Optimal dividend and risk control in diffusion models with linear costs*

Hiroaki Morimoto

Abstract We consider the optimization problem of dividends and risk exposures of a firm in the diffusion model with linear costs. The variational inequality associated with this problem is given by the nonlinear form of elliptic type. Using the viscosity solutions technique, we solve the corresponding penalty equation and show the existence of a classical solution to the variational inequality. The optimal policy of dividend payment and risk exposure is shown to exist.

1 Introduction

We consider the optimal dividend and risk control problem of a firm in the diffusion model with linear costs. Let R_t be the risk process refered to the reserve of the firm at time $t \geq 0$. The risk process evolves according to the stochastic differential equation:

$$dR_t = \mu dt + \sigma dB_t, \quad R_0 = x > 0,$$

on a complete probability space $(\Omega, \mathscr{F}, P)$, carrying a one-dimensional standard Brownian motion $\{B_t\}$, endowed with the natural filtration $\mathscr{F}_t$ generated by $\sigma(B_s, s \leq t)$ for $t \geq 0$, where $\mu > 0$ denotes the profit per unit time and $\sigma \neq 0$ is a diffusion coefficient.

A control policy (a, L) is described by a pair of the risk exposure $a = \{a_t\}$ and the flow $L = \{L_t\}$ of dividend payments. The portion $1 - a_t$ of the reserve is paid for reinsurance and L_t denotes the total amount of dividend paid out up to time t. The policy (a, L) is said to be admissible if $\{a_t\}$ is an $\{\mathscr{F}_t\}$-progressively measurable process such that

$$0 \leq a_t \leq 1, \quad t \geq 0,$$

and $\{L_t\}$ is a nonnegative, nondecreasing, continuous $\{\mathscr{F}_t\}$-adapted process with $x - L_0 > 0$. We respectively denote by $\mathscr{A}$ and $\mathscr{L}$ the class of all admissible risk exposures a and dividends L. Given $(a, L) \in \mathscr{A} \times \mathscr{L}$, the dynamics of the risk process $\{R_t\}$ is given by

Department of Mathematics, Ehime University
Matsuyama 790-0826, Japan
e-mail: morimoto@mserv.sci.ehime-u.ac.jp

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$$dR_t = a_t(\mu dt + \sigma dB_t) - \nu R_t dt - dL_t, \quad R_0 = x - L_0 > 0, \tag{1}$$

where νx represents the linear cost function of the reserve x or the debt payment for $\nu \geq 0$.

The objective is to find an optimal policy $(a^*, L^*) = \{(a_t^*, L_t^*)\}$ so as to maximize the expected present value of dividends up to bankruptcy:

$$J(a,L) = E\Big[\int_0^{\vartheta} e^{-\alpha t} dL_t\Big], \tag{2}$$

over all $(a,L) \in \mathscr{A} \times \mathscr{L}$, where $\vartheta = \vartheta(R) := \inf\{t \geq 0 : R_t = 0\}$ and $\alpha > 0$ is a discount factor. In case of $\nu = 0$, this problem has been studied by Højgaard and Taksar [4] and Taksar [7]. We refer to Morimoto [5, 6] for the viscosity solutions technique in the stochastic optimization problem.

Our approach consists in finding a classical solution v of the following variational inequality associated with the problem:

$$v'(x) \geq 1, \quad x > 0, \tag{3}$$

$$-\alpha v + \max_{0 \leq a \leq 1}(\frac{1}{2}a^2\sigma^2 v'' + a\mu v') - \nu x v' \leq 0, \quad x > 0, \tag{4}$$

$$\{-\alpha v + \max_{0 \leq a \leq 1}(\frac{1}{2}a^2\sigma^2 v'' + a\mu v') - \nu x v'\}(v' - 1)^+ = 0, \quad x > 0, \tag{5}$$

$$v(0) = 0. \tag{6}$$

In order to solve the variational inequality (3) - (6), we study the penalty equation of the form:

$$-\alpha u + \frac{1}{2}\varepsilon^2 x^2 u'' + \mathscr{M}u - \nu x u' + \frac{1}{\varepsilon}(u' - 1)^- = 0, \quad x > 0, \tag{7}$$

$$u(0) = 0, \tag{8}$$

where $\mathscr{M}u = \max_{0 \leq a \leq 1}(\frac{1}{2}a^2\sigma^2 u'' + a\mu u')$ and $\varepsilon \in (0,1)$. We show the existence of a solution $u \in C^2(0,\infty) \cap C[0,\infty)$ to the penalty equation (7), (8). By using the penalization method, we prove the convergence of u to a concave viscosity solution $v \in C^2(0,\infty) \cap C[0,\infty)$ of (3) - (6) as $\varepsilon \to 0$. Furthermore, we present the optimal policy (a^*, L^*) with the reflecting barrier at the free boundary x^* for v.

2 The penalized problem

For each $(a,c) \in \mathscr{C} \times \mathscr{C}$, there exists a unique nonnegative solution $\{X_t\}$ of

$$dX_t = 1_{\{t \leq \vartheta(X)\}}\{a_t(\mu dt + \sigma dB_t) + \varepsilon X_t d\bar{B}_t - \nu X_t dt - \frac{c_t}{\varepsilon}dt\}, \quad X_0 = x \geq 0, \tag{9}$$

where $\{\bar{B}_t\}$ is a Brownian motion, mutually independent of $\{B_t\}$, and $\mathscr{C}$ denotes the class $\mathscr{A}$ for $\{(B_t, \bar{B}_t)\}$. Since $(u'-1)^- = \max_{0\le c\le 1}(1-u')c$, we observe that the penalty equation (7) is the Hamilton-Jacobi-Bellman equation associated with the maximization problem:

$$u(x) := \sup_{(a,c)\in\mathscr{C}\times\mathscr{C}} E\Big[\int_0^{\theta} e^{-\alpha t}\frac{c_t}{\varepsilon}dt\Big], \tag{10}$$

subject to (9), where $\theta := \vartheta(X)$ and the supremum is taken over all systems $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}; \{(B_t, \bar{B}_t)\}, \{a_t\}, \{c_t\})$.

Lemma 1. *There exists a concave supersolution $f \in C^1(0,\infty)\cap C^2((0,\infty)\setminus\{m\})$ of (7), (8), independent of ε, for some $m > 0$.*

Proof. Define

$$f(x) = \begin{cases} k(x), & x \le m, \\ x - m + k(m), & x \ge m, \end{cases}$$

where $k(x) = Kx^{\lambda}$ and $m = \sigma^2(1-\lambda)/\mu$. For a suitable choice of $0 < \lambda < 1 < K$, we see that f is a supersolution, i.e.,

$$-\alpha f + \frac{1}{2}\varepsilon^2 x^2 f'' + \mathscr{M} f - \nu x f' + \frac{1}{\varepsilon}(f'-1)^- \le 0, \quad x > 0,\ x \ne m.$$

Theorem 1. *We have*

$$\begin{aligned} &0 \le u(x) \le f(x), \quad x \ge 0, \\ &|u(x) - u(y)| \le f(|x-y|), \quad x, y \ge 0. \end{aligned}$$

Proof. By (9) and (10), we see $\theta \ge \vartheta(Y)$ and $u(x) \ge u(y)$ if $x \ge y$. Applying the generalized Ito formula for convex functions to f, we can show the assertions.

Definition 1. Let $\zeta \in C[0,\infty)$ satisfy (8). Then ζ is called a viscosity subsolution (resp., supersolution) of (7), (8) if, whenever for $\phi \in C^2$, $\zeta - \phi$ attains its local maximum (resp., minimum) at $z > 0$, then

$$-\alpha\zeta + \frac{1}{2}\varepsilon^2 x^2 \phi'' + \mathscr{M}\phi - \nu x\phi' + \frac{1}{\varepsilon}(\phi'-1)^-\Big|_{x=z} \ge 0 \quad (resp., \le 0).$$

We call ζ a viscosity solution of (7), (8) if it is both a viscosity sub- and supersolution of (7), (8).

Theorem 2. *u is a viscosity solution of (7), (8).*

Proof. By Theorem 1, we can see that the dynamic programming principle holds for u. Therefore, we obtain the viscosity property of u.

Theorem 3. *We have*

$$u \in C^2(0,\infty)\cap C[0,\infty).$$

Proof. For any $0 < p < q$, we consider the boundary value problem:

$$-\alpha w + \frac{1}{2}\varepsilon^2 x^2 w'' + \mathscr{M} w - \nu x w' + \frac{1}{\varepsilon}(w' - 1)^- = 0, \quad x \in (p,q),$$
$$w(p) = u(p), \quad w(q) = u(q). \tag{11}$$

By uniformly ellipticity, the theory of fully nonlinear elliptic equations [3] yields that there exists a unique solution $w \in C^2(p,q) \cap C[p,q]$ of (11). By the uniqueness result on viscosity solutions, we have $w = u$ and u is smooth.

Theorem 4. *u is concave on $[0,\infty)$.*

3 Variational inequalities

3.1 Viscosity solutions

Definition 2. Let $\zeta \in C[0,\infty)$ satisfy (6). Then ζ is called a viscosity solution of (3) - (6), if the following assertions are satisfied:

(a) For any $\phi \in C^2(0,\infty)$ and any local minimum point $\bar{z} > 0$ of $\zeta - \phi$,

$$\phi'(\bar{z}) \geq 1, \quad -\alpha\zeta + \mathscr{M}\phi - \nu x \phi'\Big|_{x=\bar{z}} \leq 0,$$

(b) For any $\phi \in C^2(0,\infty)$ and any local maximum point $z > 0$ of $\zeta - \phi$,

$$\{-\alpha\zeta + \mathscr{M}\phi - \nu x \phi'\}(\phi' - 1)^+\Big|_{x=z} \geq 0.$$

Theorem 5. *There exists a subsequence $\{u_{\varepsilon_n}\}$ such that*

$$u_{\varepsilon_n} \to v \in C[0,\infty) \quad \text{locally uniformly in } (0,\infty) \text{ as } \varepsilon_n \to 0. \tag{12}$$

Furthermore, v is a viscosity solution of (3) - (6).

Proof. Let $0 < p < q$ be arbitrary. By concavity and Theorem 1, we get

$$0 \leq u_\varepsilon'(x)x \leq u_\varepsilon(x) - u_\varepsilon(0) \leq \|f\|_{C[p,q]}, \quad x \in [p,q].$$

Hence

$$\sup_\varepsilon \|u_\varepsilon'\|_{C[p,q]} < \infty. \tag{13}$$

Thus, by the Ascoli-Arzelà theorem, there exists a subsequence $\{u_{\varepsilon_n}\}$ satisfying (12). By Theorem 2, passing to the limit, we obtain the viscosity property of v.

3.2 Regularity and the free boundary

Theorem 6. *We have*

$$u_{\varepsilon_n}'(x) \geq 1 \quad \text{for} \quad x > 0.$$

Proof. By Theorems 4 and 5, we note that v is concave and twice differentiable almost everywhere. We recall that $\partial v(q) = \{v'(q)\}$ at a differentiable point $q > 0$ of v. Then, by the viscosity property of v, we have the assertion.

Now, let $A_n(x)$ denote the maximizer of $\max_{0\le a\le 1}(\frac{1}{2}a^2\sigma^2 u''_{\varepsilon_n} + a\mu u'_{\varepsilon_n})$, i.e.,

$$A_n(x) = G(M_n(x)), \quad M_n(x) = -\mu u'_{\varepsilon_n}/\sigma^2 u''_{\varepsilon_n},$$

where

$$G(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 < x. \end{cases}$$

Lemma 2. *For any* $0 < p < q$, *we have*

$$M_n(x) \ge (2\alpha p/\mu) \wedge 1, \quad x \in [p,q], \tag{14}$$

$$\sup_n \sup\{|A_n(x) - A_n(y)|/|x-y| : x,y \in [p,q], x \ne y\} < \infty. \tag{15}$$

Proof. By (7), concavity and Theorem 6, we have

$$\alpha u_{\varepsilon_n} + \nu x u'_{\varepsilon_n} = \frac{1}{2}\varepsilon_n^2 x^2 u''_{\varepsilon_n} + \max_{0\le a\le 1}(\frac{1}{2}a^2\sigma^2 u''_{\varepsilon_n} + a\mu u'_{\varepsilon_n}) \le \frac{\mu}{2} u'_{\varepsilon_n} M_n(x).$$

By concavity, $x u'_{\varepsilon_n} \le u_{\varepsilon_n}$. Thus we get $\alpha x \le (\mu/2) M_n(x)$, which implies (14).

Next, let $A_n(x) = M_n(x)$ on $(p_1, q_1) \subset [p,q]$. By (14) and (13), we get

$$0 \le \sup_{x\in[p_1,q_1]} -\sigma^2 u''_{\varepsilon_n}(x) < \infty, \quad \sup_n \|A'_n\|_{C[p_1,q_1]} < \infty,$$

which implies (15).

Theorem 7. *We have*

$$v \in C^2(0,\infty), \quad v' \ge 1 \quad \textit{on} \ (0,\infty).$$

Proof. For any $0 < p < q$, we set $(\bar{p}, \bar{q}) = (p/2, q + p/2)$. Consider the boundary value problem:

$$\begin{aligned} &\frac{1}{2}\varepsilon_n^2 x^2 \zeta'' + \frac{1}{2}A_n(x)^2\sigma^2\zeta'' + \{A_n(x)\mu - \nu x\}\zeta' = \alpha u_{\varepsilon_n}, \quad x \in (\bar{p}, \bar{q}), \\ &\zeta(\bar{p}) = u_{\varepsilon_n}(\bar{p}), \quad \zeta(\bar{q}) = u_{\varepsilon_n}(\bar{q}). \end{aligned} \tag{16}$$

By Theorems 3 and 6, we see that u_{ε_n} solves (16). By Lemma 2 and the interior Schauder estimates for (16), we have $\sup_n \|u_{\varepsilon_n}\|_{C^{2,\gamma}[p,q]} < \infty, 0 < \gamma < 1$, which completes the proof. We remark that v is a classical solution of (3) - (6).

Theorem 8. *There exists the free boundary* $x^* \in (0,\infty)$ *for* v, *which fulfills*

$$x^* = \sup\{x > 0 : v'(x) > 1\}.$$

Proof. By the contradiction arguments, we can see that $\{\cdot\}$ is non-empty and $x^* < \infty$.

4 Optimal policies

Consider the SDE with reflecting barrier conditions for the free boundary x^*:

$$dR_t^* = \bar{A}(R_t^*)(\mu dt + \sigma dB_t) - \nu R_t^* dt - dL_t^*, \quad R_0^* = x, \tag{17}$$

$$L_t^* = \int_0^t 1_{\{R_s^* = x^*\}} dL_s^*, \tag{18}$$

$$L_t^* \text{ is continous and nondecreasing}, \tag{19}$$

$$R_t^* \leq x^*, \quad t \geq 0, \tag{20}$$

$$\int_0^t 1_{\{R_s^* = x^*\}} ds = 0, \quad t \geq 0, \tag{21}$$

where $\bar{A}(x)$ is the continuous extension of $A(x) := G(-\mu v'/\sigma^2 v'')$ for $x > 0$ and $\bar{A}(x) = 0$ for $x \leq 0$.

Lemma 3. *We have* $\lim_{x \to 0+} A(x) = 0$ *and* $\bar{A}$ *is Lipschitz on* $(-\infty, x^*]$.

Theorem 9. *We assume* $0 < x \leq x^*$. *Then the optimal policy* (a^*, L^*) *for (2) subject to (1) is given by* $a_t^* = \bar{A}(R_t^*)$ *and* $\{L_t^*\}$ *of (17) - (21).*

Proof. According to [1], by Lemma 3, there exists a unique solution $\{(R_t^*, L_t^*)\}$ of (17) - (21). Applying Ito's formula to (3) - (6), we can obtain the optimality.

5 Conclusion

The optimal policy with the reflecting barrier at the free boundary is shown to exist.

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