Chapter 2
Mathematical Formulas Involving the Different Zeta Functions

In this chapter, a compendium of original formulas resulting from the zeta-regularization techniques, developed by the author and collaborators is given. Although some of the original derivations are reproduced, what follows is mainly intended as a table for practical use by the reader—the full derivations and arguments involved can be found in the accompanying bibliographical references. In particular, useful expressions are provided for the analytic continuation of Riemann, Hurwitz and Epstein zeta functions and generalizations of them, for their asymptotic expansions (including those for derivatives of Hurwitz's ζ), the zeta-function regularization theorem—and its use for multiple zeta-functions with arbitrary exponents—and, in another section, the first immediate applications of the theorem. All this is followed by a very careful study of the analytic continuation of multiple series which terms are combinations involving arbitrary coefficients and exponents, a case that is very involved and has not been treated properly in the mathematical literature. Of course this case always involves the elusive term that shows up in the correct application of the zeta-function regularization theorem. Some mistakes which regretfully appeared in a few formulas of the original papers have been corrected.

2.1 A Simple Recurrence for the Higher Derivatives of the Hurwitz Zeta Function

A recurrent formula which allows for the calculation of the asymptotic series expansion of any derivative, $\zeta^{(m)}(z,a) = \partial^m \zeta(z,a)/\partial z^m$, of the Hurwitz zeta function $\zeta(z,a)$ is here given. In particular, the first terms of the series corresponding to $\zeta''(-n,a)$ in inverse powers of a are written explicitly, for $n = 0, 1, 2, 3$. Knowledge of these expressions is basic in the zeta-function regularization procedure.

Some time ago, an asymptotic expansion for the first derivative

$$\zeta'(-n,a) \equiv \frac{\partial}{\partial z}\zeta(z,a)\bigg|_{z=-n}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

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of the Hurwitz zeta function

$$\zeta(z,a)=\sum_{n=0}^{\infty}(n+a)^{-z}, \quad \operatorname{Re} z>1,\ a\neq 0,-1,-2,\ldots \tag{2.2}$$

in inverse powers of a was derived [95] (see (2.8)), that has been found to be very useful by a number of authors—as a convenient tool, e.g., for the computation of effective actions in non-trivial backgrounds and also for the derivation of other interesting zeta function relations (see, for example, Steiner [96], Rudaz [97], and Ref. [51]). The simplicity and very quick convergence of the expressions we will give below have been recognized to be their most remarkable characteristics. They render them very useful for numerical applications [96], and for the subsequent derivation of related expressions for other zeta [97] and theta [51] functions.

In this chapter we describe the use the procedure of Refs. [95, 98] in order to obtain the asymptotic expansion corresponding to any derivative of the Hurwitz zeta function (2.2)

$$\zeta^{(m)}(z,a)\equiv\frac{\partial^m}{\partial z^m}\zeta(z,a). \tag{2.3}$$

The interest of such formulas has been manifest since some years ago, and actually a couple of attempts had been made by some authors to solve the problem, but they did not turn out to be completely successful. It is rather clear from the very beginning that, for the general case (2.3), it is not possible to obtain an expression so simple as the one derived in Ref. [95] for (2.1) (see (2.8) below).

We will not repeat here the detailed derivation of the asymptotic expansion corresponding to (2.1), given in Ref. [95]. Starting from Hermite's integral representation of the Hurwitz zeta function $\zeta(z,a)$ (in the future we will omit the subindex H, as is normal practice)

$$\zeta(z,a)=\frac{a^{-z}}{2}+\frac{a^{1-z}}{z-1}+2\int_0^\infty\left(t^2+a^2\right)^{-z/2}\sin\left(z\tan^{-1}(t/a)\right)\frac{dt}{e^{2\pi t}-1}, \tag{2.4}$$

one easily gets

$$\begin{aligned}\zeta'(z,a)={}&-\frac{a^{-z}}{2}\ln a-\frac{a^{1-z}}{z-1}\ln a-\frac{a^{1-z}}{(z-1)^2}\\&+2\int_0^\infty\left(t^2+a^2\right)^{-z/2}\cos\left(z\tan^{-1}(t/a)\right)\tan^{-1}(t/a)\frac{dt}{e^{2\pi t}-1}\\&-\int_0^\infty\left(t^2+a^2\right)^{-z/2}\sin\left(z\tan^{-1}(t/a)\right)\ln\left(t^2+a^2\right)\frac{dt}{e^{2\pi t}-1}.\end{aligned} \tag{2.5}$$

We invite the reader to read Ref. [95] for more details of the mathematical procedure employed, which is similar to the ordinary one derived from Watson's lemma and Laplace's method [84, 99] and is therefore quite a conventional one for the obtaintion of asymptotic expansions. In particular, the functions appearing in the integrands are replaced by their power series expansions near $t = 0$, e.g.

$$\begin{aligned}
\tan^{-1}\left(\frac{t}{a}\right) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\left(\frac{t}{a}\right)^{2j+1}, \\
\ln\left(t^2 + a^2\right) &= 2\ln a + \sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{j}\left(\frac{t}{a}\right)^{2j},
\end{aligned} \tag{2.6}$$

and one then checks with the residuum terms that the series that are formed by integrating term by term verify the condition of asymptoticity. Alternatively, another procedure can be employed that leads to the same result, namely repeated integration by parts.

In any way, the final result turns out to be the same [97] that one would obtain by naive derivation term by term of the asymptotic series corresponding to the Hurwitz zeta function (2.2)

$$\begin{aligned}
\zeta(z+1, a) &= \frac{1}{z}a^{-z} + \frac{1}{2}a^{-z-1} + \frac{1}{z}\Sigma_0(z, a), \\
\Sigma_0(z, a) &\equiv \sum_{k=2}^{\infty} \frac{B_k}{k!}(z)_k \, a^{-z-k}, \\
(z)_k &\equiv z(z+1)\cdots(z+k-1) = \frac{\Gamma(z+k)}{\Gamma(z)}.
\end{aligned} \tag{2.7}$$

Here $(z)_k$ is Pochhammer's symbol (the rising factorial function) and the B_k are Bernoulli's numbers. The asymptotic series corresponding to $\zeta'(z+1, a)$ can be expressed as

$$\begin{aligned}
\zeta'(z+1, a) &= -\left(\frac{1}{z} + \ln a\right)\zeta(z+1, a) + \frac{1}{2z}a^{-z-1} + \frac{1}{z}\Sigma_1(z, a), \\
\Sigma_1(z, a) &\equiv \sum_{k=2}^{\infty} B_k \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)}a^{-z-k}.
\end{aligned} \tag{2.8}$$

Notice that this is not a trivial result since—as is well known—term by term derivation of an asymptotic series is controlled by a Tauberian theorem (and *not* by an abelian one). These expansions are valid for large $|a|$ and $|\operatorname{Arg} a| < \pi$. In particular, when $z = -n$, $n \in \mathbb{N}$, the above expressions reduce to [100]

$$\zeta(1-n, a) = -\frac{1}{n}B_n(a), \tag{2.9}$$

where $B_n(a)$ is the Bernoulli polynomial of degree n, and to

$$\zeta'(1-n,a) = \frac{1}{n}\left(\ln a - \frac{1}{n}\right)B_n(a) - \frac{1}{2n}a^{n-1}$$
$$- \frac{1}{n}\sum_{k=2}^{n} B_k \sum_{j=0}^{k-1} \binom{n}{j} \frac{(-1)^j}{k-j}\, a^{n-k}$$
$$+ (-1)^{n-1}(n-1)! \sum_{k=n+1}^{\infty} \frac{B_k}{k(k-1)\cdots(k-n)} a^{n-k}, \quad (2.10)$$

respectively.

Now, starting again from Hermite's integral representation (2.4) and repeating, for ζ'', the same procedure used for ζ', in particular, the replacements (2.6) or integration by parts (quite involved), we arrive to the following expression for the second derivative

$$\zeta''(z+1,a) = -2\left(\frac{1}{z} + \ln a\right)\zeta'(z+1,a)$$
$$- \left(2\frac{\ln a}{z} + \ln^2 a\right)\zeta(z+1,a) + \frac{1}{z}\Sigma_2(z,a), \quad (2.11)$$
$$\Sigma_2(z,a) \equiv \sum_{k=2}^{\infty} B_k \sum_{j=0}^{k-1} \frac{1}{k-j} \sum_{h=0}^{j-1} \frac{(z)_h}{h!(j-h)} a^{-z-k}.$$

This is, term by term, the same result that one would have obtained by naive derivation of the preceding asymptotic series. Actually, the alternative procedure (namely that of partial integration)—which was already discussed in [95]—proves to be here the most convenient one in order to exhibit the asymptotic character of the series (2.11).

With some additional effort, the following operational recurrence can be found, in general [98]

$$\frac{\partial^m}{\partial z^m}\zeta(z+1,a)$$
$$= -\left[\left(\frac{\partial}{\partial z} + \ln a\right)^m - \left(\frac{\partial}{\partial z}\right)^m + \frac{m}{z}\left(\frac{\partial}{\partial z} + \ln a\right)^{m-1}\right]\zeta(z+1,a)$$
$$+ \frac{1}{z}\Sigma_m(z,a), \quad (2.12)$$

being

$$\Sigma_m(z,a) \equiv \sum_{k=2}^{\infty} B_k \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \cdots \sum_{j_m=0}^{j_{m-1}-1} \frac{(z)_{j_m}}{j_m!(j_{m-1}-j_m)} a^{-z-k}, \quad (2.13)$$

for large $|a|$ and $|\operatorname{Arg} a| < \pi$. Using the operational iteration (2.12), the following general recurrence is obtained, which yields the asymptotic expansion for any derivative of the Hurwitz zeta function in terms of the asymptotic expansion corresponding to the derivatives of lower order:

$$\zeta^{(m)}(z+1,a) = -\sum_{j=1}^{m} \binom{m}{j} \left(\frac{j}{z} + \ln a\right) \ln^{j-1} a\, \zeta^{(m-j)}(z+1,a) + \frac{1}{z} \Sigma_m(z,a), \tag{2.14}$$

$\Sigma_m(z,a)$ being given by (2.13).

If we restrict ourselves to the particular values $z = -n$, $n \in \mathbb{N}$, we obtain the more simple expression

$$\zeta^{(m)}(1-n,a) = \sum_{j=1}^{m} \binom{m}{j} \left(\frac{j}{n} - \ln a\right) \ln^{j-1} a\, \zeta^{(m-j)}(1-n,a) - \frac{1}{n} \Sigma_m(-n,a), \tag{2.15}$$

where now

$$\Sigma_m(-n,a) = \sum_{k=2}^{\infty} B_k \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \cdots \sum_{j_m=0}^{\mu_m} \binom{n}{j_m} \frac{(-1)^j}{j_{m-1}-j_m} a^{n-k}, \tag{2.16}$$

being $\mu_m = \min(n, j_{m-1}-1)$.

(2.12) to (2.16) constitute the main results of this section. Even if they do not provide a general explicit asymptotic expression for any derivative of the Hurwtiz zeta function but any of such asymptotic series can immediately be found by solving the very simple recurrences (2.14) or (2.15), starting from (2.7) and (2.9), and (2.8) and (2.10), respectively. These expressions are very appropriate for numerical and analytical explicit calculations, in connection with the computational software packets commonly available.

To prove this statement, let us obtain the asymptotic series for the second derivative, at non-positive integer values of z. It is given by

$$\begin{aligned}\zeta''(1-n,a) = {}& \left(-\frac{2}{n^2} + \frac{2\ln a}{n} - \ln^2 a\right) \frac{B_n(a)}{n} - \left(\frac{1}{n} - \ln a\right) \frac{a^{n-1}}{n} \\ & - \frac{1}{n} \Sigma_2(-n,a) - \frac{2}{n} \left(\frac{1}{n} - \ln a\right) \Sigma_1(-n,a),\end{aligned} \tag{2.17}$$

where $B_n(a)$ is the Bernoulli polynomial of degree n, and

$$\begin{aligned}\Sigma_2(-n,a) = {}& \sum_{k=2}^{n} B_k \sum_{j=0}^{k-1} \frac{1}{k-j} \sum_{h=0}^{j-1} \binom{n}{h} \frac{(-1)^h}{j-h}\, a^{n-k} \\ & + \sum_{k=n+1}^{\infty} B_k \sum_{j=0}^{n} \frac{1}{k-j} \sum_{h=0}^{j-1} \binom{n}{h} \frac{(-1)^h}{j-h}\, a^{n-k}\end{aligned}$$

$$+\,(-1)^n n! \sum_{k=n+1}^{\infty} B_k \sum_{j=n+1}^{k-1} \frac{1}{(k-j)j(j-1)\cdots(j-n)} a^{n-k} \quad (2.18)$$

and

$$\Sigma_1(-n,a) = \sum_{k=2}^{n} B_k \sum_{j=0}^{k-1} \binom{n}{j} \frac{(-1)^j}{k-j} a^{n-k}$$
$$+\,(-1)^n n! \sum_{k=n+1}^{\infty} \frac{B_k}{k(k-1)\cdots(k-n)} a^{n-k}. \quad (2.19)$$

It is also clear enough that these expressions for the asymptotic series are well suited for practical purposes. We have used *Mathematica* in a conventional workstation in order to obtain a number of leading terms of the above series, for different values of n. This can be done in less than a minute. Below there is a list of the first few results obtained (we will not bother the reader with the full sample):

$$\zeta''(0,a) = -a\left(\ln^2 a - 2\ln a + 1\right) + \frac{1}{2}\ln^2 a - \frac{a^{-1}}{6}\ln a$$
$$+\,\frac{a^{-3}}{60}\left(\frac{\ln a}{3} - \frac{1}{2}\right) - a^{-5}\left(\frac{\ln a}{630} - \frac{4}{1209}\right) + \cdots, \quad (2.20)$$

$$\zeta''(-1,a) = -\frac{a^2}{4}\left(2\ln^2 a - 2\ln a + 1\right) + \frac{a}{2}\ln^2 a + \frac{1}{12}\left(\ln^2 a - 2\ln a\right)$$
$$-\,\frac{a^{-2}}{360}\ln a + \frac{a^{-4}}{15120}(6\ln a - 5) + \cdots, \quad (2.21)$$

$$\zeta''(-2,a) = -\frac{a^3}{27}\left(9\ln^2 a - 6\ln a + 2\right) + \frac{a^2}{2}\ln^2 a - \frac{a}{6}\left(\ln^2 a - \ln a\right)$$
$$+\,\frac{a^{-1}}{60}\left(\frac{\ln a}{3} + \frac{1}{2}\right) - \frac{a^{-3}}{3780}\ln a$$
$$+\,\frac{a^{-5}}{1800}\left(\frac{\ln a}{7} - \frac{1}{12}\right) + \cdots, \quad (2.22)$$

and

$$\zeta''(-3,a) = -\frac{a^4}{32}\left(8\ln^2 a - 4\ln a + 1\right) + \frac{a^3}{2}\ln^2 a + \frac{a^2}{12}\left(3\ln^2 a + 2\ln a\right)$$
$$+\,\frac{1}{60}\left(\frac{\ln^2 a}{2} + \frac{11\ln a}{6} + 1\right)$$
$$+\,\frac{a^{-2}}{504}\left(\frac{\ln a}{5} + \frac{1}{6}\right) - \frac{a^{-4}\ln a}{16800} + \cdots. \quad (2.23)$$

Actually, one reaches values of n as high as $n = 20$ very quickly, what proves that the above recurrent expressions, (2.13) to (2.16), are in fact very efficient for practical applications.

2.2 The Zeta-Function Regularization Theorem

As advanced before, the zeta-function regularization procedure is a quite useful regularization tool in quantum field theory. A keystone of the method is the zeta function regularization theorem. In this section, the theorem will be illustrated while addressing the practical question of the regularization of multi-series of the general type

$$\sum_{n_1,\ldots,n_N} \left[a_1(n_1+c_1)^{\alpha_1} + \cdots + a_N(n_N+c_N)^{\alpha_N} + c\right]^{-s}, \tag{2.24}$$

with $a_1,\ldots,a_N,\alpha_1,\ldots,\alpha_N > 0$, $c_1,\ldots,c_N$ arbitrary reals and $c \geq 0$. When $c_1 = \cdots = c_N = 0$ the term with $n_1 = \cdots = n_N = 0$ must be suppressed from the sum (which is then usually denoted by Σ'). Only the most simple cases have been properly studied in the literature (e.g., $a_1 = \cdots = a_N$, $c_1 = \cdots = c_N = 0$ or $\pm 1/2$, $\alpha_1 = \cdots = \alpha_N = 1, 2$, $c = 0$, etc.). The zeta function regularization theorem in its most general form leads to an asymptotic expansion valid for arbitrary a's and α's, which is very convenient for numerical computations. In particular, useful expressions can be derived from it for the analytical continuation of Riemann, Hurwitz and Epstein zeta functions and their generalizations (see Chap. 4), and for their asymptotic expansions—including those of derivatives and integrals. Physical applications of the zeta-regularization procedure include the proper definition of the vacuum energy, the Casimir effect, spontaneous compactification in quantum gravity, stability analysis of strings and membranes, etc., and embrace also very recent experiments of solid state and condensed matter physics employing liquid helium (those will be described in the following chapters).

The method of zeta-function regularization has a rather long history. There are precedents in the use of Riemann and Epstein zeta functions as summation (i.e., regularization) procedures in the late sixties [44, 45, 47]. However, the zeta-function regularization method as such was introduced in the middle seventies [23, 24, 39–41]. The paper by Hawking [24] (of 1977) is generally considered as the first systematic description of the zeta function procedure as a useful technique in physics for providing the finite values corresponding to path integrals over fields in curved backgrounds and for the evaluation of determinants of quadratic differential operators (see, however, the other references mentioned, in which the method had already been applied before). The calculation of determinants of differential operators is a basic, multipurpose need in theoretical physics and in several branches of mathematics (such as analysis and number theory).

In the last 15 years the zeta-regularization procedure has been used more and more by the leading physicists and mathematicians and we can nowadays say that it is a basic procedure of quantum field theory. At the beginning the method was rather simple minded, but nowadays it comprises a whole set of different techniques, of increasing difficulty, to treat the several degrees of complexity of the physical (and corresponding mathematical) problems to be solved.

The list of people who have been dealing with zeta functions at one instance or other would be just non-ending. Maybe Al Actor is one of the persons that have devoted more years to this subject (at least among those of the mathematical-physicists squad). According to Actor himself [53], a milestone in the field of regularization of discrete sums of the general form (2.24) has been the proof of the so-called *zeta-function regularization theorem*. In its final formulation, it is the result of hard work of A.A. Actor, H.A. Weldon, A. Romeo, and the author [48, 101, 102]. The uses and applications of the theorem in its most general form [48]—for discrete series of the type (2.24)—are very far reaching. In particular it leads to asymptotic expansions, valid for arbitrary a's and α's, of the multi-series of this general kind, which are well suited for numerical computations. These expansions are unchallenged in its usefulness for such purposes. They will be presented later in this chapter.

The zeta function regularization theorem provides a method for the computation of expressions like (2.24)—and even more involved ones—valid for $\mathrm{Re}(s)$ big enough, in terms of their analytic (usually meromorphic) continuation to other values of s. In the zeta-function procedure they are given as combinations of the ordinary Riemann and Hurwitz zeta-functions.

A very simple case corresponds to the Hamiltonian zeta-function $\zeta(s) \equiv \sum_i E_i^{-s}$, with E_i eigenvalues of H [103, 104]. For a system of N non-interacting harmonic oscillators, one has $\alpha_j = 1,\ j = 1, 2, \ldots, N$, and the a_j are the corresponding eigenfrequencies ω_j. Another interesting case is partial toroidal compactification (spacetime $\mathbb{T}^p \times \mathbb{R}^{q+1}$). Then $\alpha_j = 2$ and, usually, $c_j = 0, \pm 1/2$. One is thus led to the Epstein zeta-functions [105–107]

$$
\begin{aligned}
Z_N(s) &= \sum_{n_1,\ldots,n_N=-\infty}^{\infty}{}' \left(n_1^2 + \cdots + n_N^2\right)^{-s}, \\
Y_N(s) &= \sum_{n_1,\ldots,n_N=-\infty}^{\infty}{}' \left[\left(n_1 + \frac{1}{2}\right)^2 + \cdots + \left(n_N + \frac{1}{2}\right)^2\right]^{-s}
\end{aligned}
\tag{2.25}
$$

(remember that the prime means omission of the term $n_1 = \cdots = n_N = 0$). Other powers α_j appear when one deals with the spherical compactification (spacetime $\mathbb{S}^p \times \mathbb{R}^{q+1}$) and with more involved ones arising, e.g., in superstring theory and their membrane and p-brane generalizations [10, 11, 13, 14]. Hence, the general expression (2.24). The only precedents in the literature (to our knowledge) of this kind of evaluations have been restricted to few special cases other than $a_1 = \cdots = a_N$ and $c_1 = \cdots = c_N = 0$. Very famous is the expression due to Hardy [17], a particular case of our final formula.

2.2.1 The Theorem (Special Form)

An interesting result concerning the interchange of the order of summation of the infinite series appearing in zeta-function regularization is due to Weldon [102]. His investigation originated in some difficulties which appeared in a paper by Actor [101] when he tried to obtain the value of the thermodynamical potential corresponding to a relativistic Bose gas by using the zeta-function regularization procedure. Unfortunately, Weldon's proof had its own limitations, and the statements in [102] concerning the extent of its validity were actually not right. This is quite easy to check in some particular cases, and was stressed in [108].

Let us briefly summarize the nice proof due to Weldon of the validity of the zeta-function regularization procedure [102] and point out its shortcomings. Using the same notation as in [102], let us consider the four series

$$S_F = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \sum_{a=0}^{\infty} m^a f(a), \tag{2.26}$$

$$S_B = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} m^a f(a), \tag{2.27}$$

$$S_{AF} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a m^a f(a), \tag{2.28}$$

$$S_{AB} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a m^a f(a), \tag{2.29}$$

where $f(a) \geq 0$ for positive integer a. They are assumed to be convergent, as they stand. The idea of the zeta-function regularization procedure begins with the interchange of the order of the summation of the two infinite series involved in each case.

Theorem 1 *Let $f(a)$ be defined in the complex a-plane, satisfying*:

1. *The function $f(a)$ is regular for* $\mathrm{Re}\, a \geq 0$.
2. *Either*
   (a) *in the case of* (2.26) *and* (2.27), $am^a f(a) \to 0$, *as* $|a| \to \infty$, *for* $\mathrm{Re}\, a \geq 0$ *and fixed m*;
   (b) *in the case of* (2.28) *and* (2.29), $am^a f(a) e^{-\pi|\mathrm{Im}\, a|} \to 0$, *as* $|a| \to \infty$, *for* $\mathrm{Re}\, a \geq 0$ *and fixed m*.

Then it turns out that, in the fermionic cases, (2.26) *and* (2.28), *one can naively interchange the order of the summations, to get*

$$S_F = \sum_{a=0}^{\infty} \eta(s+1-a) f(a), \qquad S_{AF} = \sum_{a=0}^{\infty} (-1)^a \eta(s+1-a) f(a), \tag{2.30}$$

while in the bosonic cases, (2.27) *and* (2.29), *one obtains the additional contributions*

$$\begin{aligned} S_B &= \sum_{a=0}^{\infty} \zeta(s+1-a) f(a) - \pi \operatorname{ctg}(\pi s) f(s), \quad s \notin \mathbb{N}, \\ S_B &= \sum_{\substack{a=0 \\ a \neq s}}^{\infty} \zeta(s+1-a) f(a) + \gamma f(s) - f'(s), \quad s \in \mathbb{N}, \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} S_{AB} &= \sum_{a=0}^{\infty} (-1)^a \zeta(s+1-a) f(a) - \pi \csc(\pi s) f(s), \quad s \notin \mathbb{N}, \\ S_{AB} &= \sum_{\substack{a=0 \\ a \neq s}}^{\infty} (-1)^a \zeta(s+1-a) f(a) + (-1)^s \big[\gamma f(s) - f'(s)\big], \quad s \in \mathbb{N}, \end{aligned} \tag{2.32}$$

respectively. Here $\zeta(s)$ *and* $\eta(s)$ *are the Riemann ordinary and alternating zeta functions*:

$$\begin{aligned} \zeta(s) &= \sum_{m=1}^{\infty} m^{-s}, \quad \operatorname{Re} s > 1, \\ \eta(s) &= \sum_{m=1}^{\infty} (-1)^{m+1} m^{-s}, \quad \operatorname{Re} s > 0, \\ \eta(s) &= \big(1 - 2^{1-s}\big) \zeta(s), \end{aligned} \tag{2.33}$$

γ *is Euler–Mascheroni's constant, and* $f'(s)$ *means derivative of* f *with respect to* s.

The proof of the preceding theorem proceeds by integration in the complex a-plane. One writes (2.26) to (2.29) under the form of contour integrals

$$S_F = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \cot(\pi a), \tag{2.34}$$

$$S_B = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \cot(\pi a), \tag{2.35}$$

$$S_{AF} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \csc(\pi a), \tag{2.36}$$

$$S_{AB} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \csc(\pi a), \tag{2.37}$$

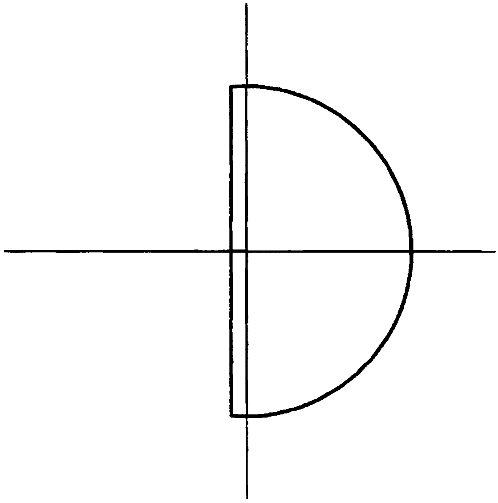

Fig. 1 The *closed contour C* (that one should always follow counterclockwise, i.e., with the positive orientation) consists of the *straight line* $\mathrm{Re}\, a = a_0$, $0 < a_0 < 1$, and of the *semicircumference* 'at infinity' on its right, *K*

where $\mathcal{C}$ is the closed contour defined by the straight line $\mathrm{Re}\, a = -a_0$—for fixed a_0 such that $0 < a_0 < 1$—and by the semicircumference at infinity on the right (see Fig. 1). The contribution from the semicircumference is zero in every case, due to the asymptotic behavior of $f(a)$ and, as long as $\mathrm{Re}\, s > -1$, integration extended to the line $\mathrm{Re}\, a = -a_0$ can be interchanged with the remaining summation over m. The final step is to close the contour $\mathcal{C}$ again with the semicircumference at infinity. In the cases (2.35) and (2.37) there appears an additional contribution from the pole of the zeta function $\zeta(s+1-a)$ at $a = s$. On the contrary, in the cases (2.34) and (2.36) the alternating zeta function $\eta(s+1-a)$ has no pole in the region enclosed by $\mathcal{C}$. All the steps in this procedure are quite simple and one obtains (2.30) to (2.32).

However it was further explicitly stated by Weldon in [102] that the results for the alternating fermionic and for the alternating bosonic cases, S_{AF} and S_{AB}, respectively, could be naively extended to the following types of series

$$\begin{aligned} S_{AF}^{(N)} &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a m^{Na} f(a), \\ S_{AB}^{(N)} &= \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a m^{Na} f(a), \end{aligned} \tag{2.38}$$

with N any positive integer. By going through the same proof once more, he just obtained a trivial modification of the above results. That this generalization of (2.30) and (2.32) and for any positive integer N is in error is easy to check. In particular, it was noticed by Actor in [108]. As a clear example, let us study the simplest case after the (only correct) one $N = 1$ (explicitly considered in [102]), i.e. $N = 2$. Let

$$S \equiv \sum_{m=1}^{\infty} e^{-m^2} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a \frac{m^{2a}}{a!} \bigg|_{s=-1}, \tag{2.39}$$

where the last operation consists in doing the analytic continuation of the resulting series to $s = -1$. The function $f(a)$ is here $f(a) = \frac{1}{\Gamma(a+1)}$ and all the hypotheses of the theorem are fulfilled. Use of Weldon's formula gives

$$S = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \zeta(-2a) - \frac{\frac{\pi}{2} \csc(-\frac{\pi}{2})}{\Gamma(1-\frac{1}{2})} = -\frac{1}{2} + \frac{\sqrt{\pi}}{2}, \tag{2.40}$$

which is false, though numerically almost undetectable, because

$$S = 0.3863186, \qquad \frac{\sqrt{\pi}-1}{2} = 0.3862269, \qquad \Delta \equiv \frac{\sqrt{\pi}-1}{2} - S = -9.17 \times 10^{-5}. \tag{2.41}$$

Going on to $N = 2, 3, 4, \ldots$, it is not difficult to see that, if N is constrained to be a positive integer, Weldon's formula is true only for $N = 1$ ((2.30) and (2.32)).

As the author managed to demonstrate in [48], the step which fails to be correct in Weldon's proof for general N is the last one, namely, even if the asymptotic behavior (2b) of the function $f(a)$ allows us to suppress the contribution from the curved contour in the second step, this will be no longer true when we try to close again the circuit $\mathcal{C}$ in the last step. *There is in fact a contribution coming from the integral of* $\zeta(s+1-Na)f(a)$ *over the semicircumference at infinity* (due to the asymptotic behavior of the zeta-function). And this is so whatever it be the value we choose for s. The study of the asymptotic behavior of $\zeta(s+1-Na)$ immediately distinguishes the case $N \le 1$ from $N > 1$. It is, however, misleading in some sense, because the fact that the zeta-function diverges for $N > 1$ does *not* necessarily mean that the contour actually provides a non-zero contribution invalidating Weldon's proof (that had been conjectured by Actor, at a first instance). Things must be done with great care due to the presence of highly oscillating factors.

Let us restrict the argument to the case $f(a) = \frac{1}{\Gamma(a+1)}$. This is enough for many applications and the generalization to other situations proceeds by analogy. In this case, the fact that the poles of Γ are the non-positive integers and a suitable application of the zeta function reflection formula allow us to write the additional contribution as a contour integral over a curved path in the complex left half-plane. Besides, by using the relation

$$\Gamma\left(\frac{z}{2}\right)\zeta(z) = \int_0^{\infty} dt\, t^{z/2-1} S_2(t), \quad \operatorname{Re} z > 0, \tag{2.42}$$

where

$$S_\alpha(t) \equiv \sum_{m=1}^{\infty} e^{-m^\alpha t}, \tag{2.43}$$

and owing to the behavior of the complex function $\Gamma(z)$ which has simple poles at $z = -n$ for $n = 0, 1, 2, \ldots$, with residues

$$\mathrm{Res}_{z=-n}\,\Gamma(z) = \frac{(-1)^n}{n!}, \tag{2.44}$$

and with the aid of

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z), \tag{2.45}$$

we can write

$$S_{AB}^{(\alpha)} \equiv \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a \frac{m^{\alpha a}}{\Gamma(a+1)}, \quad \alpha \in \mathbb{R}, \tag{2.46}$$

as

$$S_{AB}^{(\alpha)} \equiv \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \oint_{\bar{C}} \frac{da}{2\pi i} m^{-\alpha a} \Gamma(a), \tag{2.47}$$

where now the contour $\bar{C}$ consists of the line $\mathrm{Re}\, a = a_0$, with a_0 fixed, $0 < a_0 < 1$, and of the semicircumference at infinity on the left. For $s = -1$,

$$S_{AB}^{(\alpha)}(s=-1) = \sum_{m=1}^{\infty} \sum_{a=0}^{\infty} (-1)^a \frac{m^{\alpha a}}{a!} = \sum_{m=1}^{\infty} e^{-m^{\alpha}} = S_{\alpha}(1). \tag{2.48}$$

Finally, after correctly making the last step in the above proof, we end up with

$$S_{AB}^{(\alpha)} = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \zeta(s+1-\alpha a) + \frac{1}{\alpha} \Gamma\left(-\frac{s}{\alpha}\right) - \Delta_{AB}^{(\alpha)}, \quad \frac{s}{\alpha} \notin \mathbb{N}, \tag{2.49}$$

$$\begin{aligned} S_{AB}^{(\alpha)} = &\sum_{\substack{a=0 \\ a \neq s/\alpha}}^{\infty} \frac{(-1)^a}{a!} \zeta(s+1-\alpha a) + (-1)^{\frac{s}{\alpha}} \left[\frac{\gamma}{\Gamma(\frac{s}{\alpha}+1)} + \frac{\Gamma'(\frac{s}{\alpha}+1)}{\alpha \Gamma^2(\frac{s}{\alpha}+1)} \right] \\ &- \Delta_{AB}^{(\alpha)}, \quad \frac{s}{\alpha} \in \mathbb{N}, \end{aligned} \tag{2.50}$$

where $\Delta_{AB}^{(\alpha)}$ is the contribution of the curved part K of the contour $\bar{C}$—which consists now of the line $\mathrm{Re}\, a = a_0$, for fixed a_0 such that $0 < a_0 < 1$ and by the semicircumference at infinity on the left (see Fig. 2)

$$\Delta_{AB}^{(\alpha)} \equiv \int_K \frac{da}{2\pi i} \zeta(s+1+\alpha a) \Gamma(a). \tag{2.51}$$

This contribution is non-zero for any value of s. We can check that it actually provides the term missing from (2.40). Before proceeding with the actual calculation of (2.51), one can, as an illustrating exercise, close back the contour on the right instead of on the left, and see that the same series is obtained (a desired identity!).

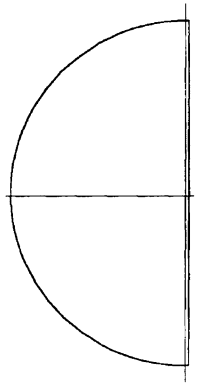

Fig. 2 The *closed contour* $\overline{C}$ (also counterclockwise) consists of the *straight line* $\operatorname{Re} a = a_0$, $0 < a_0 < 1$, and of the *semicircumference* 'at infinity' on its left

Coming back to (2.51) and doing the same for $s = -1$ and $\alpha = 2$, we must use first the reflection formula for the zeta function

$$\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{z-1/2}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z), \tag{2.52}$$

what yields

$$\Delta_{AB}^{(2)}(s=-1) = \int_K \frac{da}{2i\sqrt{\pi}} \int_0^\infty dt\, t^{-a-1/2} S_2\big(\pi^2 t\big) = -\sqrt{\pi} S_2\big(\pi^2\big), \tag{2.53}$$

that is

$$S_2(1) = -\frac{1}{2} + \frac{\sqrt{\pi}}{2} + \sqrt{\pi} S_2\big(\pi^2\big). \tag{2.54}$$

This result happens to be just a particular case of Jacobi's theta function identity

$$\theta_3(z,\tau) = \tau^{-1/2} e^{\pi z^2/\tau} \theta_3\left(\frac{z}{i\tau}, \frac{1}{\tau}\right), \tag{2.55}$$

θ_3 being the elliptic function

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 \tau + 2\pi n z}, \quad z \in \mathbb{C},\ \tau \in \mathbb{R}^+. \tag{2.56}$$

Notice that $S_2(\pi t) = \frac{1}{2}[\theta(0,t) - 1]$. (2.53) is an exact expression. Once more, we observe that the contribution of the contour provides, in fact, the missing term.

Let us now again consider (2.44) for general α and, $s = -1$. (2.42) and (2.43) read, in this case,

$$\Gamma(z)\zeta(\alpha z) = \int_0^\infty dt\, t^{z-1} S_\alpha(t), \tag{2.57}$$

$S_\alpha(t)$ being the function given in (2.48). No simple reflection formula like (2.52) exists for $\alpha \neq 2$. We have, instead,

$$\zeta(\alpha z) = \frac{2\Gamma(1-\alpha z)}{(2\pi)^{1-\alpha z}} \sin\left(\frac{\pi \alpha z}{2}\right) \zeta(1-\alpha z), \tag{2.58}$$

and we get

$$S_\alpha \equiv S_\alpha(1) = \sum_{m=1}^{\infty} e^{-m^\alpha} = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \zeta(-\alpha a) + \frac{1}{\alpha} \Gamma\left(\frac{1}{\alpha}\right) - \Delta_\alpha, \tag{2.59}$$

being the contribution of the contour

$$\Delta_\alpha = \int_K \frac{da}{2\pi i} \zeta(\alpha a) \Gamma(a). \tag{2.60}$$

Putting everything together, we have proven the following

Theorem 2 (Zeta function regularization theorem, particular case) *Under the hypothesis* (1), (2a) *and* (2b) *above*, *we have that*:

1. *For* $-\infty < \alpha < 2$, *the contribution of the semicircumference at infinity is zero*, *i.e.*
$$\Delta_\alpha = 0, \quad \alpha < 2. \tag{2.61}$$
2. *For* $\alpha = 2$, *the contribution of the semicircumference at infinity is given by*
$$\Delta_2 = -\sqrt{\pi}\, S_2\left(\pi^2\right). \tag{2.62}$$

The result for $\alpha \leq 1$ was known already and constitutes Weldon's proof of zeta-function regularization. The result for $\alpha = 2$ is due to the author. It shows very clearly that, on the contrary, the statements in [102] about the validity of the proof for any positive integer α were false, the reason being that the semicircumference at infinity does *not* yield a zero contribution. It was precisely the last step of the proof in [102] that was wrong. The fact that the numerical value of Δ_α is so small (it can be thought of as an infinitesimal correction, see (2.62)) as compared with the rest of the terms in (2.49) and (2.50) gives sense to the whole procedure of zeta-function regularization.

However, this is strictly true only for small α. For large α, Δ_α ceases to be an infinitesimal contribution. Actually, in the case considered,

$$\begin{aligned} \Delta_\alpha &= 0, \quad \alpha < 2; \\ \Delta_2 &= 9.17 \times 10^{-5}; \qquad \Delta_4 = 0.04; \qquad \Delta_6 = 0.07; \\ \Delta_\alpha &\to 0.13, \quad \alpha \to \infty, \end{aligned} \tag{2.63}$$

which represent, respectively, contributions of the 0%, 0.02%, 11%, 19%, and 36% to the final value of $S_\alpha(1)$. A more precise statement on this point—together with a substantial extension of the theorem to general situations—will be given in the next chapter.

2.3 Immediate Application of the Theorem

The most interesting and simple of the new cases is when $0 < c < 1$, e.g.

$$S_c \equiv S_c^{(2)}(-1) = \sum_{m=0}^{\infty} e^{-(m+c)^2}. \tag{2.64}$$

It can be expressed in terms of Hurwitz zeta-functions, as

$$S_c = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \zeta(-2m, c) + \frac{\sqrt{\pi}}{2} + \sqrt{\pi} \cos(2\pi c) S\left(\pi^2\right), \tag{2.65}$$

where $S(t) \equiv \sum_{m=0}^{\infty} e^{-tm^2}$. For $c \neq 0, 1/2$, this series is asymptotic.

1. Particular case $c = 1$: we recover the known equality (a special case of Jacobi's one)

$$S(1) = \frac{\sqrt{\pi} - 1}{2} + \sqrt{\pi} S\left(\pi^2\right). \tag{2.66}$$

2. For $c = 1/2$ we have $\zeta(-2m, 1/2) = 0,\ m = 0, 1, 2, \ldots$, and the result

$$\sum_{m=0}^{\infty} \exp\left[-\left(m + \frac{1}{2}\right)^2\right] = \frac{\sqrt{\pi}}{2} - \sqrt{\pi} \sum_{m=1}^{\infty} \exp\left(-m^2\pi^2\right) \tag{2.67}$$

permits us to obtain the value of the series with 10^{-10} accuracy, with just two terms

$$\sum_{m=0}^{\infty} \exp\left[-\left(m + \frac{1}{2}\right)^2\right] = \frac{\sqrt{\pi}}{2} - \sqrt{\pi} e^{-\pi^2} + \mathcal{O}\left(10^{-10}\right). \tag{2.68}$$

3. For $c = 0$ we obtain the previous result (2.54)

$$\sum_{m=0}^{\infty} e^{-m^2} = \frac{1}{2} + \frac{\sqrt{\pi}}{2} + \sqrt{\pi} S\left(\pi^2\right). \tag{2.69}$$

4. For $c = 1/4$, we have

$$\sum_{m=0}^{\infty} \exp\left[-\left(m + \frac{1}{4}\right)^2\right] \sim \frac{\sqrt{\pi}}{2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \zeta\left(-2m, \frac{1}{4}\right). \tag{2.70}$$

The series is now asymptotic. It stabilizes between the 8th and the 12th summand and provides its best value (with $\simeq 10^{-7}$ accuracy) when we add the 10 first terms (optimal truncation of the asymptotic series).

5. For $c = 1/3$ and $c = 1/6$,

$$\sum_{m=0}^{\infty} \exp\left[-\left(m + \frac{1}{3j}\right)^2\right] \sim \frac{\sqrt{\pi}}{2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \zeta\left(-2m, \frac{1}{3j}\right) + (-1)^j \frac{\sqrt{\pi}}{2} \sum_{m=1}^{\infty} \exp\left(-m^2\pi^2\right), \quad j = 1, 2. \quad (2.71)$$

6. Some more relations are

$$\sum_{m=-\infty}^{\infty} \exp\left[-(m+c)^2\right] \sim \sqrt{\pi} + 2\sqrt{\pi}\cos(2\pi c) S\left(\pi^2\right),$$

$$\begin{aligned} \sum_{m=0}^{\infty} m \exp\left[-(m+c)^2\right] &\sim \frac{1}{2} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[\zeta(-2m-1, c) - c\zeta(-2m, c)\right] \\ &\quad - \frac{\sqrt{\pi}}{2} c + \sqrt{\pi}\left[\pi \sin(2\pi c) - c\cos(2\pi c)\right] S\left(\pi^2\right), \end{aligned} \quad (2.72)$$

$$\sum_{m=0}^{\infty} \exp\left[-a(m+c)^2\right] \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^m \zeta(-2m, c) + \frac{1}{2}\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{a}} \cos(2\pi c) S\left(\frac{\pi^2}{a^2}\right).$$

We get the general expression (of Epstein–Hurwitz type with $N = 2$)

$$\begin{aligned} E_2(s; a_1, a_2; c_1, c_2) &\sim \sum_{n_1, n_2 = 0}^{\infty} \left[a_1(n_1 + c_1)^2 + a_2(n_2 + c_2)^2\right]^{-s} \\ &= \frac{a_2^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(s+m)}{m!} \left(\frac{a_1}{a_2}\right)^m \\ &\quad \cdot \zeta(-2m, c_1)\zeta(2s + 2m, c_2) \\ &\quad + \frac{a_2^{-s}}{2}\left(\frac{\pi a_2}{a_1}\right)^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(2s - 1, c_2) \\ &\quad \cdot \frac{2\pi^s}{\Gamma(s)} \cos(2\pi c_1) a_1^{-\frac{s}{2}-\frac{1}{4}} a_2^{-\frac{s}{2}+\frac{1}{4}} \end{aligned}$$

$$\cdot \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} n_1^{s-\frac{1}{2}} (n_2 + c_2)^{-s+\frac{1}{2}} K_{s-\frac{1}{2}} \left[2\pi \sqrt{\frac{a_2}{a_1}} n_1 (n_2 + c_2) \right], \tag{2.73}$$

K_ν being the modified Bessel function of the second kind. It constitutes the general analytic continuation formula for two-dimensional series. In particular, for $s = 0$,

$$E_2(0; a_1, a_2; c_1, c_2) = \left(c_1 - \frac{1}{2} \right) \left(c_2 - \frac{1}{2} \right), \tag{2.74}$$

for $s = -1$,

$$\begin{aligned} &E_2(-1; a_1, a_2; c_1, c_2) \\ &\quad = a_2 \left(\frac{1}{2} - c_1 \right) \zeta(-2, c_2) + a_1 \left(\frac{1}{2} - c_2 \right) \zeta(-2, c_1) \\ &\quad \cdot \frac{1}{3} \left(c_1 - \frac{1}{2} \right) \left(c_2 - \frac{1}{2} \right) \left[a_1 c_1 (1 - c_1) + a_2 c_2 (1 - c_2) \right], \end{aligned} \tag{2.75}$$

and for $s = 2$,

$$\begin{aligned} &E_2(2; a_1, a_2; c_1, c_2) \\ &\sim \frac{1}{a_2^2} \sum_{m=0}^{\infty} (-1)^m (m+1) \left(\frac{a_1}{a_2} \right)^m \\ &\cdot \zeta(-2m, c_1) \zeta(2m+4, c_2) + \frac{\pi}{4a_2} \frac{1}{\sqrt{a_1 a_2}} \zeta(3, c_2) \\ &+ \frac{\pi^2 \cos(2\pi c_1)}{a_1 a_2} \sum_{n=0}^{\infty} \left\{ (n + c_2)^{-2} \left[\exp\left(2\pi \sqrt{\frac{a_2}{a_1}} (n + c_2) \right) - 1 \right]^{-2} \right. \\ &\left. + \left[(n + c_2)^{-2} + \sqrt{\frac{a_1}{a_2}} \frac{(n + c_2)^{-3}}{2\pi} \right] \left[\exp\left(2\pi \sqrt{\frac{a_2}{a_1}} (n + c_2) \right) - 1 \right]^{-1} \right\}. \end{aligned} \tag{2.76}$$

The general expression for arbitrary N turns out to be

$$\begin{aligned} &E_N(s; a_1, \ldots, a_N; c_1, \ldots, c_N) \\ &\sim \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_1^m \zeta(-2m, c_1) \\ &\cdot \Gamma(s+m) E_{N-1}(s+m; a_2, \ldots, a_N; c_2, \ldots, c_N) \\ &+ \frac{1}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} E_{N-1}\left(s - \frac{1}{2}; a_2, \ldots, a_N; c_2, \ldots, c_N \right) + \sqrt{\frac{\pi}{a_1}} \frac{\cos(2\pi c_1)}{\Gamma(s)} \end{aligned}$$

$$\cdot \sum_{n_1=1}^{\infty} \sum_{n_2,\ldots,n_N=0}^{\infty} \int_0^{\infty} dt\, t^{s-3/2} \exp\left[-\frac{\pi^2 n_1^2}{a_1 t} - t \sum_{j=2}^{N} a_j (n_j + c_j)^2 \right]. \qquad (2.77)$$

A word about the notation. In this section we have tried to be consistent with the sign $\sim$ to denote 'asymptotic expansion'. Usually, however, the equality sign is also employed (as for ordinary Taylor series). Normally this does not turn out to be a problem since—at least in the situations to be considered here—the asymptotic (resp. convergent) character of the series on the r.h.s. is not difficult to recognize.

2.4 Expressions for Multi-series on Combinations Involving Arbitrary Constants and Exponents

We shall now make use of the zeta function regularization theorem in order to obtain expressions for the most general multi-series of the type presented in the introduction, which would be impossible to derive by other means (at least with comparable easiness and universality). The same notation which has commonly been used in other references of the author will be employed here, e.g.,

$$\begin{aligned} M_N^c(s; \vec{a}; \vec{\alpha}; \vec{c}) &\equiv M_N^c(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N; c_1, \ldots, c_N) \\ &\equiv \sum_{n_1,\ldots,n_N=0}^{\infty} \left[a_1 (n_1 + c_1)^{\alpha_1} + \cdots + a_N (n_N + c_N)^{\alpha_N} + c \right]^{-s}, \end{aligned} \qquad (2.78)$$

and for the generalized Epstein-like case:

$$\begin{aligned} E_N^c(s; \vec{a}; \vec{c}) &\equiv M_N^c(s; a_1, \ldots, a_N; 2, \ldots, 2; c_1, \ldots, c_N) \\ &= \sum_{n_1,\ldots,n_N=0}^{\infty} \left[a_1 (n_1 + c_1)^2 + \cdots + a_N (n_N + c_N)^2 + c \right]^{-s}. \end{aligned} \qquad (2.79)$$

Consider the case of M_2^c. We need the result of the regularization theorem as applied to the double series

$$S_\alpha(t, s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} n^{\alpha k}, \quad \alpha \in R, \qquad (2.80)$$

which converges for $\mathrm{Re}(s) > 0$ large enough. We can write

$$S_\alpha(t, s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \oint_C \frac{dk}{2\pi i} t^k n^{-\alpha k} \Gamma(k), \qquad (2.81)$$

where the contour $\mathcal{C}$ consists of the straight line $\mathrm{Re}(k) = k_0$, with k_0 fixed, $0 < k_0 < 1$, and of the semicircumference at infinity on the left of this line (see Fig. 2). The regularization theorem tells us in this case that [109, 110]

$$S_\alpha(t,s) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}\zeta(s+1-\alpha k) + \frac{1}{\alpha}\Gamma\left(-\frac{s}{\alpha}\right)t^{-1/\alpha} - \Delta_\alpha(t,s), \qquad \frac{s}{\alpha} \notin N, \tag{2.82}$$

where $\Delta_\alpha(t,s)$ is the contribution of the curved part K of the contour $\mathcal{C}$:

$$\Delta_\alpha(t,s) \equiv \int_K \frac{dk}{2\pi i}\zeta(s+1+\alpha k)\Gamma(k)t^k. \tag{2.83}$$

With this, we obtain

$$\begin{aligned} M_2^c(s;\vec{a};\vec{\alpha};\vec{c}) = \frac{a_2^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} &\frac{(-1)^m \Gamma(s+m)}{m!}\left(\frac{a_1}{a_2}\right)^m \\ &\cdot \zeta(-\alpha_1 m, c_1) M_1^{c/a_2}(s+m;1;\alpha_2;c_2) \\ &+ \frac{a_2^{-s}}{\alpha_1}\Gamma\left(\frac{1}{\alpha_1}\right)\left(\frac{a_2}{a_1}\right)^{1/\alpha_1}\frac{\Gamma(s-\frac{1}{\alpha_1})}{\Gamma(s)} M_1^{c/a_2}(s-1/\alpha_1;1;\alpha_2;c_2) \\ &+ \frac{a_2^{-s}}{\Gamma(s)}\left(\frac{a_2}{a_1}\right)^{1/\alpha_1}\int_K \frac{da}{2\pi i}\zeta(s+1+\alpha_1 a, c_1) \\ &\cdot M_1^{c/a_2}(s+a;1;\alpha_2;c_2)\Gamma(a)\Gamma(s+a), \end{aligned} \tag{2.84}$$

and also

$$\begin{aligned} M_1^c(s;a_1;\alpha_1;c_1) = \frac{c^{-s}}{\Gamma(s)} &\sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(s+m)}{m!}\left(\frac{a_1}{c}\right)^m \zeta(-\alpha_1 m, c_1) \\ &+ \frac{c^{-s}}{\alpha_1}\Gamma\left(\frac{1}{\alpha_1}\right)\left(\frac{c}{a_1}\right)^{1/\alpha_1}\frac{\Gamma(s-\frac{1}{\alpha_1})}{\Gamma(s)} \\ &+ \frac{c^{-s}}{\Gamma(s)}\left(\frac{c}{a_1}\right)^{1/\alpha_1}\int_K \frac{da}{2\pi i}\zeta(s+1+\alpha_1 a, c_1)\Gamma(a)\Gamma(s+a). \end{aligned} \tag{2.85}$$

It is not difficult to build, from these two expressions, a recurrence leading to the calculation of M_N^c from the knowledge of M_{N-1}^c, and starting with the formula for M_1^c. At each step, this involves a complex integration over a curved contour at infinity, a term which is in general very small compared with the rest.

As the full calculation is rather involved and lengthy, let us here—for the benefit of the reader—just show in detail the first two steps leading to the formula which corresponds to the case when the c's are zero. We shall accumulate the contributions coming from the series commutation into a single (small) term that we will call simply Δ. So this part will not be really taken care of in the course of the calculation, owing to its smallness (see the relevant numerics in the next chapter). The starting point is again the use of the Mellin transform on the power terms of the series considered

$$M_3(s;\vec{a};\vec{\alpha};\vec{c}=\vec{0}) = \sum_{n_1,n_2,n_3=1}^{\infty} \left(a_1 n_1^{\alpha_1} + a_2 n_2^{\alpha_2} + a_3 n_3^{\alpha_3}\right)^{-s}, \tag{2.86}$$

followed by an expansion on the n_1-terms in the integrand exponent. We thus get

$$\begin{aligned}
&\sum_{n_1,n_2,n_3=1}^{\infty} \left(a_1 n_1^{\alpha_1} + a_2 n_2^{\alpha_2} + a_3 n_3^{\alpha_3}\right)^{-s} \\
&\quad = \frac{1}{a_3^s \Gamma(s)} \left\{ \sum_{n_2,n_3=1}^{\infty} \sum_{k_1=0}^{\infty} (-1)^{k_1} \frac{b_1^{k_1}}{k_1!} \zeta(-\alpha_1 k_1) \right. \\
&\qquad \cdot \int_0^\infty dt\, t^{s+k_1-1} \exp\left[-t\left(b_2 n_2^{\alpha_2} + b_3 n_3^{\alpha_3}\right)\right] \\
&\qquad \left. + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \sum_{n_2,n_3=1}^{\infty} \int_0^\infty dt\, t^{s-(1/\alpha_1)-1} \exp\left[-t\left(b_2 n_2^{\alpha_2} + b_3 n_3^{\alpha_3}\right)\right] \right\},
\end{aligned} \tag{2.87}$$

where $b_j \equiv a_j/a_3$, $j = 1, 2$. By proceeding again in the same way with the n_2-terms of the exponents, we obtain

$$\begin{aligned}
&\sum_{n_1,n_2,n_3=1}^{\infty} \left(a_1 n_1^{\alpha_1} + a_2 n_2^{\alpha_2} + a_3 n_3^{\alpha_3}\right)^{-s} \\
&\quad = \frac{1}{a_3^s \Gamma(s)} \left\{ \sum_{n_3=1}^{\infty} \sum_{k_1,k_2=0}^{\infty} (-1)^{k_1+k_2} \frac{b_1^{k_1}}{k_1!} \frac{b_2^{k_2}}{k_2!} \zeta(-\alpha_1 k_1) \zeta(-\alpha_2 k_2) \right. \\
&\qquad \cdot \int_0^\infty dt\, t^{s+k_1+k_2-1} \exp\left(-t n_3^{\alpha_3}\right) \\
&\qquad + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \sum_{n_3=1}^{\infty} \sum_{k_2=0}^{\infty} (-1)^{k_2} \frac{b_2^{k_2}}{k_2!} \zeta(-\alpha_2 k_2) \int_0^\infty dt\, t^{s+k_2-(1/\alpha_1)-1} \exp\left(-t n_3^{\alpha_3}\right) \\
&\qquad + \frac{\Gamma(1/\alpha_2)}{\alpha_2 b_2^{1/\alpha_2}} \sum_{n_3=1}^{\infty} \sum_{k_1=0}^{\infty} (-1)^{k_1} \frac{b_1^{k_1}}{k_1!} \zeta(-\alpha_1 k_1) \int_0^\infty dt\, t^{s+k_1-(1/\alpha_2)-1} \exp\left(-t n_3^{\alpha_3}\right) \\
&\qquad \left. + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \frac{\Gamma(1/\alpha_2)}{\alpha_2 b_2^{1/\alpha_2}} \sum_{n_3=1}^{\infty} \int_0^\infty dt\, t^{s-(1/\alpha_1)-(1/\alpha_2)-1} \exp\left(-t n_3^{\alpha_3}\right) \right\}.
\end{aligned} \tag{2.88}$$

Performing now the series commutation and resumming the corresponding zeta functions, we finally get the asymptotic series:

$$
\begin{aligned}
&M_3(s;\vec{a};\vec{\alpha};\vec{c}=\vec{0}) \\
&\quad = \sum_{n_1,n_2,n_3=1}^{\infty} \left(a_1 n_1^{\alpha_1} + a_2 n_2^{\alpha_2} + a_3 n_3^{\alpha_3}\right)^{-s} \\
&\quad \sim \frac{1}{a_3^s \Gamma(s)} \left\{ \sum_{k_1,k_2=0}^{\infty} (-1)^{k_1+k_2} \frac{b_1^{k_1}}{k_1!} \frac{b_2^{k_2}}{k_2!} \Gamma(s+k_1+k_2) \right. \\
&\qquad \cdot \zeta(-\alpha_1 k_1)\zeta(-\alpha_2 k_2)\zeta\left(\alpha_3(s+k_1+k_2)\right) \\
&\qquad + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \sum_{k_2=0}^{\infty} (-1)^{k_2} \frac{b_2^{k_2}}{k_2!} \Gamma(s+k_2-1/\alpha_1)\zeta(-\alpha_2 k_2)\zeta\left(\alpha_3(s+k_2-1/\alpha_1)\right) \\
&\qquad + \frac{\Gamma(1/\alpha_2)}{\alpha_2 b_2^{1/\alpha_2}} \sum_{k_1=0}^{\infty} (-1)^{k_1} \frac{b_1^{k_1}}{k_1!} \Gamma(s+k_1-1/\alpha_2)\zeta(-\alpha_1 k_1)\zeta\left(\alpha_3(s+k_1-1/\alpha_2)\right) \\
&\qquad \left. + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \frac{\Gamma(1/\alpha_2)}{\alpha_2 b_2^{1/\alpha_2}} \Gamma(s-1/\alpha_1-1/\alpha_2)\zeta\left(\alpha_3(s-1/\alpha_1-1/\alpha_2)\right) \right\} + \Delta.
\end{aligned}
\tag{2.89}
$$

By repeating this analysis, after N steps we easily obtain the general formula (for the $\vec{c}=\vec{0}$, $c=0$ case, with $b_j \equiv a_j/a_N$, $j=1,\ldots,N-1$):

$$
\begin{aligned}
&M_N(s;\vec{a};\vec{\alpha};\vec{c}=\vec{0}) \\
&\quad = \sum_{n_1,\ldots,n_N=1}^{\infty} \left(a_1 n_1^{\alpha_1} + \cdots + a_N n_N^{\alpha_N}\right)^{-s} \\
&\quad \sim \frac{1}{a_N^s \Gamma(s)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^{p} \frac{b_{i_r}^{-1/\alpha_{i_r}}}{\alpha_{i_r}} \Gamma\left(\frac{1}{\alpha_{i_r}}\right) \\
&\qquad \cdot \sum_{k_{j_1},\ldots,k_{j_{N-p-1}}=0}^{\infty} \Gamma\left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right) \prod_{l=1}^{N-p-1} \frac{(-b_{j_l})^{k_{j_l}}}{k_{j_l}!} \\
&\qquad \cdot \zeta(-\alpha_{j_l} k_{j_l})\zeta\left(\alpha_N\left[s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right]\right) + \Delta_{ER},
\end{aligned}
\tag{2.90}
$$

where $1 \le i_1 < \cdots < i_p \le N-1$, $1 \le j_1 < \cdots < j_{N-p-1} \le N-1$, being $i_1,\ldots,i_p$, $j_1,\ldots,j_{N-p-1}$ a permutation of $1,2,\ldots,N-1$. The sum on $C_{N-1,p}$ means sum over the $\binom{N-1}{p}$ choices of the indices $i_1, \ldots, i_p$ among the $1, 2, \ldots, N-1$, and the term Δ_{ER} includes all the Δ corrections which appear at each step of the recurrence.

Proceeding in the same way when we are confronted with the general case $\vec{c} \neq \vec{0}$ and $c \neq 0$, the recurrence can be solved explicitly also, the result being ([109], corrected)

$$\begin{aligned}
M_N^c(s;\vec{a};\vec{\alpha};\vec{c}) = &\sum_{n_1,\ldots,n_N=1}^{\infty} \left(a_1 n_1^{\alpha_1} + \cdots + a_N n_N^{\alpha_N} + c\right)^{-s} \\
&\cdot \frac{a_N^{-s}}{\Gamma(s)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^{p} \frac{b_{i_r}^{-1/\alpha_{i_r}}}{\alpha_{i_r}} \Gamma\left(\frac{1}{\alpha_{i_r}}\right) \\
&\cdot \sum_{k_{j_1},\ldots,k_{j_{N-p-1}}=0}^{\infty} \Gamma\left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right) \\
&\cdot \prod_{l=1}^{N-p-1} \frac{(-b_{j_l})^{k_{j_l}}}{k_{j_l}!} \zeta(-\alpha_{j_l} k_{j_l}, c_{j_l}) \\
&\cdot M_1^{c/a_N}\left(\alpha_N\left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right); 1; \alpha_N\right) + \Delta_{ER}, \qquad (2.91)
\end{aligned}$$

(notice that a small mistake in (3.22) and (3.23) of Ref. [109] has been corrected).

Going down to the particular case when the $\alpha_i = 2$ (2.79) things become much more concrete. As mentioned before, then the expression giving our additional corrections to the series commutation reduces to a theta function identity [109], with the result

$$\begin{aligned}
\sum_{m=0}^{\infty} \exp\left[-a(m+c)^2\right] \sim &\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a^m \zeta(-2m, c) + \frac{1}{2}\sqrt{\frac{\pi}{a}} \\
&+ \sqrt{\frac{\pi}{a}} \cos(2\pi c) S\left(\frac{\pi^2}{a^2}\right), \qquad (2.92)
\end{aligned}$$

and this yields the recurrence

$$\begin{aligned}
E_N^c(s;\vec{a};\vec{c}) = &\frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_1^m \zeta(-2m, c_1) \Gamma(s+m) \\
&\cdot E_{N-1}^c(s+m; a_2, \ldots, a_N; c_2, \ldots, c_N) \\
&+ \frac{1}{2}\sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s-1/2)}{\Gamma(s)} E_{N-1}^c(s-1/2; a_2, \ldots, a_N; c_2, \ldots, c_N) \\
&+ \frac{2\pi^s}{\Gamma(s)} \cos(2\pi c_1) a_1^{-s/2-1/4}
\end{aligned}$$

$$\cdot \sum_{n_1=1}^{\infty} \sum_{n_2,\ldots,n_N=0}^{\infty} n_1^{s-1/2} \left[c + \sum_{j=2}^{N} a_j (n_j + c_j)^2 \right]^{-s/2+1/4}$$

$$\cdot K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a_1}} \sqrt{ + \sum_{j=2}^{N} a_j (n_j + c_j)^2 } \right), \tag{2.93}$$

where K_ν is the modified Bessel function of the second kind. The recurrence starts with

$$E_1^c(s; a_1; c_1) \sim \frac{c^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(s+m)}{m!} \left(\frac{a_1}{c} \right)^m \zeta(-2m, c_1)$$

$$+ \frac{c^{1/2-s}}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} + \frac{2\pi^s}{\Gamma(s)} \cos(2\pi c_1) a_1^{-s/2-1/4} c^{-s/2+1/4}$$

$$\cdot \sum_{n_1=1}^{\infty} n_1^{s-1/2} K_{s-1/2}\left(2\pi n_1 \sqrt{\frac{c}{a_1}} \right). \tag{2.94}$$

Then

$$E_2^c(s; a_1, a_2; c_1, c_2)$$

$$\sim \frac{a_2^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(s+m)}{m!} \left(\frac{a_1}{a_2} \right)^m \zeta(-2m, c_1)$$

$$\cdot E_1^{c/a_2}(s+m; 1; c_2) + \frac{a_2^{1/2-s}}{2} \sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} E_1^{c/a_2}(s-1/2; 1; c_2)$$

$$+ \frac{2\pi^s}{\Gamma(s)} \cos(2\pi c_1) a_1^{-s/2-1/4} a_2^{-s/2+1/4}$$

$$\cdot \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} n_1^{s-1/2} \left[a_2 (n_2 + c_2)^2 + c \right]^{-s/2+1/4}$$

$$\cdot K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a_1}} \sqrt{a_2 (n_2 + c_2)^2 + c} \right), \tag{2.95}$$

and so on. Expressions for the special case $c = 0$ are given in Ref. [109] (see also the preceding section and equations below).

The very particular case, $a_1 = \cdots = a_N = 1$, $c_1 = \cdots = c_N = 1$ and $\alpha_1 = \cdots = \alpha_N = 2$, simplifies considerably. For $c = 0$, we get

$$E_N(s) = \frac{(-1)^{N-1}}{2^{N-1}} \frac{1}{\Gamma(s)} \sum_{j=0}^{N-1} (-1)^j \binom{N-1}{j} \Gamma(2s-j) \zeta\left(s - \frac{j}{2} \right) + \Delta_{ER}, \tag{2.96}$$

and, for $c \neq 0$,

$$E_N^c(s) = \frac{(-1)^{N-1}}{2^{N-1}} \frac{1}{\Gamma(s)} \sum_{j=0}^{N-1} (-1)^j \binom{N-1}{j} \Gamma\left(s - \frac{j}{2}\right) E_1^c\left(s - \frac{j}{2}\right) + \Delta_{ER}. \tag{2.97}$$

The poles of this last function arise from those of $E_1^c(s - j/2)$, which are obtained for the values of s such that $s - j/2 = 1/2, -1/2, -3/2, \ldots$. They are poles of order one at $s = N/2, (N-1)/2, N/2 - 1, \ldots$, except for $s = 0, -1, -2, \ldots$, since then the function is finite (owing to the $\Gamma(s)$ in the denominator). These poles are directly removed by zeta-function regularization.

In the particular case $c_1 = \cdots = c_N = 0$, we have

$$\begin{aligned}
&E_N^c(s; a_1, \ldots, a_N) \\
&\quad \equiv \sum_{n_1,\ldots,n_N=1}^{\infty} \left(a_1 n_1^2 + \cdots + a_N n_N^2 + c^2\right)^{-s} \\
&\quad = -\frac{1}{2} E_{N-1}^c(s; a_2, \ldots, a_N) + \frac{1}{2}\sqrt{\frac{\pi}{a_1}} \frac{\Gamma(s-1/2)}{\Gamma(s)} E_{N-1}^c(s-1/2; a_2, \ldots, a_N) \\
&\qquad + \frac{\pi^s}{\Gamma(s)} a_1^{-s/2} \sum_{k=0}^{\infty} \frac{a_1^{k/2}}{k!(16\pi)^k} \prod_{j=1}^{k} \left[(2s-1)^2 - (2j-1)^2\right] \sum_{n_1,\ldots,n_N=1}^{\infty} n_1^{s-k-1} \\
&\qquad \cdot \left(a_2 n_2^2 + \cdots + a_N n_N^2 + c^2\right)^{-(s+k)/2} \\
&\qquad \cdot \exp\left[-\frac{2\pi}{\sqrt{a_1}} n_1 \left(a_2 n_2^2 + \cdots + a_N n_N^2 + c^2\right)^{1/2}\right].
\end{aligned} \tag{2.98}$$

The recurrence starts from expression

$$\begin{aligned}
E_1^c(s; 1) = -\frac{c^{-2s}}{2} + \frac{\sqrt{\pi}}{2} \frac{\Gamma(s-1/2)}{\Gamma(s)} c^{-2s+1} \\
+ \frac{2\pi^s c^{-s+1/2}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n c).
\end{aligned} \tag{2.99}$$

We get, for $c \neq 0$,

$$\begin{aligned}
E_2^c(s) &= -\frac{1}{2} E_1^c(s) + \frac{\sqrt{\pi}}{2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} E_1^c\left(s - \frac{1}{2}\right) + \Delta_{ER}, \\
E_3^c(s) &= \frac{1}{4} E_1^c(s) - \frac{\sqrt{\pi}}{2} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} E_1^c\left(s - \frac{1}{2}\right) + \frac{\pi}{4(s-1)} E_1^c(s-1) + \Delta_{ER},
\end{aligned} \tag{2.100}$$

and similar expressions for $c = 0$. This case can be obtained from the former by analytically continuing in the parameter c. Δ_{ER} is again the well-known term coming from (additional) series commutation. Actually, for numerical evaluations we do not need to consider exponentially small terms in the asymptotic expansions above, which give a very good and quick approximation.

To finish, another couple of particularly useful expressions are

$$\begin{aligned}
&\sum_{n_1,n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2} \\
&\quad = \frac{1}{24}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) - \frac{\zeta(3)}{8\pi^2}\left(\frac{a_1}{a_2^2} + \frac{a_2}{a_1^2}\right) \\
&\qquad - \frac{\pi^{3/2}}{2\sqrt{a_1 a_2}}\left[\exp\left(-2\pi\frac{a_1}{a_2}\right)\left(1 + \mathcal{O}\left(10^{-3}\right)\right)\right],
\end{aligned} \tag{2.101}$$

and (this one obtained after additional regularization)

$$\begin{aligned}
&\sum_{n_1,n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2 + c^2} \\
&\quad = \frac{c}{4} - \frac{\pi}{6}a_1 a_2 c^3 + \left(\frac{1}{4\pi}\sqrt{\frac{c}{a_1}} - \frac{c a_2}{4\pi a_1}\right)\left[\exp(-2\pi c a_1)\left(1 + \mathcal{O}\left(10^{-3}\right)\right)\right].
\end{aligned} \tag{2.102}$$

In both cases we have assumed (this is, of course, no restriction) that $a_2 \le a_1$.

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