Chapter 2
Fourier Series

2.1 Introduction

The formula

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x\ldots \tag{2.1}$$

was published by Leonhard Euler (1707–1783) before Fourier's work began, so you might like to ponder the question why Euler did not receive the credit of Fourier series [1]. The Eq. 2.1 is really interesting in terms of types of the functions placed in left and right hand side. The function of x in left hand side is just a definite number, whereas functions used in the right hand side are all infinite series. Sum of all the sinusoids of x in *radian* will be equal to exactly $x/2$ (*if* $x = 1$, $x/2 = 0.5$). As we allow the number of terms for addition in right hand side, it will closer to the value of $x/2$. It can be easily checked by putting any finite value for x.

Basic interpretation of Fourier series also suggests to express the periodic signals as summation of sine and cosines of integral multiple of frequencies. The concept is therefore to some extent analogous to Eq. 2.1. The detail understanding will be obtained from discussion in the subsequent sections of the present chapter.

We, as the readers of communication engineering, shall look at Fourier series as an efficient tool for signal conditioning. From Chap. 1 it is understood that in the world of signals, only sinusoids (and obviously co sinusoids) are mono-tone (single frequency) signals. If we can successfully express any composite periodic signal into sinusoids, we can directly analyze the components of different signal, i.e., we can obtain the mono tone signal components with definite amount of amplitude (either voltage or current or energy) and phase. This representation is called as *spectral representation* like amplitude spectrum, phase spectrum, energy spectrum respectively. The representation can solve a lot of problems in communication engineering like choice of efficient bandwidth of a composite signal transmission. During transmission, the signal components with high energy

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are chosen depending upon the *power spectral density* (*PSD*) representation, which gives the average power distribution of signal with respect to unit bandwidth.

In this chapter, we will discuss the foundation of Fourier series with elegant support of mathematics and our own understanding. In the light of Fourier series, the concept of signal translation, scaling and many more geometric operations are discussed. The applicability of Fourier series in terms of Dirichlet's condition is also discussed. Signal energy and power are also measured in frequency domain and in time domain, as well using Perseval's theorem. The reasons behind the physical representation of Fourier spectrum of periodic signals or power signal to be (i.e., Fourier series representation) a discrete line representation is also discussed in the following sections.

Some interesting worked out problems are presented in this chapter for better understanding of the subject. The visual attacks on the problems are from the knowledge of basic Euclidian geometry also presented with supporting logics. Finally, an application on feature extraction from image using phase congruency concept of Fourier series is discussed.

2.2 Statement and Interpretation

Fourier series states that,

> Any periodic function (or signal) can be expressed as a summation of orthogonal pair of matrices with one fundamental frequency and infinite number of harmonics.

From the statement, the first thing which needs to be clarified is the term "orthogonality". In one sentence, the word orthogonality means the measure of similarity or the correlation coefficient equal to zero, i.e., two vectors or *phasors* (vector representation of signal) are said to be *orthogonal*, if the cross-correlation coefficient between those two vectors is zero. The concept is discussed incorporating boundary conditions as follows.

From Fig. 2.1, we can see two vectors $\vec{x}$ and $\vec{g}$ of same magnitude make an angle θ between them. If we apply our knowledge of engineering drawing and graphics, we can see the vector $\vec{g}$ from the top to make a "plan" or top-view. If the angle θ is very small, we can say, $\vec{x}$ and $\vec{g}$ are more or less similar. As the angle θ increases, the amount of similarity decreases. We can also measure the amount of similarity by the factor c, when the *projection* (or plan) of vector $\vec{g}$ on $\vec{x}$ is defined by $c\vec{x}$. It is also observed that, when $\vec{x}$ and $\vec{g}$ are perfectly aligned, the value of $c = 1$. As the angle increases, c goes far from 1. It physically signifies that the length of the line given by projection of $\vec{g}$ on $\vec{x}$, decreases. At an instant, when $\theta = 90°$, projection of $\vec{g}$ on $\vec{x}$ would be a point instead of a line. As $\vec{x}$ is a vector of finite magnitude, whereas the approximated/projected vector becomes a point, i.e. a vector of length zero, therefore the measure of similarity or correlation coefficient becomes $c = 0$. Hence, we can say, $\vec{g}$ and $\vec{x}$ are *orthogonal* to each other. If we now keep on increasing the angle θ,

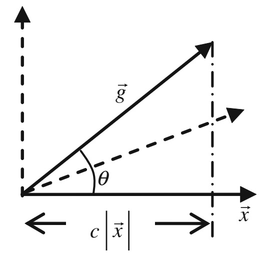


Fig. 2.1 Measure of similarity between two vectors

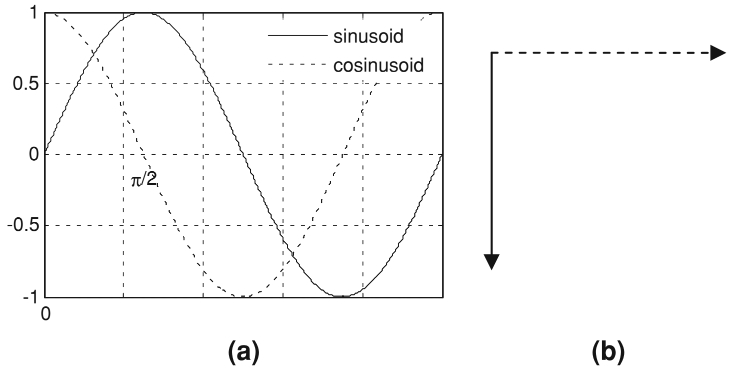


Fig. 2.2 Trigonometric orthogonality: **a** sine and cosine waveforms, **b** sine and cosine phasors

the amount of similarity will again increase but in opposite direction. Therefore, range of correlation coefficient is always $-1 < c < 1$.

Let's consider now a pair of signals/functions which are orthogonal. As shown in Fig. 2.2, one sinusoid signal is just $\pi/2$ behind the cosinusoid signal of same frequency. The phasor (vector representation of signal) also shows the same effect. Therefore, from the previous discussion, we can choose Sine and Cosine of same angle, i.e., frequency as a pair of *orthogonal set*. When a number of harmonics are used to form the set of frequencies and thereafter angles, the orthogonal set becomes *orthogonal set of matrices*. Table 2.1 completes our understanding of Fourier series applying trigonometric functions.

It means any periodic function/signal $f(t)$ can be expressed using trigonometric orthogonal signals sine and cosine from the Table 2.1, as

$$\begin{aligned} f(t) = & a_0 \cos 2\pi(0)t + b_0 \sin 2\pi(0)t \\ & + a_1 \cos 2\pi(f_0)t + b_1 \sin 2\pi(f_0)t \\ & + a_2 \cos 2\pi(2f_0)t + b_2 \sin 2\pi(2f_0)t \\ & + a_3 \cos 2\pi(3f_0)t + b_3 \sin 2\pi(3f_0)t \\ & \cdots \end{aligned} \tag{2.2}$$

Table 2.1 Expression of Fourier series

Frequency	Coefficient of in-phase (cos) component	Coefficient of quadrature-phase (sin) component	In-phase (cos) component	Quadrature-phase (sin) component
0	a_0	b_0	$a_0\cos 2\pi(0)t$	$b_0\sin 2\pi(0)t$
f_0	a_1	b_1	$a_1\cos 2\pi(f_0)t$	$b_1\sin 2\pi(f_0)t$
$2f_0$	a_2	b_2	$a_2\cos 2\pi(2f_0)t$	$b_2\sin 2\pi(2f_0)t$
$3f_0$	a_3	b_3	$a_3\cos 2\pi(3f_0)t$	$b_3\sin 2\pi(3f_0)t$
–	–	–	–	–
∞	a_∞	b_∞	$a_\infty\cos 2\pi(\infty f_0)t$	$b_\infty\sin 2\pi(\infty f_0)t$

i.e.,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n\cos\, \mathrm{n}\omega_0 t + b_n\sin\, \mathrm{n}\omega_0 t) \tag{2.3}$$

It is clear from Fig. 2.3 that the synthesized signal by adding a number of sine and/or cosine waves becomes different depending upon the amplitudes/coefficients of the sinusoids and co sinusoids. In other words, the amplitudes a_0, a_n and b_n should be function of the given periodic signal $f(t)$, in the process of signal analysis.

2.3 Fourier Coefficients

Signals are not *like* vectors, instead signals are vectors. There is a strong analogy between signals and vectors [2]. As vectors, signals also have measurable magnitude and direction. As vector, we can also express signal as a sum of components. The vector representation of signal is called as *phasor*. The representation of signals as summation its components in various ways, eventually supports the statement of Fourier series.

2.3.1 Component of a Vector

A vector is defined using its magnitude and direction. According to the Fig. 2.4, a vector $\vec{x}$ is presented. The direction is aligned with the X-axis. Now, if we virtually rotate the vector by an angle θ, it forms another vector $\vec{g}$ which is having equal magnitude with that of vector $\vec{x}$, whereas the directions are not same. From the preliminary concept of engineering drawing/graphics/drafting, if we see from top to take a *plan* (top view), we have a reduced magnitude vector. Physically, taking top view of the vector is nothing but taking the component of vector $\vec{g}$ along X-axis or along the direction of $\vec{x}$. It means, approximating the magnitude of the vector, if it would be of direction along X-axis. From the Fig. 2.4, it is clear that the vector $\vec{g}$ is approximated by vector $c\vec{x}$, where $c < 1$. The approximation is not

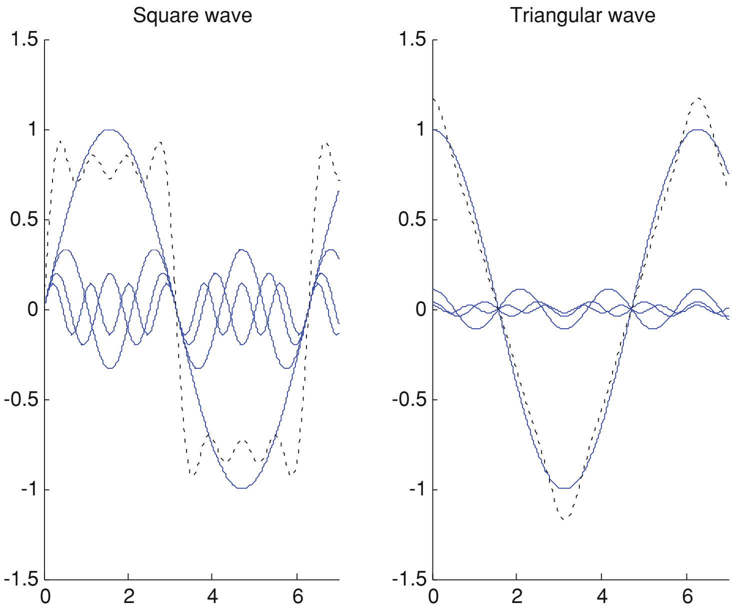


Fig. 2.3 Fourier synthesized signals by varying the amplitudes of the component signals

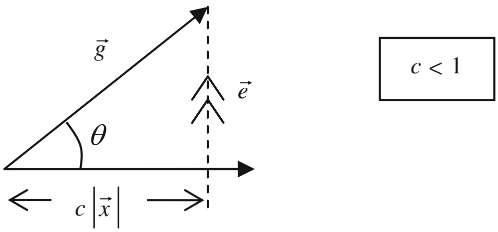


Fig. 2.4 Approximation of a vector in terms of another vector

cent percent correct. The correction can be done using triangle law for vector summation. The figure says,

$$\vec{g} = c\vec{x} + \vec{e} \tag{2.4}$$

In other words, we can interpret the observation. If we approximate the vector $\vec{g}$ by vector $c\vec{x}$,

$$\text{i.e.,}\, \vec{g} \approx c\vec{x} \tag{2.5}$$

the error in the approximation is the vector $\vec{e} = \vec{g} - c\vec{x}$. Here, one may ask, why to find out the component of a vector, the looking direction must be exactly 90°. The answer is, we need to minimize the *error*; and from a fixed point the minimum distance to a line is always the perpendicular distance.

2.3.2 Component of a Signal

The concepts of vector component and orthogonality may be extended to signals. Making analogy with the statement in the previous sub-section, let's consider the problem of approximating a real signal g(t) in terms of x(t) over an interval $[t_1, t_2]$:

$$g(t) \approx cx(t) \qquad t_1 < t < t_2 \tag{2.6}$$

The error $e(t)$ in this approximation is

$$e(t) = \begin{cases} g(t) - cx(t) & t_1 < t < t_2 \\ 0; & otherwise \end{cases} \tag{2.7}$$

For best approximation, we need to minimize the error. As the error is expressed here essentially as a signal, for minimum error, the error energy, E_e in the interval (t_1, t_2) needs to be minimized.
The error energy is given by:

$$E_e = \int_{t_1}^{t_2} |e(t)|^2 dt$$

or

$$\begin{aligned} E_e &= \int_{t_1}^{t_2} \{g(t) - cx(t)\}^2 dt \\ &= \int_{t_1}^{t_2} g^2(t)dt - 2c\int_{t_1}^{t_2} g(t)x(t)dt + c^2\int_{t_1}^{t_2} x^2(t)dt \end{aligned} \tag{2.8}$$

The error is minimized with respect to the correlation coefficient c. therefore, to minimize the error energy, the derivative $\frac{dE_g}{dc}$ must be equal to zero.

$$\frac{dE_g}{dc} = 0 - 2\int_{t_1}^{t_2} g(t)x(t)dt + 2c\int_{t_1}^{t_2} x^2(t)dt = 0 \tag{2.9}$$

$$\therefore c = \frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt} \tag{2.10}$$

The Eq. 2.10 suggests the final formation of correlation coefficient i.e., similarity measure between these two signals $g(t)$ and $x(t)$. This expression for deriving the similarity coefficient c will be useful for derivation of Fourier coefficients in trigonometric (a_0, a_n, b_n), compact (C_0, C_n, θ_n) and complex (D_n) Fourier series. In the next section, the Trigonometric Fourier coefficients are derived using Eq. 2.10.

2.3.3 Coefficients of Trigonometric Fourier Series

Due to the property of orthogonality, just like unit vectors, we can express any periodic signal by summation of sine and cosine, i.e., choosing the sinusoids and cosinusoids as the *basis.* For orthogonality, we find the following properties between sine and cosine given by Eq. 2.11a–2.11d [3].

$$\int_T \cos m\theta \cos n\theta d\theta = \begin{cases} 0; & m \neq n \\ T/2; & m = n \neq 0 \\ T; & m = n = 0 \end{cases} \tag{2.11a}$$

$$\int_T \sin m\theta \sin n\theta d\theta = \begin{cases} 0; & m \neq n \\ T/2; & m = n \neq 0 \\ 0; & m = n = 0 \end{cases} \tag{2.11b}$$

$$\int_T \cos m\theta \sin n\theta d\theta = 0; \tag{2.11c}$$

$$\int_T e^{j(n-m)\theta} d\theta = \begin{cases} 0; & m \neq n \\ T; & m = n \end{cases} \tag{2.11d}$$

where, $\int_T$ signifies definite integral with T distance between upper and lower bound. One time period signal can be taken out from any time window, but the width of the window must be *T,* i.e., the range can be $[0, T]$ or $[-T/2, T/2]$ or $[-T/4, 3T/4]$ or any other window with width *T.* Depending on the pattern and formation of the signal, the window is chosen. The choice of time window is discussed using a worked out problem in Example 2.1.

From Eq. 2.3, let's take out the terms in the right hand side one by for minimum error of approximation as shown below to find the expressions of correlation coefficients. The terms can be taken out one by one as sine and cosine holds the criterion of orthogonality as shown in Eq. 2.11a–2.11d.

$$f(t) \approx a_0 \times 1 \tag{2.12}$$

Comparing Eq. 2.6 and 2.12, from Eq. 2.10, we get

$$a_0 = \frac{\int_T f(t) \times 1 dt}{\int_T 1 dt} = \frac{\int_T f(t) dt}{T} = \frac{1}{T} \int_T f(t) dt \tag{2.13}$$

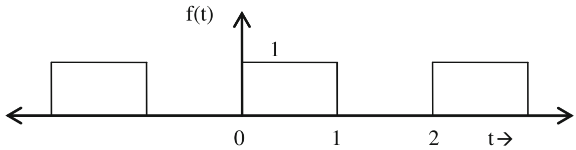


Fig. 2.5 The signal f(t) in time domain

If we approximate

$$f(t) \approx a_n \times \cos n\omega_0 t \tag{2.14}$$

Comparing Eq. 2.6 and 2.14, from Eq. 2.10, we get

$$a_n = \frac{\int_T f(t) \times \cos n\omega_0 t dt}{\int_T \cos^2 n\omega_0 t dt} = \frac{\int_T f(t) \cos n\omega_0 t dt}{T/2}$$

or,

$$a_n = \frac{2}{T} \int_T f(t) \cos n\omega_0 t dt \tag{2.15}$$

Similarly,

$$b_n = \frac{\int_T f(t) \times \sin n\omega_0 t dt}{\int_T \sin^2 n\omega_0 t dt} = \frac{\int_T f(t) \sin n\omega_0 t dt}{T/2}$$

or,

$$b_n = \frac{2}{T} \int_T f(t) \sin n\omega_0 t dt \tag{2.16}$$

Example 2.1 Express the signal f(t) in Fourier series and draw the amplitude spectrum.

From the Fig. 2.5, we can define the function f(t) as

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

and,

$$f(t+2) = f(t)$$

Therefore, the periodicity of the function f(t) is T = 2. To chose one time period (T) window, we should find the time region, where minimum number of pieces are to be handles during 'T'. If we take the region [0 T], it is having two regions. From 0 to T/2, f(t) is 1 and from T/2 to T, the value is 0.
Then, we obtain the coefficients a_0, a_n and b_n

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\int_0^2 f(t)dt = 0.5\int_0^1 1dt + 0.5\int_1^2 0dt = 0.5 - 0 = 0.5$$

Or, since $\int_a^b f(t)dt$ total area under the graph $y = f(t)$ over the interval $[a, b]$. Hence,

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T} \times \begin{pmatrix} \text{Area under graph} \\ \text{over } [0, T] \end{pmatrix} = \frac{1}{2} \times (1 \times 1) = 0.5$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^2 f(t)\omega_0 t dt \\ &= \int_0^1 1 \cos n\pi t dt + \int_1^2 0dt = \left[\frac{\sin n\pi t}{n\pi}\right]_0^1 = \frac{\sin n\pi}{n\pi} \end{aligned}$$

It is to be noted that, n is an integer which leads $\sin n\pi = 0$ since $\sin \pi = \sin 2\pi = \sin 3\pi = \ldots = 0$
Therefore,

$$a_n = 0$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^2 f(t)\omega_0 t dt \\ &= \int_0^1 1 \sin n\pi t dt + \int_1^2 0dt = \left[-\frac{\cos n\pi t}{n\pi}\right]_0^1 = \frac{1 - \cos n\pi}{n\pi} \end{aligned}$$

It can be viewed that,

$$\cos \pi = \cos 3\pi = \cos 5\pi = \ldots = -1$$
$$\cos 2\pi = \cos 4\pi = \cos 6\pi = \ldots = 1$$

Or,

$$\cos n\pi = (-1)^n$$

Therefore,

$$b_n = \frac{1 - (-1)^n}{n\pi} = \begin{cases} 2/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Finally,

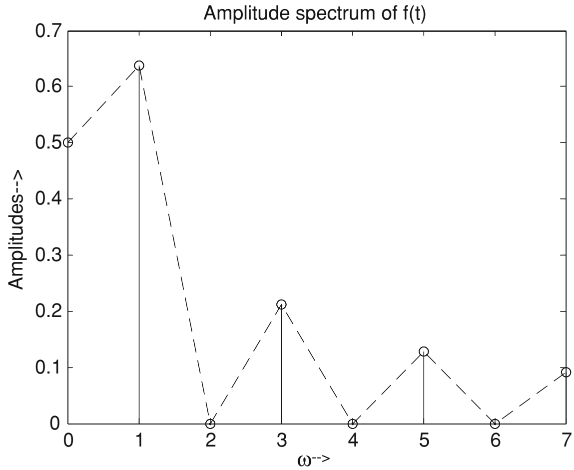


Fig. 2.6 Amplitude spectrum of f(t)

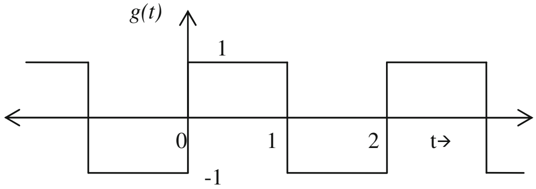


Fig. 2.7 The signal g(t) in time domain

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n\pi}\right]\sin n\omega_0 t \\ &= \frac{1}{2} + \frac{2}{\pi}\sin\omega_0 t + \frac{2}{3\pi}\sin 3\omega_0 t + \frac{2}{5\pi}\sin 5\omega_0 t + \ldots \end{aligned} \tag{2.17}$$

Example 2.2 Express the signal g(t) in Fourier series and draw the amplitude spectrum.

We can approach to the problem similarly as approached to the previous problem. Otherwise, we can also solve the problem utilizing the result of the previous problem (Fig. 2.6).
From the Fig. 2.7, we can define the function $g(t)$ as

$$g(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$$

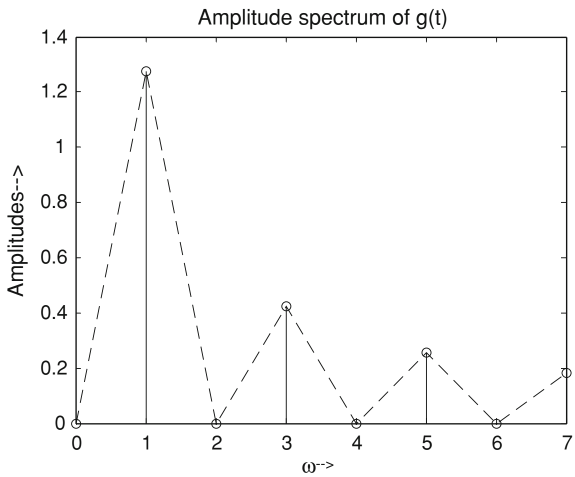


Fig. 2.8 Amplitude spectrum of $g(t)$

If we just take a look over the two function $f(t)$ and $g(t)$ (Figs. 2.6 and 2.7), we can observe some similarities and dissimilarities as follows.
Similarities:

(1) Time periods, i.e., fundamental frequencies for both the signals are same.
(2) Geometric patterns of both the signals are similar, i.e., square wave.

Dissimilarities:

(1) Peak-to-peak amplitude of $g(t)$ is just double with respect to that of $f(t)$.
(2) The average value over one time period is 0.5 for $f(t)$ whereas, average value over time period is zero for $g(t)$

Now, to relating the dissimilarities we can express $g(t)$ in terms of $f(t)$ as, they are having some important signal parameter similarities.

We have just removed the DC component 0.5 and then multiplied the entire signal by 2 (Fig. 2.8).

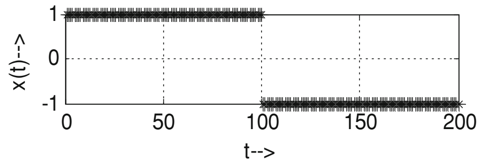
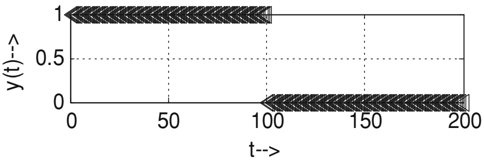
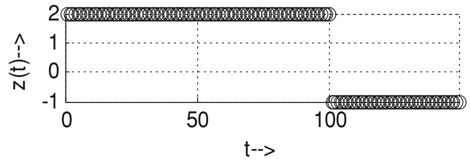
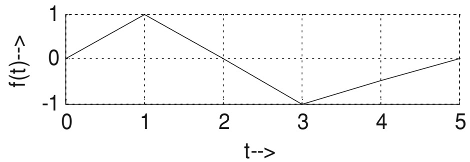
$$g(t) = 2\{f(t) - 0.5\}$$

From Eq. 2.17

$$g(t) = 2\left\{\left(\frac{1}{2} + \frac{2}{\pi}\sin\omega_0 t + \frac{2}{3\pi}\sin 3\omega_0 t + \frac{2}{5\pi}\sin 5\omega_0 t + \ldots\right) - \frac{1}{2}\right\}$$

$$\therefore g(t) = \left(\frac{4}{\pi}\sin\omega_0 t + \frac{4}{3\pi}\sin 3\omega_0 t + \frac{4}{5\pi}\sin 5\omega_0 t + \ldots\right) \tag{2.18}$$

Table 2.2 Derivation of a_0 as extraction of DC offset

Signal	a_0
	$a_0 = 0$ as, areas over and under x-axis over one time period (T = 200) are exactly equal.
	$a_0 = 0.5$ as, if we shift the entire signal down by subtracting a DC of amount 0.5 (or shifting the time axis up to 0.5), we can get equal areas over and under the curve.
	$a_0 = \frac{100 \times 2 + (-50)}{150} = 1$
	$a_0 = \frac{0.5 \times 1 \times 2 + (-0.5 \times 3 \times 1)}{5} = \frac{1 - 1.5}{5} = -\frac{0.5}{5} = -\frac{1}{10}$

2.3.4 Physical Existences of the Coefficients

The existence and measurement of the coefficients of Fourier series is dependent on the shape or pattern of the signal. If we can recognize the pattern in a signal, we can infer to some extent on the existence of coefficients a_0, a_n and b_n. In this subsection we will discus the physical meaning of the coefficients and then we will infer the possible existence of those coefficients. As seen in the Example 2.1 and 2.2, in both the cases, only sine components are there, i.e., a_n for all n is 0. It is to be noted that in the spectral representation (Figs. 2.6 and 2.8) we have used the index 'n' of frequency. In Example 2.1, $a_0 = 0.5$, whereas $a_0 = 0$ in Example 2.2. The reasons behind will be thoroughly understood after going through this subsection.

2.3.4.1 Physical interpretation of a_0

If we look at the Eq. 2.13, it clearly says, this coefficient is actually the average magnitude of the signal over one time period. In other words, this is the DC component of the signal. From Table 2.1 and Eq. 2.3 it is observed that, no *cosine* or *sine* component of a signal is attached with the coefficient a_0. Moreover, it is also seen from Eq. 2.13 that the derivation of a_0 indicates the average area formed by the signal in both the polarities over one complete time period. Therefore, when a signal comes for Fourier analysis, the first step employed is to describe the signal mathematically over one time period. If the areas formed by the signal over and under X-axis are same, the value of a_0 is obviously zero. If the areas are unequal, then we should try to predict the amount of up or down shift of the signal to make the areas over and under x-axis equal. The amount of shift of the time axis is actually the value of a_0. In some cases, the geometric prediction or visualization is not so easy. Then we need to calculate the areas over and below the x-axis over one time period. The total area is then averaged by the time period to get a_0. Some examples of getting a_0 without much calculations are given in Table 2.2.In the four chosen different examples, it has been shown how a_0 is derived by extracting the DC offset of the signal.

2.3.4.2 Physical interpretation of a_n and b_n

From understanding of signal, we can classify signal into three types like (1) odd, (2) even and (3) odd + even. We can also understand that, the odd part of the signal contains sinusoids; even part of the signal contains co-sinusoids. Therefore, if a signal can be clearly identified as odd or even using the two-mirror process as discussed in Chap. 1, we may need to find either the *cos* or the *sin* components for even and odd signals respectively. To measure the fitness of sine or cos, we need to make the signal DC free. Then we need to try to fit cos or sine wave into the signal for closest approximation. As for an example, we can refer Table 2.2. The signal f(t) is of some DC value. If we remove the DC and look at the signal, drawing at least three periods in right and left side of the Y-axis, we can easily feel that *sine* can be fitted into the signal suitably. As obtained from the Eqs. 2.15 and 2.16, a_n and b_n are the cosine and sine components of a signal. Therefore, b_n will have some definite values, whereas $a_n = 0$ as no cosine components should be there in trigonometric Fourier series expression, as a_n be the coefficient of cosine component of the signal. It also inferred that, for odd signals, $a_n = 0$ and in even signal $b_n = 0$, only for the signal which is neither odd nor even, i.e., which can be expressed as summation of odd and even signal, have non-zero a_n and b_n values. The idea would be more clear when we discuss odd and even symmetry in the next section of study.

2.4 Even and Odd Symmetry

Any function *f*(*t*) is even if its plot is symmetrical about the vertical axis, i.e. $f(t) = f(-t)$. We can guess a mirror placed in the y-axis to get the image on the negative x-plane, as shown in the examples below.

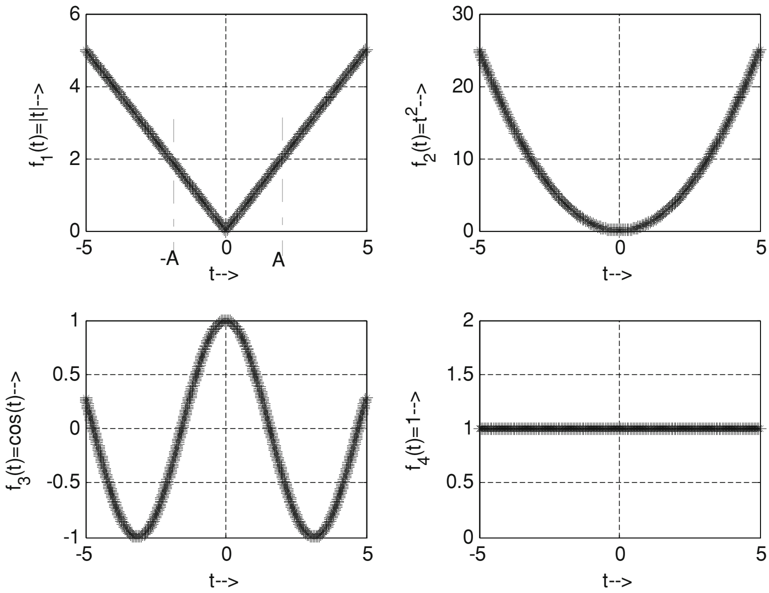


Fig. 2.9 Examples of even functions

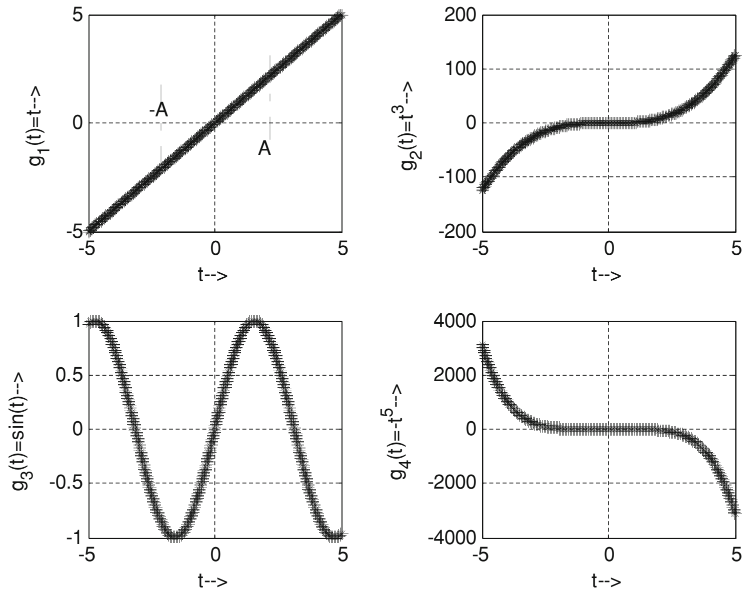


Fig. 2.10 Examples of odd functions

The integral of an **even** function from $-A$ to $+A$ is twice the integral from 0 to $+A$ as shown in $f_I(t)$.

$$\int_{-A}^{+A} f_{\text{even}}(t)dt = 2\int_{0}^{+A} f_{\text{even}}(t)dt \tag{2.19}$$

On the other hand, any function $f(t)$ is **odd** if its plot is anti-symmetrical about the vertical axis, i.e. $f(t) = -f(-t)$. We can understand the formation of the function as double reflection by one mirror placed on the y-axis followed by another mirror placed on the x-axis, as shown in some examples in Figs. 2.9 and 2.10.

The integral of an **odd** function from $-A$ to $+A$ is zero as shown in $g_I(t)$.

$$\int_{-A}^{+A} g_{\text{odd}}(t)dt = 0 \tag{2.20}$$

Therefore, the calculations for derivation of Fourier coefficients become easy (and in some cases unnecessary), after understanding the physical interpretations. The ***even–odd properties*** of functions are as follows. It differs from the general concept of even–odd properties of numbers.

$$\begin{aligned} Even \times Even &= Even \\ Odd \times Odd &= Even \\ Odd \times Even &= Odd \\ Even \times Odd &= Odd \end{aligned} \tag{2.21}$$

From the even–odd properties of even symmetry,

$$a_n = \frac{2}{T}\underbrace{\int_{-T/2}^{T/2} f(t)}_{Even\times}\underbrace{\cos n\omega t dt}_{Even=Even} = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t dt \tag{2.22}$$

$$b_n = \frac{2}{T}\underbrace{\int_{-T/2}^{T/2} f(t)}_{Even\times}\underbrace{\sin n\omega t dt}_{Odd=Odd} = 0 \tag{2.23}$$

Similarly, from the even–odd properties of odd symmetry,

$$a_0 = \frac{2}{T}\underbrace{\int_{-T/2}^{T/2} f(t)dt}_{Odd} = 0 \tag{2.24}$$

$$a_n = \frac{2}{T} \underbrace{\int_{-T/2}^{T/2} f(t) \cos n\omega t dt}_{Odd \times Even = Odd} = 0 \tag{2.25}$$

$$b_n = \frac{2}{T} \underbrace{\int_{-T/2}^{T/2} f(t) \sin n\omega t dt}_{Odd \times Odd = Even} = \frac{4}{T} \int_{0}^{T/2} f(t) \sin n\omega t dt \tag{2.26}$$

2.5 Compact Fourier Series

In basic formation of trigonometric Fourier series, as seen from the previous section, even function gives the valid a_n magnitudes whereas, odd function (signal) gives valid b_n magnitudes only. So, while drawing the magnitude and phase spectrum, we can just take a_n or b_n with respect to the odd or even property of the signal, and then the set (either a_n or b_n) are plotted with respect to n (As Figs. 2.6 and 2.8). But, when we look at those signals which are neither odd nor even, both the magnitudes are non-zero, i.e., valid. Then we need to think about another coefficient which is the sole representative of both a_n and b_n. For this purpose, another now coefficient is proposed as $C_n = \sqrt{a_n^2 + b_n^2}$. The new Fourier series expression incorporating the new coefficient is called as compact Fourier series.

From Eq. 2.3,

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos \text{n}\omega_0 t + b_n \sin \text{n}\omega_0 t) \\ &= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \text{n}\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \text{n}\omega_0 t \right) \end{aligned} \tag{2.27}$$

Now, taking

$$C_0 = a_0 \tag{2.28}$$

$$C_n = \sqrt{a_n^2 + b_n^2} \tag{2.29}$$

$$\cos \theta_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}$$

$$\therefore \sin \theta_n = \sqrt{1 - \cos^2 \theta_n} = \sqrt{1 - \frac{a_n^2}{a_n^2 + b_n^2}} = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}$$

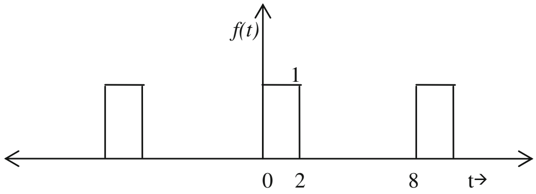


Fig. 2.11 A signal which is neither odd nor even

Therefore, Eq. 2.17 becomes

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n\left(\cos\theta_n \cos\, n\omega_0 t + \sin\theta_n \sin\, n\omega_0 t\right)$$
$$= C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t - \theta_n\right) \tag{2.30}$$

Now,

$$\tan\theta_n = \frac{\sin\theta_n}{\cos\theta_n} = \frac{b_n}{a_n}$$
$$\therefore \theta_n = \tan^{-1}\frac{b_n}{a_n} \tag{2.31}$$

Therefore, Eq. 2.30 can be re-written as

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t - \tan^{-1}\frac{b_n}{a_n}\right) \tag{2.32}$$

This is the generalized or compact form of the trigonometric Fourier series.

Example 2.3 Express the function f(t) is Fourier series and draw the magnitude and phase spectrum (Fig. 2.11)

One period of the signal f(t) is taken out. It is obvious from the figure that, the time period is $T = 8$.

Therefore, fundamental frequency $\omega_0 = 2\pi/8 = 0.7854$ rad/s. From the signal pattern, we have chosen the limit of integration for finding out the coefficients of Fourier series, to be [0,8].

One time period window of the signal now can be defined as

$$f(t) = \begin{cases} 1; & (0 \geq t > 2) \\ 0; & (2 \geq t > 8) \end{cases} \tag{2.33}$$

From the definition of Fourier coefficients,

$$a_0 = \frac{1 \times 2 + 0 \times 6}{8} = \frac{1}{4} \tag{2.34}$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^8 f(x) \times \cos n\omega_0 t dt \\
&= \frac{2}{T}\left[\int_0^2 1 \times \cos n\omega_0 t dt + \int_2^8 0 \times \cos n\omega_0 t dt\right] \\
&= \frac{2}{T} \times \frac{1}{n\omega_0}\left\{\sin n\omega_0 t|_0^2 + 0\right\} \\
&= \frac{2}{T} \times \frac{T}{2\pi n}\sin\frac{4\pi n}{8} \\
&= \frac{1}{n\pi}\sin\frac{n\pi}{2} = \frac{1}{2} sinc\frac{n\pi}{2}
\end{aligned} \tag{2.35}
$$

Thus the a_n values will be zero for even values of n and non-zero with alternative sign for odd values of n.

Therefore, alternatively a_n can be expressed as

$$
a_n = \begin{cases} 0; & for \quad n \rightarrow Even \\ \frac{1}{n\pi}; & for \quad n = 1, 5, 9, 13, \ldots \\ -\frac{1}{n\pi}; & for \quad n = 3, 7, 11, 15, \ldots \end{cases} \tag{2.36}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^8 f(x) \times \sin n\omega_0 t dt \\
&= \frac{2}{T}\left[\int_0^2 1 \times \sin n\omega_0 t dt + \int_2^8 0 \times \sin n\omega_0 t dt\right] \\
&= \frac{2}{T} \times \frac{-1}{n\omega_0}\left\{\cos n\omega_0 t|_0^2 + 0\right\} \\
&= \frac{2}{T} \times \frac{-T}{2\pi n}\left\{\cos\frac{4\pi n}{8} - 1\right\} \\
&= \frac{1}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)
\end{aligned} \tag{2.37}
$$

The b_n coefficients will be zero whenever $\cos(n\pi/2) = 1$, which occurs for $n = 4, 8, 12, 16, \ldots..$

The signal given in this example is following neither the single mirror concept nor the double mirror concept to satisfy the evenness or oddness of a signal respectively. Therefore from the discussion in Chap. 1, we can conclude that the given signal f(t) must be expressed as a summation of an odd and an even signal. a_n and b_n are the representatives of the even and odd component of the given signal, respectively. Now, to draw the amplitude spectrum, we need to consider

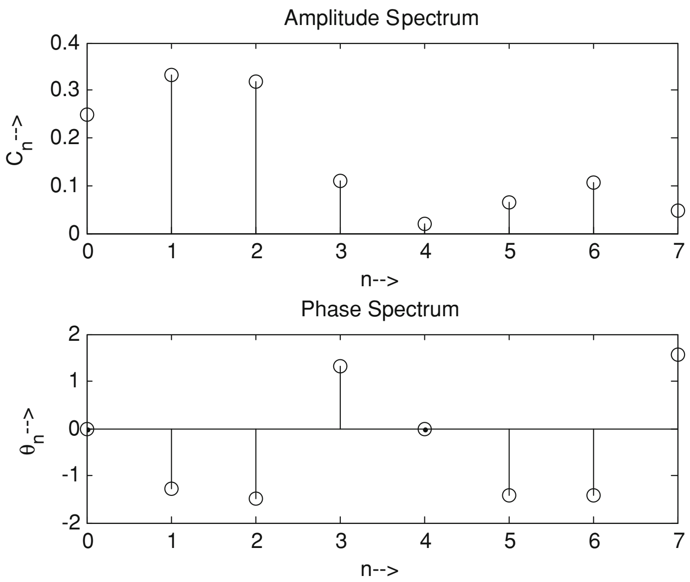


Fig. 2.12 Amplitude and phase spectrum of a general signal (neither odd nor even) using compact Fourier series

both a_n and b_n. As they are orthogonal pair, we can express resultant amplitude using Pythagoras theorem for right angle triangle and express the compact amplitude as hypotenuse (Eq. 2.29).

From Eqs. 2.28, 2.29 and 2.31 we can compute the compact Fourier series coefficients as

$$C_0 = a_0$$
$$C_n = \sqrt{a_n^2 + b_n^2}$$
$$\theta_n = \tan^{-1}\frac{b_n}{a_n}$$

Next, C_n and θ_n are plotted with respect to n to represent amplitude spectrum and phase spectrum respectively as shown in Fig. 2.12.

2.6 Dirichlet Conditions

For the existence of Fourier series, there are two basic conditions as follows.

1. **Necessary condition:** The series exists if and only if the coefficients $a_0, a_n, b_n$are finite (Eqs. 2.13, 2.15, 2.16). From Eq. 2.13 it is understandable that, the existence of the coefficients are guaranteed if $g(t)$ is absolutely integrable over one time period, i.e.,

$$\int_T |g(t)|dt < \infty \tag{2.38}$$

This is known as the *weak Dirichlet condition.*

If a function satisfies weak Dirichlet condition, the existence of Fourier series is guaranteed, but the series may not converge at every point.

As for an example, if a function has infinite number of maxima and minima in one time period, then the function contains an appreciable amount of components of frequencies approaching to infinity. Thus, the higher coefficients in the series do not decay rapidly. Therefore the series will not converge uniformly. For convergence of the Fourier series, we need another condition over weak Dirichlet condition as follows.

2. **Sufficient condition:** The function have only finite number of maxima and minima in one time period, and it may have only finite number of discontinuities in one time period.
   Combining these two conditions we can form complete condition for applicability and convergence of Fourier series named as *strong Dirichlet condition.* Signal obeying the strong Dirichlet condition can be expressed in Fourier series and the series will be convergent. This is the sufficient condition for convergence of Fourier series.

2.7 Exponential Fourier Series

Exponential Fourier series is the modified form of trigonometric Fourier series. This expression can also be called as *most generalized* or *double sided Fourier series.* As we know sine and cosine can be expressed using summation of exponential functions. From Eq. 2.11d, it is also observed that, exponential function also obeys the rule of orthogonality. From trigonometric Fourier series,

$$\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos \mathrm{n}\omega_0 t + b_n \sin \mathrm{n}\omega_0 t) \\
&= a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{\mathrm{e}^{\mathrm{jn}\omega_0 t} + \mathrm{e}^{-\mathrm{jn}\omega_0 t}}{2} + b_n \frac{\mathrm{e}^{\mathrm{jn}\omega_0 t} - \mathrm{e}^{-\mathrm{jn}\omega_0 t}}{2\mathrm{j}} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \frac{(a_n - jb_n)}{2} \mathrm{e}^{\mathrm{jn}\omega_0 t} + \frac{(a_n + jb_n)}{2} \mathrm{e}^{-\mathrm{jn}\omega_0 t}
\end{aligned} \tag{2.39}$$

Now, if we assume that, the new coefficients for this new type of Fourier series representation are given by

$$D_0 = a_0 \tag{2.40}$$

$$D_n = \frac{(a_n - jb_n)}{2} \tag{2.41}$$

$$D_{-n} = D_n^* = \frac{(a_n + jb_n)}{2} \tag{2.42}$$

Therefore, Eq. 2.3 can be re-written as

$$f(t) = D_0 + \sum_{n=1}^{\infty} \left(D_n \mathrm{e}^{\mathrm{jn}\omega_0 t} + D_{-n} \mathrm{e}^{-\mathrm{jn}\omega_0 t} \right) \tag{2.43}$$

The first term within the summation signifies positive going of ordered variable *n*. Similarly, the second term within summation signifies positive going of –*n*, i.e., negative going of *n*. We can therefore express Eq. 2.43 as

$$f(t) = D_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} D_n \mathrm{e}^{\mathrm{jn}\omega_0 t} \tag{2.44}$$

We can also include the zero$^{\text{th}}$ index easily. The first term D_0 can be written as $D_0 e^{j0\omega_0 t}$. Therefore, the final equation is

$$f(t) = \sum_{n=-\infty}^{\infty} D_n \mathrm{e}^{\mathrm{jn}\omega_0 t} \tag{2.45}$$

Eq. 2.45 is the expression of *complex Fourier series*. Similar to the other Fourier series expression, in complex Fourier series also we can determine the Fourier coefficient. From the relationship between the coefficients of classical trigonometric Fourier series and that of the complex Fourier series illustrated in Eq. 2.35, and the derived expression from Eq. 2.13, 2.15 and 2.16, we have

$$\begin{aligned}
D_n &= \frac{1}{2}(a_n - jb_n) \\
&= \frac{1}{2}\left\{ \frac{2}{T}\int_T f(t)\cos n\omega_0 t dt - j\frac{2}{T}\int_T f(t)\sin n\omega_0 t dt \right\} \\
&= \frac{1}{\mathrm{T}}\left\{ \int_T f(t)\cos n\omega_0 t dt - \int_T f(t) j \sin n\omega_0 t dt \right\} \\
&= \frac{1}{\mathrm{T}}\int_T f(t)\{\cos n\omega_0 t - j\sin n\omega_0 t\} dt \\
&= \frac{1}{\mathrm{T}}\int_T f(t) e^{-jn\omega_0 t} dt
\end{aligned} \tag{2.46}$$

This is really interesting to see that (Eq. 2.39, 2.40), unlike to the other forms of Fourier series; here the sequence index n can run from $-\infty$ to ∞. So, this form of Fourier series representation is defined as *double sided Fourier series* or universal Fourier series.

2.8 Parseval's Theorem for Power

Parserval's theorem states that, the average power in a periodic signal is equal to the sum of the average power in its DC component and the average powers in its harmonics.

Let's consider a periodic signal $x(t)$.

As we know that

$$|x(t)|^2 = x(t)x^*(t)$$

($x^*(t)$ is the complex conjugate of $x(t)$)
The power of a signal $x(t)$ over a cycle is expressed as

$$P = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 \tag{2.47}$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)x^*(t)dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x^*(t)\left[\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}\right]dt$$

Here, $\omega_0 = 2\pi/T$
Interchanging the order of integration and summation, we get

$$P = \frac{1}{T}\sum_{n=-\infty}^{\infty} D_n.\int_{-\frac{T}{2}}^{+\frac{T}{2}} x^*(t)e^{jn\omega_0 t}dt \tag{2.48}$$

As,

$$D_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt \tag{2.49}$$

$$D_n^* = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{jn\omega_0 t} dt \tag{2.50}$$

So, from above three equations, power can be expressed as

$$\begin{aligned} P &= \frac{1}{T} \sum_{n=-\infty}^{\infty} D_n . \left(TD_n^*\right) \\ &= \sum_{n=-\infty}^{\infty} D_n . \left(D_n^*\right) \\ &= \sum_{n=-\infty}^{\infty} |D_n|^2 \end{aligned} \tag{2.51}$$

From the expressions given in Eqs. 2.47 and 2.51, Perseval's Theorem of power is validated.

2.9 Phase Congruency: Application of Fourier Series in 1D and 2D Signal Processing (Image Processing)

Till the previous section we have discussed a lot on frequency domain representation of periodic signals. As understood from the statement of Fourier series, periodic signals always hold sinusoidal components of different frequencies which are integral multiple of a fundamental frequency. In the present section we'll concentrate on the phase criteria or rather the relative phase property of the component signals.

If we just take a look on the Example 2.1, it has derived infinite number of sinusoid components from a given square wave. To look closely, we see that the sinusoid components are all odd harmonics of the fundamental. The amplitudes of the even harmonics are zero, in this case. In the present section we will investigate the "why" of the result and from the understanding we will discuss a real life application using this concept.

In Fig. 2.13 we have shown the two sets of sinusoidal components and their respective synthesized signal. In Fig. 2.13a, a circular region is shown. For both the signals (with frequency f_0 and $2f_0$), the circular region is the region of zero crossing. The interesting thing to observe in Fig. 2.13a is that, the lower frequency signal goes from positive to negative polarity (i.e., negative going) at zero-crossing where as the higher frequency signal is positive going at the region of zero crossing. These opposite going of the component signals form a cross-like (X pattern) appearance at zero crossing. Therefore these two signals cannot contribute in synthesizing a new signal like square wave which is further negative going at the region of zero crossing. In other words all the component signals must support the fundamental frequency sinusoid to increase the sloe so that it approaches towards a perfect vertical line.

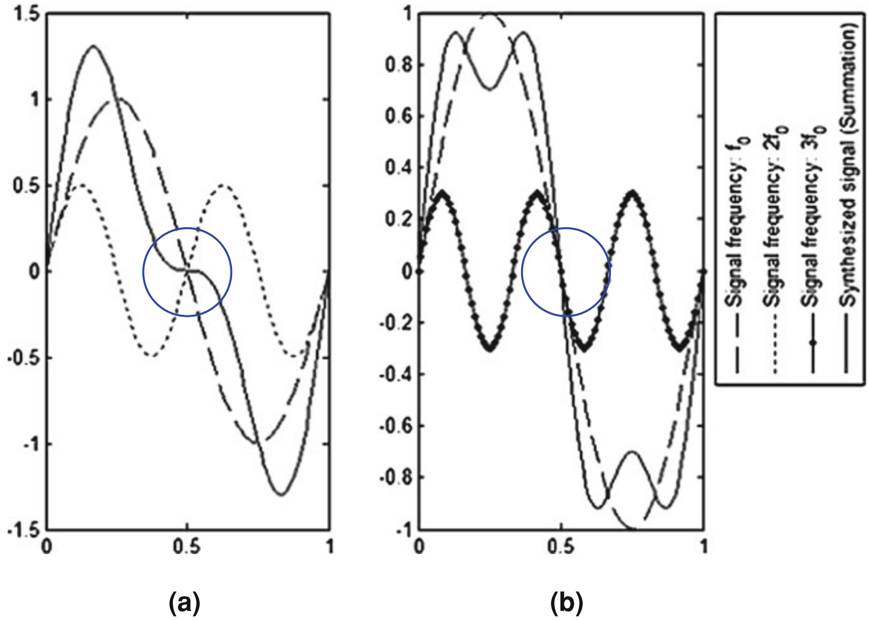


Fig. 2.13 Illustration of phase congruency in terms of Fourier components

On the contrary, as seen in Fig. 2.13b, we have used the third harmonic $(3f_0)$ along with the fundamental signal for Fourier synthesis. It has successfully generated a square like signal with higher slope with respect to the fundamental signal. Actually at the zero crossing, both the component signal are negative going. Therefore in synthesis the higher frequency component strengthens the square wave synthesis. That's why we did not get any even harmonics in Fourier synthesis of a square wave. This is the concept of *phase congruency*. In the present example, the region of amplitude transition of the square wave from +1 to -1 is detected in terms of congruent phase, i.e., the region where are the component signals would meet in same phase. The idea can also be realized from Fig. 2.15. The readers can also design their own experiments for the realization of phase congruency by using the GUI based Fourier synthesizer (Version 1.1), enclosed herewith as supplementary electronic material.

The step-like transition in one dimensional signal is equivalent to sharp intensity transition with respect to space in image like two dimensional signals. This sharp intensity-transition can also be characterized as an "edge" in image. Therefore, from the concept of phase congruency we can also detect edge like features from image.

In the classical approach of edge detection researchers always seared for high intensity gradients in an image. Rather than think of features in differential terms an alternative approach is to think of features in the *frequency domain*. Image profiles can be thought of as being formed by a Fourier series as shown in

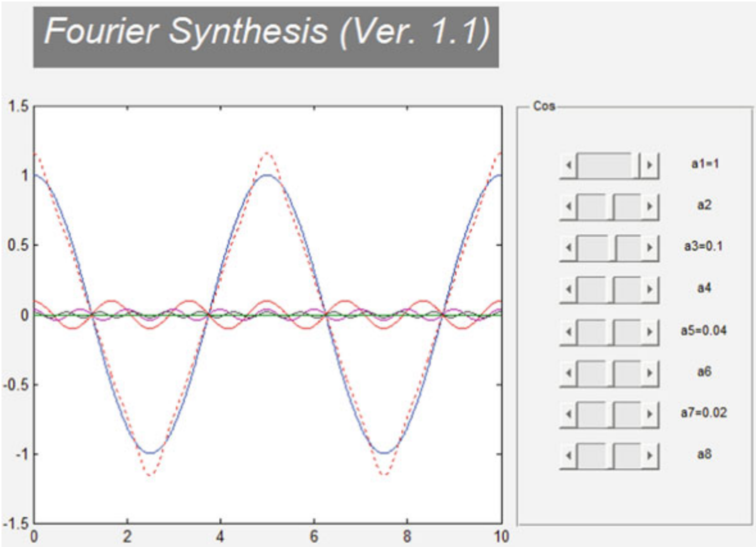


Fig. 2.14 Generation of triangular wave by Fourier synthesis

Figs. 2.14 and 2.15. The triangular and square spatial waveforms can be expressed in terms of Fourier components as I_1 and I_2 respectively as follows. Here only one direction (either row wise or column wise) is considered for the ease of understanding.

$$
\begin{aligned}
I_1(x) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\{(2n+1)x\} \\
I_1(x) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sin\left\{(2n+1)x + \frac{\pi}{2}\right\}
\end{aligned} \tag{2.52}
$$

and

$$
I_2(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\{(2n+1)x\} \tag{2.53}
$$

Here also it is to be noted that the Fourier components are all *in phase* at the point of the step in the square wave (Fig. 2.15), and at the peaks and troughs of the triangular wave (Fig. 2.14). Congruency of phase at any angle produces a clearly perceived feature. We can generalize our Fourier series expression to generate a wide range of waveforms with the equation

$$
I(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^p} \sin\{(2n+1)x + \phi\} \tag{2.54}
$$

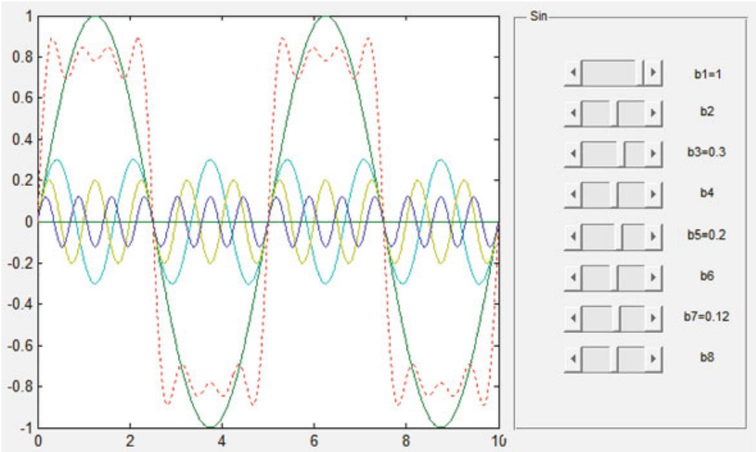


Fig. 2.15 Generation of square wave by Fourier synthesis

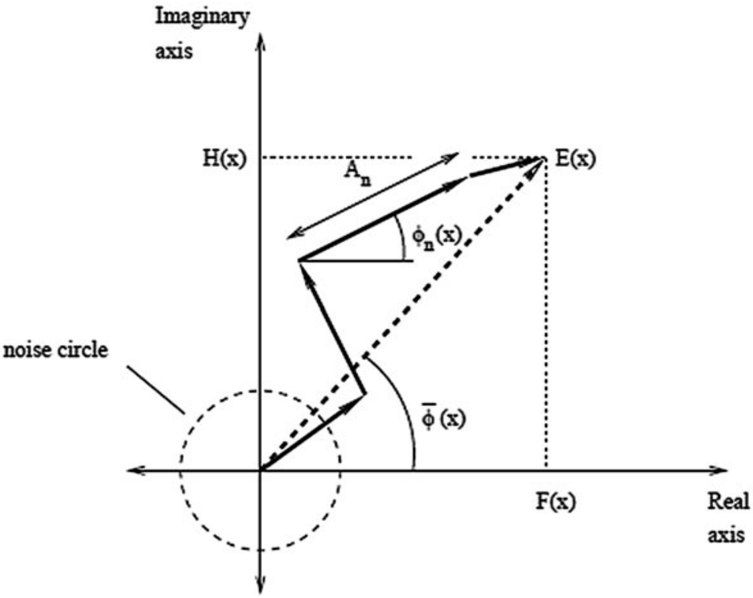


Fig. 2.16 Polar diagram showing the Fourier components at a location in the signal plotted head to tail. The weighted mean phase angle is given by $A(x)$. The noise circle represents the level of $E(x)$ one can expect just from the noise in the signal

where ϕ is the phase offset defining the angle at which phase congruency occurs at features, and p is the exponent that describes the rate of amplitude decay with frequency in the Fourier series.

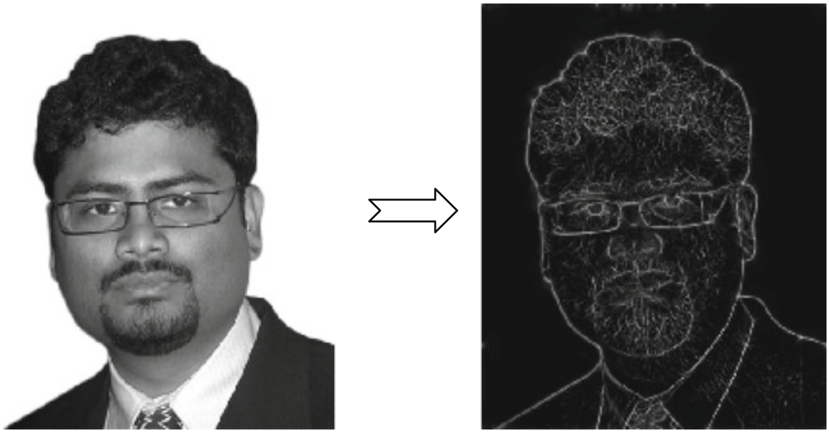


Fig. 2.17 Edge-like feature extraction from an image using Kovesi's measure of phase congruency

The measurement of phase congruency at a point in a signal can be seen geometrically in Fig. 2.16. The local Fourier components at a location X in the signal will each have amplitude $A_n(x)$ and a phase angle $\phi_n(x)$. Figure 2.16 plots these local Fourier components as complex vectors adding head to tail. The sum of these components projected onto the real axis represents $F(x)$, the original signal. The magnitude of the vector from the origin to the end point is the *Local Energy*, $|E(x)|$.

The measure of phase congruency developed by Morrone et al. [4] is

$$PC_{Morrone} = \frac{|E(x)|}{\sum_n A_n(x)} \tag{2.55}$$

Under this definition phase congruency is the ratio of $|E(x)|$ to the overall path length taken by the local Fourier components in reaching the end point. If all the Fourier components are in phase all the complex vectors would be aligned and the ratio $\frac{|E(x)|}{\sum_n A_n(x)}$ would be one. If there is no coherence of phase, the ratio falls to a minimum of zero. Phase congruency provides a measure that is independent of the overall magnitude of the signal making it invariant to variations in image illumination and/or contrast. Fixed threshold values of feature significance can then be used over wide classes of images. It can be shown that this measure of phase congruency is a function of the cosine of the deviation of each phase component from the mean

$$PC_{derived} = \frac{\sum_n A_n\left(\cos\left(\phi_n(x) - \bar{\phi}(x)\right)\right)}{\sum_n A_n(x)} \tag{2.56}$$

This measure of phase congruency (Eq. 2.55) does not provide good localization and it is also sensitive to noise. Kovesi [5–8] developed a modified measure consisting of the cosine minus the magnitude of the sine of the phase deviation as given in the following equation; this produces a more localized response.

$$PC_{Kovesi} = \frac{\sum_n W(x)\left\lfloor A_n\left(\cos\left(\phi_n(x) - \bar{\phi}(x)\right) - \left|\sin\left(\phi_n(x) - \bar{\phi}(x)\right)\right|\right) - T\right\rfloor}{\sum_n A_n(x) + \varepsilon} \tag{2.57}$$

The term $W(x)$ is a factor that weights for frequency spread (congruency over many frequencies is more significant than congruency over a few frequencies). A small constant ε is incorporated to avoid division by zero. Only energy values that exceed the threshold T, the estimated noise influence, are counted in the result. The symbols $\lfloor\ \rfloor$ denote that the enclosed quantity is equal to itself when its value is positive, and zero otherwise. Edge detection using Kovesi's measure of phase congruency [5–8] is presented in Fig. 2.17.

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