2

The Elements of Euclid

"At age eleven, I began Euclid, with my brother as my tutor. This was one of the greatest events of my life, as dazzling as first love. I had not imagined that there was anything as delicious in the world." (B. Russell, quoted from K. Hoechsmann, *Editorial, π in the Sky*, Issue 9, Dec. 2005. A few paragraphs later K. H. added: An innocent look at a page of contemporary theorems is no doubt less likely to evoke feelings of "first love".)

"At the age of 16, Abel's genius suddenly became apparent. Mr. Holmboë, then professor in his school, gave him private lessons. Having quickly absorbed the *Elements*, he went through the *Introductio* and the *Institutiones calculi differentialis* and *integralis* of Euler. From here on, he progressed alone."
(Obituary for Abel by Crelle, *J. Reine Angew. Math.* 4 (1829) p. 402; transl. from the French)

"The year 1868 must be characterised as [Sophus Lie's] breakthrough year. ... as early as January, he borrowed [from the University Library] Euclid's major work, *The Elements* ..." (*The Mathematician Sophus Lie* by A. Stubhaug, Springer 2002, p. 102)

"There never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures ... from the plan laid down by Euclid."
(A. De Morgan 1848; copied from the *Preface* of Heath, 1926)

"Die Lehrart, die man schon in dem ältesten auf unsere Zeit gekommenen Lehrbuche der Mathematik (den Elementen des Euklides) antrifft, hat einen so hohen Grad der Vollkommenheit, dass sie von jeher ein Gegenstand der Bewunderung [war] ... [The style of teaching, which we already encounter in the oldest mathematical textbook that has survived (the *Elements* of Euclid), has such a high degree of perfection that it has always been the object of great admiration ...]" (B. Bolzano, *Grössenlehre*, p. 18r, 1848)

A. Ostermann and G. Wanner, *Geometry by Its History*,
Undergraduate Texts in Mathematics, DOI: 10.1007/978-3-642-29163-0_2,

Euclid's *Elements* are considered by far the most famous mathematical *oeuvre*. Comprising about 500 pages organised in 13 books, they were written around 300 B.C. All the mathematical knowledge of the period is collected there and presented with a rigour which remained unequalled for the following two thousand years.

Over the years, the *Elements* have been copied, recopied, modified, commented upon and interpreted unceasingly. Only the painstaking comparison of all available sources allowed Heiberg in 1888 to essentially reconstruct the original version. The most important source (M.S. 190; this manuscript dates from the 10th century) was discovered in the treasury[1] of the Vatican, when Napoleon's troops invaded Rome in 1809. Heiberg's text has been translated into all scientific languages. The English translation by *Sir Thomas L. Heath* in 1908 (second enlarged edition 1926) is completed by copious comments.

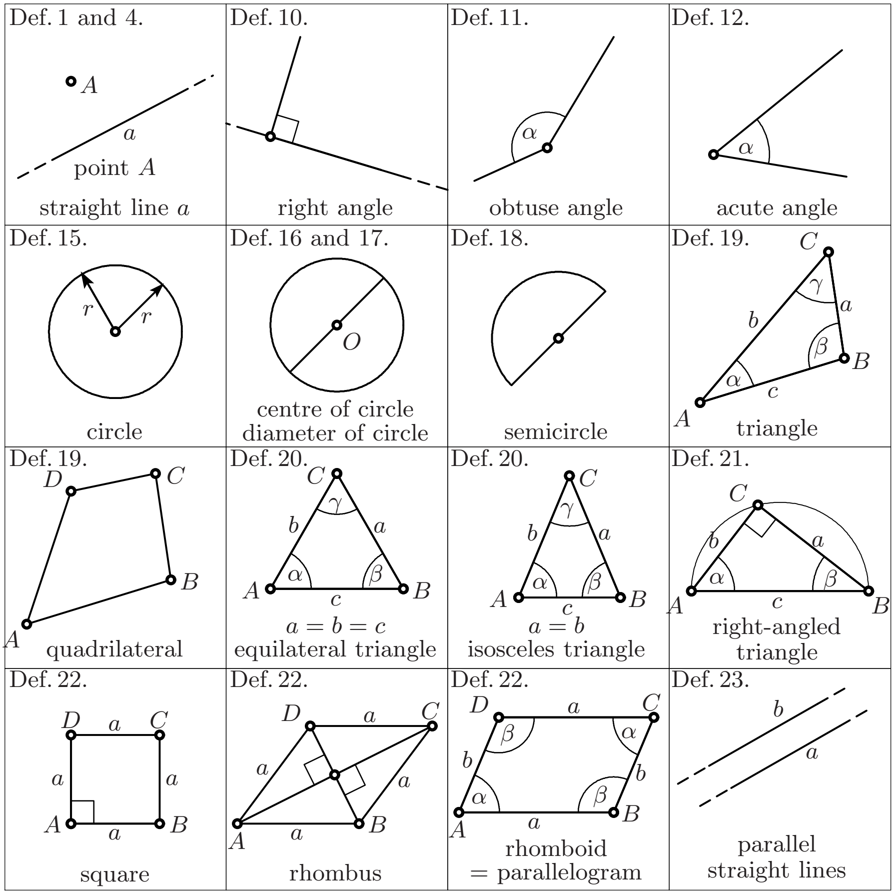


Fig. 2.1. Euclid's definitions from Book I

[1]That's where invading troops go first ...

2.1 Book I

The definitions. The *Elements* start with a long list of 23 *definitions*, which begins with

Σημεῖόν ἐστιν, οὗ μέρος οὐθέν   (A *point* is that which has no part)

and goes on until the definition of parallel lines (see the quotation on p. 36).

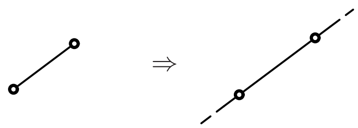
Euclid's definitions avoid figures; in Fig. 2.1 we give an overview of the most interesting definitions in the form of pictures. Euclid does not distinguish between straight lines and segments. For him, two segments are apparently "equal to one another" if their *lengths* are the same. So, for example, a circle is defined to be a plane figure for which all radius lines are "equal to one another".

The postulates.[2] Let the following be postulated:

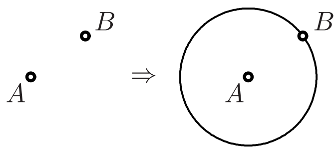
1. To draw a straight line from any point to any point.

Post. 1. 

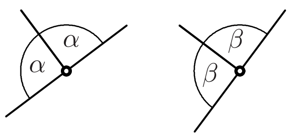
2. To produce a finite straight line continuously in a straight line.

Post. 2. 

3. To describe a circle with any centre and distance.

Post. 3. 

4. That all right angles are equal to one another.

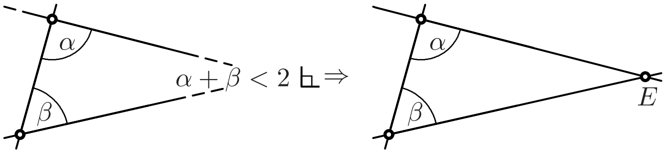
Post. 4. 

$\Rightarrow \qquad \alpha = \beta = \llcorner$

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

[2] English translation from Heath (1926).

Post. 5.



Remark. The first three postulates raise the usual constructions with *ruler*[3] (Post. 1 and 2) and *compass* (Post. 3) to an intellectual level. The fourth postulate expresses the homogeneity of space in all directions by using the right angle as a universal measure for angles; the fifth postulate, finally, is the celebrated *parallel postulate*. Over the centuries, it gave rise to many discussions.

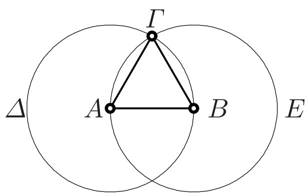
The postulates are followed by *common notions* (also called *axioms* in some translations) which comprise the usual rules for equations and inequalities.

The propositions. Then starts the sequence of *propositions* which develops the entire geometry from the definitions, the five postulates, the axioms and from propositions already proved. Among others, the *trivialities* of Chap. 1 now become real propositions. A characteristic of Euclid's approach is that the alphabetic order of the points indicates the order in which they are constructed during the proof.

In order to give the flavour of the old text, we present the first two propositions in full and with the original Greek letters; but we will soon abandon this cumbersome style[4] and turn to a more concise form with lower case letters for side lengths (Latin alphabet) and angles (Greek alphabet), as has become standard, for good reason, in the meantime.

Eucl. I.1. *On a given finite straight line AB to construct an equilateral triangle.*

The construction is performed by describing a circle $\varDelta$ centred at A and passing through B (Post. 3) and another circle E centred at B and passing through A (Post. 3). Their point of intersection $\varGamma$ is then joined to A and to B (Post. 1). The distance $A\varGamma$ is equal to $B\varGamma$ and to AB, which makes the triangle equilateral.

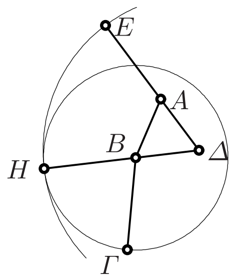


Remark. The fact that Euclid assumes without hesitating the existence of the intersection point $\varGamma$ of two circles has repeatedly been criticised (Zeno, Proclus, ...). Obviously, a *postulate of continuity* is required. For a detailed discussion we refer the reader to Heath (1926, vol. I, p. 242).

[3] In order to emphasise that this ruler has no markings on it, some authors prefer to use the expression straightedge instead.

[4] "... statt der grässlichen Euklidischen Art, nur die Ecken mit Buchstaben zu markieren; [... instead of the horrible Euclidean manner of denoting only the vertices by letters;]" (F. Klein, *Elementarmathematik, Teil II*, 1908, p. 507; in the third ed., 1925, p. 259 the adjective *horrible* is omitted).

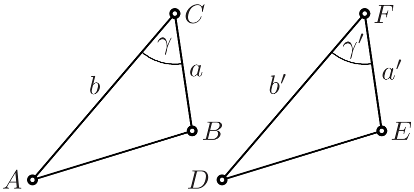
Eucl. I.2. *To place at a given point A a straight line AE equal to a given straight line $B\Gamma$.*



For the construction, one erects an equilateral triangle $AB\Delta$ on the segment AB (Eucl. I.1), produces the lines ΔB and ΔA (Post. 2) and describes the circle with centre B passing through Γ (Post. 3) to find the point H on the line ΔB. Then one draws the circle with centre Δ passing through H (Post. 3). The intersection point E of this circle with the line ΔA has the required property. Indeed, the distance $B\Gamma$ equals the distance BH, and the distance ΔH equals the distance ΔE. Hence, the distance AE equals the distance BH, since the distance ΔB equals ΔA.

Remark. Post. 3 only allows one to draw a circle with given centre A and passing through a given point B. The aim of this proposition is to show that one is now allowed to draw a circle with a *compass-carried* radius. This proof also was criticised by Proclus. Depending on different positions of the points A, B and Γ, various cases must be distinguished, with a slightly different argument in each case. To prove all particular cases separately already here becomes cumbersome. Therefore, Euclid's method will henceforth be our model: as soon as *one* case is understood, the others are left to the intelligent reader.

Eucl. I.4. *Given two triangles with $a = a'$, $b = b'$, $\gamma = \gamma'$, then all sides and angles are equal.*



This result is a cornerstone for all that follows. In its proof, Euclid speaks vaguely of *applying* the triangle ACB onto the triangle DFE, of *placing* the point C on the point F, of placing the line a on the line a', etc. Of course, this lack of precision attracted much criticism.[5] Note that in Hilbert's axiomatic formulation of geometry, see Sect. 2.7, this *proposition* becomes an *axiom*.

Eucl. I.5 (commonly known as *Pons Asinorum*, i.e. asses' bridge). *If in a triangle $a = b$, then $\alpha = \beta$.*

One of the *trivialities* of the previous section thus becomes a real theorem. Let us see how Euclid proved this proposition. One produces (see Fig. 2.2,

[5] "Betrachten wir aber andererseits - das scheint noch die einzig mögliche Lösung in diesem Wirrwarr - diese Nr. 4 als ein späteres Einschiebsel ... [If we consider on the other hand — and this seems to be the only possible solution in this chaos — this No. 4 as a later insertion ...]" (F. Klein, *Elementarmathematik, Teil II*, 1908, p. 416; third ed., 1925, p. 217 with a modified wording).

left) CA and CB (Post. 2) to the points F and G with $AF = BG$ (Eucl. I.2), and joins F to B and A to G (Post. 1). Thus the triangles FCB and GCA are equal by Eucl. I.4, i.e. $\alpha + \delta = \beta + \varepsilon$, $\eta = \zeta$ and $FB = GA$. Now, by Eucl. I.4, the triangles AFB and BGA are equal and thus $\delta = \varepsilon$. Using the above identity, one has $\alpha = \beta$. This seems to be a brilliant proof, but is in fact needlessly complicated. Pappus remarked 600 years later that it would be sufficient to apply Eucl. I.4 to the triangles ACB and BCA with A and B interchanged, see Fig. 2.2, centre and right.

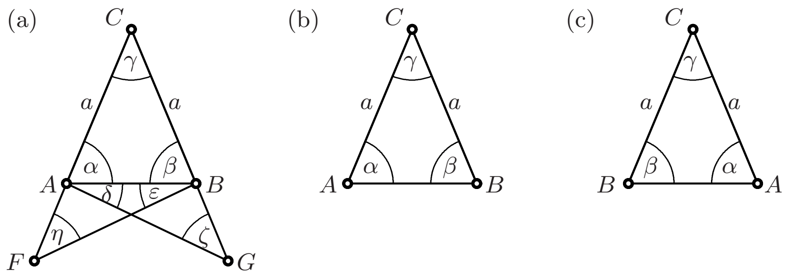


Fig. 2.2. Angles in an isosceles triangle

This proposition is immediately followed by Eucl. I.6, where the converse implication is proved: $\alpha = \beta$ implies $a = b$.

The next two propositions treat the problem of uniquely determining a triangle by prescribing the length of the three sides.

Eucl. I.7. *Consider the two triangles of Fig.* 2.3 (a)*, erected on the same base AB and on the same side of it. If $a = a'$ and $b = b'$, then $C = D$.*

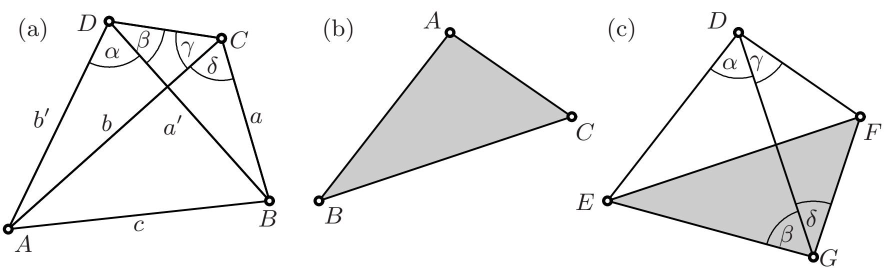


Fig. 2.3. Triangles with equal sides

Proof by Euclid. Suppose that $C \neq D$. Since DAC is isosceles by hypothesis, $\alpha + \beta = \gamma$ (Eucl. I.5). Since DBC is isosceles, $\beta = \gamma + \delta$ (Eucl. I.5). Thus we have on the one hand $\gamma > \beta$, and on the other hand $\gamma < \beta$, which is impossible. $\square$

This is our first *indirect proof*. More than two thousand years later, a school of mathematics rejected this kind of reasoning, because "one can not prove something true with the help of something false" (L.E.J. Brouwer, 1881–1966).

Eucl. I.8. *If two triangles ABC and DEF have the same sides, they also have the same angles.*

The proof of Philo of Byzantium, which is given here, is more elegant than Euclid's. We apply the triangle ABC (see Fig. 2.3 (b)) onto the triangle DEF in such a manner that the line BC is placed on EF and the point A which becomes G lies on the opposite side of EF to D (see Fig. 2.3 (c)). By hypothesis, DEG is isosceles and thus $\alpha = \beta$ (Eucl. I.5). But DFG is also isosceles and hence $\gamma = \delta$ (Eucl. I.5). Thus the angle at A $(= \beta + \delta)$ is equal to the angle at D $(= \alpha + \gamma)$. For the other angles, one repeats the same reasoning, placing first AC on DF, then AB on DE.

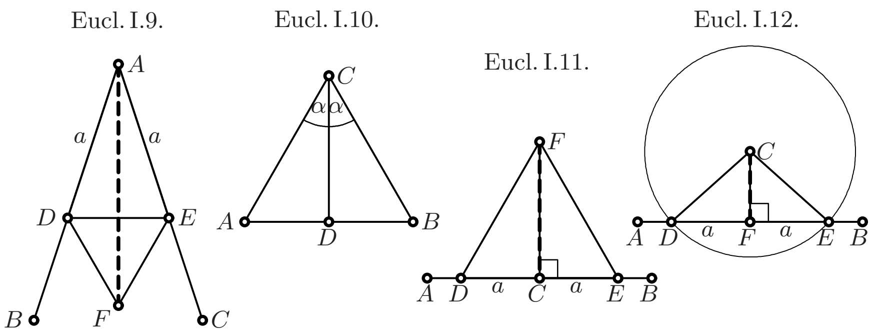


Fig. 2.4. Propositions I.9–I.12

Eucl. I.9–I.12. These propositions treat the bisection of an angle BAC (see Fig. 2.4.I.9), the bisection of a line AB (see Fig. 2.4.I.10) and the erection of the perpendicular to a line AB at a point C on it (see Fig. 2.4.I.11). The common tool for solving these three problems is the equilateral triangle (Eucl. I.1). Finally, the construction of a perpendicular to a line AB *from* a point C outside of it (see Fig. 2.4.I.12) is achieved with the help of a circle (Post. 3) and the midpoint of DE (Eucl. I.10).

The entrance of Postulate 4.

> "When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands".
>
> (Def. 10 of Euclid's first book in the transl. of Heath, 1926).

The fourth postulate expresses the homogeneity of the plane, the absence of any privileged direction, and allows one to compare, add and subtract the angles around a point. It does this by defining *the right angle* as a universal unit. We denote this angle (90°) by the symbol $\llcorner$.

Eucl. I.13. *Let the line AB cut the line CD (Fig. 2.5). Then $\alpha + \beta = 2\llcorner$.*

Proof. Draw the perpendicular BE, which divides the angle β into $\llcorner + \eta$. Thus

$$\left.\begin{aligned} \beta &= \llcorner + \eta \\ \alpha + \eta &= \llcorner \end{aligned}\right\} \quad \Rightarrow \quad \alpha + \beta + \eta = 2\llcorner + \eta$$

which proves the assertion. □

Fig. 2.5. Eucl. I.13 (left) and its proof (right)

Eucl. I.14. *In the situation of Fig. 2.6 (left), let $\alpha + \beta = 2\llcorner$. Then C lies on the line DB.*

Fig. 2.6. Eucl. I.14 (left) and its proof (right)

Proof. Let E lie on the line DB, i.e. by Eucl. I.13, let $\gamma + \beta = 2\llcorner$. By hypothesis, $\alpha + \beta = 2\llcorner$. These angles are equal by the fourth postulate, hence $\gamma = \alpha$. Therefore, E and C lie on the same line. □

Eucl. I.15. *If two straight lines cut one another, they make the opposite angles equal to one another, i.e. $\alpha = \beta$ in Fig. 2.7 (left).*

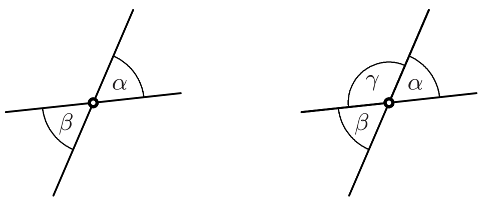


Fig. 2.7. Eucl. I.15 (left) and its proof (right)

Proof. By Eucl. I.13, we have $\alpha + \gamma = 2\llcorner$ and also $\gamma + \beta = 2\llcorner$. By Post. 4, $\alpha + \gamma = \gamma + \beta$. The result then follows from subtracting γ from each side. □

Eucl. I.16. *If one side of a triangle is produced at C (see Fig. 2.8), the exterior angle δ satisfies $\delta > \alpha$ and $\delta > \beta$.*

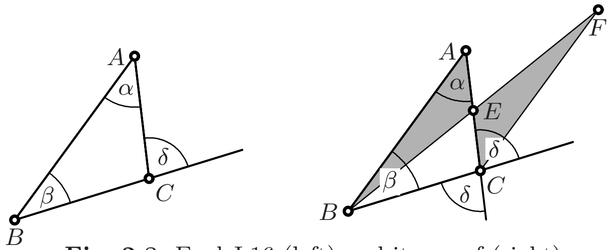


Fig. 2.8. Eucl. I.16 (left) and its proof (right)

Proof. Let E be the midpoint of AC (Eucl. I.10). We produce BE (Post. 2) and cut off the distance EF such that $EF = BE$ (Post. 3). The grey angles at E are equal (Eucl. I.15), hence the two grey triangles are identical (Eucl. I.4). Thus the grey angle at C is α, which is obviously smaller than δ. For the second inequality, one proceeds similarly with the angle on the other side of C (which is equal to δ by Eucl. I.15). □

Remark. In the geometry on the *sphere*, which we will discuss in more detail in Section 5.6, Eucl. I.16 is the first of Euclid's propositions which does not remain valid. Suppose, for example, that B is at the North Pole and A, E and C lie on the Equator. Then $\alpha = \llcorner$ and $\delta = \llcorner$, hence the inequality $\delta > \alpha$ is false. The reason is that the point F, which in our example becomes the South Pole, is no longer certain to remain in the open sector between the produced lines CA and BC.

Eucl. I.17–I.26. Various theorems of Euclid on the congruence of triangles determined by certain side lengths or angles (see Fig. 2.9). The ambiguous case ASS (last picture) is not mentioned by Euclid. For an inequality involving the angles and sides of a triangle (Eucl. I.18), see Exercise 11 below.

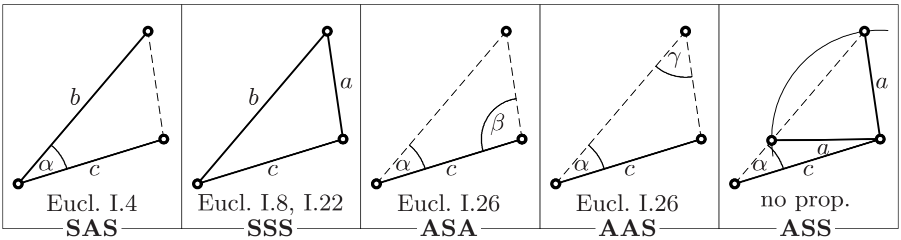


Fig. 2.9. Congruence theorems for triangles

Eucl. I.20 states the famous *triangle inequality*

$$a < b + c, \qquad b < c + a, \qquad c < a + b \tag{2.1}$$

(see Exercise 12 below). This result has been ridiculed as being evident even to an ass. For if one puts the ass at one vertex of the triangle and hay at another, the ass will follow the side that joins the two vertices and will not make the detour through the third vertex (*digni ipsi, qui cum Asino foenum essent*, Heath, 1926, vol. I, p. 287). Proclus gave a long logical-philosophical answer. Instead, he could have said briefly: "The *Elements* were not written for asses".

> "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction".
>
> (Def. 23 of Euclid's first book in the transl. of Heath, 1926).

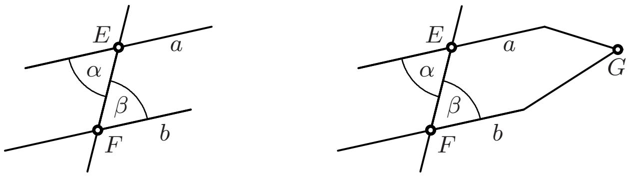


Fig. 2.10. Eucl. I.27 (left) and its proof (right)

Eucl. I.27. *If some line cuts two lines a and b under angles α and β (see Fig. 2.10), then $\alpha = \beta$ implies that the lines are parallel. In this case, we write $a \parallel b$ for short.*

Proof. If a and b were not parallel, they would meet in a point G, see Fig. 2.10. Then EGF would be a triangle having α as exterior angle. Therefore, α would be greater than β (Eucl. I.16), which contradicts the assumption. □

The entrance of Postulate 5. Eucl. I.27, which ensures the *existence* of parallels (simply take $\alpha = \beta$ and you have a parallel), is the last of the propositions, carefully collected by Euclid at the beginning of his treatise, which do not require the fifth postulate for its proof. This part of geometry is called *absolute geometry*. For all that follows we need the *uniqueness* of parallels, which requires the fifth postulate.

Eucl. I.29. *If $a \parallel b$ (see Fig. 2.11), then $\alpha = \beta$.*

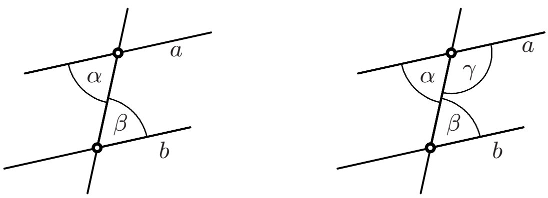


Fig. 2.11. Eucl. I.29 (left) and its proof (right)

Proof. Suppose $\alpha > \beta$. By Eucl. I.13, $\alpha + \gamma = 2\llcorner$, hence $\beta + \gamma < 2\llcorner$. By the fifth postulate, these lines have to meet, which is a contradiction. A similar reasoning shows that $\alpha < \beta$ is also impossible. $\square$

Remark. Combined with Eucl. I.15, the propositions Eucl. I.27 and Eucl. I.29 give variants, one of which formulates the *trivial properties of parallel angles* of Fig. 1.7 (Eucl. I.28).

Remark. For more than 2000 years, geometers conjectured that Eucl. I.29 could be established without appealing to the fifth postulate. Many attempts were made to prove this conjecture, without success. We shall return to this question in Section 2.7.

Eucl. I.30. *For any three lines a, b, c with $a \parallel b$ and $b \parallel c$, we have $a \parallel c$.*

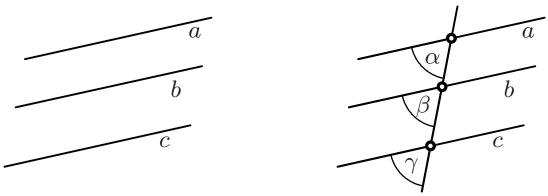


Fig. 2.12. Eucl. I.30 (left) and its proof (right)

Proof. By Eucl. I.27 and Eucl. I.29, the lines a and b are parallel if and only if the angles α and β are equal. $\square$

Eucl. I.31. *Drawing a parallel to a given line through a given point A.*

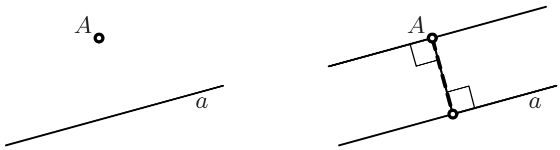


Fig. 2.13. Eucl. I.31 (left) and the proposed construction (right)

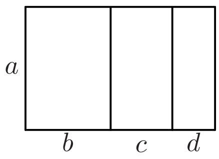
Proof. Euclid's proof makes use of Eucl. I.23 which is itself a consequence of Eucl. I.22. One can also use two orthogonal lines (Eucl. I.12 followed by Eucl. I.11). $\square$

Remark. Proclus made the following statement in his commentary: *There exists* at most one *line through a given point A which is parallel to a given line.* This statement turns out to be equivalent to the fifth postulate. In the form just given, it is called *Playfair's axiom* (1795).

Eucl. I.32 gives the formula $\alpha + \beta + \gamma = 2\llcorner$ for the three angles of an arbitrary triangle, see (1.1) and the proof in Fig. 1.8. This is a very old theorem, certainly known to Thales. It comes quite late in Euclid's list, since its proof requires the fifth postulate.

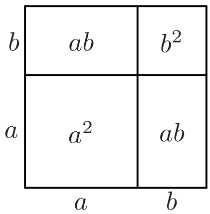
The remainder of Book I. Eucl. I.33–34 treat parallelograms; Eucl. I.35–41 the areas of parallelograms and triangles; Eucl. I.42–45 the construction of parallelograms with a prescribed area; Eucl. I.46 treats the construction of a square. The highlight of the first book, however, is Pythagoras' theorem (Eucl. I.47, see the proof on page 16 and Fig. 1.19) and its converse: *if a, b, c are the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is right-angled.*

Book II. This book contains *geometrical algebra*, i.e. algebra expressed in geometric terms. For instance, the product of two numbers a, b is represented geometrically by the area of a rectangle with sides a and b. We have for example the following relations, Eucl. II.1 and Eucl. II.4:



$$\Leftrightarrow \quad a(b + c + d) = ab + ac + ad$$

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \Leftrightarrow$$



Eucl. II.5 concerns the identity

$$a^2 - b^2 = (a + b)(a - b)$$

(see Fig. 2.14 left). The two light grey rectangles are the same. If one adds the dark rectangle to each, one obtains on the left the rectangle $(a + b) \times (a - b)$, and on the right an L-shaped "gnomon", which is the difference of a^2 and b^2.

Eucl. II.8. The identity $(a + b)^2 - (a - b)^2 = 4ab$ (see Exercise 14 below).

Eucl. II.13. The identity[6]

$$2uc = b^2 + c^2 - a^2 \tag{2.2}$$

for the segment u cut off from the side of a triangle by the altitude (see Fig. 2.14, middle). Euclid obtains this result from $c^2 + u^2 = 2cu + (c - u)^2$ (which is Eucl. II.7, a variant of Eucl. II.4), by adding h^2 on both sides and applying Eucl. I.47 twice.

[6] The original text, in Heath's translation, is as follows: "In acute-angled triangles the square of the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle." We see how complicated life was before the invention of good algebraic notation; and the case of an obtuse angle, where u becomes negative, required another proposition (Eucl. II.12).

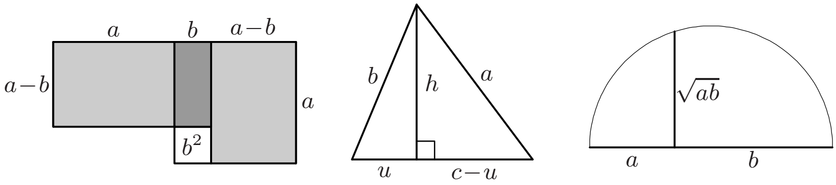


Fig. 2.14. Eucl. II.5 (left), Eucl. II.13 (middle), and Eucl. II.14 (right)

Remark. For a *direct* proof of (2.2), without using Pythagoras' theorem, see Exercise 18 below. With the advance of algebra, the above propositions can all be obtained from Eucl. II.1 by simple calculations. However, Euclid's figures remain beautiful illustrations for these algebraic identities and, moreover, pictures such as that in Fig. 2.14 (left) appeared at the very beginning of this algebra (see Fig. II.1 below).

Eucl. II.14 proves the *altitude theorem* (1.10), by using Eucl. II.8 in the same way[7] as in Exercise 22 of Chap. 1. It allows the *quadrature of a rectangle*, i.e. the construction of a square with an area equal to that of a given rectangle (see Fig. 2.14 right).

2.2 Book III. Properties of Circles and Angles

The third book is devoted to circles and angles. For instance, Eucl. III.20 is the *central angle theorem*, see Theorem 1.4 and Fig. 1.9; Eucl. III.21 is a variant of this theorem, see Exercise 3 of Chap. 1.

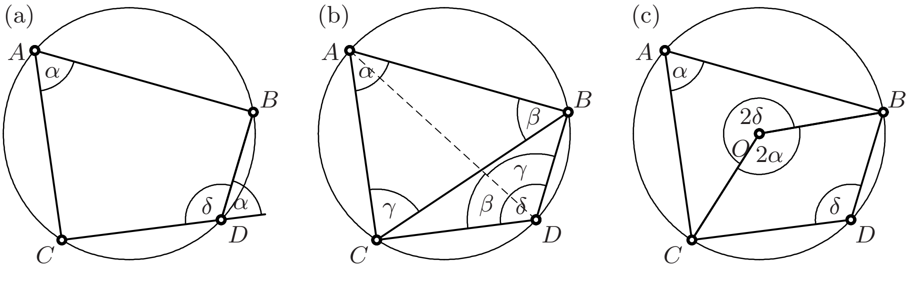


Fig. 2.15. Angles of a quadrilateral inscribed in a circle (Eucl. III.22)

Eucl. III.22. *Let ABDC be a quadrilateral inscribed in a circle, as shown in Fig. 2.15 (a). Then the sum of two opposite angles equals two right angles:*

$$\alpha + \delta = 2\llcorner\!\!\lrcorner . \tag{2.3}$$

[7]It also follows from Eucl. III.35 below, for the particular case where AB is a diameter and CD is orthogonal to AB.

Proof by Euclid. We consider the triangle ABC in Fig. 2.15 (b). By Eucl. III.21, we have the two angles β and γ at the point D. This shows that $\delta = \beta + \gamma$. The result is thus a consequence of Eucl. I.32. □

Another proof of Eucl. III.22. It is clear from Fig. 2.15 (c) that the central angles cover the four right angles around O, i.e., by applying Eucl. III.20, we have $2\alpha + 2\delta = 4\,\llcorner$. (Euclid did not consider angles greater than $2\,\llcorner$; hence he would not have presented such a proof.) □

Eucl. III.35. *If two chords AB and CD of a circle intersect in a point E inside the circle (see Fig. 2.16 (a)), then*

$$AE \cdot EB = CE \cdot ED\,. \tag{2.4}$$

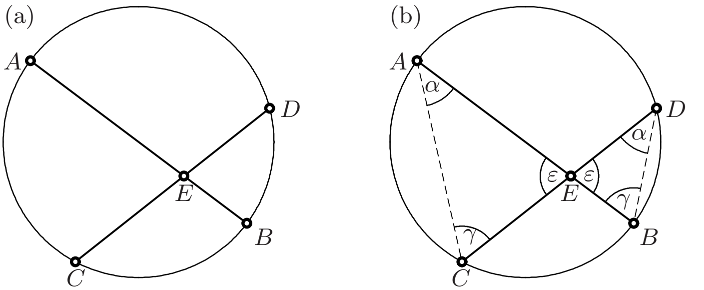


Fig. 2.16. Eucl. III.35 (a) and its proof by Thales' theorem

Proof. Concerned by rigour, Euclid persistently refuses to use Thales' theorem. Hence his proof, repeatedly using Pythagoras' theorem (Eucl. I.47), requires $1\frac{1}{2}$ pages. Being less scrupulous, we see by Eucl. III.21 that the triangles AEC and DEB are similar, see Fig. 2.16 (b). Hence (2.4) follows from Thales' theorem. □

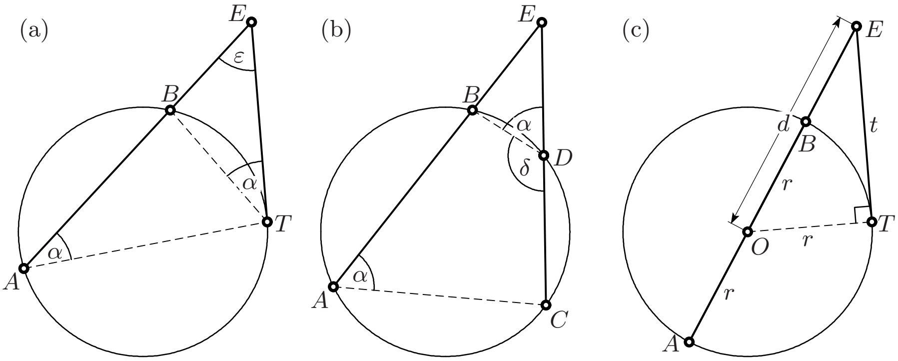


Fig. 2.17. Eucl. III.36 (a); Clavius' corollary (b); relation with Pythagoras' theorem and Steiner's power of a point with respect to a circle (c).

Eucl. III.36. *Let E be a point outside a circle and consider a line through E that cuts the circle in two points A and B. Further let T be the point of tangency of a tangent through E (see Fig. 2.17 (a)). Then*

$$AE \cdot BE = (TE)^2 \,. \tag{2.5}$$

Proof. The two angles marked α in Fig. 2.17 (a) are equal by Eucl. III.21, because they are inscribed angles on the arc BT (the second one is a limiting case as in Eucl. III.32, cf. Exercise 17 on page 57). Hence ATE is similar to TBE and the result follows from Thales' theorem. This, again, is not Euclid's original proof. □

Corollary (Clavius 1574). *Let A, B, C and D denote four points on a circle. If the line AB meets the line CD in a point E outside the circle (see Fig. 2.17 (b)), then*

$$AE \cdot BE = CE \cdot DE \,. \tag{2.6}$$

Proof. This is clear from Eucl. III.36, because $AE \cdot BE$ and $CE \cdot DE$ are both equal to $(TE)^2$.

We can also prove this corollary directly by Eucl. III.22, because the triangles AEC and DEB are similar. Then Eucl. III.36, as well as the picture Fig. 2.17 (a), would be limiting cases where C and D coincide. □

Remark. The particular case of Eucl. III.36, in which AB is a diameter of the circle (see Fig. 2.17 (c)), leads to $t^2 = (d+r)(d-r) = d^2 - r^2$. This is in accordance with Pythagoras' theorem since the angle at T is right by Eucl. III.18 (see Exercise 16). The quantity $d^2 - r^2$ is called the *power of the point E with respect to the circle*, an important concept introduced by Steiner (1826a, §9).

Book IV. This book treats circles, inscribed in or circumscribed to triangles, squares, regular pentagons (Eucl. IV.11), hexagons (Eucl. IV.15). Without Thales' theorem, the treatment of the pentagon is still unwieldy. The more elegant proof that we gave in Chap. 1 appears much later in the *Elements* (Eucl. XIII.9). The book ends with the construction of the regular 15-sided polygon (Eucl. IV.16, see Fig. 2.18).

Fig. 2.18. Eucl. IV.16 (left); application to modern car technology (right).

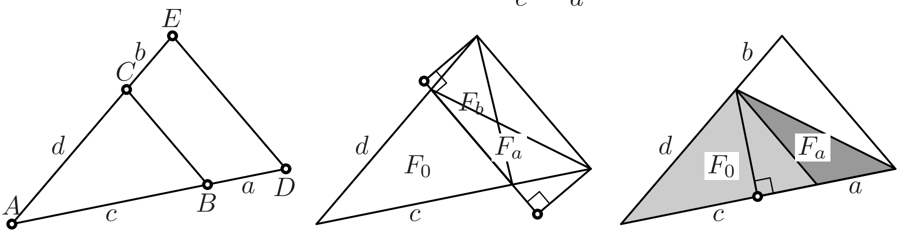
2.3 Books V and VI. Real Numbers and Thales' Theorem

> "There is nothing in the whole body of the *Elements* of a more subtile invention, nothing more solidly established, and more accurately handled than the doctrine of proportionals."
>
> (I. Barrow; see Heath, 1926, vol. II, p. 186)

Book V. The theory of proportions. This theory is due to Eudoxus and has been greatly admired. It concerns ratios of irrational quantities and their properties. One constantly works with inequalities that are multiplied by integers. One thereby *squeezes* irrational quantities between rational ones, somewhat in the style of *Dedekind cuts* 2200 years later.

Book VI. Thales-like theorems. Once the theory of proportions is established, one can finally give a rigorous proof of Thales' theorem.

Eucl. VI.2. *If BC is parallel to DE, then* $\frac{a}{c} = \frac{b}{d}$ (see the figure on the left).

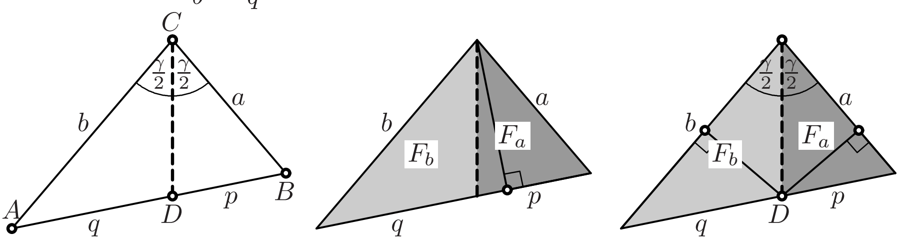


Proof. One joins B to E and C to D. This gives two triangles with the same base CB and the same altitude, hence with the same area $F_a = F_b$, see the second figure. Thus, if F_0 denotes the area of ABC,

$$F_a = F_b \quad \Rightarrow \quad \frac{F_a}{F_0} = \frac{F_b}{F_0} \quad \Rightarrow \quad \frac{a}{c} = \frac{b}{d}$$

since $\frac{F_a}{F_0} = \frac{a}{c}$. (We use here the fact that both triangles have the same altitude on AD, see the figure on the right.) □

Eucl. VI.3 (Theorem of the angle bisector). *Let CD be the bisector of the angle γ. Then* $\frac{a}{b} = \frac{p}{q}$ (see the figure on the left).



Proof. Euclid proves this theorem as an application of Eucl. VI.2. We, however, use the spirit of the above proof and consider the areas F_a and F_b of the triangles DBC and ADC, respectively. These triangles have the same altitude on AB (see second figure). As the points on the angle bisector have the same distance from both sides (a consequence of Eucl. I.26), the triangles have the same altitude on AC and BC, respectively, see the figure on the right. Thus we have on the one hand

$$\frac{F_a}{F_b} = \frac{p}{q}, \quad \text{and on the other hand} \quad \frac{F_a}{F_b} = \frac{a}{b}. \qquad \square$$

The subsequent propositions are variants of Thales' theorem and their converses; Eucl. VI.9 explains how to cut off a rational length from a line, see Fig. 1.6; Eucl. VI.19 proves Theorem 1.6 on the areas of similar triangles. It is only now that Euclid is fully prepared for Naber's proof of the Pythagorean theorem, see Fig. 1.21.

2.4 Books VII and IX. Number Theory

These books introduce a completely different subject, the theory of numbers (divisibility, prime numbers, composite numbers, even and odd numbers, square numbers, perfect numbers). The later development of this theory, now called *number theory*, with results that are simple to enunciate, but whose proofs require the deepest thought and the most difficult considerations, became the favourite subject of the greatest among the mathematicians (Fermat, Euler, Gauss[8]) and is still full of mysteries and open problems.

The results are not geometrical, but the way of thinking is, at least for Euclid.

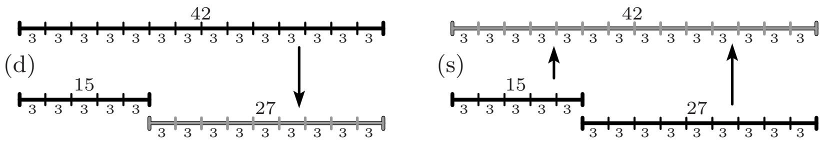


Fig. 2.19. Measure of difference (d) and sum (s) of two numbers

The book starts with propositions about the divisibility of numbers. The main tool is the observation, already known from Book V (in particular Eucl. V.1 and V.5), that if a number divides (Euclid says "measures") *two* quantities, it also divides their *difference* (see Fig. 2.19, (d)), and their *sum* (Fig. 2.19, (s)). This leads to Eucl. VII.2, better known as the Euclidean algorithm.

[8] "Die schönsten Lehrsätze der höheren Arithmetik ... haben das Eigne, dass ... ihre Beweise ... äusserst versteckt liegen, und nur durch sehr tief eindringende Untersuchungen aufgespürt werden können. Gerade diess ist es, was der höheren Arithmetik jenen zauberischen Reiz gibt, der sie zur Lieblingswissenschaft der ersten Geometer gemacht hat." (Gauss, 1809; *Werke*, vol. 2, p. 152)

Eucl. VII.2. *Given two numbers not relatively prime, to find their greatest common measure.*

The Euclidean algorithm.[9] Given a pair of distinct positive integers, say a, b with $a > b$, subtract the smaller from the larger. Then repeat this with the new pair $a - b$, b. Any common divisor of a and b also divides $a - b$ and b, and conversely. Therefore, the last non-zero difference is divisible by the greatest common divisor of a and b, and divides it. Hence it is their *greatest common divisor*.

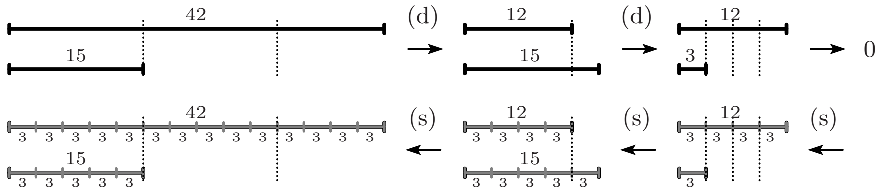


Fig. 2.20. Euclidean algorithm for the greatest common measure of two numbers

Other highlights of these books are Eucl. VII.34 on the least common multiple of two numbers and Eucl. IX.20 on the fact that the number of primes is infinite.

Book X. A classification of irrational numbers

This book is the culmination of the mathematical theory of the *Elements*, using the tools from analysis (Books V and VI) and number theory (Books VII–IX) in order to set up an immense classification of irrationals (with 115 propositions in all).

Eucl. X.1. This is the first convergence result in history, telling us that for n sufficiently large, $a \cdot 2^{-n}$ becomes smaller than any number $\varepsilon > 0$.[10] The main advantage of this proposition is to terminate proofs which otherwise would go on indefinitely (see e.g. Eucl. X.2 and Eucl. XII.2 below).

Eucl. X.2 applies the algorithm of Eucl. VII.2 to *real* numbers. If the algorithm never terminates, the ratio of the two initial numbers $a > b$ is *irrational*.[11] Two thousand years later, this led to the theory of *continued fractions* (see e.g. Hairer and Wanner, 1997, p. 67).

Example. In Fig. 2.21 we see the Euclidean algorithm applied to $a = \Phi$ (resp. $a = \sqrt{2}$) and $b = 1$. We see that we obtain an infinite sequence of similar triangles (resp. squares) and an unending sequence of remainders $c = a - b$,

[9] The Arabic word "algorithm" only appeared some thousand years later.

[10] The ε, though a Greek letter, came into use for this purpose only with Weierstrass many many centuries later. If you want to know, Euclid used a capital Γ at this place.

[11] In Euclid's words: a and b are *incommensurable*.

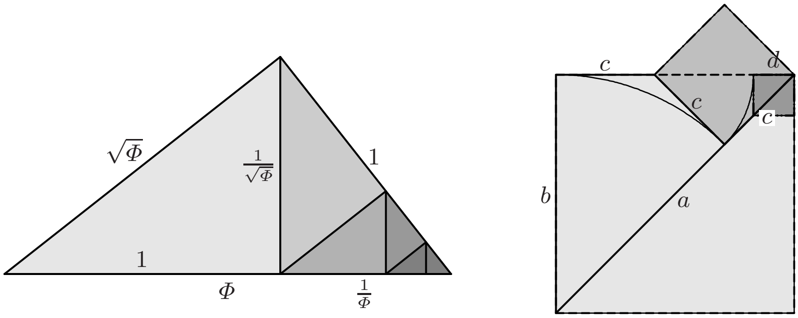


Fig. 2.21. Euclidean algorithm for Φ and $\sqrt{2}$

$d = b - c$, $e = c - d$ (resp. $c = a - b$, $d = b - 2c$, $e = c - 2d$), etc. Hence, both Φ and $\sqrt{2}$ must be irrational. The second picture is inspired by a drawing in Chrystal (1886, vol. I, p. 270), the first by a result of Viète (1600), who discovered that Φ, $\sqrt{\Phi}$ and 1 form a Pythagorean triple.

Other highlights of this book are Eucl. X.9, which shows that numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, etc. are irrational, and Eucl. X.28, which contains the construction of Pythagorean triples.

2.5 Book XI. Spatial Geometry and Solids

Book XI introduces solids (στερεός). Euclid gives the definition of a *pyramid* (πῡρᾰμίς; a solid formed by a polygon, an apex and triangles; see Fig. 2.22),

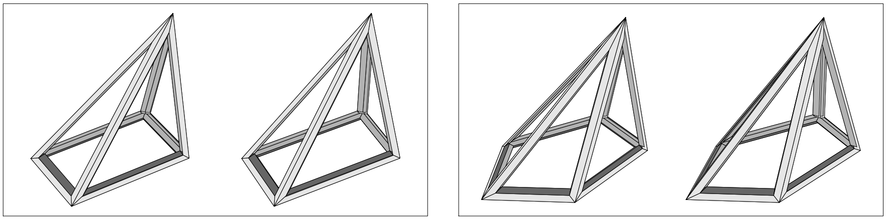


Fig. 2.22. Pyramids over a rectangle and over a pentagon, respectively

a *prism* (πρῖσμα; a solid formed by a polygon, a second identical polygon parallel to the first one, and parallelograms; see Fig. 2.23, left),

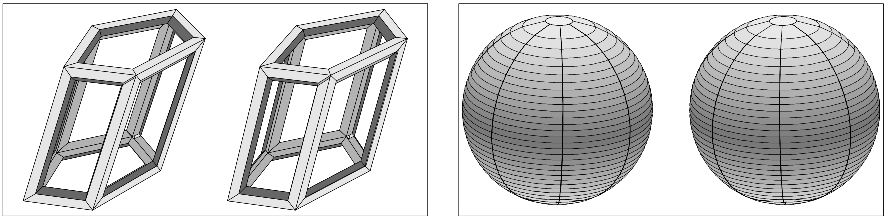


Fig. 2.23. Prism over a pentagon (left) and sphere (right)

a *sphere* (σφαῖρᾰ; a solid obtained by rotating a semicircle around the diameter; see Fig. 2.23, right), a *cone* (κῶνός; a solid formed by rotating a right-angled triangle around a leg; see Fig. 2.24, left), a *cylinder* (κύλινδρος, rotation of a rectangle around a side; see Fig. 2.24, right),

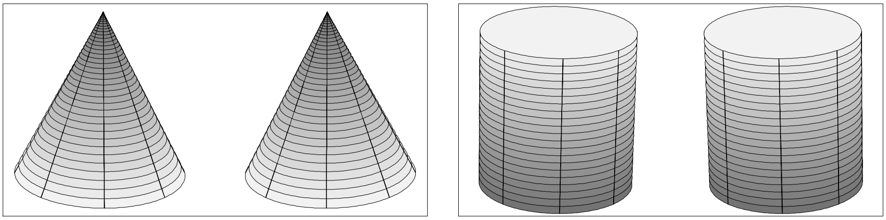


Fig. 2.24. Cone and cylinder

a *cube* (κῠβος; see Fig. 2.25, left), an *octahedron* (ὀκτάεδρον from ὀκτάεδρος – eight-sided; see Fig. 2.25, right)

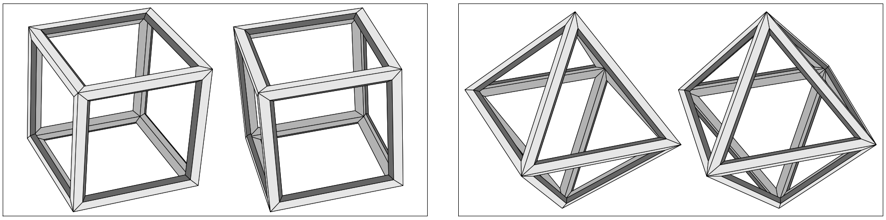


Fig. 2.25. Cube and octahedron

an *icosahedron* (εἰκοσάεδρον; see Fig. 2.26, left), and finally a *dodecahedron* (δωδεκάεδρον; see Fig. 2.26, right).

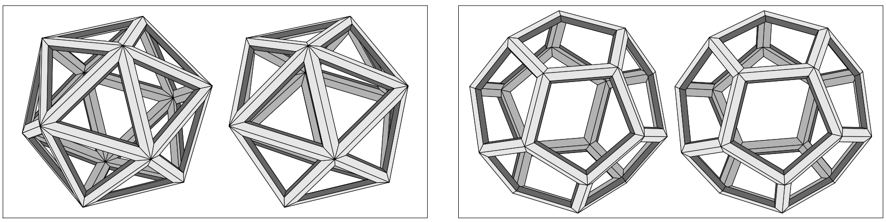


Fig. 2.26. Icosahedron and dodecahedron

The four last ones, together with the *tetrahedron* (τετράεδρον, with four faces) which Euclid does not define, form the class of regular polyhedra. This class is identical to that of the *Platonic solids* or *cosmic figures*; Plato described them in his *Timæus* and associated them to the *five elements* (cube $\leftrightarrow$ earth, icosahedron $\leftrightarrow$ water, octahedron $\leftrightarrow$ air, tetrahedron $\leftrightarrow$ fire, dodecahedron $\leftrightarrow$ ether). An illustration by Kepler is reproduced in Fig. 2.27.

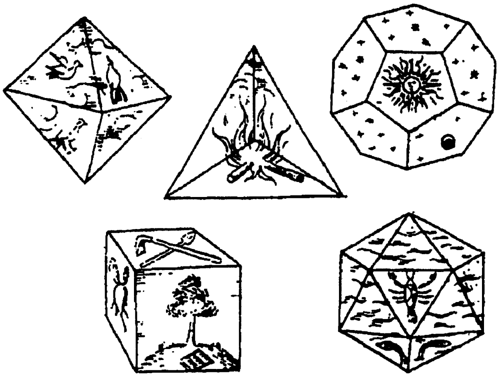


Fig. 2.27. Platonic solids (drawings by Kepler, *Harmonices mundi*, p. 79, 1619)

We further note the interesting fact that tetrahedron $\leftrightarrow$ tetrahedron, octahedron $\leftrightarrow$ cube, and dodecahedron $\leftrightarrow$ icosahedron are seen to be dual by joining the *centres* of the faces of the regular polyhedra, see Figs. 2.28–2.30.

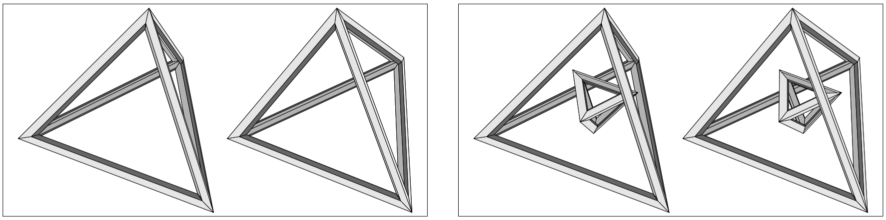


Fig. 2.28. Self-duality of tetrahedron

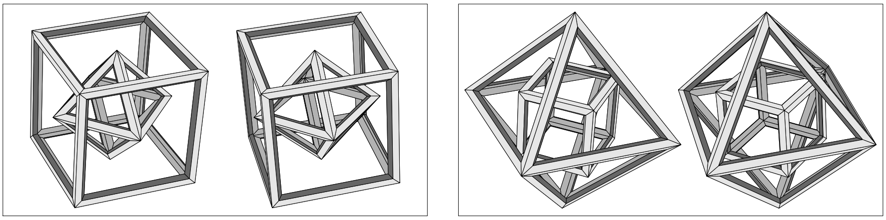


Fig. 2.29. Duality between cube and octahedron

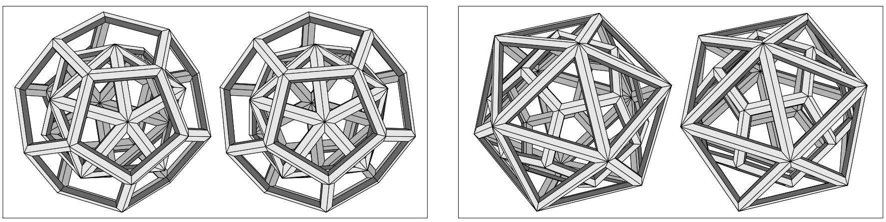


Fig. 2.30. Duality between icosahedron and dodecahedron

Euclid omitted the definition of the *parallelepiped* (παραλληλεπίπεδον, a solid with parallel surfaces) and of the *right-angled parallelepiped* (where all angles are right), see Fig. 2.31.

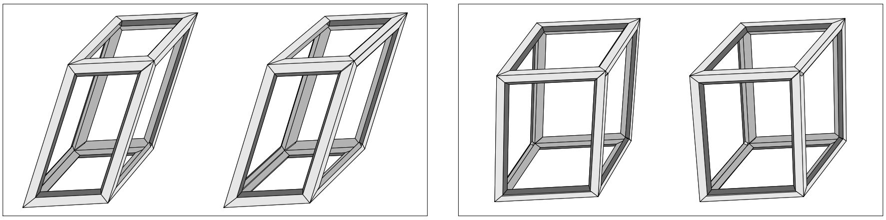


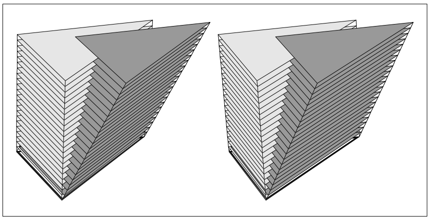
Fig. 2.31. Parallelepiped and right-angled parallelepiped

Eucl. XI.1–XI.26. Properties of planes, lines and angles in space. We postpone these questions to Part II where we will discuss them using tools from linear algebra.

Eucl. XI.27 ff. Volume of prisms and parallelepipeds. We have

$$\mathcal{V} = \mathcal{A} \cdot h \qquad \text{where} \quad \mathcal{A} = \text{area of the base}; \quad h = \text{altitude}. \tag{2.7}$$

The proofs are in the style of the second figure of Fig. 1.11 (cut off a piece and add it onto the other side). An alternative proof — in the spirit of Archimedes — can be given by cutting the solid into thin slices (*exhaustion method*); for an illustration, see Fig. 2.32).



oblique prism → right prism

Fig. 2.32. Transformation of an oblique prism into a right prism

2.6 Book XII. Areas and Volumes of Circles, Pyramids, Cones and Spheres

Areas and volumes of more complicated figures are the topic of Book XII. Euclid starts with circles.

Eucl. XII.2. *The areas $\mathcal{A}_1$ and $\mathcal{A}_2$ of two circles C_1 and C_2 of radii r_1 and r_2, respectively, satisfy*

$$\frac{r_2}{r_1} = q \qquad \Rightarrow \qquad \frac{\mathcal{A}_2}{\mathcal{A}_1} = q^2. \tag{2.8}$$

Proof. The proof is based upon Eucl. VI.19, see Theorem 1.6. Its rigour is impressive.
Suppose that $\frac{\mathcal{A}_2}{\mathcal{A}_1} > q^2$, i.e.

$$q^2 \mathcal{A}_1 < \mathcal{A}_2 . \tag{2.9}$$

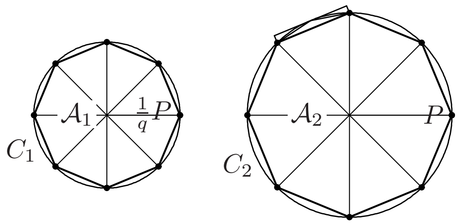


Fig. 2.33. Proof of Eucl. XII.2

We now apply an idea, called the *method of exhaustion* and attributed by Archimedes to Eudoxus: we inscribe in the circle C_2 a polygon P whose area fits in the gap given by (2.9). In order to see that this is possible, one shows that by doubling the number of points of P, the difference of the areas diminishes by at least the factor $\frac{1}{2}$ (see the small rectangle in Fig. 2.33, right). One then applies Eucl. X.1 and obtains for the area of P

$$q^2 \mathcal{A}_1 < \mathcal{P} < \mathcal{A}_2 . \tag{2.10}$$

The polygon P is then divided by q and transferred into C_1. Then, by Eucl. VI.19, and because $\frac{1}{q}P$ is contained in C_1,

$$\frac{1}{q^2}\,\mathcal{P} < \mathcal{A}_1 .$$

If this inequality is multiplied by q^2, we obtain a contradiction with (2.10).

For the assumption $\frac{\mathcal{A}_2}{\mathcal{A}_1} < q^2$ one exchanges the roles of C_1 and C_2 and arrives at a similar contradiction. Thus, the only possibility is $\frac{\mathcal{A}_2}{\mathcal{A}_1} = q^2$. □

Euclid, with his disdain for all practical applications, says not a word about the actual value of the similarity factor, which is today denoted by π. With the famous estimate (1.11) we obtain

$$\mathcal{A} = r^2\pi \qquad \text{where } \pi \text{ is a number satisfying} \quad 3\,\frac{10}{71} < \pi < 3\,\frac{1}{7} \tag{2.11}$$

(see Exercise 22 below).

Eucl. XII.3–XII.9. Volumes of pyramids. The result is

$$\mathcal{V} = \frac{\mathcal{A}\cdot h}{3} \qquad \text{where} \quad \mathcal{A} = \text{area of the base}, \quad h = \text{altitude}. \tag{2.12}$$

We again prefer to give a proof by using thin slices, see Fig. 2.34. To make the factor 1/3 convincing, Euclid decomposes a triangular prism into *three* pyramids which have — two by two — the same base and altitude. Thus, all three have the same volume (see upper picture of Fig. 2.35). A simpler proof (Clairaut, 1741) is obtained by cutting a cube into *six* pyramids of altitude $\frac{h}{2}$ (see lower left picture of Fig. 2.35). Cavalieri (1647, Exercitatio Prima,

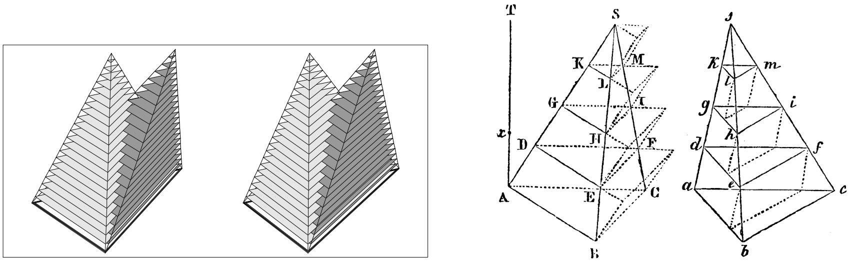


Fig. 2.34. Volume of a pyramid; on the right: drawing by Legendre (1794), p. 203

Prop. 24) shows by calculus that, in modern notation, $\int_0^1 x^2\,dx = \frac{1}{3}$. This is illustrated by a skew quadratic pyramid which, when assembled as in the lower right picture of Fig. 2.35, shows once again that the volumes of the solids “erunt in ratione tripla”.

Eucl. XII.10–XII.15. (Volumes of cylinders and cones.) We have:

$$\mathcal{V}_{\text{cylinder}} = r^2\pi h, \qquad \mathcal{V}_{\text{cone}} = \frac{r^2\pi h}{3}. \tag{2.13}$$

Eucl. XII.17. *The volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ of two spheres with radius r_1 and r_2, respectively, satisfy*

$$\frac{r_2}{r_1} = q \qquad \Rightarrow \qquad \frac{\mathcal{V}_2}{\mathcal{V}_1} = q^3. \tag{2.14}$$

The proof is similar to that of Eucl. XII.2, but more involved.

Later, Archimedes (see *On conoids and spheroids*, Prop. XXVII) found that

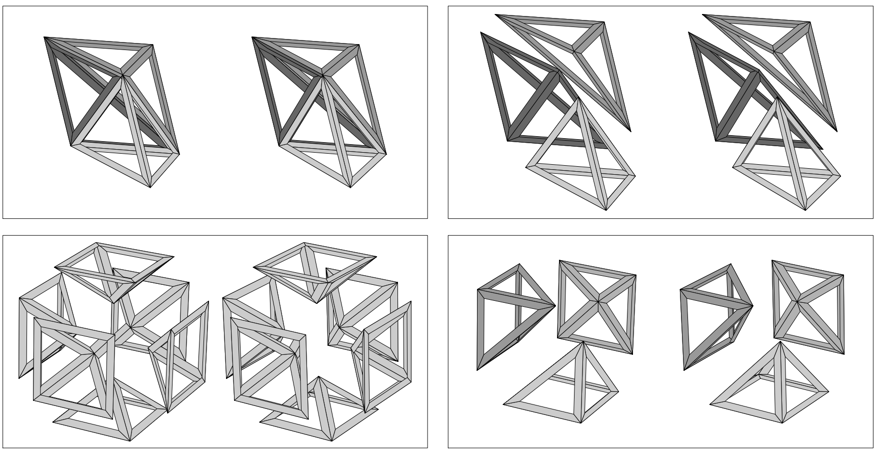


Fig. 2.35. Proof of Eucl. XII.7 (above); proof by Clairaut (below left), Cavalieri (below right)

$$\mathcal{V}_{\text{sphere}} = \frac{4\pi r^3}{3}$$

and the beautiful relation

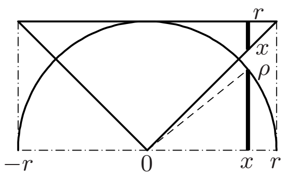
$$\boxed{\mathcal{V}_{\text{cone}} : \mathcal{V}_{\text{sphere}} : \mathcal{V}_{\text{cylinder}} = 1 : 2 : 3} \tag{2.15}$$

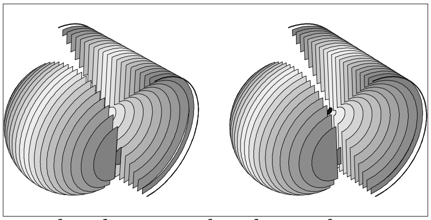
for a cylinder circumscribing the sphere, and a double-cone with the same radius and altitude as the cylinder.

Archimedes' proof uses *slim slices* by observing that, slice by slice, the area $\mathcal{A}$ of the cross-section of the sphere

$$\mathcal{A}_{\text{sphere}} = \rho^2\pi = r^2\pi - x^2\pi = \mathcal{A}_{\text{cylinder}} - \mathcal{A}_{\text{cone}}$$

equals that of the cylinder minus that of the cone. This is obvious from Fig. 2.36, which shows that $\rho = \sqrt{r^2 - x^2}$.





vol. sphere = vol. cyl. − vol. cone

Fig. 2.36. Volume of sphere, cylinder and cone

Book XIII. Construction and properties of the Platonic solids

Eucl. XIII.1–12 are concerned with the golden ratio, the regular pentagon and isosceles triangles, see Chap. 1.

Eucl. XIII.13–18. Euclid constructs the tetrahedron, octahedron, cube, icosahedron and dodecahedron. For the dodecahedron, he starts from a cube by adding *hipped roofs* on each face, as shown in Fig. 2.37, see also Exercise 16 in Sect. 1.9.

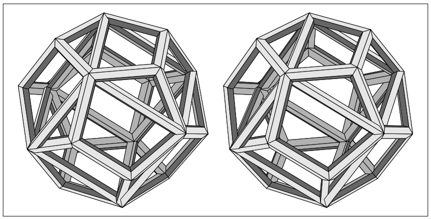


Fig. 2.37. The dodecahedron built on a cube

2.7 Epilogue

> "Some time ago in Berlin, a brilliant young man from a respected family was dining with an elderly man, to whom he explained enthusiastically all the research he was carrying out in geometry, which is so easy at the beginning and becomes difficult only later. 'For me', said the elderly man, 'the first principles are very difficult and contain complications which I cannot resolve'. The young man smiled sarcastically, until someone whispered in his ear: 'Do you know to whom you are talking? To Euler!' "
> (Testimony of L. Hoffmann 1786; quoted from Pont, 1986, p. 467)

> "Die vorliegende Untersuchung ist ein neuer Versuch, für die Geometrie ein *vollständiges* und *möglichst einfaches* System von Axiomen aufzustellen und aus denselben die wichtigsten geometrischen Sätze ... abzuleiten, ... [The following investigation is a new attempt to choose for geometry a *simple* and *complete* set of independent axioms and to deduce from them the most important geometrical theorems ...]"
> (D. Hilbert, 1899, p. 1; Engl. trans. by E.J. Townsend, 1902)

> "Studying the foundations is not an easy task. If the reader encounters difficulties when reading the first chapter ... he may skip the proofs ... "
> (M. Troyanov, 2009, p. 3; transl. from the French)

> "Ich habe noch einen kurzen Schlusssatz hinzugefügt – für ungläubige und formale Gemüther. [I have also added a short closing sentence — for unbelieving and formal minds.]"
> (D. Hilbert, letter to F. Klein, 4. 3. 1891)

For more than 2000 years, the *Elements* of Euclid have served as a basic text in geometry. Their austere beauty has fascinated readers throughout the ages. However, the *Elements* have also received much critical attention from the very beginning, examples of which we have already seen in our discussions following Eucl. I.1 and Eucl. I.4. Authors have repeatedly tried to improve on Euclid's axioms. A particularly thorough contribution was Legendre's book (1794), which was reprinted in many editions during more than a century. But only during the 19th century were final breakthroughs made in two directions: (a) in relaxing one of Euclid's postulates, creating *non-Euclidean geometry*; (b) in laying firmer foundations for classical geometry by a complete reorganisation and strengthening of the axioms (Hilbert).

Non-Euclidean geometry. During all these 2000 years, Euclid's Postulate 5 on parallel lines was suspected of being superfluous; this caused an enduring discussion with innumerable attempts to deduce it from the other postulates. The continued failure of all these efforts finally aroused the suspicion that such a proof is impossible. Gauss expressed in several letters to his friends, but not in print, the idea that one could create an entirely new geometry which does *not* satisfy Postulate 5. The construction of this so-called *hyperbolic geometry* was carried out and published independently by Bolyai (1832) and Lobachevsky (1829/30) and was the origin of non-Euclidean geometry. The originally very complicated theory was later simplified by the models of Beltrami (see Fig. 7.25 on page 213), Klein and Poincaré. For more details we refer to the textbooks by Gray (2007, Chaps. 9, 10, 11), Hartshorne (2000) and the article Milnor (1982). Many interesting details are given in Klein (1926, pp. 151–155). Very careful historical notes accompany the advanced text Ratcliff (1994) and a complete epistemological account of all the actors of this long development is given in Pont (1986).

Hilbert's axioms. The ongoing formalisation of mathematics in the second half of the 19th century also called for firmer foundations of classical geometry. In 1899, Hilbert came up with a new and "simple" system of 21 axioms, later reduced to 20, because the axiom II.4 was seen to be redundant. This system of axioms characterises plane and solid Euclidean geometry. Many of Euclid's vague definitions for the principal objects of Euclidean geometry, namely *points*, *straight lines* and *planes*, are simply omitted[12] and Hilbert characterises them by their mutual relations, such as *situated*, *between*, *parallel*, and *congruent*. The actual calculations are based on a so-called segment arithmetic, leading first to Pappus' theorem (see Thm. 11.3 on page 325), and then to Thales' theorem as a consequence.

During the 20th century, attempts were made to reduce the large number of Hilbert's axioms. The main idea for this was to assume the real numbers to be known, which allowed, for example in Birkhoff (1932), the introduction of a set of four postulates to axiomatically describe plane Euclidean geometry. His postulates are based on the use of a (scaled) ruler and a protractor; this is made possible by accepting the fundamental properties of the real numbers. In this approach, Thales' theorem is simply postulated.

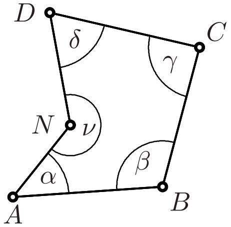
Despite the great importance of axiomatic systems, their austere character often discourages beginners (see the quotation above). We will therefore abandon at this point the axiomatic bones and turn our attention to a meatier fare. It is interesting to note that Hilbert himself, in his later book written with Cohn-Vossen, *Geometry and the Imagination* (1932), did not mention his own system of axioms at all.

[12]In Hilbert's own words, such basic objects may be replaced by *tables*, *chairs* and *beer mugs*, as long as they meet the required relations.

2.8 Exercises

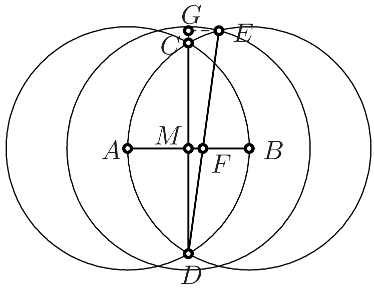
1. Prove the extension by Proclus of Eucl. I.32 (cf. Heath, 1926, vol. I, p. 322): *for any polygon with n vertices the sum of the interior angles satisfies*

$$\alpha + \beta + \gamma + \ldots + \nu = 2 \llcorner (n-2)\,. \qquad (2.16)$$



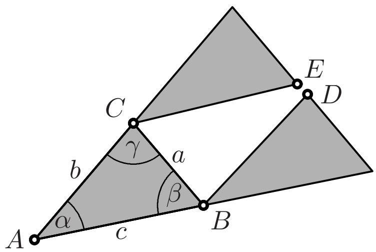
2. The assertion of the first two pictures of Fig. 1.7 (see Chap. 1) for *parallel* angles are Eucl. I.29 together with I.15. Prove the last assertion, for *orthogonal* angles.

3. (Golden ratio with ruler and rusty compass; Hofstetter, 2005.) Extend the construction of Eucl. I.1 and Eucl. I.10, by adding another circle of the same radius centred at the midpoint M (see figure at right), to obtain the point F which divides the segment AB in the golden ratio.



4. Let ABC be a triangle with right angle at C. Show that the vertex C lies on the Thales circle of the hypotenuse AB.

5. Close a gap in the "Stone Age proof" of Thales's theorem in Chap. 1 (see Fig. 1.2): It is *not* evident that the points D and E, after the parallel translations of the triangle ABC, must *really* coincide.



"Figures don't lie, but liars figure."
(Mark Twain [from an e-mail by Jerry Becker])

6. Criticise the "proof" by W.W. Rouse Ball (see Hartshorne, 2000, p. 36) of a wrong variant of Eucl. I.5: *Every triangle is isosceles*, which goes as follows: Let E be the intersection of the angle bisector at A and the perpendicular bisector of BC, see Fig. 2.38, left. Drop the perpendiculars EF and EG. Then use all the valid propositions of Euclid to show that $AF = AG$ and $FB = GC$. From this the "result" follows.

   A clever student might object that the intersection point E could be *outside* the triangle. However, this situation is not much better, see Fig. 2.38, right.

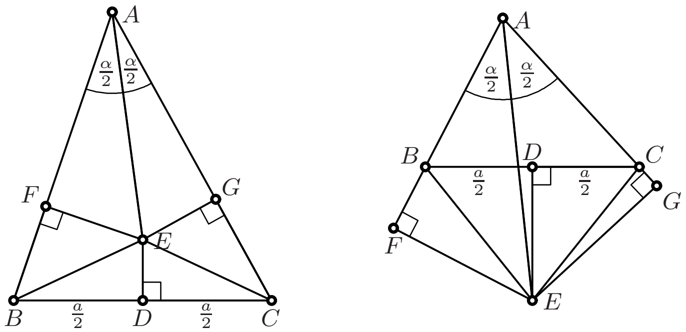


Fig. 2.38. The proof that every triangle is isosceles

7. Let ABC be an isosceles triangle and D the midpoint between B and C (see Fig. 2.39 (a)). Use judiciously chosen propositions of Euclid to prove that the line AD is perpendicular to BC. In the language of Chap. 4 below, we say that the median through A, the bisector of the angle BAC, the perpendicular bisector of BC and the altitude through A coincide.

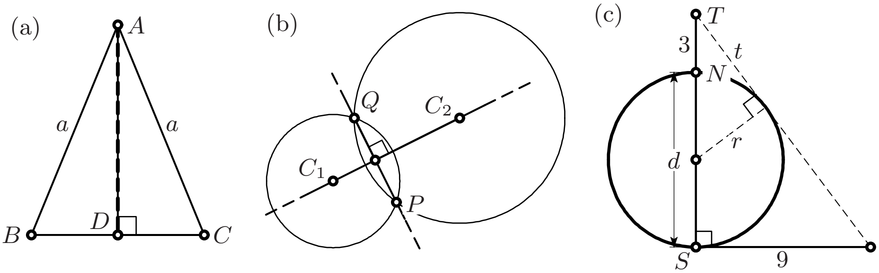


Fig. 2.39. Median of an isosceles triangle (a); radical axis of two circles (b); the problem of Qin Jiushao (c)

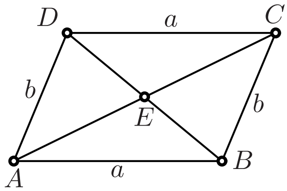
8. Use the result of the previous exercise to show that the *radical axis* QP of two circles (see Fig. 2.39 (b)) is perpendicular to the line joining the two centres.

9. Solve a problem by Qin Jiushao, China 1247:[13] Given a circular walled city of unknown diameter with four gates, one at each of the four cardinal points. A tree T lies 3 li[14] north of the northern gate N. If one turns and walks eastwards for 9 li immediately on leaving the southern gate S, the tree just comes into view. Find the diameter of the city wall (see Fig. 2.39 (c) and Dörrie, 1943, §262).

[13] English wording by J.J. O'Connor and E.F. Robertson, The MacTutor History of Mathematics Archive, `http://www-history.mcs.st-andrews.ac.uk/index.html`

[14] A *li* is a traditional Chinese unit of length, nowadays 500 m.

10. Prove that the diagonals of a parallelogram bisect each other and that, in addition, the diagonals of a rhombus are perpendicular to each other (see the figure to the right, and Def. 22 of Fig. 2.1).



11. Reconstruct Euclid's proof for Eucl. I.18: *In any triangle the greater side subtends the greater angle*, i.e. show that if in a triangle AC is greater than AB, then β is greater than γ.
Hint. Insert a point D such that $AB = AD$; see Fig. 2.40 (a).

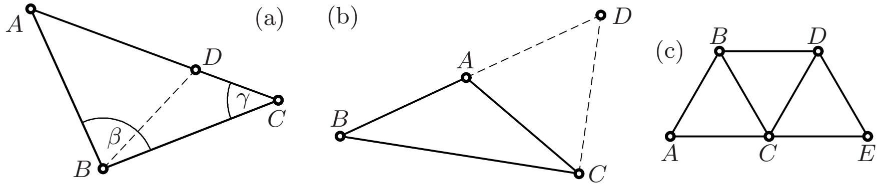


Fig. 2.40. Eucl. I.18; Eucl. I.20 and Eucl. IV.15

12. Give Euclid's proof of the triangle inequality (Eucl. I.20) with the help of Fig. 2.40 (b); i.e. show that $AB + AC$ is greater than BC. The auxiliary point D is found by producing line AB so that $AD = AC$.

13. The following exercise is the basis for understanding the regular hexagon (Eucl. IV.15): if three equal equilateral triangles are as in Fig. 2.40 (c), then ACE is a straight line.

14. Find a geometric proof for Eucl. II.8, which expresses the algebraic identity

$$(a + b)^2 - (a - b)^2 = 4ab$$

and was a key relation in the search for Pythagorean triples. (*Hint.* A look at Fig. 12.1 might help.)

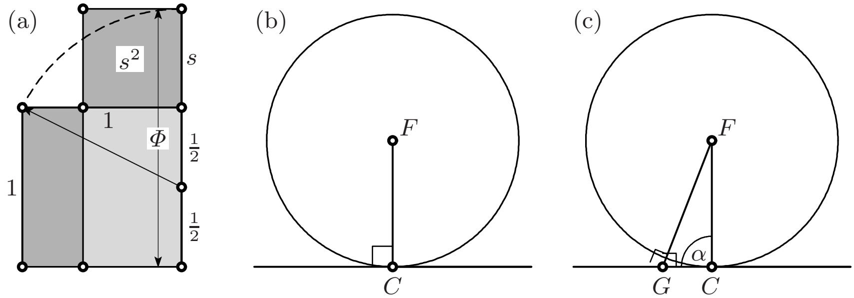


Fig. 2.41. Proof of Eucl. II.11 (a); property of the tangent to a circle (b); Euclid's proof of Eucl. III.18 (c)

15. Explain the solution of Eucl. II.11 in Fig. 2.41 (a) for the computation of the golden ratio Φ determined by equation (1.4).

16. Discover Euclid's proof for Eucl. III.18: *If a straight line touches a circle with centre F at a point C, then FC is perpendicular to this line* (see Fig. 2.41 (b)). (*Hint.* A look at Fig. 2.41 (c) might help.)

17. Find a proof of Eucl. III.32, which states that *if a line EF touches a circle at B, and if C and D are points on this circle, then the angle DCB is equal to the angle DBE* (see Fig. 2.42, left).

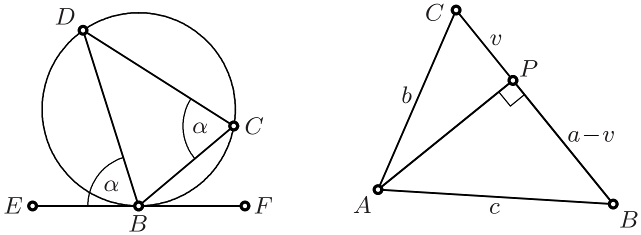


Fig. 2.42. Eucl. III.32 (left); Eucl. II.13 (right)

18. Eucl. II.13, i.e. formula (2.2), written for the situation of Fig. 2.42 (right), reads as

$$a^2 + b^2 - 2av = c^2 , \qquad (2.17)$$

and is a direct extension of Pythagoras' theorem (1.8). *Question:* can you, inspired by Euclid's proof of Fig. 1.19, find a *direct* proof of (2.17)?

19. Let two circles intersect in two points P and Q (see Fig. 2.43 (a)). From a point T on one of the circles, produce TP and TQ to cut the other circle at A and B. Show that the tangent at T is parallel to AB.

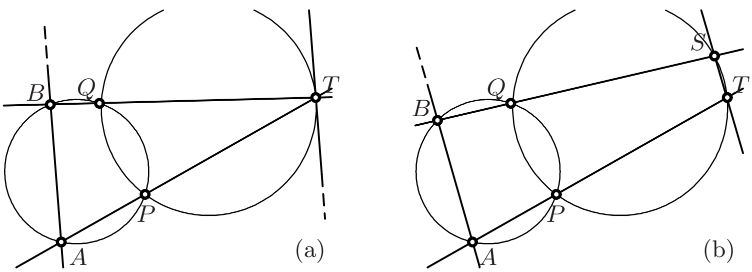


Fig. 2.43. Property of the tangent to a circle (left); two secants to two circles (right)

20. Prove a beautiful result, generally attributed to Jacob Steiner, the *four-circles theorem*: Suppose that four circles intersect in points A, A', B, B', C, C' and D, D' as shown in Fig. 2.44 (a). Show then that A, B, C, D are concyclic (i.e. lie on a circle) if and only if A', B', C', D' are.

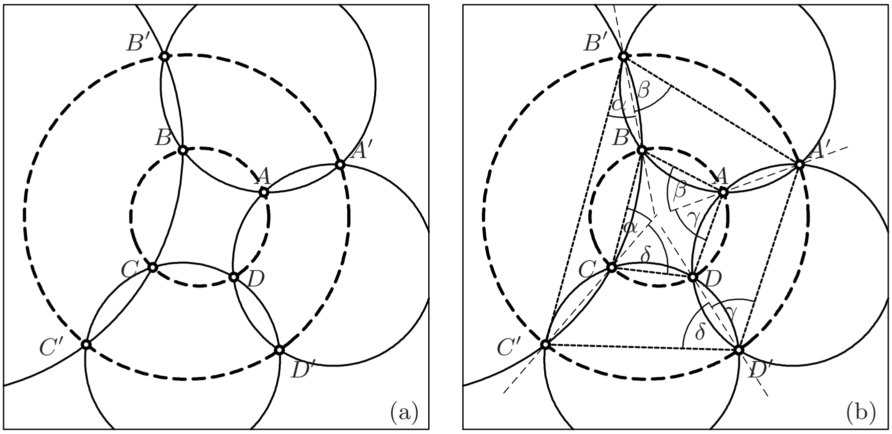


Fig. 2.44. The four-circles theorem (left); its proof (right)

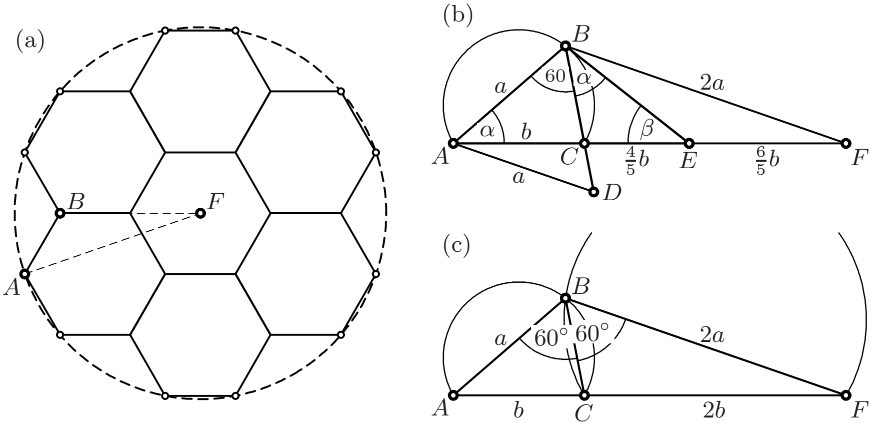


Fig. 2.45. Pappus' hexagon problem

21. Solve "Pappus' last mathematical problem" (from *Collection*, Book VIII, Prop. 16, see Fig. 2.45 (a)): Inscribe in a given circle with radius AF seven identical regular hexagons of maximal size. The problem reduces to the question: Given a segment AF, find a point B such that $BF = 2 \cdot AB$ and the angle ABF is $120°$.
(a) Verify Pappus' construction (Fig. 2.45 (b)): Insert on the segment AF points C and E such that $AC = \frac{1}{3} \cdot AF$ and $CE = \frac{4}{5} \cdot AC$. Draw on AC a circle containing an angle of $60°$ (by Eucl. III.21), and draw EB, tangent to the circle at B. Then B is the required point.
(b) Is there an easier solution?

22. (Archimedes' calculation of π.) Compute the perimeters of the regular inscribed and circumscribed 96-gons of a circle of radius 1 to show that

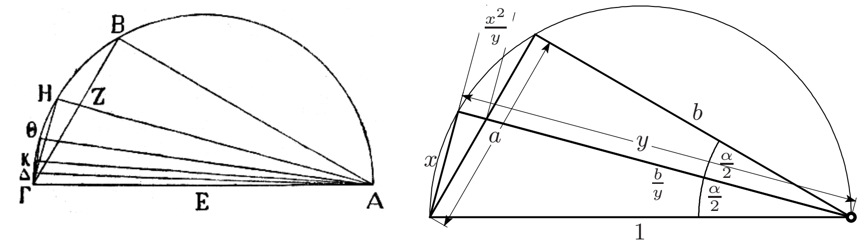


Fig. 2.46. Archimedes' computation of the regular inscribed 96-gon

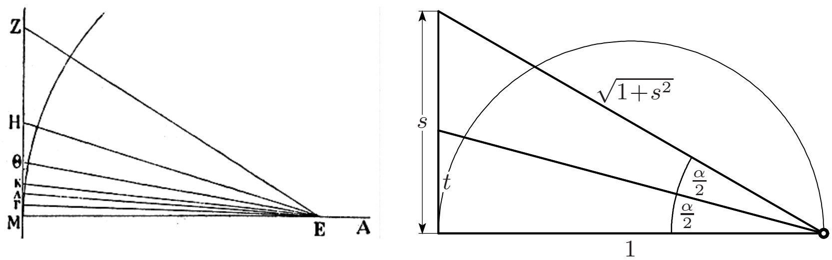


Fig. 2.47. Archimedes' computation of the regular circumscribed 96-gon

$$3\,\frac{10}{71} < \pi < 3\,\frac{1}{7}\,.$$

(a) Apply Pythagoras, Thales and Eucl. III.20 to find $x = H\Gamma$ if $a = B\Gamma$ is known (see Fig. 2.46) and H is the midpoint of the arc $B\Gamma$. This allows one to compute successively, starting from the hexagon, the perimeters of the regular dodecagon, 24-gon, 48-gon and 96-gon.
Hint. The triangles ABZ, $AH\Gamma$ and ΓHZ are similar.

(b) Apply Eucl. VI.3 to find $t = H\Gamma$ if $s = Z\Gamma$ is known (see Fig. 2.47). This will lead similarly to the perimeters of the circumscribed regular n-gons.

23. (Another of the divine discoveries of Euler.) Count, for each of the polyhedra from Euclid's Book XI drawn above,

$$\begin{array}{lll} s_0 & \ldots & \text{the number of vertices,} \\ s_1 & \ldots & \text{the number of edges,} \\ s_2 & \ldots & \text{the number of faces.} \end{array}$$

Make a list of these values and discover Euler's famous relation (Euler, 1758).

http://www.springer.com/978-3-642-29162-3

Geometry by Its History
Ostermann, A.; Wanner, G.
2012, XII, 440 p., Hardcover
ISBN: 978-3-642-29162-3