J : H \to \mathbf{R}$ is locally Lipschitz then $J\circ y : [\tau,T] \to \mathbf{R}$ is globally Lipschitz and therefore it is absolutely continuous. Hence for a.e. $t \in (\tau,T)$ $\exists \frac{d(J\circ y)(t)}{dt}$, $\frac{d(J\circ y)(\cdot)}{dt}$ $\in L_1(\tau,T)$ and $\int_s^t \frac{d}{d\xi}(J\circ y)(\xi)d\xi = (J\circ y)(t) - (J\circ y)(s)\ \forall \tau \le s < t \le T$. At that taking into account the regularity of $J\circ y$, note that $(J\circ y)^\circ(t_0,v) = (J\circ y)'(t_0,v) = \frac{d(J\circ y)(t_0)}{dt}\cdot v$ for a.e. $t_0 \in (\tau,T)$, $\forall v \in \mathbf{R}$. This implies that for a.e. $t_0 \in (\tau,T)$ $\partial(J\circ y)(t_0) = \left\{\frac{d(J\circ y)(t_0)}{dt}\right\}$.

A weak solution of the problem (2.120) with initial data

$$u(\tau) = a, \quad u'(\tau) = b \tag{2.123}$$

on the interval $[\tau, T]$ exists for any $a \in V, b \in H$. It follows from [23, Theorem 1.4]. Thus, the next lemma holds true (see [33, Lemma 4.1, p. 78] and [33, Lemma 3.1, p. 71]).

Lemma 2.17. *For any $\tau < T$, $a \in V$, $b \in H$ the Cauchy problem (2.120), (2.123) has a weak solution $(y, y')^T \in L_\infty(\tau, T; E)$. Moreover, each weak solution $(y, y')^T$ of the Cauchy problem (2.120), (2.123) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; E)$ and $y'' \in L_2(\tau, T; V^*)$.*

Let us consider the next denotations: $\forall \varphi_\tau = (a, b)^T \in E$ we consider $\mathscr{D}_{\tau,T}(\varphi_\tau) = \{ (u(\cdot), u'(\cdot))^T \mid (u, u')^T$ is a weak solution of (2.120) on $[\tau, T]$, $u(\tau) = a$, $u'(\tau) = b \}$. From Lemma 2.17 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; E) = W_\tau^T$. Let us complete the given subsection by checking that translation and concatenation of weak solutions is a weak solution too.

Lemma 2.18. *If $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, then $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$ $\forall s$. If $\tau < t < T$, $\varphi_\tau \in E$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi_\tau)$, then $\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases}$ belongs to $\mathscr{D}_{\tau,T}(\varphi_\tau)$.*

Proof. The proof is trivial.

Let $\varphi = (a, b)^T \in E$ and

$$\mathscr{V}(\varphi) = \frac{1}{2}\|\varphi\|_E^2 + J(a). \tag{2.124}$$

Lemma 2.19. *Let $\tau < T$, $\varphi_\tau \in E$, $\varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$. Then $\mathscr{V} \circ \varphi : [\tau, T] \to \mathbf{R}$ is absolutely continuous and for a.e. $t \in (\tau, T)$ $\frac{d}{dt}\mathscr{V}(\varphi(t)) = -\gamma\|y'(t)\|_H^2$.*

Proof. Let $-\infty < \tau < T < +\infty$, $\varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in W_\tau^T$ be an arbitrary weak solution of (2.120) on (τ, T). As $\partial J(y(\cdot)) \subset L_2(\tau, T; H)$ then from [33, Lemma 4.1, p. 78] and [33, Lemma 3.1, p. 71] we get that the function $t \to \|y'(t)\|_H^2 + \|y(t)\|_V^2$ is absolutely continuous and for a.e. $t \in (\tau, T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\|y'(t)\|_H^2 + \|y(t)\|_V^2\right] &= (y''(t) + Gy(t), y'(t))_H = -\gamma\|y'(t)\|_H^2 \\ &\quad -(d(t), y'(t))_H, \end{aligned} \tag{2.125}$$

where $d(t) \in \partial J(y(t))$ for a.e. $t \in (\tau, T)$ and $d(\cdot) \in L_2(\tau, T; H)$. As $y(\cdot) \in C^1([\tau, T]; H)$ and $J : H \to \mathbf{R}$ is regular and locally Lipschitz, due to Lemma 2.16 we obtain that for a.e. $t \in (\tau, T)$ $\exists \frac{d}{dt}(J \circ y)(t)$. Moreover, $\frac{d}{dt}(J \circ y)(\cdot) \in L_1(\tau, T)$

and for a.e. $t \in (\tau, T)$, $\forall p \in \partial J(y(t))$ $\frac{d}{dt}(J \circ y)(t) = (p, y'(t))_H$. In particular, for a.e. $t \in (\tau, T)$ $\frac{d}{dt}(J \circ y)(t) = (d(t), y'(t))_H$. Taking into account (2.125) we finally obtain the necessary statement.

The lemma is proved.

Lemma 2.20. *Let* $T > 0$. *Then any weak solution of Problem (2.120) on* $[0, T]$ *can be extended to a global one defined on* $[0, +\infty)$.

Proof. The statement of this lemma follows from Lemmas 2.17–2.19, Conditions (2.121), (2.122) and from the next estimates: $\forall \tau < T$, $\forall \varphi_\tau \in E$, $\forall \varphi(\cdot) = (y(\cdot), y'(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, $\forall t \in [\tau, T]$ $2c_3 + \left(1 + \frac{2c_3}{\lambda_1}\right) \|y(\tau)\|_V^2 + \|y'(\tau)\|_H^2 \geq 2\mathscr{V}(\varphi(\tau)) \geq 2\mathscr{V}(\varphi(t)) = \|y(t)\|_V^2 + \|y'(t)\|_H^2 + 2J(y(t)) \geq \left(1 - \frac{\mu}{\lambda_1}\right) \|y(t)\|_V^2 + \|y'(t)\|_H^2 - 2c_4$.

The lemma is proved.

For an arbitrary $\varphi_0 \in E$ let $\mathscr{D}(\varphi_0)$ be the set of all weak solutions (defined on $[0, +\infty)$) of problem (2.120) with initial data $\varphi(0) = \varphi_0$. We remark that from the proof of Lemma 2.20 we obtain the next corollary.

Corollary 2.2. *For any* $\varphi_0 \in E$ *and* $\varphi \in \mathscr{D}(\varphi_0)$ *the next inequality is fulfilled:*

$$\|\varphi(t)\|_E^2 \leq \frac{\lambda_1 + 2c_3}{\lambda_1 - \mu} \|\varphi(0)\|_E^2 + \frac{2(c_3 + c_4)\lambda_1}{\lambda_1 - \mu} \quad \forall t > 0. \tag{2.126}$$

From Corollary 2.2 and Conditions $\underline{H(R)}$, $\underline{H(G)}$, $\underline{H(J)}$, $(\underline{H_0})$ in a standard way we obtain such proposition.

Theorem 2.5. *Let* $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq1} \subset W_\tau^T$ *be an arbitrary sequence of weak solutions of (2.120) on* $[\tau, T]$ *such that* $\varphi_n(\tau) \to \varphi_\tau$ *weakly in* E, $n \to +\infty$, *and let* $\{t_n\}_{n\geq1} \subset [\tau, T]$ *be a sequence such that* $t_n \to t_0$, $n \to +\infty$. *Then there exist* $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ *such that up to a subsequence* $\varphi_n(t_n) \to \varphi(t_0)$ *weakly in* E, $n \to +\infty$.

Proof. We prove this theorem in several steps.

Step 1. Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))\}_{n\geq1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (2.120) on $[\tau, T]$ and $\{t_n\}_{n\geq1} \subset [\tau, T]$:

$$\varphi_n(\tau) \to \varphi_\tau \text{ weakly in } E, \quad t_n \to t_0, \quad n \to +\infty. \tag{2.127}$$

In virtue of Corollary 2.2 we have that $\{\varphi_n(\cdot)\}_{n\geq1}$ is bounded on $W_\tau^T \subset L_\infty(\tau, T; E)$. Therefore up to a subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq1} \subset \{\varphi_n(\cdot)\}_{n\geq1}$ we have:

$$
\begin{aligned}
&u_{n_k} \to u \text{ weakly star in } L_\infty(\tau, T; V),\\
&u'_{n_k} \to u' \text{ weakly star in } L_\infty(\tau, T; H),\\
&u''_{n_k} \to u'' \text{ weakly star in } L_\infty(\tau, T; V^*),\\
&d_{n_k} \to d \text{ weakly star in } L_\infty(\tau, T; H),\\
&u_{n_k} \to u \text{ in } L_2(\tau, T; H),\\
&u_{n_k}(t) \to u(t) \text{ in } H \text{ for a.e. } t \in [\tau, T],\\
&u'_{n_k}(t) \to u'(t) \text{ in } V^* \text{ for a.e. } t \in (\tau, T),\\
&Ru'_{n_k} \to Ru' \text{ weakly in } L_2(\tau, T; H),\\
&Gu_{n_k} \to Gu \text{ weakly in } L_2(\tau, T; V^*),\ k \to +\infty,
\end{aligned}
\tag{2.128}
$$

where $\forall n \geq 1$ $d_n \in L_2(\tau, T; H)$,

$$
u''_n(t) + Ru'_n(t) + d_n(t) + Gu_n(t) = F,\ d_n(t) \in \partial j(u_n(t)) \text{ for a.e. } t \in (\tau, T).
\tag{2.129}
$$

Step 2. As ∂j is demiclosed is a standard way we get that $d(\cdot) \in \partial j(u(\cdot))$, $\varphi := (u, u') \in \mathscr{D}_{\tau,T}(\varphi_\tau) \subset W_\tau^T$.

Step 3. For a fixed $h \in V$ from (2.128) it follows that the sequence of real functions $(u_{n_k}(\cdot), h)$, $(u'_{n_k}(\cdot), h) : [\tau, T] \to \mathbf{R}$ is uniformly bounded and equipotentionally continuous one. Taking into account (2.128), (2.126) and density of the embedding $V \subset H$ we obtain that $u'_{n_k}(t_{n_k}) \to u'(t_0)$ weakly in H and $u_{n_k}(t_{n_k}) \to u(t_0)$ weakly in V, $k \to +\infty$, whence it follows that the first part of this theorem is fulfilled.

The theorem is proved.

Theorem 2.6. *Let $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (2.120) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ strongly in E, $n \to +\infty$, then up to a subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $C([\tau, T]; E)$, $n \to +\infty$.*

Proof. Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u'_n(\cdot))^T\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (2.120) on $[\tau, T]$ and $\{t_n\}_{n\geq 1} \subset [\tau, T]$:

$$
\varphi_n(\tau) \to \varphi_\tau \text{ strongly in } E,\ n \to +\infty.
\tag{2.130}
$$

From Theorem 2.5 we have that there exist $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq 1} \subset \{\varphi_n(\cdot)\}_{n\geq 1}$, $\varphi_{n_k}(\cdot) \to \varphi(\cdot)$ weakly in E uniformly on $[\tau, T]$, $k \to +\infty$. Let us prove that

$$
\varphi_{n_k} \to \varphi \text{ in } W_\tau^T, \quad k \to +\infty.
\tag{2.131}
$$

By contradiction suppose the existence of $L > 0$ and subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_{n_k}\}_{k\geq 1}$ such that $\forall j \geq 1$ $\max\limits_{t\in[\tau,T]} \|\varphi_{k_j}(t) - \varphi(t)\|_E = \|\varphi_{k_j}(t_j) - \varphi(t_j)\|_E \geq L$. Without loss of generality we suggest that $t_j \to t_0 \in [\tau, T]$, $j \to +\infty$. Therefore, by virtue of the continuity of $\varphi : [\tau, T] \to E$, we have

$$
\varliminf_{j\to+\infty} \|\varphi_{k_j}(t_j) - \varphi(t_0)\|_E \geq L.
\tag{2.132}
$$

On the other hand we prove that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ in } E, \;\; j \to +\infty. \tag{2.133}$$

Firstly we remark that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ weakly in } E, \quad j \to +\infty \tag{2.134}$$

(see Theorem 2.5 for details). Secondly let us prove that

$$\overline{\lim_{j\to+\infty}} \|\varphi_{k_j}(t_j)\|_E \leq \|\varphi(t_0)\|_E. \tag{2.135}$$

Since J is sequentially weakly continuous, $\mathscr{V}$ is sequentially weakly lower semicontinuous on E. Hence, we obtain

$$\mathscr{V}(\varphi(t_0)) \leq \underline{\lim_{j\to+\infty}} \mathscr{V}(\varphi_{k_j}(t_j)), \quad \int\limits_{\tau}^{t_0} \|u'(s)\|_H^2 ds \leq \underline{\lim_{j\to+\infty}} \int\limits_{\tau}^{t_j} \|u'_{k_j}(s)\|_H^2 ds, \tag{2.136}$$

and hence

$$\mathscr{V}(\varphi(t_0)) + \gamma \int\limits_{\tau}^{t_0} \|u'(s)\|_H^2 ds \leq \underline{\lim_{j\to+\infty}} \left(\mathscr{V}(\varphi_{k_j}(t_j)) + \gamma \int\limits_{\tau}^{t_j} \|u'_{k_j}(s)\|_H^2 ds \right). \tag{2.137}$$

Since by the energy equation both sides of (2.137) equal $\mathscr{V}(\varphi(\tau))$ (see Lemma 2.19), it follows from (2.136) that $\mathscr{V}(\varphi_{k_j}(t_j)) \to \mathscr{V}(\varphi(t_0))$, $j \to +\infty$ and (2.135). Convergence (2.133) directly follows from (2.134), (2.135) and [16, Chap. I]. To finish the proof of the theorem we remark that (2.133) contradicts (2.132). Therefore, (2.131) is true.

The theorem is proved.

We define the m-semiflow $\mathscr{G}$ as $\mathscr{G}(t,\xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}$, $t \geq 0$. Denote the set of all nonempty (nonempty bounded) subsets of E by $P(E)$ ($\beta(E)$). We remark that the multivalued map $\mathscr{G} : \mathbf{R}_+ \times E \to P(E)$ is *strict m-semiflow*, i.e. (see Lemma 2.18) $\mathscr{G}(0,\cdot) = \mathrm{Id}$ (the identity map), $\mathscr{G}(t+s,x) = \mathscr{G}(t,\mathscr{G}(s,x))$ $\forall x \in E, t, s \in \mathbf{R}_+$. Further $\varphi \in \mathscr{G}$ will mean that $\varphi \in \mathscr{D}(\xi_0)$ for some $\xi_0 \in E$.

Definition 2.2. The m-semiflow $\mathscr{G}$ is called *asymptotically compact*, if for any sequence $\varphi_j \in \mathscr{G}$ with $\varphi_j(0)$ bounded, and for any sequence $t_j \to +\infty$, the sequence $\varphi_j(t_j)$ has a convergent subsequence.

Theorem 2.7. *The m-semiflow $\mathscr{G}$ is asymptotically compact.*

Proof. Let $\xi_n \in \mathscr{G}(t_n, v_n)$, $v_n \in B \in \beta(E)$, $n \geq 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check the precompactness of $\{\xi_n\}_{n\geq 1}$ in E. In order to do that without loss of the generality it is sufficiently to extract a convergent in E subsequence from $\{\xi_n\}_{n\geq 1}$. From Corollary 2.2 we obtain that there exist such $\{\xi_{n_k}\}_{k\geq 1}$ and $\xi \in E$ that $\xi_{n_k} \to \xi$ weakly in E, $\|\xi_{n_k}\|_E \to a \geq \|\xi\|_E$, $k \to +\infty$. Show that $a \leq \|\xi\|_E$. Let us fix an arbitrary $T_0 > 0$. Then for rather big $k \geq 1$ $\mathscr{G}(t_{n_k}, v_{n_k}) \subset \mathscr{G}(T_0, \mathscr{G}(t_{n_k} - T_0, v_{n_k}))$. Hence $\xi_{n_k} \in \mathscr{G}(T_0, \beta_{n_k})$, where $\beta_{n_k} \in \mathscr{G}(t_{n_k} - T_0, v_{n_k})$ and $\sup\limits_{k\geq 1} \|\beta_{n_k}\|_E < +\infty$ (see Corollary 2.2). From Theorem 2.5 for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\geq 1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\geq 1}$, $\beta_{T_0} \in E$ we obtain:

$$\xi \in \mathscr{G}(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } E, \quad j \to +\infty. \tag{2.138}$$

From the definition of $\mathscr{G}$ we set: $\forall j \geq 1$ $\xi_{k_j} = (y_j(T_0), y_j'(T_0))^T$, $\beta_{k_j} = (y_j(0), y_j'(0))^T$, $\xi = (y_0(T_0), y_0'(T_0))^T$, $\beta_{T_0} = (y_0(0), y_0'(0))^T$, where $\varphi_j = (y_j, y_j')^T \in C([0, T_0]; E)$, $y_j'' \in L_2(0, T_0; V^*)$, $d_j \in L_\infty(0, T_0; H)$,

$$y_j''(t) + Ry_j'(t) + Gy_j(t) + d_j(t) = \bar{0}, \quad d_j(t) \in \partial J(y_j(t)) \quad \text{for a.e.} \quad t \in (0, T_0).$$

Let for each $t \in [0, T_0]$ $I(\varphi_j(t)) := \frac{1}{2}\|\varphi_j(t)\|_E^2 + J(y_j(t)) + \frac{\gamma}{2}(y_j'(t), y_j(t))$. Then, in virtue of Lemma 2.16, [33, Lemma 4.1,p. 78] and [33, Lemma 3.1,p. 71], $\frac{dI(\varphi_j(t))}{dt} = -\gamma I(\varphi_j(t)) + \gamma \mathscr{H}(\varphi_j(t))$, for a.e. $t \in (0, T_0)$, where $\mathscr{H}(\varphi_j(t)) = J(y_j(t)) - \frac{1}{2}(d_j(t), y_j(t))$.

From (2.126), (2.138) we have $\exists \bar{R} > 0 : \forall j \geq 0$ $\forall t \in [0, T_0]$ $\|y_j'(t)\|_H^2 + \|y_j(t)\|_V^2 \leq \bar{R}^2$. Moreover,

$$\begin{array}{ll} y_j \to y_0 \text{ weakly in } L_2(0, T_0; V), & y_j' \to y_0' \text{ weakly in } L_2(0, T_0; H), \\ y_j \to y_0 \text{ in } L_2(0, T_0; H), & d_j \to d \text{ weakly in } L_2(0, T_0; H), \\ y_j'' \to y_0'' \text{ weakly in } L_2(0, T_0; V^*), \ \forall t \in [0, T_0] & y_j(t) \to y_0(t) \text{ in } H, \ j \to +\infty. \end{array} \tag{2.139}$$

For any $j \geq 0$ and $t \in [0, T_0]$ $I(\varphi_j(t)) = I(\varphi_j(0))e^{-\gamma t} + \int\limits_0^t \mathscr{H}(\varphi_j(s))e^{-\gamma(t-s)}ds$, in particular $I(\varphi_j(T_0)) = I(\varphi_j(0))e^{-\gamma T_0} + \int\limits_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\gamma(T_0-s)}ds$. From (2.139) and Lemma 2.16 we have $\int\limits_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\gamma(T_0-s)}ds \to \int\limits_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\gamma(T_0-s)}ds$, $j \to +\infty$. Therefore, $\overline{\lim\limits_{j\to+\infty}} I(\varphi_j(T_0)) \leq \overline{\lim\limits_{j\to+\infty}} I(\varphi_j(0))e^{-\gamma T_0} + \int\limits_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\gamma(T_0-s)} ds = I(\varphi_0(T_0)) + \left[\overline{\lim\limits_{j\to+\infty}} I(\varphi_j(0)) - I(\varphi_0(0))\right]e^{-\gamma T_0} \leq I(\varphi_0(T_0)) + \bar{\bar{c}}e^{-\gamma T_0}$, where $\bar{\bar{c}}$ does not depend on $T_0 > 0$. On the other hand, from (2.139) we have $\overline{\lim\limits_{j\to+\infty}} I(\varphi_j(T_0)) \geq \frac{1}{2} \overline{\lim\limits_{j\to+\infty}} \|\varphi_j(T_0)\|_E^2 + J(y_0(T_0)) + \frac{\gamma}{2}(y_0'(T_0), y_0(T_0))$. Therefore we obtain: $\frac{1}{2}a^2 \leq \frac{1}{2}\|\xi\|_E^2 + \bar{\bar{c}}e^{-\gamma T_0}$ $\forall T_0 > 0$. Thus, $a \leq \|\xi\|_E$.

The Theorem is proved.

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in E} \mathscr{D}(y_0)$ of all weak solutions of the inclusion (2.120), defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, i.e. $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0$ $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h + s)$, $s \geq 0$. On $\mathscr{K}_+$ we set the *translation semigroup* $\{T(h)\}_{h \geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$ we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \geq 0$.

On $\mathscr{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbf{R}_+; E)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbf{R}_+; E) \iff \forall M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; E),$$

where Π_M is the restriction operator to the interval $[0, M]$ [37, p. 179]. We denote the restriction operator to $[0, +\infty)$ by Π_+.

Let us consider the autonomous inclusion (2.120) on the entire time axis. Similarly to the space $C^{loc}(\mathbf{R}_+; E)$ the space $C^{loc}(\mathbf{R}; E)$ is endowed with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbf{R}$ (cf. [37, p. 180]). A function $u \in C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E)$ is said to be *a complete trajectory* of the inclusion (2.120), if $\forall h \in \mathbf{R}$ $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$ [37, p. 180]. Let $\mathscr{K}$ be a family of *all complete trajectories* of the inclusion (2.120). Note that $\forall h \in \mathbf{R}$, $\forall u(\cdot) \in \mathscr{K}$ $u_h(\cdot) \in \mathscr{K}$. We say that the complete trajectory $\varphi \in \mathscr{K}$ is *stationary* if $\varphi(t) = z$ for all $t \in \mathbf{R}$ for some $z \in E$. Following [4, p.486] we denote the set of rest points of $\mathscr{G}$ by $Z(\mathscr{G})$. We remark that $Z(\mathscr{G}) = \{(\bar{0}, u) \mid u \in V,\ G(u) + \partial J(u) \ni \bar{0}\}$.

From Conditions $H(G)$ and $H(J)$ it follows that

Lemma 2.21. *The set $Z(\mathscr{G})$ is bounded in E.*

From Lemma 2.19 the existence of Lyapunov function (see [4, p. 486]) for $\mathscr{G}$ is follows.

Lemma 2.22. *A functional $\mathscr{V} : E \to \mathbf{R}$, defined by (2.124), is a Lyapunov function for $\mathscr{G}$.*

2.7 Asymptotic Behavior of the Second-Order Evolution Inclusions

Here, we consider at first long-time behavior for state functions of viscoelastic fields that can be described with the second-order evolution inclusion. We can obtain the similar results for piezoelectric fields, analyzing the respective proofs. Therefore, we have a chain of results for global and trajectory attractors, presented in this section.

At first, we remark that the existence of the global attractor for Second-Order Evolution Inclusions and Hemivariational Inequalities considered in Sect. 2.6 (piezoelectric fields) directly follows from Lemmas 2.17, 2.18, 2.21, and 2.22; Theorems 2.5–2.7; and [5, Theorem 2.7].

Theorem 2.8. *The m-semiflow $\mathscr{G}$ has the invariant compact in the phase space E global attractor $\mathscr{A}$. For each $\psi \in \mathscr{K}$, the limit sets*

$$\alpha(\psi) = \{z \in E \mid \psi(t_j) \to z \text{ for some sequence } t_j \to -\infty\},$$
$$\omega(\psi) = \{z \in E \mid \psi(t_j) \to z \text{ for some sequence } t_j \to +\infty\}$$

are connected subsets of $Z(G)$ on which $\mathscr{V}$ is constant. If $Z(G)$ is totally disconnected (in particular, if $Z(G)$ is countable), the limits

$$z_- = \lim_{t \to -\infty} \psi(t), \quad z_+ = \lim_{t \to +\infty} \psi(t)$$

exist and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to +\infty$ for every $\varphi \in \mathscr{K}_+$.

2.7.1 Existence of the Global Attractor

First, we consider constructions presented in [4, 24]. Denote the set of all nonempty (nonempty bounded) subsets of E by $P(E)$ ($\beta(E)$). We recall that the multivalued map $G : \mathbf{R} \times E \to P(E)$ is said to be a *m-semiflow* if:

(a) $G(0, \cdot) = \mathrm{Id}$ (the identity map).

(b) $G(t + s, x) \subset G(t, G(s, x))\ \forall x \in E, t, s \in \mathbf{R}_+$;
m-semiflow is a *strict* one if $G(t + s, x) = G(t, G(s, x))\ \forall x \in E, t, s \in \mathbf{R}_+$.

From Lemmas 2.13 and 2.15, it follows that any weak solution can be extended to a global one defined on $[0, +\infty)$. For an arbitrary $\xi_0 = (a, b)^T \in E$, let $\mathscr{D}(\xi_0)$ consists of pairs of functions $(u(\cdot), u'(\cdot))^T$, defined on $[0, +\infty)$, where $(u(\cdot), u'(\cdot))^T$ is a weak solution (defined on $[0, +\infty)$) of the problem (2.78) with initial data $u(0) = a$, $u'(0) = b$.

We define the semiflow G as $G(t, \xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}$.

Lemma 2.23. *G is the strict m-semiflow.*

Proof. Let $\xi \in G(t + s, \xi_0)$. Then $\xi = \psi(t + s)$, where $\psi(\cdot) \in \mathscr{D}(\xi_0)$. From Lemma 2.14, it follows that $\varphi(\cdot) = \psi(s + \cdot) \in \mathscr{D}(\psi(s))$. Hence, $\xi = \varphi(t) \in G(t, \psi(s)) \subset G(t, G(s, \xi_0))$.

Vice versa, if $\xi \in G(t, G(s, \xi_0))$, then $\exists \psi(\cdot) \in \mathscr{D}(\xi_0), \varphi(\cdot) \in \mathscr{D}(\psi(s))$: $\xi = \varphi(t)$. Define the map

$$\phi(\zeta) = \begin{cases} \psi(\zeta), & \zeta \in [0, s], \\ \varphi(\zeta - s), & \zeta \in [s, t + s]. \end{cases}$$

From Lemma 2.14, it follows that $\phi(\cdot) \in \mathscr{D}(\xi_0)$. Hence, $\xi = \phi(t + s) \in G(t + s, \xi_0)$.

We recall that the set $\mathscr{A}$ is said to be a *global attractor* G, if

1. $\mathscr{A}$ is negatively semiinvariant (i.e., $\mathscr{A} \subset G(t, \mathscr{A})\ \forall t \geq 0$).
2. $\mathscr{A}$ is attracting set, that is,

$$\operatorname{dist}(G(t, B), \mathscr{A}) \to 0, \quad t \to +\infty \quad \forall B \in \beta(E), \tag{2.140}$$

   where $\operatorname{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_E$ is the Hausdorff semidistance.
3. For any closed set $Y \subset H$, satisfying (2.140), we have $\mathscr{A} \subset Y$ (minimality).

The global attractor is said to be *invariant*, if $\mathscr{A} = G(t, \mathscr{A})\ \forall t \geq 0$.
Note that from the definition of the global attractor, it follows that it is unique.
We prove the existence of the invariant compact global attractor.

Theorem 2.9. *The m-semiflow G has the invariant compact in the phase space E global attractor $\mathscr{A}$.*

Proof. From Lemma 2.15, it follows that $\exists R, \bar{\alpha}, \bar{\beta} > 0$:

$$\forall \xi_0 \in E,\ \forall \xi(\cdot) \in \mathscr{D}(\xi_0),\ \forall t \geq 0 \quad \|\xi(t)\|_E^2 \leq \bar{\beta}\|\xi_0\|_E^2 e^{-\tilde{\alpha} t} + \frac{R^2}{2}. \tag{2.141}$$

Thus, the ball $B_0 = \{u \in E \mid \|u\|_E \leq R\}$ is the absorbing set, that is, $\forall B \in \beta(E)$ $\exists T(B) > 0$: $\forall t \geq T(B)\ G(t, B) \subset B_0$. In particular, from (2.141), it follows that the set $\cup_{t \geq 0} G(t, B)$ is bounded one in $E\ \forall B \in \beta(E)$. Note also that from Theorem 2.4, it follows that the map $G(t, \cdot) : E \to \beta(E)$ takes compact values.

The upper semicontinuity of the map $u_0 \to G(t, u_0)$ (cf. [2, Definition 1.4.1, p. 38]) follows from that fact that the given map is compact-valued one and Theorem 2.4 (cf. [18]). In order to do that, it is sufficient to show [3, p. 45] that $\forall \xi_0 \in E,\ \forall \varepsilon > 0\ \exists \delta(\xi_0, \varepsilon) > 0$: $\forall \xi \in B_\delta(\xi_0)\ G(t, \xi) \subset B_\varepsilon(G(t, \xi_0)) = \{z \in E \mid \operatorname{dist}(z, G(t, \xi_0)) < \varepsilon\}$. If it is not true, then there exist $\xi_0 \in E$, $\varepsilon > 0$, $\{\delta_n\}_{n \geq 1} \subset (0, +\infty)$, $\{\xi_n\}_{n \geq 1} \subset E$ such that $\forall n \geq 1\ \xi_n \in B_{\delta_n}(\xi_0)$, $G(t, \xi_n) \not\subset B_\varepsilon(G(t, \xi_0))$ and $\delta_n \to 0$, $n \to +\infty$. Then $\forall n \geq 1\ \exists \zeta_n(\cdot) \in \mathscr{D}(\xi_n)$: $\zeta_n(t) \notin B_\varepsilon(G(t, \xi_0))$. Since $\xi_n \to \xi_0$ in E, $n \to +\infty$, then from Theorem 2.4, it follows that $\zeta_n(t) \to \zeta(t) \in G(t, \xi_0)$ in E, $n \to +\infty$, for some $\zeta(\cdot) \in \mathscr{D}(\xi_0)$. We obtain a contradiction with $\forall n \geq 1\ \|\zeta_n(t) - \zeta(t)\|_E \geq \varepsilon$.

Now, we check the upper asymptotic semicompactness of the m-semiflow G. Let $\xi_n \in G(t_n, v_n)$, $v_n \in B \in \beta(E)$, $n \geq 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check the precompactness of $\{\xi_n\}_{n \geq 1}$ in E. In order to do that without loss of the generality, it is sufficiently to extract a convergent in E subsequence from $\{\xi_n\}_{n \geq 1}$.

From Lemma 2.15 and Theorem 2.4, we obtain that there exist such $\{\xi_{n_k}\}_{k \geq 1}$ and $\xi \in E$ that

$$\xi_{n_k} \to \xi \text{ weakly in } E, \quad \|\xi_{n_k}\|_E \to a \geq \|\xi\|_E, \quad k \to +\infty. \tag{2.142}$$

Show that

$$a \leq \|\xi\|_E. \tag{2.143}$$

Let us fix an arbitrary $T_0 > \sqrt{\lambda_1}$, where $\lambda_1 > 0$ is the constant from (2.86). Then for rather big $k \geq 1$ $G(t_{n_k}, v_{n_k}) \subset G(T_0, G(t_{n_k} - T_0, v_{n_k})) \subset G(T_0, B_0)$. Hence, $\xi_{n_k} \in G(T_0, \beta_{n_k})$, where $\beta_{n_k} \in G(t_{n_k} - T_0, v_{n_k})$ and $\|\beta_{k_j}\|_E \leq R$ $\forall j \geq 1$. From Lemma 2.15, Theorem 2.4, and (2.142) for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\geq 1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\geq 1}$, $\beta_{T_0} \in E$, we obtain: $\forall j \geq 1$

$$\xi \in G(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } E, \quad j \to +\infty. \tag{2.144}$$

From the definition of G, we obtain that $\forall j \geq 1$

$$\xi_{k_j} = \begin{pmatrix} y_j(T_0) \\ y_j'(T_0) \end{pmatrix}, \ \beta_{k_j} = \begin{pmatrix} y_j(0) \\ y_j'(0) \end{pmatrix}, \quad \xi = \begin{pmatrix} y_0(T_0) \\ y_0'(T_0) \end{pmatrix}, \ \beta_{T_0} = \begin{pmatrix} y_0(0) \\ y_0'(0) \end{pmatrix},$$

where $y_j \in C([0, T_0]; V)$: $y_j' \in W_{0,T_0}$ and

$$y_j'' + d_j + B_{0,T_0} y_j = \bar{0}, \quad d_j \in A_{0,T_0}(y_j) - f_{0,T_0}, \ j \geq 0. \tag{2.145}$$

Let us fix an arbitrary $\varepsilon \in (0, \sqrt{\lambda_1})$. From (2.141), we have:

$$\forall j \geq 0, \ \forall t \in [0, T_0] \quad \|y_j'(t)\|_H^2 + \|y_j(t)\|_V^2 \leq R^2(\bar{\beta} + 1/2) =: \bar{R}^2. \tag{2.146}$$

From the proof of Theorem 2.4, we obtain that

$$\begin{aligned}
&\exists C > 0: \quad \|y_j'\|_{X_{0,T_0}} + \|y_j''\|_{X^*_{0,T_0}} + \|d_j\|_{X^*_{0,T_0}} \leq C \quad \forall j \geq 0; \\
&y_j' \to y_0' \text{ in } C([\varepsilon, T_0]; H), \\
&y_j' \to y_0' \text{ weakly in } W_{0,T_0}, \\
&y_j' \to y_0' \text{ in } L_2(0, T_0; H), \\
&d_j \to d_0 \text{ weakly in } X^*_{0,T_0}, \\
&v_j \to v_0 \text{ in } C([0, T_0]; V), \quad j \to +\infty, \\
&\forall j \geq 0, \ \forall t \in [0, T_0] \quad v_j(t) = y_j(t) - y_j(0).
\end{aligned} \tag{2.147}$$

Let us consider the next denotations:

$$Y_j(t) = \frac{1}{2}\left[\|y_j(t)\|_V^2 + \|y_j'(t)\|_H^2\right] + \varepsilon(y_j'(t), y_j(t)), \quad t \in [0, T_0], \ j \geq 0.$$

Then $\forall j \geq 0$ and for a.e. $t \in (0, T_0)$

$$\begin{aligned}\frac{dY_j(t)}{dt} &= -2\varepsilon Y_j(t) + 2\varepsilon\|y_j'(t)\|_H^2 - \langle d_j(t), y_j'(t)\rangle_V - \varepsilon\langle d_j(t), y_j(t)\rangle_V \\ &\quad +2\varepsilon^2(y_j'(t), y_j(t)).\end{aligned}$$

Hence,

$$\frac{d}{dt}\left[Y_j(t)e^{2\varepsilon t}\right] = 2\varepsilon\|y_j'(t)\|_H^2 e^{2\varepsilon t} - \langle d_j(t), y_j'(t)\rangle_V e^{2\varepsilon t}$$
$$-\varepsilon\langle d_j(t), y_j(t)\rangle_V e^{2\varepsilon t}$$
$$+2\varepsilon^2(y_j'(t), y_j(t))^{2\varepsilon t}.$$

Thus, $\forall j \geq 0$

$$\begin{aligned} Y_j(T_0) = Y_j(0)e^{-2\varepsilon T_0} + 2\varepsilon \int_0^{T_0} \|y_j'(t)\|_H^2 e^{-2\varepsilon(T_0-t)} dt \\ - \int_0^{T_0} \langle d_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)} dt \\ -\varepsilon \int_0^{T_0} \langle d_j(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)} dt \\ +2\varepsilon^2 \int_0^{T_0} (y_j'(t), y_j(t)) e^{-2\varepsilon(T_0-t)} dt. \end{aligned} \tag{2.148}$$

From (2.147), for every $j \geq 1$ and for a.e. $t \in (0, T_0)$, we obtain:

$$2\varepsilon \int_0^{T_0} \|y_j'(t)\|_H^2 e^{-2\varepsilon(T_0-t)} dt \to 2\varepsilon \int_0^{T_0} \|y_0'(t)\|_H^2 e^{-2\varepsilon(T_0-t)} dt, \quad j \to +\infty. \tag{2.149}$$

In view of (2.145), $\forall j \geq 0$ and a.e. $t \in (0, T_0)$

$$\langle d_j(t), y_j'(t)\rangle_V = -\frac{1}{2}\frac{d}{dt}\left[\|y_j(t)\|_V^2 + \|y_j'(t)\|_H^2\right]$$
$$= -\frac{1}{2}\frac{d}{dt}\left[\|v_j(t)\|_V^2 + 2\langle B_0 y_j(0), v_j(t)\rangle_V + \|y_j'(t)\|_H^2\right].$$

Taking into account (2.147), we have:

$$\lim_{j\to+\infty} \int_\varepsilon^{T_0} \langle d_j(t), y_j'(t)\rangle_V dt = \int_\varepsilon^{T_0} \langle d_0(t), y_0'(t)\rangle_V dt.$$

Further, following [20, pp. 7–10], from Sect. 2.2 and (2.147), we obtain that

$$\lim_{j\to+\infty}\int_{\varepsilon}^{T_0}|\langle d_j(t), y_j'(t)-y_0'(t)\rangle_V|dt=0,$$

and due to (2.147),

$$\int_{\varepsilon}^{T_0}\langle d_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt\to\int_{\varepsilon}^{T_0}\langle d_0(t), y_0'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt,\quad j\to+\infty. \tag{2.150}$$

From Condition (A_1) and (2.146), we obtain:

$$\forall j\geq 0\quad\left|\int_0^{\varepsilon}\langle d_j(t), y_j'(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt\right|\leq c(1+\bar{R})\bar{R}e^{-2\varepsilon(T_0-\varepsilon)}\varepsilon. \tag{2.151}$$

From Condition (A_3), we have that

$$\forall j\geq 0\ \exists z_j\in L_2(0,T_0;Z^*):\quad d_j(\cdot)=A_1y_j'(\cdot)+z_j(\cdot).$$

Taking into account Condition (A_3), (2.149), and [41, 42], we obtain that $y_j\to y_0$ in $L_2(0,T_0;Z)$, $z_j\to z_0$ weakly in $L_2(0,T_0;Z^*)$, $j\to+\infty$. Therefore,

$$\int_0^{T_0}\langle z_j(t), y_j(t)\rangle_Z e^{-2\varepsilon(T_0-t)}dt\to\int_0^{T_0}\langle z_0(t), y_0(t)\rangle_Z e^{-2\varepsilon(T_0-t)}dt,\ j\to+\infty,$$

$$\begin{aligned}-\int_0^{T_0}\langle A_1y_j'(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt&=-\int_0^{T_0}e^{-2\varepsilon(T_0-t)}\frac{d}{dt}\langle A_1y_j(t), y_j(t)\rangle_V dt\\&=-\langle A_1y_j(T_0), y_j(T_0)\rangle_V+\langle A_1y_j(0), y_j(0)\rangle_V e^{-2\varepsilon T_0}\\&\quad+2\varepsilon\int_0^{T_0}\langle A_1y_j(t), y_j(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt\\&\quad\leq\|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0}+\bar{R}^2).\end{aligned}$$

Thus,

$$\overline{\lim_{j\to+\infty}}\left(-2\varepsilon\int_0^{T_0}\langle d_j(t),y_j(t)\rangle_V e^{-\varepsilon(T_0-t)}dt\right)$$
$$\le -\varepsilon\int_0^{T_0}\langle d_0(t),y_0(t)\rangle_V e^{-2\varepsilon(T_0-t)}dt+\|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0}+2\bar{R}^2)\varepsilon. \tag{2.152}$$

In virtue of (2.90) and (2.146),

$$2\varepsilon^2\left|\int_0^{T_0}(y_j'(t),y_j(t))e^{-2\varepsilon(T_0-t)}dt\right|\le\frac{\varepsilon}{2\sqrt{\lambda_1}}\bar{R}^2. \tag{2.153}$$

Finally, from (2.90) and (2.148)–(2.152), we obtain:

$$\overline{\lim_{j\to+\infty}}Y_j(T_0)\le R^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)e^{-2\varepsilon T_0}$$
$$+Y_0(T_0)+\|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0}+2\bar{R}^2)\varepsilon$$
$$+2c(1+\bar{R})\bar{R}e^{-2\varepsilon(T_0-\varepsilon)}\varepsilon+\frac{\varepsilon}{\sqrt{\lambda_1}}\bar{R}^2.$$

Thus, $\forall\varepsilon\in(0,\sqrt{\lambda_1})$, $\forall T_0>\sqrt{\lambda_1}$

$$\frac{1}{2}a^2\left(1-\frac{\varepsilon}{\sqrt{\lambda_1}}\right)\le R^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)e^{-2\varepsilon T_0}+\frac{1}{2}\|\xi\|_E^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)$$
$$+\|A_1\|_{L(V;V^*)}(R^2e^{-2\varepsilon T_0}+2\bar{R}^2)\varepsilon$$
$$+2c(1+\bar{R})\bar{R}e^{-2\varepsilon(T_0-\varepsilon)}\varepsilon+\frac{\varepsilon}{\sqrt{\lambda_1}}\bar{R}^2.$$

Rushing $T_0\to+\infty$ in the last inequality, we obtain: $\forall\varepsilon\in(0,\sqrt{\lambda_1})$

$$a^2\left(1-\frac{\varepsilon}{\sqrt{\lambda_1}}\right)\le\|\xi\|_E^2\left(1+\frac{\varepsilon}{\sqrt{\lambda_1}}\right)+4\|A_1\|_{L(V;V^*)}\bar{R}^2\varepsilon+\frac{2\varepsilon}{\sqrt{\lambda_1}}\bar{R}^2. \tag{2.154}$$

Passing to the limit (as $\varepsilon\to 0+$) in the inequality (2.154), we obtain (2.143). From (2.142) to (2.143), it follows that $\xi_{n_k}\to\xi$ in E, $k\to+\infty$.

Thus, the existence of the global attractor with required properties directly follows from results from Chap. 1.

2.7.2 Existence of the Trajectory Attractor

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in E} \mathscr{D}(y_0)$ of all weak solutions of the inclusion (2.78), defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, that is, $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0$ $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h+s)$, $s \geq 0$. On $\mathscr{K}_+$, we set the *translation semigroup* $\{T(h)\}_{h\geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$, we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \geq 0$.

We shall construct the attractor of the translation semigroup $\{T(h)\}_{h\geq 0}$, acting on $\mathscr{K}_+$. On $\mathscr{K}_+$, we consider a topology induced from the Fréchet space $C^{loc}(\mathbf{R}_+; E)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbf{R}_+; E) \iff \forall M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; E),$$

where Π_M is the restriction operator to the interval $[0, M]$ [37, p. 179]. We denote the restriction operator to $[0, +\infty)$ by Π_+.

We recall that the set $\mathscr{P} \subset C^{loc}(\mathbf{R}_+; E) \cap L_\infty(\mathbf{R}_+; E)$ is said to be *an attracting one* for the trajectory space $\mathscr{K}_+$ of the inclusion (8) in the topology of $C^{loc}(\mathbf{R}_+; E)$, if for any bounded (in $L_\infty(\mathbf{R}_+; E)$) set $\mathscr{B} \subset \mathscr{K}_+$ and an arbitrary number $M \geq 0$, the next relation

$$\mathrm{dist}_{C([0,M];E)}(\Pi_M T(t)\mathscr{B}, \Pi_M \mathscr{P}) \to 0, \quad t \to +\infty \tag{2.155}$$

holds true.

A set $\mathscr{U} \subset \mathscr{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; E)$ (cf. [37, Definition 1.2, p. 179]), if:

(i) $\mathscr{U}$ is a compact set in $C^{loc}(\mathbf{R}_+; E)$ and bounded one in $L_\infty(\mathbf{R}_+; E)$.
(ii) $\mathscr{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, that is, $T(h)\mathscr{U} = \mathscr{U}$ $\forall h \geq 0$.
(iii) $\mathscr{U}$ is an attracting set in the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbf{R}_+; E)$.

Note that from the definition of the trajectory attractor, it follows that it is unique.

Let us consider the inclusion (8) on the entire time axis. Similarly to the space $C^{loc}(\mathbf{R}_+; E)$, the space $C^{loc}(\mathbf{R}; E)$ is endowed with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbf{R}$ (cf. [37, p. 180]). A function $u \in C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E)$ is said to be *a complete trajectory* of the inclusion (8), if $\forall h \in \mathbf{R}$ $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$ [37, p. 180]. Let $\mathscr{K}$ be a family of all complete trajectories of the inclusion (8). Note that

$$\forall h \in \mathbf{R},\ \forall u(\cdot) \in \mathscr{K} \quad u_h(\cdot) \in \mathscr{K}. \tag{2.156}$$

The existence of the trajectory attractor and its structure properties follow from such theorem.

Theorem 2.10. *Let $\mathscr{A}$ be a global attractor from Theorem 2.9. Then there exists the trajectory attractor $\mathscr{P} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$. At that, the next formula takes place*

$$\mathscr{P} = \Pi_+ \mathscr{K} = \Pi_+\{y \in \mathscr{K} \mid y(t) \in \mathscr{A}\ \forall t \in \mathbf{R}\}, \tag{2.157}$$

Proof. The proof repeats the proof of corresponding statement from work [18] and, it is based on results of Theorems 2.4, 2.9.

First, we prove some auxiliary lemmas.

Lemma 2.24. *The set $\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}; E)$, and bounded one in $L_\infty(\mathbf{R}; E)$. Moreover,*

$$\forall \xi(\cdot) \in \mathscr{K}, \; \forall t \in \mathbf{R} \quad \xi(t) \in \mathscr{A}, \tag{2.158}$$

where $\mathscr{A}$ is the global attractor from Theorem 2.9.

Proof.

Step 1. Let us show that $\mathscr{K} \neq \emptyset$. Note that in view of conditions (B_1), (B_2), (H_1), it follows that $\exists v \in V$: $B_0(v) = f_0$. We set $u(t) = v \; \forall t \in \mathbf{R}$. Then $(u, u')^T \in \mathscr{K} \neq \emptyset$.

Step 2. Let us prove (2.158). For any $y \in \mathscr{K}$ $\exists d > 0$: $\|y(t)\|_E \le d \; \forall t \in \mathbf{R}$. We set $B = \cup_{t \in \mathbf{R}} \{y(t)\} \in \beta(E)$. Note that $\forall \tau \in \mathbf{R}$, $\forall t \in \mathbf{R}_+$ $y(\tau) = y_{\tau - t}(t) \in G(t, y_{\tau - t}(0)) \subset G(t, B)$. From Theorem 2.9 and from (2.140), it follows that $\forall \varepsilon > 0 \; \exists T > 0$: $\forall \tau \in \mathbf{R}$ $\operatorname{dist}(y(\tau), \mathscr{A}) \le \operatorname{dist}(G(T, B), \mathscr{A}) < \varepsilon$. Hence, taking into account the compactness of $\mathscr{A}$ in E, for any $u(\cdot) \in \mathscr{K}$, $\tau \in \mathbf{R}$, it follows that $u(\tau) \in \mathscr{A}$.

Step 3. The boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}_+; E)$ follows from (2.158) and the boundedness of $\mathscr{A}$ in E.

Step 4. Let us check the compactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; E)$. In order to do that, it is sufficient to check the precompactness and completeness.

Step 4.1. Let us check the precompactness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; E)$. If it is not true, then in view of (2.156), $\exists M > 0$: $\Pi_M \mathscr{K}$ is not precompact set in $C([0, M]; E)$. Hence, there exists a sequence $\{v_n\}_{n \ge 1} \subset \Pi_M \mathscr{K}$ that has not a convergent subsequence in $C([0, M]; E)$. On the other hand, $v_n = \Pi_M u_n$, where $u_n \in \mathscr{K}$, $v_n(0) = u_n(0) \in \mathscr{A}$, $n \ge 1$. Since $\mathscr{A}$ is compact set in E (see Theorem 2.9), then in view of Theorem 2.4, $\exists \{v_{n_k}\}_{k \ge 1} \subset \{v_n\}_{n \ge 1}$, $\exists \eta \in E$, $\exists v(\cdot) \in \mathscr{D}_{0,M}(\eta)$: $v_{n_k}(0) \to \eta$ in E, $v_{n_k} \to v$ in $C([0, T]; E)$, $k \to +\infty$. We obtained a contradiction.

Step 4.2. Let us check the completeness of $\mathscr{K}$ in $C^{loc}(\mathbf{R}; E)$. Let $\{v_n\}_{n \ge 1} \subset \mathscr{K}$, $v \in C^{loc}(\mathbf{R}; E)$: $v_n \to v$ in $C^{loc}(\mathbf{R}; E)$, $n \to +\infty$. From the boundedness of $\mathscr{K}$ in $L_\infty(\mathbf{R}; E)$, it follows that $v \in L_\infty(\mathbf{R}; E)$. From Theorem 2.9, we have that $\forall M > 0$ the restriction $v(\cdot)$ to the interval $[-M, M]$ belongs to $\mathscr{D}_{-M,M}(v(-T))$. Therefore, $v(\cdot)$ is the complete trajectory of the inclusion (8). Thus, $v \in \mathscr{K}$.

Lemma 2.25. *Let $\mathscr{A}$ be a global attractor from Theorem 2.9. Then*

$$\forall y_0 \in \mathscr{A} \quad \exists y(\cdot) \in \mathscr{K}: \quad y(0) = y_0. \tag{2.159}$$

Proof. Let $y_0 \in \mathscr{A}$, $u(\cdot) \in \mathscr{D}(y_0)$. From (2.85) and (2.140), we obtain that $\forall t \in \mathbf{R}_+$ $y(t) \in \mathscr{A}$. From Theorem 2.9, it follows that $G(1, \mathscr{A}) = \mathscr{A}$. Therefore,

$$\forall \eta \in \mathscr{A} \quad \exists \xi \in \mathscr{A}, \; \exists \varphi_\eta(\cdot) \in \mathscr{D}_{0,1}(\xi): \quad \varphi_\eta(1) = \eta.$$

For any $t \in \mathbf{R}$, we set

$$y(t) = \begin{cases} u(t), & t \in \mathbf{R}_+, \\ \varphi_{y(-k+1)}(t+k), & t \in [-k, -k+1),\ k \in \mathbf{N}. \end{cases}$$

Note that $y \in C^{loc}(\mathbf{R}; E)$, $y(t) \in \mathscr{A}$ $\forall t \in \mathbf{R}$ (hence, $y \in L_\infty(\mathbf{R}; E)$), and in view of Lemma 2.14, $y \in \mathscr{K}$. At that, $y(0) = y_0$.

Let us continue the proof of the theorem. From Lemma 2.24 and the continuity of the operator $\Pi_+ : C^{loc}(\mathbf{R}; E) \to C^{loc}(\mathbf{R}_+; E)$, it follows that the set $\Pi_+\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbf{R}_+; E)$ and bounded one in $L_\infty(\mathbf{R}_+; E)$. Moreover, the second equality in (2.157) holds true. The strict invariance of $\Pi_+\mathscr{K}$ follows from the autonomy of the inclusion (8).

Let us prove that $\Pi_+\mathscr{K}$ is the attracting set for the trajectory space $\mathscr{K}_+$ in the topology of $C^{loc}(\mathbf{R}_+; E)$. Let $B \subset \mathscr{K}_+$ be a bounded set in $L_\infty(\mathbf{R}_+; E)$, $M \geq 0$. Let us check (2.155). If it is not true, then there exist sequences $t_n \to +\infty$, $v_n(\cdot) \in B$ such that

$$\forall n \geq 1 \quad \mathrm{dist}_{C([0,T];E)}(\Pi_M v_n(t_n + \cdot), \Pi_M \mathscr{K}) \geq \varepsilon. \tag{2.160}$$

On the other hand, from the boundedness of B in $L_\infty(\mathbf{R}_+; E)$, it follows that $\exists R > 0$: $\forall v(\cdot) \in B$, $\forall t \in \mathbf{R}_+$ $\|v(t)\|_E \leq R$. Thus, $\exists N \geq 1$: $\forall n \geq N$ $v_n(t_n) \in G(t_n, v_n(0)) \subset G(1, G(t_n - 1, v_n(0))) \subset G(1, \overline{B_R})$, where $\overline{B_R} = \{u \in E \mid \|u\|_E \leq R\}$. Hence, taking into account (2.140) and the asymptotic semicompactness of m-semiflow G (see the proof of Theorem 2.9), we obtain that $\exists \{v_{n_k}(t_{n_k})\}_{k\geq 1} \subset \{v_n(t_n)\}_{n\geq 1}$, $\exists z \in \mathscr{A}$: $v_{n_k}(t_{n_k}) \to z$ in E, $k \to +\infty$. Further, $\forall k \geq 1$, we set $\varphi_k(t) = v_{n_k}(t_{n_k} + t)$, $t \in [0, M]$. Note that $\forall k \geq 1$ $\varphi_k(\cdot) \in \mathscr{D}_{0,M}(v_{n_k}(t_{n_k}))$. Then from Theorem 2.4, there exists a subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_k\}_{k\geq 1}$ and an element $\varphi(\cdot) \in \mathscr{D}_{0,M}(z)$:

$$\varphi_{k_j} \to \varphi \text{ in } C([0, M]; E), \quad j \to +\infty. \tag{2.161}$$

At that, taking into account the invariance of $\mathscr{A}$ (see Theorem 2.9), $\forall t \in [0, M]$ $\varphi(t) \in \mathscr{A}$. In consequence of Lemma 2.25, there exist $y(\cdot), v(\cdot) \in \mathscr{K}$: $y(0) = z$, $v(0) = \varphi(M)$. For any $t \in \mathbf{R}$, we set

$$\psi(t) = \begin{cases} y(t), & t \leq 0, \\ \varphi(t), & t \in [0, M], \\ v(t - M), & t \geq M. \end{cases}$$

In view of Lemma 2.14, $\psi(\cdot) \in \mathscr{K}$. Therefore, from (2.160), we obtain:

$$\forall k \geq 1 \quad \|\Pi_M v_{n_k}(t_{n_k} + \cdot) - \Pi_M \psi(\cdot)\|_{C([0,M];E)} = \|\varphi_k - \varphi\|_{C([0,M];E)} \geq \varepsilon,$$

that contradicts with (2.161).

Thus, the set $\mathscr{P}$ constructed in (2.157) is the trajectory attractor in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbf{R}_+; E)$.

2.7.3 Auxiliary Properties of the Global and Trajectory Attractors

Let $\mathscr{A}$ be a global attractor from Theorem 2.9 and $\mathscr{P}$ be a trajectory attractor from Theorem 2.10.

$$\mathscr{A} \text{ is a compact in the space } E \tag{2.162}$$

$$\mathscr{P} \text{ is a compact in the space } C^{loc}(\mathbf{R}_+; E) \tag{2.163}$$

Moreover,

$$\mathscr{P} = \Pi_+ \mathscr{K} = \Pi_+ \{y \in \mathscr{K} \mid y(t) \in \mathscr{A} \ \forall t \in \mathbf{R}\}, \tag{2.164}$$

where $\mathscr{K}$ is the family of all complete trajectories of the inclusion (8) and Π_+ is the restriction operator on $\mathbf{R}_+$. Note that from Lemma 2.24, it follows that $\mathscr{K} \neq \emptyset$;

$$\mathscr{K} \text{ is a compact in the space } C^{loc}(\mathbf{R}; E); \tag{2.165}$$

$$\forall \xi(\cdot) \in \mathscr{K} \ \forall t \in \mathbf{R} \ \xi(t) \in \mathscr{A}; \tag{2.166}$$

$$\forall y_0 \in \mathscr{K} \ \forall t_0 \in \mathbf{R} \ \exists y(\cdot) \in \mathscr{K} : \ y(t_0) = y_0. \tag{2.167}$$

2.7.3.1 "Translation Compactness" of the Trajectory Attractor

For any $y \in \mathscr{K}$, let us set

$$\mathscr{H}(y) = \mathrm{cl}_{C^{loc}(\mathbf{R};E)}\{y(\cdot + s) \mid s \in \mathbf{R}\} \subset C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E).$$

Such family is said to be a hull of function $y(\cdot)$ in $\Xi = C^{loc}(\mathbf{R}; E)$.

Definition 2.3. The function $y(\cdot) \in \Xi$ is said to be translation compact (tr.-c.) in Ξ if the hull $\mathscr{H}(y)$ is compact in Ξ.

Similar constructions for the set of functional parameters that are responsible for nonautonomy of evolution equation are considered, for example, in [9, p. 917].

Definition 2.4. The family $\mathscr{U} \in \Xi$ is said to be translation compact, if $\mathscr{H}(\mathscr{U}) = \mathrm{cl}_\Xi\{y(\cdot + s) \mid \ y(\cdot) \in \mathscr{U}, s \in \mathbf{R}\}$ is a compact in Ξ.

From autonomy of system (8), applying the Arzelá-Ascoli compactness criterion (see the proof of Proposition 6.1 from [9]), we obtain the translation compactness criterion for the family $\mathscr{U}$:

(a) The set $\{y(t) \mid \ t \in \mathbf{R}, \ \ y \in \mathscr{U}\}$ is a compact in E.
(b) There exists a positive function $\alpha(s) \to 0+$ $(s \to 0+)$ such that

$$\|y(t_1) - y(t_2)\|_E \leq \alpha(|t_1 - t_2|) \ \forall y \in \mathscr{U} \ \ \forall t_1, t_2 \in \mathbf{R}.$$

From the autonomy of problem (8) and (2.165), it follows that

Corollary 2.3. *$\mathscr{K}$ is translation-compact set in Ξ.*

Similarly, if we set $\Xi_+ = C^{loc}(\mathbf{R}_+; E)$ we obtain

Corollary 2.4. *$\mathscr{P}$ is translation-compact set in Ξ_+.*

2.7.3.2 Stability

Definition 2.5. [4, p. 487] The subset $\mathscr{A} \subset E$ is *Lyapunov stable* if for given $\varepsilon > 0$, there exists such $\delta > 0$ that if $D \subset E$ with $\mathrm{dist}(D, \mathscr{A}) < \delta$, then $\mathrm{dist}(T(t)D, \mathscr{A}) < \varepsilon$ for all $t \geq 0$.

We recall that (see [4, p.481])

$$T(t)D = \{\varphi(t) |\ \varphi(\cdot) \in \mathscr{D}(\varphi_0),\ \varphi_0 \in D\}.$$

Note also that

$$G(t, z) = T(t)\{z\} \quad \forall t \geq 0,\ \forall z \in E.$$

From [4, p. 487], it follows that a subset $\mathscr{A}$ is Lyapunov stable if and only if the given $\{\varphi_j(\cdot)\}_{j\geq 1}$ is a sequence of weak solutions (defined on $[0, +\infty)$) of problem (8) with $\mathrm{dist}(\varphi_j(0), \mathscr{A}) \to 0$, $j \to +\infty$ and $t_j \geq 0$ we have $\mathrm{dist}(\varphi_j(t_j), \mathscr{A}) \to 0$, $j \to +\infty$.

Definition 2.6. [4, p. 487] The subset $\mathscr{A}$ is uniformly asymptotically stable if $\mathscr{A}$ is Lyapunov stable and it is locally attracting (see [4, p. 482]).

Note that an attracting set is locally attracting one.

Corollary 2.5. *$\mathscr{A}$ is uniformly asymptotically stable.*

Proof. The proof follows from the definition of G, [4, Theorem 6.1], properties of solutions from Lemma 2.13, Theorem 2.4, and from the autonomy of problem (8).

Similar results are true for sets $\mathscr{P}$ and $\mathscr{K}$ in corresponding extended phase spaces.

2.7.3.3 Connectedness

Definition 2.7. [4, p. 485] M-semiflow G has Kneser's property, if $G(t, z)$ is connected for each $z \in E$, $t \geq 0$.

Corollary 2.6. *If G has Kneser's property, then $\mathscr{A}$ is connected.*

Proof. The proof follows from [4, Corollary 4.3], Lemma 2.13, and from the connectedness of the phase space E.

Note that the connectedness of G can be checked by different way (see, e.g., [34–36]). In order to do that, as a rule, it is required an auxiliary regularity of interaction functions. In the general case, Kneser's property for problem (8) can be checked using the method of proof from [36, Theorem 5], where we can consider Yosida approximation instead the proposed approximation.

Corollary 2.7. *If G has Kneser's property, then $\mathscr{K} \subset C^{loc}(\mathbf{R}; E)$ is connected and $\mathscr{P} \subset C^{loc}(\mathbf{R}_+; E)$ is connected.*

Proof. The proof is follows from (2.164) to (2.167) and from Corollary 2.6.

2.7.3.4 Behavior of Solutions on the Global Attractor

We say that the complete trajectory $\varphi \in \mathscr{K}$ is stationary if $\varphi(t) = z$ for all $t \in \mathbf{R}$ for some $z \in E$.

Following [4, p. 486], we denote the set of rest points of G by $Z(G)$.

Note that
$$Z(G) = \{(z, \bar{0}) |\ z \in B_0^{-1}(f - A_0(\bar{0}))\}.$$
Thus, $Z(G)$ is a convex, nonempty, weakly compact in $V \times V$ set.

For investigating of trajectory behavior of solutions on the attractor $\mathscr{A}$, it is necessary to consider similar definitions to [4, p. 486]:

Definition 2.8. A functional $\mathscr{V} : \mathscr{A} \to \mathbf{R}$ is a Lyapunov function for G on $\mathscr{A}$ provided

(i) $\mathscr{V}$ is continuous.
(ii) $\mathscr{V}(\varphi(t)) \le \mathscr{V}(\varphi(s))$ whenever $\varphi \in \mathscr{K}$ and $t \ge s \ge 0$.
(iii) If $\mathscr{V}(\psi(t)) = constant$ for some $\psi \in \mathscr{K}$ and all $t \in \mathbf{R}$.

Then, ψ is stationary.

As a consequence of Theorem in the presence of a Lyapunov function, the behavior of such complete orbits can be characterized.

Theorem 2.11. *Suppose that there exists a Lyapunov function $\mathscr{V}$ for G on $\mathscr{A}$. Then for each $\psi \in \mathscr{K}$, the limit sets*

$$\alpha(\psi) = \{z \in E |\ \psi(t_j) \to z \textit{ for some sequence } t_j \to -\infty\},$$

$$\omega(\psi) = \{z \in E |\ \psi(t_j) \to z \textit{ for some sequence } t_j \to +\infty\}$$

are connected subsets of $Z(G)$ on which $\mathscr{V}$ is constant.

Proof. The proof follows from the proof of [4, Theorem 5.1], asymptotic compactness of G and from properties of solutions of problem (8).

2.7.3.5 The Estimate of the Fractal Dimension

As a rule, the finite dimension of the global attractor demands an auxiliary differentiability in some sense by initial data from the m-semiflow G that for one's turn involves an auxiliary regularity of interaction functions (in the case of problem (3)–(7), it involves an auxiliary regularity of functional j).

Let us show that, generally speaking, for problem (3)–(7), the fractal dimension of the attractor $\mathscr{A}$ can be equal to $+\infty$. In order to show that, we consider a particular case of problem (3)–(7). Let $N = 2$, $\Omega = (0,1) \times (0,\pi)$, $\Gamma_C = \{(x_1, x_2)|\ x_1 = 1, x_2 \in (0,\pi)\}$, $\Gamma_D = \partial\Omega \backslash \Gamma_C$.

First, we consider the auxiliary problem

$$\begin{cases} \triangle y = 0 \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma_D, \\ \frac{\partial y}{\partial x_1} \in [-1, 1] \text{ on } \Gamma_C. \end{cases} \tag{2.168}$$

For each $n \in \mathbf{N}$, let us set

$$y_n(x_1, x_2) = \frac{1}{n \cdot \cosh(n)} \sinh(n \cdot x_1) \cdot \sin(n \cdot x_2),\ \ (x_1, x_2) \in \bar{\Omega}.$$

Then $\forall c \in [-1, 1]$ $\forall n \in \mathbf{N}$ $c \cdot y_n(\cdot)$ is a solution of (2.168).

Note that $\forall n \neq m$

$$(y_n, y_m)_{L_2(\Omega)} = 0,$$

$$\forall n \geq 1\ \|y_n\|^2_{L_2(\Omega)} = \frac{\pi}{4n^3} \cdot \frac{1 - e^{-4n} - 4ne^{-2n}}{1 + e^{-4n} + 2e^{-2n}} \geq \frac{\bar{\alpha^*}^2}{n^3},$$

where $\bar{\alpha^*}$ does not depend on $n \in \mathbf{N}$.

In this case, if we set

$$z_n(\cdot) = \frac{\bar{\alpha^*}}{\|y_n\|_{L_2(\Omega)} \cdot n^{\frac{3}{2}}} \cdot y_n(\cdot),\ \ n \geq 1,$$

we obtain that the set

$$K = \{y \in L_2(\Omega)|\ y(x_1, x_2) = \sum_{k=1}^{\infty} \alpha_k z_k(x_1, x_2),\ \ (x_1, x_2) \in \Omega,\ \sum_{k=1}^{\infty} |\alpha_k| = 1\}$$

consists of solutions of problem (2.168).

For $N \geq 1$, we set $\varepsilon_N = \frac{\bar{\alpha^*}}{2N^{\frac{3}{2}+1}}$, $M(\varepsilon_N)$ is a minimal quantity of balls with radius ε_N, by the help of which we can cover K. Then

$$M(\varepsilon_N) \geq \bar{C} \cdot \frac{N^{\frac{3N}{2}}}{(N!)^3}$$

and

$$\forall d > 0\ \ \overline{\lim_{N \to +\infty}} M(\varepsilon_N)\varepsilon_N^d = +\infty.$$

Therefore, the fractal dimension of the set K in the space $L_2(\Omega)$ as well as in the space $H^1(\Omega)$ is equal to $+\infty$.

Thus, the fractal dimension of the global attractor $\mathscr{A} \supset Z(G) \supset K \times K \times \{0\} \times \{0\}$ in the space E for the m-semiflow constructed on solutions of problem (3)–(7) in the case when $N = 2$,

$$B_0((y_1, y_2)^T) = (-\triangle y_1, -\triangle y_2),$$

Ω, Γ_D, Γ_C as in (2.168), $\partial j(x, 0) = [-1, 1]^2$ is equal to $+\infty$. Thus, we can see that the dimension of the attractor in the given case sufficiently depends on the differentiability of the functional $j(x, u)$ for $u = 0$.

2.8 Applications

As applications, we can consider new classes of problems with degenerations, problems on a manifold, problems with delay, stochastic partial differential equations, etc., [2–5, 7, 9–39, 41, 42] with differential operators of pseudomonotone type as corresponding choice of the phase space. Let us consider some particular classes of examples, when we can obtain stronger results for resolving operator.

2.8.1 Climate Energy Balance Model

We now consider a climate energy balance model (see Example 4). The problem is the following:

$$\begin{array}{ll} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + Bu \in QS(x)\beta(u) + h(x), & (t, x) \in \mathbf{R}_+ \times (-1, 1), \\ u_x(-1, t) = u_x(1, t) = 0, & t \in \mathbf{R}_+, \\ u(x, 0) = u_0(x), & x \in (-1, 1), \end{array} \tag{2.169}$$

where B and Q are positive constants, $S, h \in L_\infty(-1, 1)$, $u_0 \in L_2(-1, 1)$, and β is a maximal monotone graph in $\mathbf{R}^2$, which is bounded, that is, there exist $m, M \in \mathbf{R}$ such that

$$m \le z \le M, \quad \text{for all } z \in \beta(s), s \in \mathbf{R}. \tag{2.170}$$

We also assume that

$$0 < S_0 \le S(x) \le S_1, \quad \text{a.e. } x \in (-1, 1). \tag{2.171}$$

The unknown $u(t, x)$ represents the averaged temperature of the Earth surface, Q is the so-called solar constant, which is the average (over a year and over the surface of the Earth) value of the incoming solar radiative flux, and the function $S(x)$ is the insolation function given by the distribution of incident solar radiation at the top of the atmosphere. When the averaging time is of the order of 1 year or longer, the function $S(x)$ satisfies (2.171); for shorter periods, we must assume that $S_0 = 0$. The term β represents the so-called co-albedo function, which can be possibly discontinuous. It represents the ratio between the absorbed solar energy and the incident solar energy at the point x on the Earth surface. Obviously, $\beta(u(x,t))$ depends on the nature of the Earth surface. For instance, it is well known that on ice sheets, $\beta(u(x,t))$ is much smaller than on the ocean surface because the white color of the ice sheets reflects a large portion of the incident solar energy, whereas the ocean, due to its dark color and high heat capacity, is able to absorb a larger amount of the incident solar energy. We point out that this model is the particular case of the first-order evolution inclusion, considered in Sects. 2.3 and 2.4. All results from this subsection are fulfilled for state function of this problem.

2.8.2 Application for General Classes High-Order Nonlinear PDEs

Consider an example of the class of nonlinear boundary-value problems for which we can investigate the dynamics of solutions as $t \to +\infty$. Note that in discussion, we do not claim generality.

Let $n \geq 2$, $m \geq 1$, $p \geq 2$, $1 < q \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbf{R}^n$ be a bounded domain with rather smooth boundary $\Gamma = \partial\Omega$. We denote a number of differentiations by x of order $\leq m-1$ (correspondingly of order $= m$) by N_1 (correspondingly by N_2). Let $A_\alpha(x,\eta;\xi)$ be a family of real functions ($|\alpha| \leq m$), defined in $\Omega \times \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$ and satisfying the next properties:

(C_1) For a.e. $x \in \Omega$ the function $(\eta,\xi) \to A_\alpha(x,\eta,\xi)$ is continuous one in $\mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$.

(C_2) $\forall(\eta,\xi) \in \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$ the function $x \to A_\alpha(x,\eta,\xi)$ is measurable one in Ω.

(C_3) Exist such $c_1 \geq 0$ and $k_1 \in L_q(\Omega)$ that for a.e. $x \in \Omega$, $\forall(\eta,\xi) \in \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$

$$|A_\alpha(x,\eta,\xi)| \leq c_1[|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)].$$

(C_4) Exist such $c_2 > 0$ and $k_2 \in L_1(\Omega)$ that for a.e. $x \in \Omega$, $\forall(\eta,\xi) \in \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$

$$\sum_{|\alpha|=m} A_\alpha(x,\eta,\xi)\xi_\alpha \geq c_2|\xi|^p - k_2(x).$$

(C_5) For a.e. $x \in \Omega$, $\forall\eta \in \mathbf{R}^{N_1}$, $\forall\xi,\xi^* \in \mathbf{R}^{N_2}$, $\xi \neq \xi^*$, the inequality

$$\sum_{|\alpha|=m} (A_\alpha(x,\eta,\xi) - A_\alpha(x,\eta,\xi^*))(\xi_\alpha - \xi_\alpha^*) > 0$$

takes place.

Consider such denotations: $D^k u = \{D^\beta u, |\beta| = k\}$, $\delta u = \{u, Du, \ldots, D^{m-1}u\}$ (see [22, c. 194]).

For an arbitrary fixed interior force $f \in L_2(\Omega)$, we investigate the dynamics of all weak (generalized) solutions defined on $[0, +\infty)$ of such problem:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta y(x,t), D^m y(x,t)) \right) = f(x) \text{ on } \Omega \times (0, +\infty), \tag{2.172}$$

$$D^\alpha y(x,t) = 0 \text{ on } \Gamma \times (0, +\infty), \quad |\alpha| \leq m - 1. \tag{2.173}$$

as $t \to +\infty$.

Consider such denotations (see for detail [22, c. 195]): $H = L_2(\Omega)$, $V = W_0^{m,p}(\Omega)$ is a real Sobolev space,

$$a(u, \omega) = \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, \delta u(x), D^m u(x)) D^\alpha \omega(x) dx, \quad u, \omega \in V.$$

Note that Condition (H_2) takes place in view of Sobolev theorem on compactness of embedding. Taking into account Conditions (C_1)–(C_5) and [22, p. 192–199], the operator $A : V \to V^*$, defined by the formula $\langle A(u), \omega \rangle_V = a(u, \omega)$ $\forall u, \omega \in V$, satisfies Conditions (A_1)–(A_3). Hence, we can pass from problem (2.172)–(2.173) to corresponding problem in "generalized" setting (6.5). Note that

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta u, D^m u) \right) \quad \forall u \in C_0^\infty(\Omega).$$

Therefore, all statements from previous subsections, in particular, Theorems 2.1–2.3 and Lemmas 2.5–2.11, are fulfilled for weak (generalized) solutions of problem (2.172)–(2.173).

Remark 2.4. As applications, we can also consider new classes of problems with degenerations, problems on a manifold, problems with delay, stochastic partial differential equations, etc. [10, 14, 22, 32], with differential operators of pseudomonotone type as corresponding choice of the phase space.

2.8.3 Application for Chemotaxis Processes

Let us consider the problem from Example 5. This problem connected with the movement of biological cells or organisms in response to chemical gradients. If properly interpreting the derivative and correspondingly choosing phase spaces, all models can be reduced to the first-order autonomous evolution equation. For example, let us consider a particular case and examine asymptotical behavior of solutions. We consider the problem that described by the following initial-boundary problem for a quasi-linear parabolic system of equations

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = b\Delta\sigma - c\rho + du \quad \text{in} \quad \Omega \times (0,\infty), \\ &\frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ &u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) \quad \text{in} \quad \Omega. \end{aligned} \tag{2.174}$$

Here, $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance at a position $x \in \Omega \subset \mathbf{R}^2$ and a time $t \in [0,\infty)$, respectively. The mobility of individuals consists of two effects: one is random walking and the other is the directed movement in a sense that they have a tendency to move toward higher concentration of the chemical substance. This is called chemotaxis in biology [1,7,13,27]. $a > 0$ and $b > 0$ are the diffusion rates of u and ρ, respectively. $c > 0$ and $d > 0$ are the degradation and production rates of ρ, respectively. $\chi(\rho)$ is the sensitivity function due to chemotaxis. It is a real smooth function of $\rho \in [0,\infty)$ with uniformly bounded derivatives up to the third-order

$$\sup_{\rho \geq 0} \left| \frac{d^i \chi}{d\rho^i}(\rho) \right| < \infty \quad \text{for} \quad i = 1,2,3. \tag{2.175}$$

$f(u)$ is a growth term of u. It is a real smooth function of $u \in [0,\infty)$ such that $f(0) = 0$ and

$$f(u) = (-\mu u + \nu)u \quad \text{for sufficiently large } u \tag{2.176}$$

with $\mu > 0$ and $-\infty < \nu < \infty$. Let $f(u) = f_1(u)u$, then $f_1(u)$ is a smooth function of $u \in [0,\infty)$ such that $f_1(u) = -\mu u + \nu$ for sufficiently large u.

For the abstract setting of the problem, we set the product space $H = L_2(\Omega) \times H^1(\Omega)$, and consider (2.174) as an initial value problem of an evolution equation

$$\begin{aligned} &\frac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ &U(0) = U_0 \end{aligned} \tag{2.177}$$

in H. Here, A and $F(U)$ are defined by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{with} \quad \mathscr{D}(A) = H_N^2(\Omega) \times H_N^3(\Omega),$$

where $A_1 = -a\triangle + 1$ and $A_2 = -b\triangle + c$, and

$$F(U) = \begin{pmatrix} -\nabla\{u\nabla\chi(\rho)\} + \{1 + f_1(u)\}u \\ du \end{pmatrix}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathscr{D}(A).$$

The set of initial values is set by

$$K = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in L_2(\Omega) \times H^{1+\varepsilon_0}(\Omega) : u_0 \geq 0,\ \rho_0 \geq 0 \right\}$$

where $0 < \varepsilon_0 < \frac{1}{2}$ is some fixed exponent and $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix}$ is in K.

In [29], it is proved that there exists a unique global solution $U = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.177) and that the solution is continuous with respect to the initial value. Therefore, a continuous semigroup $\{S(t)\}_{t\geq 0}$ can be defined on K by $S(t)U_0 = U(t)$. For $t > 0$, $S(t)$ maps K into $K \cap \mathscr{A}$.

Proposition 2.6. *[29] There exists a universal constant $C > 0$ such that the following statement holds for each bounded ball $B_r = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in K : \|u_0\|_{L_2} + \|\rho_0\|_{H^{1+\varepsilon_0}} \leq r \right\}$, there exists a time $t_r > 0$ depending on B_r such that*

$$\sup_{t\geq t_r} \sup_{U_0 \in B_r} \|S(t)U_0\|_{H^2\times H^3} \leq C.$$

The set $\mathscr{B} = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix} \in H^2(\Omega) \times H^3(\Omega);\ \|u\|_{H^2} + \|\rho\|_{H^3} \leq C \right\} \cap K$, where C is the constant appearing in Proposition 2.6, is a compact absorbing set for $(\{S(t)\}_{t\geq 0}, K)$. Hence, by virtue of [33, Chap. 1, Theorem 1.1], there exists a global attractor $\mathscr{A} \subset K$, $\mathscr{A}$ being a compact and connected subset of K.

We set

$$\mathscr{X} = \overline{\bigcup_{t\geq t_{\mathscr{B}}} S(t)\mathscr{B}} \quad \text{(closure in the topology of} \quad K)$$

using a time $t_{\mathscr{B}}$ such that $S(t)\mathscr{B} \subset \mathscr{B}$ for every $t \geq t_{\mathscr{B}}$. We note that $\mathscr{X}$ is a compact set of K with the relation $\mathscr{A} \subset \mathscr{X} \subset \mathscr{B}$ and is absorbing and positively invariant for $\{S(t)\}_{t\geq 0}$.

Definition 2.9. A subset $\mathscr{M} \subset \mathscr{X}$ is called the *exponential attractor* for $(\{S(t)\}_{t\geq 0}, \mathscr{X})$ if (i) $\mathscr{A} \subset \mathscr{M} \subset \mathscr{X}$, (ii) $\mathscr{M}$ is a compact subset of H and is a positively invariant set for $S(t)$, (iii) $\mathscr{M}$ has finite fractal dimension $d_F(\mathscr{M})$, and (iv) $h(S(t)\mathscr{X}, \mathscr{M}) \leq c_0 exp(-c_1 t)$ for $t \geq 0$ with some constants $c_0, c_1 > 0$, where

$$h(B_0, B_1) = \sup_{U\in B_0} \inf_{V\in B_1} \|U - V\|_H$$

denotes the Hausdorf pseudodistance of two sets B_0 and B_1.

Now, it is sufficiently to apply the next theorem to the dynamical system $(\{S(t)\}_{t\geq 0}, \mathscr{X})$.

Theorem 2.12. *[29] Let $F(U)$ satisfy the Lipschitz condition*

$$\|F(U) - F(V)\|_H \leq C\|A^{1/2}(U - V)\|_H, \quad U, V \in \mathscr{X} \tag{2.178}$$

and let the mapping $G(t, U_0) = S(t)U_0$ from $[0, T]\times\mathscr{X}$ into $\mathscr{X}$ satisfy the Lipschitz condition

$$\|G(t, U_0) - G(s, V_0)\|_H \leq C_T\{|t - s| + \|U_0 - V_0\|_H\}, \quad t, s \in [0, T], U_0, V_0 \in \mathscr{X} \tag{2.179}$$

for each $T > 0$.

Then there exists an exponential attractor $\mathscr{M}$ for $(\{S(t)\}_{t\geq 0}, \mathscr{X})$.

Thus, we arrive at the main result. This result is borrowed from [29].

Theorem 2.13. *There exists an exponential attractor $\mathscr{M}$ of the dynamical system $(\{S(t)\}_{t\geq 0}, \mathscr{X})$ in H.*

Remark 2.5. For simplicity, we have assumed the condition (2.176) on $f(u)$. But this is not essential; indeed, the theorem can be proved under the conditions that

$$-\eta'(u^m + 1)u \leq f(u) \leq (-\mu' u + \nu')u, \quad u \geq 0,$$
$$-\eta'(u^m + 1)u \leq f'(u)u \leq (-\mu' u + \nu')u, \quad u \geq 0$$

with some constants $\mu', \nu', \eta' > 0$ and some positive integer m.

Remark 2.6. If the stronger decay condition

$$f(u) = -\mu u^3 + \nu u^2 + \lambda u \quad \text{for sufficiently large } u \tag{2.180}$$

with $\mu > 0$ and $-\infty < \nu, \lambda < \infty$, is assumed instead of (2.176), then $\chi(\rho)$ is allowed only to satisfy

$$\left|\frac{d^i\chi}{d\rho^i}(\rho)\right| \leq \chi_0(\rho^m + 1), \quad \rho \geq 0 \quad \text{for} \quad i = 1, 2, 3 \tag{2.181}$$

with some constant χ_0 and some positive integer m. For example, a sensitivity function $\chi(\rho) = \chi_0\rho^2$ can be taken.

2.8.4 Applications for Damped Viscoelastic Fields with Short Memory

We consider a linear viscoelastic body occupying the bounded domain Ω in $\mathbf{R}^N$ $(N = 2, 3)$ in a strainless state which is acted upon by volume forces and surface tractions and which may come in contact with a foundation on the part $\varGamma_C$ of the boundary $\partial\varOmega$ (see Example 1). The boundary $\partial\varOmega$ of the set $\varOmega$ is supposed to be a regular one, and point data of $x \in \bar{\varOmega}$ is considered in some fixed Cartesian system of coordinates. We assume that the body is endowed with short memory, that is, the state of the stress at the instant t depends only on the strain at the instant t and at the immediately preceding instants. In this case, the equation of state has the next form:

$$\sigma_{ij}(u) = b_{ijhk}\varepsilon_{kh}(u) + a_{ijhk}\frac{\partial}{\partial t}\varepsilon_{kh}(u), \quad i, j = 1, \dots, N, \tag{2.182}$$

where $u : \varOmega \times (0, +\infty) \to \mathbf{R}^N$ denotes the displacement field, $\sigma = \sigma(u)$ is the stress tensor, and $\varepsilon = \varepsilon(u)$ is the strain tensor, $\varepsilon_{hk}(u) = \frac{1}{2}(u_{k,h} + u_{h,k})$. The viscosity coefficients a_{ijhk} and the elasticity coefficients b_{ijhk} satisfy the well-known symmetry and ellipticity conditions. The dynamic behavior of the body is described by the equilibrium equation:

$$\sigma_{ij,j}(u) + f_i = \frac{\partial^2}{\partial t^2}u_i \quad \text{in} \quad \varOmega \times (0, +\infty), \tag{2.183}$$

where $f = \{f_i\}_{i=1}^N \in L_2(\varOmega; \mathbf{R}^N)$ denotes the density of body force. We suppose that the boundary $\partial\varOmega$ is divided into three parts: $\varGamma_D$, $\varGamma_N$, and $\varGamma_C$. Exactly, let $\varGamma_D$, $\varGamma_N$, and $\varGamma_C$ be disjoint sets and $\partial\varOmega = \overline{\varGamma_D} \cup \overline{\varGamma_N} \cup \overline{\varGamma_C}$. We assume that $\varGamma_C \subset \partial\varOmega$ is an open subset with positive surface measure (cf. [30, p. 196]). The displacements

$$u_i = 0 \text{ on } \varGamma_D \times (0, +\infty), \tag{2.184}$$

are prescribed on $\varGamma_D$, and surface tractions

$$S_i = \sigma_{ij}n_j = F_i \ \ (F_i = F_i(x)) \text{ on } \varGamma_N \times (0, +\infty). \tag{2.185}$$

are prescribed on $\varGamma_N$, where $F = \{F_i\}_{i=1}^N \in L_2(\varGamma_N; \mathbf{R}^N)$ denotes the vector of surface traction, $S = \{S_i\}_{i=1}^N$ is the stress vector on $\varGamma_N$, and $n = \{n_i\}_{i=1}^N$ is the outward unit normal to $\partial\varOmega$.

On $\varGamma_C$, we specify nonmonotone multivalued boundary "reaction-velocity" conditions:

$$-S \in \partial j\left(x, \frac{\partial u}{\partial t}\right) \text{ on } \varGamma_C \times (0, +\infty), \tag{2.186}$$

where $j : \varGamma_C \times \mathbf{R}^N \to \mathbf{R}$ satisfies the next conditions:

1. $j(\cdot, \xi)$ is a measurable function for each $\xi \in \mathbf{R^N}$ and $j(\cdot, 0) \in L_1(\varGamma_C)$.

2. $j(x, \cdot)$ is a locally Lipschitz function for each $x \in \varGamma_C$.

3. $\exists \bar{c} > 0$: $\|\eta\|_{\mathbf{R}^N} \le \bar{c}(1 + \|\xi\|_{\mathbf{R}^N})\ \forall x \in \varGamma_C$, $\forall \xi \in \mathbf{R}^N$, $\forall \eta \in \partial j(x, \xi)$,
   where for $x \in \varGamma_C$,

$$\partial j(x,\xi) = \{\eta \in \mathbf{R}^N \mid (\eta, v)_{\mathbf{R}^N} \le j^0(x,\xi; v)\ \forall v \in \mathbf{R}^N\}$$

is the generalized gradient of the functional $j(x,\cdot)$ at point $\xi \in \mathbf{R}^N$ and

$$j^0(x,\xi; v) = \varlimsup_{\zeta\to\xi,\, t\searrow 0} \frac{j(x,\zeta + tv) - j(x,\zeta)}{t}$$

is the generalized upper derivative of $j(x,\cdot)$ at point $\xi \in \mathbf{R}^N$ and the direction $v \in \mathbf{R}^N$.

Note that all nonconvex superpotential graphs from, in particular, the functions j, defined as a minimum and as a maximum of quadratic convex functions, satisfy the upper considered conditions on Γ_C.

For the variational formulation of the problem (2.182)–(2.186), we set: $H = L_2(\Omega; \mathbf{R}^N)$, $Z = H^\delta(\Omega; \mathbf{R}^N)$, $V = \{v \in H^1(\Omega; \mathbf{R}^N) : v_i = 0 \text{ on } \Gamma_D\}$, where $\delta \in \left(\frac{1}{2}; 1\right)$. Let $\forall u, v \in V$

$$\langle f_0, v\rangle_V = \int_\Omega f_i v_i\, dx + \int_{\Gamma_N} F_i v_i\, d\sigma(x),$$

$$a(u,v) = \int_\Omega a_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)\, dx,$$

$$b(u,v) = \int_\Omega b_{ijhk}\varepsilon_{ij}(u)\varepsilon_{ij}(v)\, dx,$$

$\bar\gamma : Z \to L_2(\partial\Omega; \mathbf{R}^N)$ be a trace operator and $\bar\gamma^* : L_2(\partial\Omega; \mathbf{R}^N) \to Z^*$ be a conjugate operator,

$$\bar\gamma^* u(z) = \int_{\partial\Omega} u(x)\bar\gamma z(x)\, d\sigma(x), \quad z \in Z,\ u \in L_2(\partial\Omega; \mathbf{R}^N).$$

Let us consider a locally Lipschitz functional $J : L_2(\Gamma_C; \mathbf{R}^n) \to \mathbf{R}$,

$$J(z) = \int_{\Gamma_C} j(x, z(x))\, d\sigma(x), \quad z \in L_2(\Gamma_C; \mathbf{R}^n).$$

Then the interaction functions A_1, A_2, and B_0 can be defined by the next relations:

$$\forall z \in Z \quad A_2(z) = \bar\gamma^* \left(\partial J(\bar\gamma z)\right),$$

$$\forall u, v \in V \quad \langle A_1 u, v\rangle_V = a(u,v), \quad \langle B_0 u, v\rangle_V = b(u,v),\ A_0(u) = A_1 u + A_2(u).$$

If we supplementary have $\bar{\alpha} > \bar{c}\bar{\beta}^2\|\bar{\gamma}\|^2$, where $\bar{\beta}$ is the embedding constant of V into Z, $\bar{\alpha}$ is the constant from the ellipticity condition for a_{ijhk}, or

$$\forall x \in \Gamma_C,\ \forall \xi \in \mathbf{R}^N,\ \forall \eta \in \partial j(x,\xi) \quad (\eta,\xi)_{\mathbf{R}^N} \geq 0,$$

then from [25], it follows that the next condition hold true:

(H_1) V, Z, H are Hilbert spaces; $H^* \equiv H$ and we have such chain of dense and compact embeddings:

$$V \subset Z \subset H \equiv H^* \subset Z^* \subset V^*.$$

(H_2) $f_0 \in V^*$.
(A_1) $\exists c > 0 : \forall u \in V, \forall d \in A_0(u)\ \|d\|_{V^*} \leq c(1 + \|u\|_V)$.
(A_2) $\exists \alpha, \beta > 0 : \forall u \in V, \forall d \in A_0(u)\ \langle d, u\rangle_V \geq \alpha\|u\|_V^2 - \beta$.
(A_3) $A_0 = A_1 + A_2$, where $A_1 : V \to V^*$ is linear, selfconjugated, positive operator and $A_2 : V \overset{\to}{\to} V^*$ satisfies such conditions:

(a) There exists such Hilbert space Z that the embedding $V \subset Z$ is dense and compact one and the embedding $Z \subset H$ is dense and continuous one.
(b) For any $u \in Z$, the set $A_2(u)$ is nonempty, convex, and weakly compact one in Z^*.
(c) $A_2 : Z \overset{\to}{\to} Z^*$ is a bounded map, that is, A_2 converts bounded sets from Z into bounded sets in the space Z^*.
(d) $A_2 : Z \overset{\to}{\to} Z^*$ is a demiclosed map, that is, if $u_n \to u$ in Z, $d_n \to d$ weakly in Z^*, $n \to +\infty$, and $d_n \in A_2(u_n)$ $\forall n \geq 1$, then $d \in A_2(u)$.

(B_1) $B_0 : V \to V^*$ is a linear selfconjugated operator.

(B_2) $\exists \gamma > 0 : \langle B_0 u, u\rangle_V \geq \gamma\|u\|_V^2$.
Here, $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbf{R}$ is the duality in $V^* \times V$, coinciding on $H \times V$ with the inner product $(\cdot,\cdot)$ in Hilbert space H.

Note that from (A_1)–(A_3) and results of this chapter, it follows that the map A_0 satisfies such condition:
$(A_3)'$ $A_0 : V \overset{\to}{\to} V^*$ is (generalized) λ_0-pseudomonotone, that is:

(a) For any $u \in V$, the set $A_0(u)$ is nonempty, convex, and weakly compact one in V^*.
(b) If $u_n \to u$ weakly in V, $n \to +\infty$, $d_n \in A_0(u_n)$ $\forall n \geq 1$ and $\varlimsup_{n\to\infty} \langle d_n, u_n - u\rangle_V \leq 0$, then $\forall \omega \in V\ \exists d(\omega) \in A_0(u) :$

$$\varliminf_{n\to+\infty} \langle d_n, u_n - \omega\rangle_V \geq \langle d(\omega), u - \omega\rangle_V.$$

(c) The map A_0 is upper semicontinuous one that acts from an arbitrary finite-dimensional subspace of V into V^*, endowed with weak topology.

Thus, in the next chapter, we investigate the dynamic of all weak solutions of the second-order nonlinear autonomous differential-operator inclusion

$$y''(t) + A_0(y'(t)) + B_0(y(t)) \ni f_0, \tag{2.187}$$

as $t \to +\infty$, which are defined as $t \geq 0$, where parameters of the problem satisfy conditions (H_1), (H_2), (A_1)–(A_3) and (B_1)–(B_2).

As a *weak solution* of the evolution inclusion (2.187) on the interval $[\tau, T]$, we consider such pair of elements $(u(\cdot), u'(\cdot))^T \in L_2(\tau, T; V \times V)$ that for some $d(\cdot) \in L_2(\tau, T; V^*)$

$$\begin{gathered} d(t) \in A_0(u'(t)) \quad \text{for almost every (a.e.)} \quad t \in (\tau, T), \\ -\int_\tau^T (\zeta'(t), u'(t))dt + \int_\tau^T \langle d(t), \zeta(t)\rangle_V dt + \int_\tau^T \langle B_0 u(t), \zeta(t)\rangle_V dt \\ = \int_\tau^T \langle f_0, \zeta(t)\rangle_V \quad \forall \zeta \in C_0^\infty([\tau, T]; V), \end{gathered} \tag{2.188}$$

where u' is the derivative of the element $u(\cdot)$ in the sense of the space of distributions $\mathscr{D}^*([\tau, T]; V^*)$.

As a *generalized solution* of the problem (2.182)–(2.186), we consider the weak solution of the corresponding problem (2.187). All results from Sects. 2.5 and 2.7 for state functions of this problem are fulfilled.

Corollary 2.8. *The m-semiflow G constructed on all generalized solutions of (2.182)–(2.186) has the compact invariant global attractor $\mathscr{A}$. For all generalized solutions (2.182)–(2.186), defined on $[0, +\infty)$, there exists the trajectory attractor $\mathscr{P}$. At that,*

$$\mathscr{A} = \mathscr{P}(0) = \{y(0) \mid y \in \mathscr{K}\}, \quad \mathscr{P} = \Pi_+ \mathscr{K},$$

where $\mathscr{K}$ is the family of all complete trajectories of corresponding differential-operator inclusion in $C^{loc}(\mathbf{R}; E) \cap L_\infty(\mathbf{R}; E)$. Moreover, global attractors are equal in the sense of [24, Definition 6, p. 88] as well as in the sense of [37, Definition 2.2, p. 182].

Thus, all statements of previous sections hold true for all generalized solutions of problem (2.182)–(2.186). In particular, all displacements and velocities are "attracted" as $t \to +\infty$ to all complete (defined on the entire time axis), globally bounded trajectories of corresponding "generalized" problem, which belongs to compact sets in corresponding phase and extended phase spaces. Questions concerning the connection and dimension of constructed attractors in the general case are opened. Note that approaches proposed in works [24, 37] are based on properties of solutions of evolution objects. Our approaches are based on properties of interaction function A from inclusion and properties of phase spaces.

2.8.5 Applications for Nonsmooth Autonomous Piezoelectric Fields

We consider a mathematical model which describes the contact between a piezoelectric body and a foundation (see Example 2). The physical setting is formulated as in [23]. We consider a plane electro-elastic material which in its unreformed state occupies an open bounded subset Ω of $\mathbf{R}^d$, $d = 2$. We agree to keep this notation since the mathematical results hold for $d \geq 2$. The boundary Γ of the piezoelectric body Ω is assumed to be Lipschitz continuous. We consider two partitions of Γ. A first partition is given by two disjoint measurable parts Γ_D and Γ_N such that $m(\Gamma_D) > 0$, and a second one consists of two disjoint measurable parts Γ_a and Γ_b such that $m(\Gamma_a) > 0$. We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Moreover, a surface tractions of density g act on Γ_N, and the electric potential vanishes on Γ_a.

The body Ω is lying on another medium (the so-called support) which introduce frictional effects. The interaction between the body and the support is described, due to the adhesion or skin friction, by a nonmonotone possibly multivalued law between the bonding forces and the corresponding displacements. In order to formulate the skin effects, we suppose that the body forces of density f consist of two parts: f_1 which is prescribed external loading and f_2 which is the reaction of constrains introducing the skin effects, that is, $f = f_1 + f_2$. Here, f_2 is a possibly multivalued function of the displacement u. We consider the reaction-displacement law of the form

$$-f_2(x) \in \partial j(x, u(x,t)) \text{ in } \Omega,$$

where $j : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ is locally Lipschitz function in its last variable and ∂j represents the Clarcke subdifferential.

The governing equations of piezoelectricity consist (see Example 2) of the equation of motion, equilibrium equation, constitutive relations, strain-displacement, and electric field-potential relations. We suppose that the accelerations in the system are not negligible, and therefore, the process is dynamic.

The equation of motion for the stress field and *the equilibrium equation* for the electric displacement field are, respectively, given by

$$\rho u'' - \mathrm{Div}\sigma = \rho f - \gamma u' \text{ in } \Omega \times (0, +\infty),$$

$$\mathrm{div} D = 0 \text{ in } \Omega \times (0, +\infty),$$

where ρ is the constant mass density (normalized as $\rho = 1$), $\gamma \in L_\infty(\Omega)$, $\gamma(x) \geq \gamma_0 > 0$ for a.e. $x \in \Omega$ is a nonnegative function characterizing the viscosity (damping) of the medium, and $\sigma : \Omega \times (0, +\infty) \to S_d$, $\sigma = (\sigma_{ij})$, and $D : \Omega \to \mathrm{R}^d$, $D = (D_i)$, $i, j = 1, \ldots, d$ represent the stress tensor and the electric displacement field, respectively. Here, S_d is the linear space of second-order symmetric tensors on R^d with the inner product and the corresponding norm $\sigma : \tau = \sum_{ij} \sigma_{ij} \tau)ij$, $\|\tau\|^2_{S_d} = \tau : \tau$, respectively. Recall also that Div is the divergence operator for tensor valued functions given by $\mathrm{Div}\sigma = (\sigma_{ij,j})$ and div stands for the divergence operator for vector-valued functions, that is, $\mathrm{div}D = (D_{i,i})$.

The stress-charge form of piezoelectric constitutive relations describes the behavior of the material and are the following:

$$\sigma = \mathscr{A}\varepsilon(u) - \mathscr{P}E(\varphi) \text{ in } \Omega \times (0, +\infty) \text{ (converse effect)},$$
$$D = \mathscr{P}\varepsilon(u) + \mathscr{B}E(\varphi) \text{ in } \Omega \times (0, +\infty) \text{ (direct effect)},$$

where $\mathscr{A} : \Omega \times S_d \to S_d$ is a linear elasticity operator with the elasticity tensor $a = (a_{ijkl})$, $\mathscr{P} : \Omega \times S_d \to \mathbf{R}^d$ is a linear piezoelectric operator represented by the piezoelectric coefficients $p = (p_{ijk})$, $i, j, k = 1, \dots d$ (third order tensor field), $\mathscr{P}^T : \Omega \times \mathbf{R}_d \to S_d$ is its transpose represented by $p^T = (p^T_{ijk}) = (p_{kij})$, and $\mathscr{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is a linear electric permittivity operator with the dielectric constants $\beta = (\beta_{ij})$ (second order tensor field). The decoupled state (purely elastic and purely electric deformations) can be obtained by setting the piezoelectric coefficients $p_{ijk} = 0$. The elasticity coefficients $a(x) = (a_{ijkl}(x))$, $i, j, k, l = 1, \dots d$ (fourth-order tensor field) are functions of position in a nonhomogeneous material. We use here notation p^T to denote the transpose of the tensor p given $p\tau \cdot v = \tau : p^T v$ for $\tau \in S_d$ and $v \in \mathbf{R}^d$.

The elastic strain-displacement and electric field-potential relations are given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \text{ in } \Omega \times (0, +\infty),$$
$$E(u) = -\nabla\varphi \text{ in } \Omega,$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))$ and $E(\varphi) = (E_i(\varphi))$ denote the linear strain tensor and the electric vector field, respectively. Here, $u : \Omega \times (0, +\infty) \to \mathbf{R}^d$, $u = (u_i)$, $i = 1, \dots, d$ and $\varphi : \Omega \to \mathbf{R}$ are the displacement vector field and the electric potential (scalar field), respectively.

Denoting by u_0 and u_1, the initial displacement and initial velocity, respectively, the classical formulation of the mechanical model can be stated as follows: find a displacement field $u : \Omega \times (0, +\infty) \to \mathbf{R}^d$ and an electric potential $\varphi : \Omega \to \mathbf{R}^d$ such that

$$u'' - \mathrm{Div}\sigma = f_1 + f_2 - \gamma u' \text{ in } \Omega \times (0, +\infty) \tag{2.189}$$

$$\mathrm{div}D = 0 \text{ in } \Omega \times (0, +\infty) \tag{2.190}$$

$$\sigma = \mathscr{A}\varepsilon(u) + \mathscr{P}^T\nabla\varphi \text{ in } \Omega \times (0, +\infty), \tag{2.191}$$

$$D = \mathscr{P}\varepsilon(u) - \mathscr{B}\nabla\varphi \text{ in } \Omega \times (0, +\infty), \tag{2.192}$$

$$u = 0 \text{ on } \Gamma_D \times (0, +\infty) \tag{2.193}$$

$$\sigma n = g \text{ on } \Gamma_D \times (0, +\infty) \tag{2.194}$$

$$\varphi = 0 \text{ on } \Gamma_a \times (0, +\infty) \tag{2.195}$$

$$D \cdot n = 0 \text{ on } \Gamma_b \times (0, +\infty) \tag{2.196}$$

$$- f_2(x) \in \partial j(x, u(x,t)) \text{ in } \Omega \times (0, +\infty) \tag{2.197}$$

$$u(0) = u_0, \ u'(0) = u_1 \text{ in } \Omega, \tag{2.198}$$

where n denotes the outward unit normal to Γ. Because of the Clarke subdifferential in (2.197), the problem will be formulated as a hemivariational inequality and then it will be embedded into a more general class of second-order evolution inclusions. Due to the multivalued term in the problem, the uniqueness of weak solutions is not expected.

Let $j : \Omega \times \mathbf{R} \to \mathbf{R}$ be defined as a minimum of two convex functions, that is, $j(x, s) = h(x) \min\{j_1(s), j_2(s)\}$ for $x \in \Omega$ and $s \in \mathbf{R}$, where $h \in L_\infty(\Omega)$, $j_1(s) = as^2$ and $j_2(s) = \frac{a}{2}(s^2 + 1)$ with $a > 0$. Then

$$\partial j(x,s) = h(x) \times \begin{cases} as & \text{if } s \in (-\infty, -1) \cup (1, +\infty) \\ 2as & \text{if } s \in (-1, 1) \\ [a, 2a] & \text{if } s = 1 \\ [-2a, -a] & \text{if } s = -1. \end{cases}$$

The model example can be modified to obtain nonmonotone zigzag relations which describe the adhesive contact laws for a granular material and a reinforced concrete, for example, the stick-slip law and the fiber bundle model law.

Another example which satisfies $H(j)$ is a superpotential of d.c. (difference of convex functions) type, that is, $j(s) = j_1(s) - j_2(s)$, where $j_1, j_2 : \mathbf{R} \to \mathbf{R}$ are convex functions.

We now turn to the variational formulation of the problem (2.189)–(2.198). We introduce the spaces for the displacement and electric potential:

$$V = \{v \in H^1(\Omega; \mathbf{R}^d) : v = 0 \text{ on } \Gamma_D\}, \tag{2.199}$$

$$\Phi = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_a\}$$

which are closed subspaces of $H^1(\Omega; \mathbf{R}^d)$ and $H^1(\Omega)$, respectively. Let $H = L_2(\Omega; \mathbf{R}^d)$ and $\mathscr{H} = L_2(\Omega; S_d)$ be Hilbert spaces equipped with the inner products $\langle u, v\rangle_H = \int_\Omega u \cdot v dx$, $\langle \sigma, \tau\rangle_{\mathscr{H}} = \int_\Omega \sigma : \tau dx$. Then the spaces (V, H, V^*) form an evolution triple of spaces. On V, we consider the inner product and the corresponding norm given by $\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}}$, $\|v\|_V = \|\varepsilon(v)\|_{\mathscr{H}}$ for $u, v \in V$. From the Korn inequality $\|v\|_{H^1(\Omega;\mathbf{R}^d)} \leq C\|\varepsilon(v)\|_{\mathscr{H}}$ for $v \in V$ with $C > 0$, it follows that $\|\cdot\|_{H^1(\Omega;\mathbf{R}^d)}$ and $\|\cdot\|_V$ are equivalent norms on V. Thus, $(V, \|\cdot\|_V)$ is a Hilbert space. On Φ, we consider the inner product $(\varphi, \psi)_\Phi = (\varphi, \psi)_{H^1(\Omega)}$ for $\varphi, \psi \in \Phi$. Then, $(\Phi, \|\cdot\|_\Phi)$ is also a Hilbert space.

Assuming sufficient regularity of the functions involved in the problem (2.189)–(2.198), multiplying (2.189) by $v \in V$ and using integration by parts, we have

$$\langle u''(t), v\rangle + \langle \sigma(u), \varepsilon(v)\rangle_{\mathscr{H}} - \int_{\Gamma} \sigma n \cdot v d\Gamma(x) = \langle f_1(t) + f_2(t), v\rangle - \langle \gamma u'(t), v\rangle$$

for a.e. $t > 0$. Since, by (2.194), we have $\int_{\Gamma} \sigma n \cdot v d\Gamma = \int_{\Gamma_N} g(t) \cdot v d\Gamma$ and (2.197) implies

$$-\int_{\Omega} f_2(x) \cdot v(x) dx \le \int_{\Omega} j^0(x, u(x,t); v(x)) dx,$$

we obtain

$$\langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + \langle \sigma(u), \varepsilon(v)\rangle_{\mathscr{H}} + \int_{\Omega} j^0(x, u(x,t); v(x)) dx \ge \langle F, v\rangle \tag{2.200}$$

where

$$\langle F, v\rangle = \langle f_1, v\rangle + \int_{\Gamma_N} g \cdot v d\Gamma \text{ for } v \in V.$$

Let $\psi \in \Phi$. From (2.190), again by using integration by parts and the condition (2.196), we have

$$-\langle D, \nabla\psi\rangle_H = 0. \tag{2.201}$$

Now inserting (2.191) into (2.200) and (2.192) into (2.201), we obtain the following variational formulation: for $-\infty < \tau < T < +\infty$, find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ and $\varphi \in L_2(\tau, T; H)$ such that $u'' \in \mathscr{V}^*_{,\mathscr{T}}$, where $\mathscr{V}^*_{\tau,T} = L_2(\tau, T; V^*)$ and

$$\begin{cases} \langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + \langle \mathscr{A}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} + \langle \mathscr{P}^T \nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} + \\ + \int_{\Omega} j^0(x, u; v) dx \ge \langle F, v\rangle \quad \text{a.e. } t, \text{ for all } v \in V \\ \langle \mathscr{B}\nabla, \nabla\rangle_H = \langle \mathscr{P}\varepsilon(u), \nabla\psi\rangle_H \quad \text{a.e. } t, \text{ for all } \psi \in \Phi \\ u(0) = u_0, \; u'(0) = u_1. \end{cases} \tag{2.202}$$

We need the following hypotheses for the constitutive tensors.

$\underline{H(a)}$: The elasticity tensor field $a = (a_{ijkl})$ satisfies $a_{ijkl} \in L_\infty(\Omega)$, $a_{ijkl} = a_{klij}$, $a_{ijkl} = a_{jikl}$, $a_{ijkl} = a_{ijlk}$ and $a_{ijkl}(x)\tau_{ij}\tau_{kl} \ge \alpha\tau_{ij}\tau_{ij}$ for a.e $x \in \Omega$ and all $\tau = (\tau_{ij}) \in S_d$ with $\alpha > 0$.

$\underline{H(p)}$: The piezoelectric tensor field $p = (p_{ijk})$ satisfies $p_{ijk} = p_{ikj} \in L_\infty(\Omega)$.

$\underline{H(\beta)}$: The dielectric tensor field $\beta = (\beta_{ij})$ satisfies $\beta_{ij} = \beta_{ji} \in L_\infty(\Omega)$ and $\beta_{ij}(x)\xi_i\xi_j \ge m_\beta|\xi|^2_{\mathbf{R}^d}$ for a.e. $x \in \Omega$ and all $\xi = (\xi_i) \in \mathbf{R}^d$ with m_β¿0.

We define the following bilinear forms $a : V \times V \to \mathbf{R}$, $b : V \times \Phi \to \mathbf{R}$, $b^T : \Phi \times V \to \mathbf{R}$ and $c : \Phi \times \Phi \to \mathbf{R}$ by

$$a(u,v) = \int_\Omega a_{ijkl}(x)\frac{\partial u_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}dx \quad \text{for} \quad u, v \in V,$$

$$b(u,\varphi) = \int_\Omega p_{ijk}(x)\frac{\partial u_i}{\partial x_j}\frac{\partial \varphi}{\partial x_k}dx \quad \text{for} \quad u \in V, \quad \varphi \in \Phi,$$

$$b^T(\varphi,u) = \int_\Omega p_{kij}(x)\frac{\partial \varphi}{\partial x_k}\frac{\partial u_i}{\partial x_j}dx \quad \text{for} \quad \varphi \in \Phi, \quad u \in V,$$

$$c(\varphi,\psi) = \int_\Omega \beta_{ij}(x)\frac{\partial \varphi}{\partial x_i}\frac{\partial \psi}{\partial x_j}dx \quad \text{for} \quad \varphi, \psi \in \Phi.$$

Then we have

$$\begin{aligned} a(u,v) &= \langle \mathscr{A}\varepsilon(u), \varepsilon(v)\rangle_{\mathscr{H}} \quad \text{for } u, v \in V,\\ b(u,\varphi) &= \langle \mathscr{P}\varepsilon(u), \nabla\varphi\rangle_H \quad \text{for } u \in V,\ \varphi \in \Phi,\\ b^T(\varphi,u) &= \langle \mathscr{P}\nabla\varphi, \varepsilon(u)\rangle_{\mathscr{H}} \quad \text{for } \varphi \in \Phi,\ u \in V,\\ c(\varphi,\psi) &= \langle \mathscr{B}\nabla\varphi, \nabla\psi\rangle_H \quad \text{for } \varphi, \psi \in \Phi, \end{aligned}$$

where the elasticity operator $\mathscr{A} : \Omega \times S_d \to S_d$ is given by $\mathscr{A}(x,\varepsilon) = a(x)\varepsilon$, $a(x) = (a_{ijkl}(x))$, the piezoelectric operator $\mathscr{P} : \Omega \times S_d \to \mathbf{R}^d$ is given by $\mathscr{P}(x,\varepsilon) = p(x)\varepsilon$, $p(x) = (p_{ijk}(x))$, the transpose to $\mathscr{P}$, $\mathscr{P}^T : \Omega \times \mathbf{R}^D \to S_d$ is given by $\mathscr{P}^T(x,\xi) = p^T(x)\xi$, $p^T(x) = (p^T_{ijk}(x)) = (p_{kij})$, and the electric permittivity operator $\mathscr{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is defined by $\mathscr{B}(x,\xi) = \beta(x)\xi$, $\beta(x) = (\beta_{ij(x)})$.

Using the above notation, the problem (2.202) is formulated as follows: find $u \in C([\tau,T];V) \cap C^1([\tau,T];V)$ and $\varphi \in L_2(\tau,T;H)$ such that $u'' \in \mathscr{V}_{\tau,T^*}$ and

$$\begin{cases} \langle u''(t), v\rangle + \langle \gamma u'(t), v\rangle + a(u(t),v) + b^T(\varphi(t),v) + \\ + \displaystyle\int_\Omega j^0(x,t,u;v)dx \geq \langle F, v\rangle \quad \text{a.e. } t, \quad \text{for all } v \in V \\ c(\varphi(t),\psi) = b(u(t),\psi) \quad \text{a.e. } t, \quad \text{for all } \psi \in \Phi \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.203}$$

The problem (2.203) is a system coupled with the hemivariational inequality for the displacement and a time-dependent stationary equation for the electric potential. We need now some auxiliary results and notation.

We remark that under hypotheses $H(p)$ and $H(\beta)$, for any $z \in V$, there exists a unique element $\varphi_z \in \Phi$ such that

$$c(\varphi_z,\psi) = b(z,\psi) \quad \text{for all} \quad \psi \in \Phi$$

and the map $C : V \to \Phi$ given by $Cz = \varphi_z$ is linear and continuous.

As a corollary, we obtain the following result: if $H(p)$, $H(\beta)$ hold and $u \in \mathscr{V}$, where $\mathscr{V} = L_2(\tau, T; V)$, then the problem

$$\begin{cases} \text{find } \varphi \in L_2(\tau, T; \Phi) \quad \text{such that} \\ c(\varphi(t), \psi) = b(u(t), \psi) \quad \text{for a.e. } t \in (\tau, T), \quad \text{for all } \psi \in \Phi \end{cases}$$

admits a unique solution $\varphi \in L_2(\tau, T; \Phi)$ and $\|\varphi\|_{L_2(\tau,T;\Phi)} \le c\|u\|_{\mathscr{V}_{\tau,T}}$ with $c > 0$. For a.e. $t \in (\tau, T)$, we have $\varphi(t) = Cu(t)$, where the operator C is defined in Lemma 3.1 of [23].

Next, since for every $\varphi \in \Phi$, the linear form $v \to b^T(\varphi, v)$ is continuous on V, so there exists a unique element $B\varphi \in V^*$ such that $b^T(\varphi, v) = \langle B\varphi, v\rangle_{V^*\times V}$ for all $v \in V$ and $B \in \mathscr{L}(\Phi, V^*)$. We observe that

$$b^T(\varphi, v) = \langle \mathscr{P}\nabla\varphi, \varepsilon(v)\rangle_{\mathscr{H}} = \int_\Omega \mathscr{P}^T\nabla\varphi : \varepsilon(v)dx \tag{2.204}$$

$$= \int_\Omega \mathscr{P}\varepsilon(v)\cdot\nabla\varphi dx = \langle \mathscr{P}\varepsilon(v), \nabla\varphi\rangle_H = b(v, \varphi) \quad \text{for all } v \in V, \quad \text{and } \varphi \in \Phi.$$

Analogously, we introduce the operator $A \in \mathscr{L}(V, V^*)$ such that $a(u, v) = \langle Au, v\rangle$ for all $u, v \in V$.

We are now in a position to reformulate the system (2.203). Since for a fixed $u \in \mathscr{V}$, the second equation in (2.203) is uniquely solvable (cf. Corollary 1 in [23]), we have

$$b^T(\varphi(t), v) = \langle B\varphi(t), v\rangle = \langle BCu(t), v\rangle \quad \text{for all } v \in V \quad \text{and a.e. } t \in (\tau, T).$$

Thus, the problem (2.203) takes the form: find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ such that $u'' \in \mathscr{V}^*_{,\mathscr{T}}$ and

$$\begin{cases} \langle u''(t) + Ru'(t) + Gu(t), v\rangle + \int_\Omega j^0(x, u; v)dx \ge \langle F, v\rangle \\ \text{a.e. } t, \quad \text{for all } v \in V \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{2.205}$$

where $R : H \to V^*$ and $G : V \to V^*$ are given by $Rv = \gamma v$ for $v \in H$ and $Gv = Av + BCv$ for $v \in V$, respectively.

The existence of solutions to the hemivariational inequality (2.205) will be a consequence of a more general result provided in [23]. We remark that operators R and G satisfy such properties: if $\gamma \in L_\infty(\Omega)$, $\gamma \ge \gamma_0 > 0$, then the operator $R : H \to H$ defined by $Rv = \gamma v$ is linear continuous symmetric and coercive. Under the hypotheses $H(a)$, $H(p)$, and $H(\beta)$, the operator $G : V \to V^*$ defined by $G = A + BC$ is linear, bounded, symmetric, and coercive.

Finally, we obtain the following second-order evolution inclusion: find $u \in C([\tau, T]; V) \cap C^1([\tau, T]; V)$ such that $u'' \in \mathscr{V}^*_{,\mathscr{T}}$ and

$$\begin{cases} u''(t) + Ru'(t) + Gu(t) + \partial J(t, u(t)) \ni F(t) \quad \text{a.e.} \quad t \in (0, T) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.206}$$

We need the following hypotheses:

$\underline{H(R)}$ $R : H \to H$ is a linear symmetric such that $\exists \gamma > 0 : (Rv, v)_H = \gamma \|v\|_H^2$ $\forall v \in H$.

$\underline{H(G)}$ $G : V \to V^*$ is linear and symmetric and $\exists c_G > 0 : \langle Gv, v \rangle_V \geq c_G \|v\|_V^2$ $\forall v \in V$.

$\underline{H(J)}$ $J : H \to \mathbf{R}$ is a function such that:

(i) $J(\cdot)$ is locally Lipschitz and regular one [12].

(ii) $\exists c_1 > 0$: $\|\partial J(v)\|_+ \leq c_1(1 + \|v\|_H)$ $\forall v \in H$.

(iii) $\exists c_2 > 0$:

$$[\partial J(v), v]_- \geq -\lambda \|v\|_H^2 - c_2 \quad \forall v \in H,$$

where $\partial J(v) = \{p \in H \mid (p, w)_H \leq J^\circ(v; w) \ \forall w \in H\}$ denotes the Clarke subdifferential of $J(\cdot)$ at a point $v \in H$ (see [12] for details), $\lambda \in (0, \lambda_1)$, $\lambda_1 > 0$: $c_G \|v\|_V^2 \geq \lambda_1 \|v\|_H^2$ $\forall v \in V$.

$\underline{(H_0)}$ V is a Hilbert space.

We remark that Condition $H(j)$(iii) is technical condition provides only dissipation of multivalued (in general case) dynamical system constructed on all weak solutions of Problem (2.189). This condition is not connected with the nonsmoothness of j.

In [23], it is proved that if hypotheses $H(R)$, $H(G)$, $H(J)$, and (H_0) hold, then the problem (2.206) has a solution. Due to the previous results, we can investigate a long-time behavior of all weak solutions of the problem (2.206) under similar but some stronger (providing a dissipation) conditions. In particular, we study the structure of the global and trajectory attractors.

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