

Chapter 2

Unramified Sheaves and Strongly \mathbb{A}^1 -Invariant Sheaves

2.1 Unramified Sheaves of Sets

We let \tilde{Sm}_k denote the category of smooth k -schemes and whose morphisms are the smooth morphisms. We start with the following standard definition.

Definition 2.1. An unramified presheaf of sets \mathcal{S} on Sm_k (resp. on \tilde{Sm}_k) is a presheaf of sets \mathcal{S} such that the following holds:

- (0) For any $X \in Sm_k$ with irreducible components X_α 's, $\alpha \in X^{(0)}$, the obvious map $\mathcal{S}(X) \rightarrow \prod_{\alpha \in X^{(0)}} \mathcal{S}(X_\alpha)$ is a bijection.
- (1) For any $X \in Sm_k$ and any open subscheme $U \subset X$ the restriction map $\mathcal{S}(X) \rightarrow \mathcal{S}(U)$ is injective if U is everywhere dense in X ;
- (2) For any $X \in Sm_k$, irreducible with function field F , the injective map $\mathcal{S}(X) \hookrightarrow \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x})$ is a bijection (the intersection being computed in $\mathcal{S}(F)$).

Remark 2.2. An unramified presheaf \mathcal{S} (either on Sm_k or on \tilde{Sm}_k) is automatically a sheaf of sets in the Zariski topology. This follows from (2). We also observe that with our convention, for \mathcal{S} an unramified presheaf, the formula in (2) also holds for X essentially smooth over k and irreducible with function field F . We will use these facts freely in the sequel. \square

Example 2.3. It was observed in [52] that any strictly \mathbb{A}^1 -invariant sheaf on Sm_k is unramified in this sense. The \mathbb{A}^1 -invariant sheaves with transfers of [79] as well as the cycle modules¹ of Rost [68] give such unramified sheaves. In characteristic $\neq 2$ the sheaf associated to the presheaf of Witt groups $X \mapsto W(X)$ is unramified by [63] (the sheaf associated in the Zariski topology is in fact already a sheaf in the Nisnevich topology). \square

¹These two notions are indeed closely related by [21].

Remark 2.4. Let \mathcal{S} be a sheaf of sets in the Zariski topology on Sm_k (resp. on \tilde{Sm}_k) satisfying properties **(0)** and **(1)** of the previous definition. Then it is unramified if and only if, for any $X \in Sm_k$ and any open subscheme $U \subset X$ the restriction map $\mathcal{S}(X) \rightarrow \mathcal{S}(U)$ is bijective if $X - U$ is everywhere of codimension ≥ 2 in X . We left the details to the reader. \square

Remark 2.5. Base change. Let \mathcal{S} be a sheaf of sets on \tilde{Sm}_k or Sm_k , let $K \in \mathcal{F}_k$ be fixed and denote by $\pi : Spec(K) \rightarrow Spec(k)$ the structural morphism. One may pull-back \mathcal{S} to the sheaf $\mathcal{S}|_K := \pi^* \mathcal{S}$ on \tilde{Sm}_K (or Sm_K accordingly). One easily checks that the sections on a separable (finite type) field extension F of K is nothing but $\mathcal{S}(F)$ when F is viewed in \mathcal{F}_k . If \mathcal{S} is unramified so is $\mathcal{S}|_K$: indeed $\pi^* \mathcal{S}$ is a sheaf and satisfies properties **(0)** and **(1)**. We prove **(3)** using the previous remark. \square

Our aim in this subsection is to give an explicit description of unramified sheaves of sets both on \tilde{Sm}_k and on Sm_k in terms of their sections on fields $F \in \mathcal{F}_k$ and some extra structure. As usual we will say that a functor $\mathcal{S} : \mathcal{F}_k \rightarrow Set$ is *continuous* if $\mathcal{S}(F)$ is the filtering colimit of the $\mathcal{S}(F_\alpha)$'s, where the F_α run over the set of subfields of F of finite type over k .

We start with the simplest case, that is to say unramified sheaves of sets on \tilde{Sm}_k .

Definition 2.6. An unramified $\tilde{\mathcal{F}}_k$ -datum consists of:

- (D1) A continuous functor $\mathcal{S} : \mathcal{F}_k \rightarrow Set$;
- (D2) For any $F \in \mathcal{F}_k$ and any discrete valuation v on F , a subset

$$\mathcal{S}(\mathcal{O}_v) \subset \mathcal{S}(F)$$

The previous data should satisfy the following axioms:

- (A1) If $i : E \subset F$ is a separable extension in \mathcal{F}_k , and v is a discrete valuation on F which restricts to a discrete valuation w on E with ramification index 1 then $\mathcal{S}(i)$ maps $\mathcal{S}(\mathcal{O}_w)$ into $\mathcal{S}(\mathcal{O}_v)$ and moreover if the induced extension $\bar{i} : \kappa(w) \rightarrow \kappa(v)$ is an isomorphism, then the following square of sets is cartesian:

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \rightarrow & \mathcal{S}(\mathcal{O}_v) \\ \bigcap & & \bigcap \\ \mathcal{S}(E) & \rightarrow & \mathcal{S}(F) \end{array}$$

- (A2) Let $X \in Sm_k$ be irreducible with function field F . If $x \in \mathcal{S}(F)$, then x lies in all but a finite number of $\mathcal{S}(\mathcal{O}_x)$'s, where x runs over the set $X^{(1)}$ of points of codimension one.

Remark 2.7. The Axiom (A1) is equivalent to the fact that for any discrete valuation v on $F \in \mathcal{F}_k$ with discrete valuation ring \mathcal{O}_v , then the following square in which \mathcal{O}_v^h is the henselization and F^h the fraction field of \mathcal{O}_v^h should be cartesian:

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_v) & \rightarrow & \mathcal{S}(\mathcal{O}_v^h) \\ \bigcap & & \bigcap \\ \mathcal{S}(F) & \rightarrow & \mathcal{S}(F^h) \end{array}$$

□

We observe that an unramified sheaf of sets \mathcal{S} on $\tilde{S}m_k$ defines in an obvious way an unramified $\tilde{\mathcal{F}}_k$ -datum. First, evaluation on the field extensions (of finite transcendence degree) of k yields a functor:

$$\mathcal{S} : \mathcal{F}_k \rightarrow \mathcal{S}et, \quad F \mapsto \mathcal{S}(F)$$

For any discrete valuation v on $F \in \mathcal{F}_k$, then $\mathcal{S}(\mathcal{O}_v)$ is a subset of $\mathcal{S}(F)$. We now claim that these data satisfy the axioms **(A1)** and **(A2)** of unramified $\tilde{\mathcal{F}}_k$ -datum.

Axiom **(A1)** is easily checked by choosing convenient smooth models over k for the essentially smooth k -schemes $\text{Spec}(F)$, $\text{Spec}(\mathcal{O}_v)$. To prove axiom **(A2)** one observes that any $x \in \mathcal{S}(F)$ comes, by definition, from an element $x \in \mathcal{S}(U)$ for $U \in Sm_k$ an open subscheme of X . Thus any $\alpha \in \mathcal{S}(F)$ lies in all the $\mathcal{S}(\mathcal{O}_x)$ for $x \in X^{(1)}$ lying in U . But there are only finitely many $x \in X^{(1)}$ not lying in U .

This construction defines a “restriction” functor from the category of unramified sheaves of sets on $\tilde{S}m_k$ to that of unramified $\tilde{\mathcal{F}}_k$ -data.

Proposition 2.8. *The restriction functor from unramified sheaves on $\tilde{S}m_k$ to unramified $\tilde{\mathcal{F}}_k$ -data is an equivalence of categories.*

Proof. Given an unramified $\tilde{\mathcal{F}}_k$ -datum \mathcal{S} , and $X \in Sm_k$ irreducible with function field F , we define the subset $\mathcal{S}(X) \subset \mathcal{S}(F)$ as the intersection $\bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_x) \subset \mathcal{S}(F)$. We extend it in the obvious way for X not irreducible so that property **(0)** is satisfied. Given a smooth morphism $f : Y \rightarrow X$ in Sm_k we define a map: $\mathcal{S}(f) : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ as follows. By property **(0)** we may assume X and Y are irreducible with field of fractions E and F respectively and f is dominant. The map $\mathcal{S}(f)$ is induced by the map $\mathcal{S}(E) \rightarrow \mathcal{S}(F)$ corresponding to the fields extension $E \subset F$ and the observation that if $x \in X^{(1)}$ then $f^{-1}(x)$ is a finite set of points of codimension 1 in Y . We check that it is a sheaf in the Nisnevich topology using Axiom **(A1)** and the characterization of Nisnevich sheaves from [59]. It is unramified. Finally to check that one has constructed the inverse to the restriction functor, one uses axiom **(A2)**. □

Definition 2.9. An unramified \mathcal{F}_k -datum \mathcal{S} is an unramified $\tilde{\mathcal{F}}_k$ -data together with the following additional data:

(D3) For any $F \in \mathcal{F}_k$ and any discrete valuation v on F , a map $s_v : \mathcal{S}(\mathcal{O}_v) \rightarrow \mathcal{S}(\kappa(v))$, called the specialization map associated to v .

These data should satisfy furthermore the following axioms:

- (A3) (i) If $i : E \subset F$ is an extension in \mathcal{F}_k , and v is a discrete valuation on F which restricts to a discrete valuation w on E , then $\mathcal{S}(i)$ maps $\mathcal{S}(\mathcal{O}_w)$ to $\mathcal{S}(\mathcal{O}_v)$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}(\mathcal{O}_w) & \rightarrow & \mathcal{S}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \mathcal{S}(\kappa(w)) & \rightarrow & \mathcal{S}(\kappa(v)) \end{array}$$

- (ii) If $i : E \subset F$ is an extension in \mathcal{F}_k , and v a discrete valuation on F which restricts to 0 on E then the map $\mathcal{S}(i) : \mathcal{S}(E) \rightarrow \mathcal{S}(F)$ has its image contained in $\mathcal{S}(\mathcal{O}_v)$ and if we let $j : E \subset \kappa(v)$ denotes the induced fields extension, the composition $\mathcal{S}(E) \rightarrow \mathcal{S}(\mathcal{O}_v) \xrightarrow{s_v} \mathcal{S}(\kappa(v))$ is equal to $\mathcal{S}(j)$.
- (A4) (i) For any $X \in Sm'_k$ local of dimension 2 with closed point $z \in X^{(2)}$, and for any point $y_0 \in X^{(1)}$ with $\bar{y}_0 \in Sm'_k$, then $s_{y_0} : \mathcal{S}(\mathcal{O}_{y_0}) \rightarrow \mathcal{S}(\kappa(y_0))$ maps $\bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y)$ into $\mathcal{S}(\mathcal{O}_{\bar{y}_0, z}) \subset \mathcal{S}(\kappa(y_0))$.
- (ii) The composition

$$\bigcap_{y \in X^{(1)}} \mathcal{S}(\mathcal{O}_y) \rightarrow \mathcal{S}(\mathcal{O}_{\bar{y}_0, z}) \rightarrow \mathcal{S}(\kappa(z))$$

doesn't depend on the choice of y_0 such that $\bar{y}_0 \in Sm'_k$.

Remark 2.10. When we will construct unramified Milnor-Witt K-theory in Sect. 3.2 below, the axiom **(A4)** will appear to be the most difficult to check. In fact the Sect. 2.3 is devoted to develop some technic to check this axiom in special cases. In Rost's approach [68] this axiom follows from the construction of the Rost's complex for two-dimensional local smooth k -scheme. However the construction of this complex (even for dimension 2 schemes) requires transfers, which we don't want to use at this point. \square

Now we claim that an unramified sheaf of sets \mathcal{S} on Sm_k defines an unramified \mathcal{F}_k -datum. From what we have done before, we already have in hand an unramified $\tilde{\mathcal{F}}_k$ -datum \mathcal{S} . Now, for any discrete valuation v on $F \in \mathcal{F}_k$ with residue field $\kappa(v)$, there is an obvious map $s_v : \mathcal{S}(\mathcal{O}_v) \rightarrow \mathcal{S}(\kappa(v))$, obtained by choosing smooth models over k for the closed immersion $\text{Spec}(\kappa(v)) \rightarrow \text{Spec}(\mathcal{O}_v)$. This defines the datum **(D3)**. We now claim that these data satisfy the previous axioms for unramified \mathcal{F}_k -datum. Axiom **(A3)** is checked by choosing convenient smooth models for $\text{Spec}(F)$, $\text{Spec}(\mathcal{O}_v)$ and/or $\text{Spec}(\kappa(v))$.

To check the axiom **(A4)** we use property **(2)** and the commutative square:

$$\begin{array}{ccc} \mathcal{S}(X) & \subset & \mathcal{S}(\mathcal{O}_{y_0}) \\ \downarrow & & \downarrow \\ \mathcal{S}(\bar{y}_0) = \mathcal{S}(\mathcal{O}_z) & \subset & \mathcal{S}(\kappa(y_0)) \end{array}$$

The following now is the main result of this section:

Theorem 2.11. *The functor just constructed from unramified sheaves of sets on Sm_k to unramified \mathcal{F}_k -data is an equivalence of categories.*

The Theorem follows from the following more precise statement:

Lemma 2.12. *Given an unramified \mathcal{F}_k -datum \mathcal{S} , there is a unique way to extend the unramified sheaf of sets $\mathcal{S} : (Sm_k)^{op} \rightarrow Set$ to a sheaf $\mathcal{S} : (Sm_k)^{op} \rightarrow Set$, such that for any discrete valuation v on $F \in \mathcal{F}_k$ with separable residue field, the map $\mathcal{S}(\mathcal{O}_v) \rightarrow \mathcal{S}(\kappa(v))$ induced by the sheaf structure is the specialization map $s_v : \mathcal{S}(\mathcal{O}_v) \rightarrow \mathcal{S}(\kappa(v))$. This sheaf is automatically unramified.*

Proof. We first define a restriction map $s(i) : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ for a closed immersion $i : Y \hookrightarrow X$ in Sm_k of codimension 1. If $Y = \Pi_\alpha Y_\alpha$ is the decomposition of Y into irreducible components then $\mathcal{S}(Y) = \Pi_\alpha \mathcal{S}(Y_\alpha)$ and $s(i)$ has to be the product of the $s(i_\alpha) : \mathcal{S}(X) \rightarrow \mathcal{S}(Y_\alpha)$. We thus may assume Y (and X) irreducible. We then claim there exists a (unique) map $s(i) : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{S}(X) & \xrightarrow{s(i)} & \mathcal{S}(Y) \\ \cap & & \cap \\ \mathcal{S}(\mathcal{O}_{X,y}) & \xrightarrow{s_y} & \mathcal{S}(\kappa(y)) \end{array}$$

where y is the generic point of Y . To check this it is sufficient to prove that for any $z \in Y^{(1)}$, the image of $\mathcal{S}(X)$ through s_y is contained in $\mathcal{S}(\mathcal{O}_{Y,z})$. But z has codimension 2 in X and this follows from the first part of axiom **(A4)**.

Now we have the following:

Lemma 2.13. *Let $i : Z \rightarrow X$ be a closed immersion in Sm_k of codimension $d > 0$. Assume there exists a factorization $Z \xrightarrow{j_1} Y_1 \xrightarrow{j_2} Y_2 \rightarrow \dots \xrightarrow{j_d} Y_d = X$ of i into a composition of codimension 1 closed immersions, with the Y_i closed subschemes of X each of which is smooth over k . Then the composition*

$$\mathcal{S}(X) \xrightarrow{s(j_d)} \dots \rightarrow \mathcal{S}(Y_2) \xrightarrow{s(j_2)} \mathcal{S}(Y_1) \xrightarrow{s(j_1)} \mathcal{S}(Z)$$

doesn't depend on the choice of the above factorization of i . We denote this composition by $\mathcal{S}(i)$.

Proof. We proceed by induction on d . For $d = 1$ there is nothing to prove. Assume $d \geq 2$. We may easily reduce to the case Z is irreducible with generic point z . We have to show that the composition

$$\mathcal{S}(X) \xrightarrow{s(j_d)} \dots \rightarrow \mathcal{S}(Y_2) \xrightarrow{s(j_2)} \mathcal{S}(Y_1) \xrightarrow{s(j_1)} \mathcal{S}(Z) \subset \mathcal{S}(\kappa(z))$$

doesn't depend on the choice of the flag $Z \rightarrow Y_1 \rightarrow \cdots \rightarrow \cdots \rightarrow X$. We may thus replace X by any open neighborhood Ω of z if we want or even by $\text{Spec}(A)$ with $A := \mathcal{O}_{X,z}$, which we do.

We first observe that the case $d = 2$ follows directly from the Axiom **(A4)**.

In general as A is regular of dimension d there exists a sequence of elements $(x_1, \dots, x_d) \in A$ which generates the maximal ideal \mathcal{M} of A and such that the flag

$$\begin{aligned} \text{Spec}(A/(x_1, \dots, x_d)) &\rightarrow \text{Spec}(A/(x_2, \dots, x_d) \rightarrow \cdots \rightarrow \text{Spec}(A/(x_d)) \\ &\rightarrow \text{Spec}(A) \end{aligned}$$

is the induced flag $Z = \text{Spec}(\kappa(z)) \subset Y_1 \subset Y_2 \subset \cdots \subset \text{Spec}(A)$.

We have thus reduced to proving that under the above assumptions the composition

$$\mathcal{S}(A) \rightarrow \mathcal{S}(\text{Spec}(A/(x_d))) \rightarrow \cdots \rightarrow \mathcal{S}(\text{Spec}(A/(x_2, \dots, x_d)) \rightarrow \mathcal{S}(\kappa(z)))$$

doesn't depend on the choice of (x_1, \dots, x_d) .

By [30, Corollary **(17.12.2)**] the conditions on smoothness on the members of the associated flag to the sequence (x_1, \dots, x_d) is equivalent to the fact the family (x_1, \dots, x_d) reduces to a basis of the $\kappa(z)$ -vector space $\mathcal{M}/\mathcal{M}^2$.

If $M \in GL_d(A)$, the sequence $M.(x_i)$ also satisfies this assumption. For instance any permutation on the (x_1, \dots, x_d) yields an other such sequence. By the case $d = 2$ which was observed above, we see that if we permute x_i and x_{i+1} the compositions $\mathcal{S}(A) \rightarrow \mathcal{S}(\kappa(v))$ are the same before or after permutation. We thus get by induction that we may permute as we wish the x_i 's.

Now assume that (x'_1, \dots, x'_d) is an other sequence in A satisfying the same assumption. Write the x'_i as linear combination in the x_j . There is a matrix $M \in M_d(A)$ with $(x'_i) = M.(x_j)$. This matrix reduces in $M_d(\kappa)$ to an invertible matrix by what we just observed above; thus M itself is invertible. One may multiply in a sequence (x_1, \dots, x_d) by a unit of A an element x_i of the sequence without changing the flag (and thus the composition). Thus we may assume $\det(M) = 1$. Now for a local ring A we know that the group $SL_d(A)$ is the group $E_d(A)$ of elementary matrices in A (see [39, Chap. VI Corollary 1.5.3] for instance). Thus M can be written as a product of elementary matrices in $M_d(A)$.

As we already know that our statement doesn't depend on the ordering of a sequence, we have reduced to the following claim: given a sequence (x_1, \dots, x_d) as above and $a \in A$, the sequence $(x_1 + ax_2, x_2, \dots, x_d)$ induces the same composition $\mathcal{S}(A) \rightarrow \mathcal{S}(\kappa(v))$ as (x_1, \dots, x_d) . But in fact the flags are the same. This proves our claim. \square

Now we come back to the proof of the Lemma 2.12. Let $i : Z \rightarrow X$ be a closed immersion in Sm_k . By what has been recalled above, X can be

covered by open subsets U such that for every U the induced closed immersion $Z \cap U \rightarrow U$ admits a factorization as in the statement of the previous Lemma 2.13. Thus for each such U we get a canonical map $s_U : \mathcal{S}(U) \rightarrow \mathcal{S}(Z \cap U)$. But applying the same Lemma to the intersections $U \cap U'$, with U' an other such open subset, we see that the s_U are compatible and define a canonical map: $s(i) : \mathcal{S}(X) \rightarrow \mathcal{S}(Z)$.

Let $f : Y \rightarrow X$ be any morphism between smooth (quasi-projective) k -schemes. Then f is the composition $Y \hookrightarrow Y \times_k X \rightarrow X$ of the closed immersion (given by the graph of f) $\Gamma_f : Y \hookrightarrow Y \times_k X$ and the smooth projection $p_X : Y \times_k X \rightarrow X$. We set

$$s(f) := \mathcal{S}(X) \xrightarrow{s(p_X)} \mathcal{S}(Y \times_k X) \xrightarrow{s(\Gamma_f)} \mathcal{S}(Y)$$

To check that this defines a functor on $(Sm_k)^{op}$ is not hard. First given a smooth morphism $\pi : X' \rightarrow X$ and a closed immersion $i : Z \rightarrow X$ in Sm_k , denote by $i'' : Z' \rightarrow X'$ the inverse image of i through π and by $\pi' : Z' \rightarrow Z$ the obvious smooth morphism. Then the following diagram is commutative

$$\begin{array}{ccc} \mathcal{S}(X) & \xrightarrow{s(\pi)} & \mathcal{S}(X') \\ \downarrow s(i) & & \downarrow s(i') \\ \mathcal{S}(Z) & \xrightarrow{s(\pi')} & \mathcal{S}(Z') \end{array}$$

Then, to prove the functoriality, one takes two composable morphism $Z \xrightarrow{g} Y \xrightarrow{f} X$ and contemplates the diagram

$$\begin{array}{ccccc} Z \hookrightarrow Z \times_k Y & \hookrightarrow & Z \times_k Y \times_k X \\ \parallel & & \downarrow & & \downarrow \\ Z \rightarrow Y & \hookrightarrow & Y \times_k X \\ \parallel & & \parallel & & \downarrow \\ Z \rightarrow Y & \rightarrow & X \end{array}$$

Then one realizes that applying \mathcal{S} and s yields a commutative diagram, proving the claim. Now the presheaf \mathcal{S} on Sm_k is obviously an unramified sheaf on Sm_k as these properties only depend on its restriction to \tilde{Sm}_k . \square

Remark 2.14. From now on in this paper, we will not distinguish between the notion of unramified \mathcal{F}_k -datum and that of unramified sheaf of sets on Sm_k . If \mathcal{S} is an unramified \mathcal{F}_k -datum we still denote by \mathcal{S} the associated unramified sheaf of sets on Sm_k and *vice versa*.

Also, one may in an obvious fashion describe unramified sheaves of groups, abelian groups, etc. on Sm_k in terms of corresponding \mathcal{F}_k -group data, \mathcal{F}_k -abelian group data, etc., where in the given \mathcal{F}_k -datum, everything is endowed with the corresponding structure and each map is a morphism for that structure. \square

Remark 2.15. The proof of Lemma 2.12 also shows the following. Let \mathcal{S} and \mathcal{E} be sheaves of sets on Sm_k , with \mathcal{S} unramified and \mathcal{E} satisfying conditions (0) and (1) of unramified presheaves. Then to give a morphism of sheaves $\Phi : \mathcal{E} \rightarrow \mathcal{S}$ is equivalent to give a natural transformation $\phi : \mathcal{E}|_{\mathcal{F}_k} \rightarrow \mathcal{S}|_{\mathcal{F}_k}$ such that:

- 1) For any discrete valuation v on $F \in \mathcal{F}_k$, the image of $\mathcal{E}(\mathcal{O}_v) \subset \mathcal{E}(F)$ through ϕ is contained in $\mathcal{S}(\mathcal{O}_v) \subset \mathcal{S}(F)$;
- 2) The induced square commutes:

$$\begin{array}{ccc} \mathcal{E}(\mathcal{O}_v) & \xrightarrow{s_v} & \mathcal{E}(\kappa(v)) \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{S}(\mathcal{O}_v) & \rightarrow & \mathcal{S}(\kappa(v)) \end{array}$$

We left the details to the reader. □

\mathbb{A}^1 -Invariant Unramified Sheaves

Lemma 2.16. 1) Let \mathcal{S} be an unramified sheaf of sets on \tilde{Sm}_k . Then \mathcal{S} is \mathbb{A}^1 -invariant if and only if it satisfies the following:

For any k -smooth local ring A of dimension ≤ 1 the canonical map $\mathcal{S}(A) \rightarrow \mathcal{S}(\mathbb{A}_A^1)$ is bijective.

- 2) Let \mathcal{S} be an unramified sheaf of sets on Sm_k . Then \mathcal{S} is \mathbb{A}^1 -invariant if and only if it satisfies the following:

For any $F \in \mathcal{F}_k$ the canonical map $\mathcal{S}(F) \rightarrow \mathcal{S}(\mathbb{A}_F^1)$ is bijective.

Proof. 1) One implication is clear. Let's prove the other one. Let $X \in Sm_k$ be irreducible with function field F . In the following commutative square

$$\begin{array}{ccc} \mathcal{S}(X) & \rightarrow & \mathcal{S}(\mathbb{A}_X^1) \\ \downarrow & & \downarrow \\ \mathcal{S}(F) & \rightarrow & \mathcal{S}(F(T)) \end{array}$$

each map is injective. We observe that $\mathcal{S}(\mathbb{A}_X^1) \rightarrow \mathcal{S}(F(T))$ factors as $\mathcal{S}(\mathbb{A}_X^1) \rightarrow \mathcal{S}(\mathbb{A}_F^1) \rightarrow \mathcal{S}(F(T))$. By our assumption $\mathcal{S}(F) = \mathcal{S}(\mathbb{A}_F^1)$; this proves that $\mathcal{S}(\mathbb{A}_X^1)$ is contained inside $\mathcal{S}(F)$. Now it is sufficient to prove that for any $x \in X^{(1)}$ one has the inclusion $\mathcal{S}(\mathbb{A}_X^1) \subset \mathcal{S}(\mathcal{O}_{X,x}) \subset \mathcal{S}(F)$. But $\mathcal{S}(\mathbb{A}_X^1) \subset \mathcal{S}(\mathbb{A}_{\mathcal{O}_{X,x}}^1) \subset \mathcal{S}(F(T))$, and our assumption gives $\mathcal{S}(\mathcal{O}_{X,x}) = \mathcal{S}(\mathbb{A}_{\mathcal{O}_{X,x}}^1)$. This proves the claim.

- 2) One implication is clear. Let's prove the other one. Let $X \in Sm_k$ be irreducible with function field F . In the following commutative square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{A}_X^1) & \subset & \mathcal{S}(\mathbb{A}_F^1) \\ \downarrow & & \parallel \\ \mathcal{S}(X) & \subset & \mathcal{S}(F) \end{array}$$

each map is injective but maybe the left vertical one. The latter is thus also injective which implies the statement. \square

Remark 2.17. Given an unramified sheaf \mathcal{S} of sets on $\tilde{S}m_k$ with Data **(D3)**, and satisfying the property that for any $F \in \mathcal{F}_k$, the map $\mathcal{S}(F) \rightarrow \mathcal{S}(F(T))$ is injective, then \mathcal{S} is an unramified \mathcal{F}_k -datum if and only if its extension to $k(T)$ is an unramified $\mathcal{F}_{k(T)}$ -datum.

Indeed, given a smooth irreducible k -scheme X , a point $x \in X$ of codimension d , then $X|_{k(T)}$ is still irreducible $k(T)$ -smooth and $\bar{x}|_{k(T)}$ is irreducible and has codimension d in $X|_{k(T)}$. Moreover the maps $M(X) \rightarrow M(X|_{k(T)})$, $M(X_x) \rightarrow M((X|_{k(T)})_{\bar{x}|_{k(T)}})$, etc. are injective. So to check the Axioms involving equality between morphisms, etc., it suffices to check them over $k(T)$ for $M|_{k(T)}$. This allows us to reduce the checking of several Axioms like **(A4)** to the case k is infinite. \square

2.2 Strongly \mathbb{A}^1 -Invariant Sheaves of Groups

Our aim in this section is to study unramified sheaves of groups \mathcal{G} on Sm_k , their potential strong \mathbb{A}^1 -invariance property, as well as the comparison between their cohomology in Zariski and Nisnevich topology.

In the sequel, by an unramified sheaf of groups we mean a sheaf of groups on Sm_k whose underlying sheaf of sets is unramified in the sense of the previous section.

Let \mathcal{G} be such an unramified sheaf of groups on Sm_k . For any discrete valuation v on $F \in \mathcal{F}_k$ we introduce the pointed set

$$H_v^1(\mathcal{O}_v; \mathcal{G}) := \mathcal{G}(F)/\mathcal{G}(\mathcal{O}_v)$$

and we observe this is a left $\mathcal{G}(F)$ -set.

More generally for y a point of codimension 1 in $X \in Sm'_k$, we set $H_y^1(X; \mathcal{G}) = H_y^1(\mathcal{O}_{X,y}; \mathcal{G})$. By axiom **(A2)**, is X irreducible with function field F the induced left action of $\mathcal{G}(F)$ on $\Pi_{y \in X(1)} H_y^1(X; \mathcal{G})$ preserves the weak-product

$$\Pi'_{y \in X(1)} H_y^1(X; \mathcal{G}) \subset \Pi_{y \in X(1)} H_y^1(X; \mathcal{G})$$

where the weak-product $\Pi'_{y \in X(1)} H_y^1(X; \mathcal{G})$ means the set of families for which all but a finite number of terms are the base point of $H_y^1(X; \mathcal{G})$. By definition, the isotropy subgroup of this action of $\mathcal{G}(F)$ on the base point of $\Pi'_{y \in X(1)} H_y^1(X; \mathcal{G})$ is exactly $\mathcal{G}(X) = \cap_{y \in X(1)} \mathcal{G}(\mathcal{O}_{X,y})$. We will summarize this property by saying that the diagram (of groups, action and pointed set)

$$1 \rightarrow \mathcal{G}(X) \rightarrow \mathcal{G}(F) \rightrightarrows \Pi'_{y \in X(1)} H_y^1(X; \mathcal{G})$$

is “exact” (the double arrow refereing to a left action).

Definition 2.18. For any point z of codimension 2 in a smooth k -scheme X , we denote by $H_z^2(X; \mathcal{G})$ the orbit set of $\Pi'_{y \in X_z^{(1)}} H_y^1(X; \mathcal{G})$ under the left action of $\mathcal{G}(F)$, where $F \in \mathcal{F}_k$ denotes the field of functions of X_z .

Now for an irreducible essentially smooth k -scheme X with function field F we may define an obvious “boundary” $\mathcal{G}(F)$ -equivariant map

$$\Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G}) \rightarrow \Pi_{z \in X^{(2)}} H_z^2(X; \mathcal{G}) \quad (2.1)$$

by collecting together the compositions, for each $z \in X^{(2)}$:

$$\Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G}) \rightarrow \Pi'_{y \in X_z^{(1)}} H_y^1(X; \mathcal{G}) \rightarrow H_z^2(X; \mathcal{G})$$

It is not clear in general whether or not the image of the boundary map is always contained in the weak product $\Pi'_{z \in X^{(2)}} H_z^2(X; \mathcal{G})$. For this reason we will introduce the following Axiom depending on \mathcal{G} which completes **(A2)**:

(A2') For any irreducible essentially smooth k -scheme X the image of the boundary map (2.1) is contained in the weak product $\Pi'_{z \in X^{(2)}} H_z^2(X; \mathcal{G})$. \square

Remark 2.19. Given an unramified sheaf of groups \mathcal{G} , and satisfying the property that for any $F \in \mathcal{F}_k$, the map $\mathcal{G}(F) \rightarrow \mathcal{G}(F(T))$ is injective, then \mathcal{G} satisfies **(A2')** if and only if its extension to $k(T)$ does. This is done along the same lines as in Remark 2.17. \square

We assume from now on that \mathcal{G} satisfies **(A2')**. Altogether we get for X smooth over k , irreducible with function field F , a “complex” $C^*(X; \mathcal{G})$ of groups, action, and pointed sets of the form:

$$1 \rightarrow \mathcal{G}(X) \subset \mathcal{G}(F) \Rightarrow \Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G}) \rightarrow \Pi_{z \in X^{(2)}} H_z^2(X; \mathcal{G})$$

We will also set for $X \in Sm_k$: $\mathcal{G}^{(0)}(X) := \Pi'_{x \in X^{(0)}} \mathcal{G}(\kappa(x))$, $\mathcal{G}^{(1)}(X) := \Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G})$ and $\mathcal{G}^{(2)}(X) := \Pi'_{z \in X^{(2)}} H_z^2(X; \mathcal{G})$. The correspondence $X \mapsto \mathcal{G}^{(i)}(X)$, $i \leq 2$, can be extended to an unramified presheaf of groups on \tilde{Sm}_k , which we still denote by $\mathcal{G}^{(i)}$. Note that $\mathcal{G}^{(0)}$ is a sheaf in the Nisnevich topology. However for $\mathcal{G}^{(i)}$, $i \in \{1, 2\}$ it is not the case in general, these are only sheaves in the Zariski topology, as any unramified presheaf.

The complex $C^*(X; \mathcal{G}) : 1 \rightarrow \mathcal{G}(X) \rightarrow \mathcal{G}^{(0)}(X) \Rightarrow \mathcal{G}^{(1)}(X) \rightarrow \mathcal{G}^{(2)}(X)$ of sheaves on \tilde{Sm}_k will play in the sequel the role of the (truncated) analogue for \mathcal{G} of the Cousin complex of [19] or of the complex of Rost considered in [68].

Definition 2.20. Let $1 \rightarrow H \subset G \Rightarrow E \rightarrow F$ be a sequence with G a group acting on the set E which is pointed (as a set not as a G -set), with $H \subset G$ a subgroup and $E \rightarrow F$ a G -equivariant map of sets, with F endowed with the trivial action. We shall say this sequence is exact if the isotropy subgroup of the base point of E is H and if the “kernel” of the pointed map $E \rightarrow F$ is equal to the orbit under G of the base point of E .

We shall say that it is exact in the strong sense if moreover the map $E \rightarrow F$ induces an injection into F of the (left) quotient set ${}_G \backslash E \subset F$.

By construction $C^*(X; \mathcal{G})$ is exact in the strong sense, for X (essentially) smooth local of dimension ≤ 2 .

Let us denote by $\mathcal{Z}^1(-; \mathcal{G}) \subset \mathcal{G}^{(1)}$ the sheaf theoretic orbit of the base point under the action of $\mathcal{G}^{(0)}$ in the Zariski topology on $\tilde{S}m_k$. We thus have an exact sequence of sheaves on $\tilde{S}m_k$ in the Zariski topology

$$1 \rightarrow \mathcal{G} \subset \mathcal{G}^{(0)} \Rightarrow \mathcal{Z}^1(-; \mathcal{G}) \rightarrow *$$

As it is clear that $H_{Zar}^1(X; \mathcal{G}^{(0)})$ is trivial (the sheaf $\mathcal{G}^{(0)}$ being flasque), this yields for any $X \in Sm_k$ an exact sequence (of groups and pointed sets)

$$1 \rightarrow \mathcal{G}(X) \subset \mathcal{G}^{(0)}(X) \Rightarrow \mathcal{Z}^1(X; \mathcal{G}) \rightarrow H_{Zar}^1(X; \mathcal{G}) \rightarrow *$$

in the strong sense.

Of course we may extend by passing to the filtering colimit the previous definitions for $X \in Sm'_k$. To be correct, we should introduce a name for the category of essentially smooth k -schemes and smooth morphisms. The previous diagram is then also a diagram of sheaves in the Zariski topology and yields for any $X \in Sm'_k$ an exact sequence as above, which could have been also obtained by passing to the colimit.

Remark 2.21. If X is an essentially smooth k -scheme of dimension ≤ 1 , we thus get a bijection $H_{Zar}^1(X; \mathcal{G}) = {}_{\mathcal{G}^{(0)}(X)} \backslash \mathcal{G}^{(1)}(X)$. For instance, when X is a smooth local k -scheme of dimension 2, and if $V \subset X$ is the complement of the closed point, a smooth k -scheme of dimension 1, we thus get a bijection

$$H_z^2(X; \mathcal{G}) = H_{Zar}^1(V; \mathcal{G})$$

Beware that here the Zariski topology is used. This gives a “concrete” interpretation of the “strange” extra cohomology set $H_z^2(X; \mathcal{G})$. \square

For $X \in Sm_k$ (or Sm'_k) as above, let us denote by $\mathcal{K}^1(X; \mathcal{G}) \subset \Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G})$ the kernel of the boundary map $\Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G}) \rightarrow \Pi'_{z \in X^{(2)}} H_z^2(X; \mathcal{G})$. The correspondence $X \mapsto \mathcal{K}^1(X; \mathcal{G})$ is a sheaf in the Zariski topology on $\tilde{S}m_k$. There is an obvious injective morphism of sheaves in the Zariski topology on $\tilde{S}m_k$: $\mathcal{Z}^1(-; \mathcal{G}) \rightarrow \mathcal{K}^1(-; \mathcal{G})$. As $C^*(X; \mathcal{G})$ is exact for any

k -smooth local X of dimension ≤ 2 , $\mathcal{Z}^1(-; \mathcal{G}) \rightarrow \mathcal{K}^1(-; \mathcal{G})$ induces a bijection for any (essentially) smooth k -scheme of dimension ≤ 2 .

Remark 2.22. In particular if X is an (essentially) smooth k -scheme of dimension ≤ 2 , the H^1 of the complex $C^*(X; \mathcal{G})$ is $H_{Zar}^1(X; \mathcal{G})$. \square

Now we introduce the following axiom on \mathcal{G} :

- (A5) (i) For any separable finite extension $E \subset F$ in \mathcal{F}_k , any discrete valuation v on F which restricts to a discrete valuation w on E with ramification index 1, and such that the induced extension $\bar{i} : \kappa(w) \rightarrow \kappa(v)$ is an isomorphism, the commutative square of groups

$$\begin{array}{ccc} \mathcal{G}(\mathcal{O}_w) & \subset & \mathcal{G}(E) \\ \downarrow & & \downarrow \\ \mathcal{G}(\mathcal{O}_v) & \subset & \mathcal{G}(F) \end{array}$$

induces a bijection $H_v^1(\mathcal{O}_v; \mathcal{G}) \cong H_w^1(\mathcal{O}_w; \mathcal{G})$.

- (ii) For any étale morphism $X' \rightarrow X$ between smooth local k -schemes of dimension 2, with closed point respectively z' and z , inducing an isomorphism on the residue fields $\kappa(z) \cong \kappa(z')$, the pointed map

$$H_z^2(X; \mathcal{G}) \rightarrow H_{z'}^2(X'; \mathcal{G})$$

has trivial kernel. \square

Remark 2.23. The Axiom (A5)(i) implies that if we denote by \mathcal{G}_{-1} the sheaf of groups

$$X \mapsto \text{Ker}(G(\mathbb{G}_m \times X) \xrightarrow{ev_1} G(X))$$

then for any discrete valuation v on $F \in \mathcal{F}_k$ one has a (non canonical) bijection

$$H_v^1(\mathcal{O}_v; \mathcal{G}) \cong \mathcal{G}_{-1}(\kappa(v))$$

Indeed one may reduce to the case where \mathcal{O}_v is henselian, and assume that $\kappa(v) \subset \mathcal{O}_v$. Choosing a uniformizing element then yields a distinguished square

$$\begin{array}{ccc} \text{Spec}(F) & \subset & \text{Spec}(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ (\mathbb{G}_m)_{\kappa(v)} & \subset & \mathbb{A}_{\kappa(v)}^1 \end{array}$$

which in view of Axiom (A5) (i) gives the bijection $\mathcal{G}((\mathbb{G}_m)_{\kappa(v)})/\mathcal{G}(\kappa(v)) \cong H_v^1(\mathcal{O}_v; \mathcal{G})$. \square

Lemma 2.24. *Let \mathcal{G} be as above. The following conditions are equivalent:*

- (i) *The Zariski sheaf $X \mapsto \mathcal{K}^1(X; \mathcal{G})$ is a sheaf in the Nisnevich topology on $\tilde{S}m_k$;*

- (ii) For any essentially smooth k -scheme X of dimension ≤ 2 the comparison map $H_{Zar}^1(X; \mathcal{G}) \rightarrow H_{Nis}^1(X; \mathcal{G})$ is a bijection;
- (iii) \mathcal{G} satisfies Axiom **(A5)**

Proof. (i) \Rightarrow (ii). Under (i) we know that $X \mapsto \mathcal{Z}^1(X; \mathcal{G})$ is a sheaf in the Nisnevich topology on essentially smooth k -schemes of dimension ≤ 2 (as $\mathcal{Z}^1(X; \mathcal{G}) \rightarrow \mathcal{K}^1(X; \mathcal{G})$ is an isomorphism on essentially smooth k -schemes of dimension ≤ 2). The exact sequence in the Zariski topology $1 \rightarrow \mathcal{G} \subset \mathcal{G}^{(0)} \Rightarrow \mathcal{Z}^1(-; \mathcal{G}) \rightarrow *$ considered above is then also an exact sequence of sheaves in the Nisnevich topology. The same reasoning as above easily implies (ii), taking into account that $H_{Nis}^1(X; \mathcal{G}^{(0)})$ is also trivial (left to the reader).

(ii) \Rightarrow (iii). Assume (ii). Let's prove **(A5) (i)**. With the assumptions given the square

$$\begin{array}{ccc} Spec(F) & \rightarrow & Spec(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ Spec(E) & \rightarrow & Spec(\mathcal{O}_w) \end{array}$$

is a distinguished square in the sense of [59]. Using the corresponding Mayer-Vietoris type exact sequence and the fact by (ii) that $H^1(X; \mathcal{G}) = *$ for any smooth local scheme X yields immediately that $\mathcal{G}(E)/\mathcal{G}(\mathcal{O}_w) \rightarrow \mathcal{G}(F)/\mathcal{G}(\mathcal{O}_v)$ is a bijection.

Now let's prove **(A5) (ii)**. Set $V = X - \{z\}$ and $V' = X' - \{z'\}$. The square

$$\begin{array}{ccc} V' & \subset & X' \\ \downarrow & & \downarrow \\ V & \subset & X \end{array}$$

is distinguished. From the discussion preceding the Lemma and the interpretation of $H_z^2(X; \mathcal{G})$ as $H_{Zar}^1(V; \mathcal{G})$, the kernel in question is thus the set of (isomorphism classes) of \mathcal{G} -torsors over V (indifferently in Zariski and Nisnevich topology as $H_{Zar}^1(V; \mathcal{G}) \cong H_{Nis}^1(V; \mathcal{G})$ by (ii)) which become trivial over V' ; but such a torsor can thus be extended to X' and by a descent argument in the Nisnevich topology, we may extend the torsor on V to X . Thus it is trivial because X is local.

(iii) \Rightarrow (i). Now assume Axiom **(A5)**. We claim that Axiom **(A5) (i)** gives exactly that $X \mapsto \mathcal{G}^{(1)}(X)$ is a sheaf in the Nisnevich topology. **(A5) (ii)** is seen to be exactly what is needed to imply that $\mathcal{K}^1(-; \mathcal{G})$ is a sheaf in the Nisnevich topology. \square

Now we observe that the monomorphism of Zariski sheaves $\mathcal{Z}^1(-; \mathcal{G}) \rightarrow \mathcal{K}^1(-; \mathcal{G})$ is $\mathcal{G}^{(0)}$ -equivariant.

Lemma 2.25. *Assume \mathcal{G} satisfies **(A5)**. Let X be an essentially smooth k -scheme. The following conditions are equivalent:*

- (i) For any open subscheme $\Omega \subset X$ the map $\mathcal{Z}^1(\Omega; \mathcal{G}) \rightarrow \mathcal{K}^1(\Omega; \mathcal{G})$ is bijective;

- (ii) For any localization U of X at some point, the map $\mathcal{Z}^1(U; \mathcal{G}) \rightarrow \mathcal{K}^1(U; \mathcal{G})$ is bijective;
- (iii) For any localization U of X at some point, the complex $C^*(U; \mathcal{G}) : 1 \rightarrow \mathcal{G}(U) \rightarrow \mathcal{G}(F) \Rightarrow \mathcal{G}^{(1)}(U) \rightarrow \mathcal{G}^{(2)}(U)$ is exact.

When moreover these conditions are satisfied for any Y étale over X , then the comparison map $H_{Zar}^1(X; \mathcal{G}) \rightarrow H_{Nis}^1(X; \mathcal{G})$ is a bijection.

Proof. (i) \Leftrightarrow (ii) is clear as both are Zariski sheaves. (ii) \Rightarrow (iii) is proven exactly as in the proof of (ii) in Lemma 2.24. (iii) \Rightarrow (i) is also clear using the given expressions of the two sides.

If we assume these conditions are satisfied, then

$${}_{\mathcal{G}^{(0)}(X)}\backslash \mathcal{Z}^1(X; \mathcal{G}) = H_{Zar}^1(X; \mathcal{G}) \rightarrow H_{Nis}^1(X; \mathcal{G}) = {}_{\mathcal{G}^{(0)}(X)}\backslash \mathcal{K}^1(X; \mathcal{G})$$

is a bijection. The last equality follows from the fact that $\mathcal{K}^1(\cdot; \mathcal{G})$ is a Nisnevich sheaf and the (easy) fact that $H_{Nis}^1(X; \mathcal{G}^{(0)})$ is also trivial. \square

Lemma 2.26. *Assume \mathcal{G} is \mathbb{A}^1 -invariant. Let X be an essentially smooth k -scheme. The following conditions are equivalent:*

- (i) For any open subscheme $\Omega \subset X$ the map

$${}_{\mathcal{G}^{(0)}(\Omega)}\backslash \mathcal{Z}^1(\Omega; \mathcal{G}) = H_{Zar}^1(\Omega; \mathcal{G}) \rightarrow H_{Zar}^1(\mathbb{A}_\Omega^1; \mathcal{G}) = {}_{\mathcal{G}^{(0)}(\mathbb{A}_\Omega^1)}\backslash \mathcal{Z}^1(\mathbb{A}_\Omega^1; \mathcal{G})$$

is bijective;

- (ii) For any localization U of X , ${}_{\mathcal{G}^{(0)}(\mathbb{A}_U^1)}\backslash \mathcal{Z}^1(\mathbb{A}_U^1; \mathcal{G}) = *$.

Proof. The implication (i) \Rightarrow (ii) follows from the fact that for U a smooth local k -scheme $H_{Zar}^1(U; \mathcal{G}) = {}_{\mathcal{G}^{(0)}(U)}\backslash \mathcal{Z}^1(U; \mathcal{G})$ is trivial. Assume (ii). Thus $H_{Zar}^1(\mathbb{A}_U^1; \mathcal{G}) = *$ for any local smooth k -scheme U . Fix $\Omega \subset X$ an open subscheme and denote by $\pi : \mathbb{A}_\Omega^1 \rightarrow \Omega$ the projection. To prove (i) it is sufficient to prove that the pointed simplicial sheaf of sets $R\pi_*(B(\mathcal{G}|_{\mathbb{A}_\Omega^1}))$ has trivial π_0 . Indeed, its π_1 sheaf is $\pi_*(\mathcal{G}|_{\mathbb{A}_\Omega^1}) = \mathcal{G}|_\Omega$ because \mathcal{G} is \mathbb{A}^1 -invariant. If the π_0 is trivial, $B(\mathcal{G}|_\Omega) \rightarrow R\pi_*(B(\mathcal{G}|_{\mathbb{A}_\Omega^1}))$ is a simplicial weak equivalence which implies the result. Now to prove that $\pi_0 R\pi_*((B(\mathcal{G}|_{\mathbb{A}_\Omega^1})))$ is trivial, we just observe that its stalk at a point $x \in \Omega$ is $H_{Zar}^1(\mathbb{A}_{X_x}^1; \mathcal{G})$ which is trivial by assumption. \square

Now we will use one more Axiom concerning \mathcal{G} and related to \mathbb{A}^1 -invariance properties:

(A6) For any localization U of a smooth k -scheme at some point u of codimension ≤ 1 , the “complex”:

$$1 \rightarrow \mathcal{G}(\mathbb{A}_U^1) \subset \mathcal{G}^{(0)}(\mathbb{A}_U^1) \Rightarrow \mathcal{G}^{(1)}(\mathbb{A}_U^1) \rightarrow \mathcal{G}^{(2)}(\mathbb{A}_U^1)$$

is exact, and moreover, the morphism $\mathcal{G}(U) \rightarrow \mathcal{G}(\mathbb{A}_U^1)$ is an isomorphism. \square

Observe that if \mathcal{G} satisfies **(A6)** it is \mathbb{A}^1 -invariant by Lemma 2.16 (as \mathcal{G} is assumed to be unramified). Observe also that if \mathcal{G} satisfies Axioms **(A2')** and **(A5)**, then we know by Lemma 2.24 that $H_{Nis}^1(\mathbb{A}_X^1; \mathcal{G}) = H_{Zar}^1(\mathbb{A}_X^1; \mathcal{G}) = H^1(\mathbb{A}_X^1; \mathcal{G})$ for X smooth of dimension ≤ 1 .

Our main result in this section is the following.

Theorem 2.27. *Let \mathcal{G} be an unramified sheaf of groups on Sm_k that satisfies Axioms **(A2')**, **(A5)** and **(A6)**. Then it is strongly \mathbb{A}^1 -invariant. Moreover, for any smooth k -scheme X , the comparison map*

$$H_{Zar}^1(X; \mathcal{G}) \rightarrow H_{Nis}^1(X; \mathcal{G})$$

is a bijection.

Remark 2.28. From Corollary 6.9 in Sect. 6.1 below applied to the \mathbb{A}^1 -local space BG itself, it follows that a strongly \mathbb{A}^1 -invariant sheaf of groups G on Sm_k is always unramified.

We thus obtain in this way an equivalence between the category of strongly \mathbb{A}^1 -invariant sheaves of groups on Sm_k and that of unramified sheaves of groups on Sm_k satisfying axioms **(A2')**, **(A5)** and **(A6)**. \square

To prove theorem 2.27 we fix an unramified sheaf of groups \mathcal{G} on Sm_k which satisfies the Axioms **(A2')**, **(A5)** and **(A6)**.

We introduce two properties depending on \mathcal{G} , an integer $d \geq 0$:

(H1)(d) For any localization U of a smooth k -scheme at some point u of codimension $\leq d$ with infinite residue field, the complex $1 \rightarrow \mathcal{G}(U) \subset \mathcal{G}^{(0)}(U) \Rightarrow \mathcal{G}^{(1)}(U) \rightarrow \mathcal{G}^{(2)}(U)$ is exact. \square

(H2)(d) For any localization U of a smooth k -scheme at some point u of codimension $\leq d$ with infinite residue field, the “complex”:

$$1 \rightarrow \mathcal{G}(\mathbb{A}_U^1) \subset \mathcal{G}^{(0)}(\mathbb{A}_U^1) \Rightarrow \mathcal{G}^{(1)}(\mathbb{A}_U^1) \rightarrow \mathcal{G}^{(2)}(\mathbb{A}_U^1)$$

is exact. \square

(H1)(d) is a reformulation of (ii) of Lemma 2.25. It is a tautology in case $d \leq 2$. **(H2)(1)** holds by Axiom **(A6)** and **(H2)(d)** implies (ii) of the Lemma 2.26.

Lemma 2.29. *Let $d \geq 0$ be an integer.*

- 1) **(H1)(d)** \Rightarrow **(H2)(d)**.
- 2) **(H2)(d)** \Rightarrow **(H1)(d+1)**

Proof of Theorem 2.27 assuming Lemma 2.29. Lemma 2.29 implies by induction on d that properties **(H1)(d)** and **(H2)(d)** hold for any d . It follows from Lemmas 2.25 and 2.26 above that for any essentially smooth k -scheme X with infinite residue fields, then $H_{Zar}^1(X; \mathcal{G}) \cong H_{Nis}^1(X; \mathcal{G})$ and $H^1(X; \mathcal{G}) \cong H_{Zar}^1(\mathbb{A}_X^1; \mathcal{G})$.

This implies Theorem 2.27 if k is infinite. Assume now k is finite. Let \mathcal{G}' be the sheaf $\pi_1^{\mathbb{A}^1}(B\mathcal{G}) = \pi_1(L_{\mathbb{A}^1}(B\mathcal{G}))$. By Corollary 6.9 of Sect. 6.1 below applied to the \mathbb{A}^1 -local space $L_{\mathbb{A}^1}(B\mathcal{G})$, \mathcal{G}' is unramified and by the first part of Theorem 6.11 it satisfies, as \mathcal{G} the Axioms **(A2')**, **(A5)** and **(A6)**.

By general properties of base change through a smooth morphism (see [52]) we see that for any henselian k -smooth local ring A , with infinite residue field, the morphism $G(A) \rightarrow G'(A)$ is an isomorphism. Let A be a k -smooth local ring of dimension ≥ 1 . By functoriality we see that $G(A) \subset G'(A)$ is injective, as the fraction field of A is infinite. If κ is a finite field (extension of k), $\mathcal{G}(\kappa) = \mathcal{G}(\kappa[T]) \subset \mathcal{G}'(\kappa[T]) = \mathcal{G}'(\kappa)$. We deduce that $\mathcal{G} \rightarrow \mathcal{G}'$ is always a monomorphism of sheaves, because if κ is a finite extension of k , $\mathcal{G}(\kappa) \subset \mathcal{G}(\kappa(T))$.

Thus we have the monomorphism $\mathcal{G} \subset \mathcal{G}'$ between unramified sheaves satisfying Axioms **(A2')**, **(A5)** and **(A6)** and which is an isomorphism on smooth local ring with infinite residue field. Now using Remark 2.23 and proceeding as below in the proof of Lemma 2.34, we see that, given a discrete valuation ring $A \subset F$, and a uniformizing element π , $H_v^1(A; \mathcal{G}) \rightarrow H_v^1(A; \mathcal{G}')$ can be identified to the morphism $G_{-1}(\kappa(v)) \subset \mathcal{G}'_{-1}(\kappa(v))$; but this is an injection as $\mathcal{G}(X \times \mathbb{G}_m) \subset \mathcal{G}'(X \times \mathbb{G}_m)$. This implies that $\mathcal{G}(A) = \mathcal{G}'(A)$ $H_v^1(A; \mathcal{G}) \cong H_v^1(A; \mathcal{G}')$ for any discrete valuation ring $A \subset F \in \mathcal{F}_k$. If we prove that $\mathcal{G}(\kappa) \subset \mathcal{G}'(\kappa)$ is an isomorphism for any finite extension κ of k then we conclude that $\mathcal{G} = \mathcal{G}'$, as both are unramified and coincide over each stalk (included the finite fields). To show $\mathcal{G}(\kappa) \subset \mathcal{G}'(\kappa)$ is an isomorphism, we observe that $\mathcal{G}(\kappa[T]) = \mathcal{G}'(\kappa[T])$ by what precedes.

Now that we know $\mathcal{G} = \mathcal{G}'$, we conclude from the fact that the composition $B\mathcal{G} \rightarrow L_{\mathbb{A}^1}(B\mathcal{G}) \rightarrow B\mathcal{G}'$ is a (simplicial weak-equivalence) that $B\mathcal{G}$ is \mathbb{A}^1 -local, and \mathcal{G} is thus strongly \mathbb{A}^1 -invariant, finishing the proof. \square

Remark 2.30. The only reason we have to separate the case of a finite residue field and infinite residue field is due to the point (ii) of Lemma 2.31 below. If one could prove this also with finite residue field, we could get rid of the last part of the previous proof. \square

Proof of Lemma 2.29 Let $d \geq 2$ be an integer (if $d < 2$ there is nothing to prove).

Let us prove (1). Assume that **(H1)(d)** holds. Let U be an irreducible smooth k -scheme with function field F . Let us study the following diagram whose middle row is $C^*(\mathbb{A}_U^1; \mathcal{G})$, whose bottom row is $C^*(U; \mathcal{G})$ and whose top row is $C^*(\mathbb{A}_F^1; \mathcal{G})$:

$$\begin{array}{ccccccc}
 \mathcal{G}(F) & \subset & \mathcal{G}(F(T)) & \twoheadrightarrow & \Pi'_{y \in (\mathbb{A}_F^1)^{(1)}} & H_y^1(\mathbb{A}_F^1; \mathcal{G}) & \\
 \cup & & \parallel & & \uparrow & & \\
 \mathcal{G}(\mathbb{A}_U^1) & \subset & \mathcal{G}(F(T)) & \Rightarrow & \Pi'_{y \in (\mathbb{A}_U^1)^{(1)}} & H_y^1(\mathbb{A}_U^1; \mathcal{G}) & \rightarrow \Pi'_{z \in (\mathbb{A}_U^1)^{(2)}} H_z^2(\mathbb{A}_U^1; \mathcal{G}) \quad (2.2) \\
 \parallel & & \cup & & \uparrow & & \uparrow \\
 \mathcal{G}(U) & \subset & \mathcal{G}(F) & \Rightarrow & \Pi'_{y \in U^{(1)}} & H_y^1(U; \mathcal{G}) & \rightarrow \Pi'_{z \in U^{(2)}} H_z^2(U; \mathcal{G})
 \end{array}$$

The top horizontal row is exact by Axiom **(A6)**. Assume U is local of dimension $\leq d$. The bottom horizontal row is exact by **(H1) (d)**. The middle vertical column can be explicitated as follows. The points y of codimension 1 in \mathbb{A}_U^1 are of two types: either the image of y is the generic point of U or it is a point of codimension 1 in U ; the first set is in bijection with $(\mathbb{A}_F^1)^{(1)}$ and the second one with $U^{(1)}$ through the map $y \in U^{(1)} \mapsto y[T] := \mathbb{A}_y^1 \subset \mathbb{A}_U^1$. For y of the first type, it is clear that the set $H_y^1(\mathbb{A}_U^1; \mathcal{G})$ is the same as $H_y^1(\mathbb{A}_F^1; \mathcal{G})$. As a consequence, $\Pi'_{y \in (\mathbb{A}_U^1)^{(1)}} H_y^1(\mathbb{A}_U^1; \mathcal{G})$ is exactly the product of $\Pi'_{y \in (\mathbb{A}_F^1)^{(1)}} H_y^1(\mathbb{A}_F^1; \mathcal{G})$ and of $\Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G})$.

To prove **(H2)(d)** we have exactly to prove the exactness of the middle horizontal row in (2.2) and more precisely that the action of $\mathcal{G}(F(T))$ on $\mathcal{K}^1(\mathbb{A}_U^1; \mathcal{G})$ is transitive.

Take $\alpha \in \mathcal{K}^1(\mathbb{A}_U^1; \mathcal{G})$. As the top horizontal row is exact, there is a $g \in \mathcal{G}(F(T))$ such that $g.\alpha$ lies in $\Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G}) \subset \Pi'_{y \in (\mathbb{A}_U^1)^{(1)}} H_y^1(\mathbb{A}_U^1; \mathcal{G})$, which is the kernel of the vertical $\mathcal{G}(F(T))$ -equivariant map $\Pi'_{y \in (\mathbb{A}_U^1)^{(1)}} H_y^1(\mathbb{A}_U^1; \mathcal{G}) \rightarrow \Pi'_{y \in (\mathbb{A}_F^1)^{(1)}} H_y^1(\mathbb{A}_F^1; \mathcal{G})$.

Thus $g.\alpha$ lies in $\mathcal{K}^1(\mathbb{A}_U^1; \mathcal{G}) \cap \Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G}) \subset \Pi'_{y \in (\mathbb{A}_U^1)^{(1)}} H_y^1(\mathbb{A}_U^1; \mathcal{G})$. Now the obvious inclusion $\mathcal{K}^1(U; \mathcal{G}) \subset \mathcal{K}^1(\mathbb{A}_U^1; \mathcal{G}) \cap \Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G})$ is a bijection. Indeed, from part (1) of Lemma 2.31 below, $\Pi'_{y \in U^{(1)}} H_y^1(U; \mathcal{G}) \subset \Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G})$ is injective and is exactly the kernel of the composition of the boundary map $\Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G}) \rightarrow \Pi'_{z \in (\mathbb{A}_U^1)^{(2)}} H_z^2(\mathbb{A}_U^1; \mathcal{G})$ and the projection

$$\Pi'_{z \in (\mathbb{A}_U^1)^{(2)}} H_z^2(\mathbb{A}_U^1; \mathcal{G}) \rightarrow \Pi'_{y \in U^{(1)}, z \in (\mathbb{A}_y^1)^{(1)}} H_z^2(\mathbb{A}_U^1; \mathcal{G})$$

This shows that $\mathcal{K}^1(\mathbb{A}_U^1; \mathcal{G}) \cap \Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G})$ is contained in $\Pi'_{y \in U^{(1)}} H_y^1(U; \mathcal{G})$.

But the right vertical map in (2.2), $\Pi'_{z \in U^{(2)}} H_z^2(U; \mathcal{G}) \rightarrow \Pi'_{z \in (\mathbb{A}_U^1)^{(2)}} H_z^2(\mathbb{A}_U^1; \mathcal{G})$, is induced by the correspondence $z \in U^{(2)} \mapsto \mathbb{A}_z^1 \subset \mathbb{A}_U^1$ and the corresponding maps on $H_z^2(-; \mathcal{G})$. By part (2) of Lemma 2.31 below, this map has trivial kernel. This easily implies that $\mathcal{K}^1(\mathbb{A}_U^1; \mathcal{G}) \cap \Pi'_{y \in U^{(1)}} H_{y[T]}^1(\mathbb{A}_U^1; \mathcal{G})$ is contained in $\mathcal{K}^1(U; \mathcal{G})$, proving our claim.

Thus $g.\alpha$ lies in $\mathcal{K}^1(U; \mathcal{G})$. Now by **(H1) (d)** we know there is an $h \in \mathcal{G}(F)$ with $hg.\alpha = *$ as required.

Let us now prove (2). Assume **(H2) (d)** holds. Let's prove **(H1) (d+1)**. Let X be an irreducible smooth k -scheme (of finite type) of dimension $\leq d+1$ with function field F , let $u \in X \in Sm_k$ be a point of codimension $d+1$ and denote by U its associated local scheme, F its function field. We have to check the exactness at the middle of $\mathcal{G}(F) \Rightarrow \Pi'_{y \in U^{(1)}} H_y^1(U; \mathcal{G}) \rightarrow \Pi'_{z \in U^{(2)}} H_z^2(U; \mathcal{G})$.

Let $\alpha \in \mathcal{K}^1(U; \mathcal{G}) \subset \Pi'_{y \in U^{(1)}} H_y^1(U; \mathcal{G})$. We want to show that there exists $g \in \mathcal{G}(F)$ such that $\alpha = g.*$. Let us denote by $y_i \in U$ the points of

codimension one in U where α is non trivial. Recall that for each $y \in U^{(1)}$, $H_y^1(U; \mathcal{G}) = H_y^1(X; \mathcal{G})$ where we still denote by $y \in X^{(1)}$ the image of y in X . Denote by $\alpha_X \in \Pi'_{y \in X^{(1)}} H_y^1(X; \mathcal{G})$ the canonical element with same support y_i 's and same components as α . α_X may not be in $\mathcal{K}^1(X; \mathcal{G})$, but, by Axiom **(A2')**, its boundary is trivial except on finitely many points z_j of codimension 2 in X . Clearly these points are not in $U^{(2)}$, thus we may, up to removing the closure of these z_j 's, find an open subscheme Ω' in X which contains u and the y_i 's and such that the element $\alpha_{\Omega'} \in \Pi'_{y \in \Omega'^{(1)}} H_y^1(X; \mathcal{G})$, induced by α , is in $\mathcal{K}^1(\Omega'; \mathcal{G})$.

By Gabber's presentation Lemma 1.15, there exists an étale morphism $U \rightarrow \mathbb{A}_V^1$, with V the localization of a k -smooth of dimension d , such that if $Y \subset U$ denotes the reduced closed subscheme whose generic points are the y_i , the composition $Y \rightarrow U \rightarrow \mathbb{A}_V^1$ is still a closed immersion and such that the composition $Y \rightarrow U \rightarrow \mathbb{A}_V^1 \rightarrow V$ is a finite morphism.

The étale morphism $U \rightarrow \mathbb{A}_V^1$ induces a morphism of complexes of the form:

$$\begin{array}{ccccc} \mathcal{G}(F) & - & \Pi'_{y \in U^{(1)}} H_y^1(U; \mathcal{G}) & \rightarrow & \Pi'_{z \in U^{(2)}} H_z^2(U; \mathcal{G}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{G}(E(T)) & - & \Pi'_{y \in (\mathbb{A}_V^1)^{(1)}} H_y^1(\mathbb{A}_V^1; \mathcal{G}) & \rightarrow & \Pi'_{z \in (\mathbb{A}_V^1)^{(2)}} H_z^2(\mathbb{A}_V^1; \mathcal{G}) \end{array}$$

where E is the function field of V . Let y'_i be the images of the y_i in \mathbb{A}_V^1 ; these are points of codimension 1 and have the same residue field (because $Y \rightarrow \mathbb{A}_V^1$ is a closed immersion). By the axiom **(A5)(i)**, we see that for each i , the map $H_{y'_i}^1(\mathbb{A}_V^1; \mathcal{G}) \rightarrow H_{y_i}^1(U; \mathcal{G})$ is a bijection so that there exists in the bottom complex an element $\alpha' \in \Pi'_{y \in (\mathbb{A}_V^1)^{(1)}} H_y^1(\mathcal{G})$ whose image is α . The boundary of this α' is trivial. To show this, observe that if $z \in (\mathbb{A}_V^1)^{(2)}$ is not contained in Y , then the boundary of α' has a trivial component in $H_z^2(\mathbb{A}_V^1; \mathcal{G})$. Moreover, if $z \in (\mathbb{A}_V^1)^{(2)}$ lies in the image of Y in \mathbb{A}_V^1 , there is, by construction, a unique point z' of codimension 2 in Ω , lying in Y and mapping to z . It has moreover the same residue field as z . The claim now follows from **(A5)(ii)**.

By the inductive assumption **(H2) (d)** we see that α' is of the form $h.*$ in $\Pi'_{y \in (\mathbb{A}_V^1)^{(1)}} H_y^1(\mathbb{A}_V^1; \mathcal{G})$ with $h \in \mathcal{G}(E(T))$. But if g denotes the image of h in $\mathcal{G}(F)$ we have $\alpha = g.*$, proving our claim. \square

Lemma 2.31. *Let \mathcal{G} be an unramified sheaf of groups on Sm_k satisfying **(A2')**, **(A5)** and **(A6)**.*

- 1) *Let v be a discrete valuation on $F \in \mathcal{F}_k$. Denote by $v[T]$ the discrete valuation in $F(T)$ corresponding to the kernel of $\mathcal{O}_v[T] \rightarrow \kappa(v)(T)$. Then the map*

$$H_v^1(\mathcal{O}_v; \mathcal{G}) \rightarrow H_{v[T]}^1(\mathbb{A}_{\mathcal{O}_v}^1; \mathcal{G})$$

is injective and its image is exactly the kernel of

$$H_{v[T]}^1(\mathbb{A}_{\mathcal{O}_v}^1; \mathcal{G}) \rightarrow \Pi'_{z \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}} H_z^2(\mathbb{A}_{\mathcal{O}_v}^1; \mathcal{G})$$

where we see $z \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}$ as a point of codimension 2 in $\mathbb{A}_{\mathcal{O}_v}^1$.

- 2) For any k -smooth local scheme U of dimension 2 with closed point u , and infinite residue field, the “kernel” of the map

$$H_u^2(\mathcal{G}) \rightarrow H_{u[T]}^2(\mathcal{G})$$

is trivial.

Proof. Part (1) follows immediately from the fact that we know from our axioms the exactness of each row of the Diagram (2.2) is exact for U smooth local of dimension 1.

To prove (2) we shall use the interpretation of $H_z^2(U; \mathcal{G})$, for U smooth local of dimension 2 with closed point z , as $H_{Zar}^1(V; \mathcal{G})$, with V the complement of the closed point u . By Lemma 2.24, we know that $H_{Zar}^1(V; \mathcal{G}) \cong H_{Nis}^1(V; \mathcal{G})$.

Pick up an element α of $H_u^2(U; \mathcal{G}) = H_{Nis}^1(V; \mathcal{G})$ which becomes trivial in $H_{u[T]}^2(\mathbb{A}_U^1; \mathcal{G}) = H_{Nis}^1(V_T; \mathcal{G})$, where $V_T = (\mathbb{A}_U^1)_{u[T]} - u'$, u' denoting the generic point of $A_u^1 \subset \mathbb{A}_U^1$. This means that the \mathcal{G} -torsor over V become trivial over V_T . As V_T is the inverse limit of the schemes of the form $\Omega - \Omega \cap \overline{u'}$, where Ω runs over the open subschemes of \mathbb{A}_U^1 which contains u' , we see that there exists such an Ω for which the pull-back of α to $\Omega - \Omega \cap \overline{u'}$ is already trivial. As Ω contains u' , $\Omega \cap \overline{u'} \subset \mathbb{A}_{\kappa(u)}^1$ is a non empty dense subset; in case $\kappa(u)$ is infinite, we thus know that there exists a $\kappa(u)$ -rational point z in $\Omega \cap \overline{u'}$ lying over u . As $\Omega \rightarrow U$ is smooth, it follows from [30, Corollary 17.16.3 p. 106] that there exists an immersion $U' \rightarrow \Omega$ whose image contains z and such that $U' \rightarrow U$ is étale. This immersion is then a closed immersion, and up to shrinking a bit U' we may assume that $\Omega \cap \overline{u'} \cap U' = \{z\}$. Thus the cartesian square

$$\begin{array}{ccc} U' - z & \rightarrow & U' \\ \downarrow & & \downarrow \\ V & \rightarrow & U \end{array}$$

is a distinguished square [59]. And the pull-back of α to $U' - z$ is trivial. Extending it to U' defines a descent data which defines an extension of α to U ; thus as any element of $H_{Zar}^1(U; \mathcal{G}) = H_{Nis}^1(U; \mathcal{G})$ α is trivial we get our claim. \square

\mathbb{G}_m -loop spaces. Recall the following construction, used by Voevodsky in [79]. Given a presheaf of groups G on Sm_k , we let G_{-1} denote the presheaf of groups given by

$$X \mapsto \text{Ker}(G(\mathbb{G}_m \times X) \xrightarrow{ev_1} G(X))$$

Observe that if G is a sheaf of groups, so is G_{-1} , and that if G is unramified, so is G_{-1} .

Lemma 2.32. *If G is a strongly \mathbb{A}^1 -invariant sheaf of groups, so is G_{-1} .*

Proof. One might prove this using our description of those strongly \mathbb{A}^1 -invariant sheaf of groups given in the previous section. We give here another argument. Let BG be the simplicial classifying space of G (see [59] for instance). The assumption that G is strongly \mathbb{A}^1 -invariant means that it is an \mathbb{A}^1 -local space. Choose a fibrant resolution $\mathcal{B}G$ of BG . We use the pointed function space

$$R\mathbf{Hom}_\bullet(\mathbb{G}_m, BG) := \mathbf{Hom}_\bullet(\mathbb{G}_m, \mathcal{B}G)$$

It is fibrant and automatically \mathbb{A}^1 -local, as BG is. Moreover its π_1 sheaf is G_{-1} and its higher homotopy sheaves vanish. Thus the connected component of $R\mathbf{Hom}_\bullet(\mathbb{G}_m, B(G))$ is BG_{-1} . This suffices for our purpose because, the connected component of the base point in an \mathbb{A}^1 -local space is \mathbb{A}^1 -local. This follows formally from the fact (see [59]) that the \mathbb{A}^1 -localization functor takes a 0-connected space to a 0-connected space.

In fact we may also prove directly that the space $R\mathbf{Hom}_\bullet(\mathbb{G}_m, B(G))$ is 0-connected. Its π_0 is the associated sheaf to the presheaf $X \mapsto H_{Nis}^1(X \times \mathbb{G}_m; G)$, and this amounts to checking that for X the henselization of point in a smooth k -scheme, then $H_{Nis}^1(\mathbb{G}_m \times X; G)$. This follows from the fact $H_{Nis}^1(\mathbb{A}^1 \times X; G)$ is trivial and the description of $H^1(-; G)$ in terms of our complex. \square

Remark 2.33. In fact given any pointed smooth k -scheme Z , and any strongly \mathbb{A}^1 -invariant sheaf G we may consider the pointed function object $G^{(Z)}$ which is the sheaf $X \mapsto \text{Ker}(G(Z \times X) \rightarrow G(X))$. The same argument as in the previous proof shows that the connected component of $R\mathbf{Hom}_\bullet(Z, B(G))$ is indeed $B(M^{(Z)})$. Consequently, the sheaf $G^{(Z)}$ is also strongly \mathbb{A}^1 -invariant. \square

Let F be in \mathcal{F}_k and let v be a discrete valuation on F , with valuation ring $\mathcal{O}_v \subset F$. We may choose an irreducible smooth k -scheme X with function field F and a closed irreducible subscheme $i : Y \subset X$ of codimension 1 which induces v on F . In particular the function field of Y is $\kappa(v)$. Assume furthermore that $\kappa(v)$ is separable over k . Then we may also assume up to shrinking X that Y is also smooth over k . Consider the pointed sheaf $X/(X - Y)$ which is called the Thom space of i . By the \mathbb{A}^1 -purity theorem of [59] there is a canonical pointed \mathbb{A}^1 -weak equivalence (in the pointed \mathbb{A}^1 -homotopy category)

$$X/(X - Y) \cong Th(\nu_i)$$

where $Th(\nu_i)$ is the Thom space of the normal bundle ν_i of i , that is to say the pointed sheaf $E(\nu_i)/E(\nu_i)^\times$. Let π be a uniformizing element for v ; one may see the class of π modulo $(\mathcal{M}_v)^2$ as a (non zero) basis element of

$(\nu_i)_y = \mathcal{M}_v/(\mathcal{M}_v)^2$, the fiber of the normal bundle at the generic point y of Y . Consequently, π (or its class in $\mathcal{M}_v/(\mathcal{M}_v)^2$) induces a trivialization of ν_i at least in a Zariski neighborhood of y . In case ν_i is trivialized, it follows from [59] that the pointed sheaf $Th(\nu_i)$ is canonically isomorphic to $T \wedge (Y_+)$, with $T := \mathbb{A}^1/\mathbb{G}_m$.

Lemma 2.34. *Let \mathcal{G} be a strongly \mathbb{A}^1 -invariant sheaf. Let Y be a smooth k -scheme. Then there is a canonical bijection*

$$\mathcal{G}_{-1}(Y) \cong H^1(T \wedge (Y_+); \mathcal{G})$$

which is a group isomorphism if \mathcal{G} is abelian.

Proof. We use the cofibration sequence

$$\mathbb{G}_m \times Y \subset \mathbb{A}^1 \times Y \rightarrow T \wedge (Y_+)$$

to get a long exact sequence in the usual sense

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{A}^1 \times Y; \mathcal{G}) &\rightarrow H^0(\mathbb{G}_m \times Y; \mathcal{G}) \Rightarrow H^1(T \wedge (Y_+); \mathcal{G}) \\ &\rightarrow H^1(\mathbb{A}^1 \times Y; \mathcal{G}) \rightarrow H^1(\mathbb{G}_m \times Y; \mathcal{G}) \rightarrow \dots \end{aligned}$$

The pointed map $H^1(Y; \mathcal{G}) = H^1(\mathbb{A}^1 \times Y; \mathcal{G}) \rightarrow H^1(\mathbb{G}_m \times Y; \mathcal{G})$ being split injective (use the evaluation at 1), we get an exact sequence

$$0 \rightarrow \mathcal{G}(Y) \subset \mathcal{G}(\mathbb{G}_m \times Y) \Rightarrow H^1(T \wedge (Y_+); \mathcal{G}) \rightarrow *$$

As $\mathcal{G}_{-1}(Y)$ is the kernel of $ev_1 : \mathcal{G}(\mathbb{G}_m \times Y) \rightarrow \mathcal{G}(Y)$, this exact sequence implies that the action of $\mathcal{G}_{-1}(Y)$ on the base point $*$ of $H^1(T \wedge (Y_+); \mathcal{G})$ induces the claimed bijection $\mathcal{G}_{-1}(Y) \cong H^1(T \wedge (Y_+); \mathcal{G})$. The statement concerning the abelian case is easy. \square

From what we did before, it follows at once by passing to the filtering colimit over the set of open neighborhoods of y the following:

Corollary 2.35. *Let F be in \mathcal{F}_k and let v be a discrete valuation on F , with valuation ring $\mathcal{O}_v \subset F$. For any strongly \mathbb{A}^1 -invariant sheaf of groups \mathcal{G} , a choice of a non-zero element μ in $\mathcal{M}_v/(\mathcal{M}_v)^2$ (that is to say the class a uniformizing element π of \mathcal{O}_v) induces a canonical bijection*

$$\theta_\mu : \mathcal{G}_{-1}(\kappa(v)) \cong H_v^1(\mathcal{O}_v; \mathcal{G})$$

which is an isomorphism of abelian groups in case \mathcal{G} is a sheaf of abelian groups.

Using the previous bijection, we may define in the situation of the corollary a map

$$\partial_v^\pi : \mathcal{G}(F) \rightarrow \mathcal{G}_{-1}(\kappa(v))$$

as the composition $\mathcal{G}(F) \twoheadrightarrow H_v^1(\mathcal{O}_v; \mathcal{G}) \cong \mathcal{G}_{-1}(\kappa(v))$ which we call *the residue map* associated to π . If \mathcal{G} is abelian, the residue map is a morphism of abelian groups.

2.3 \mathbb{Z} -Graded Strongly \mathbb{A}^1 -Invariant Sheaves of Abelian Groups

In this section we want to give some criteria which imply the Axioms **(A4)** in some particular cases of $\tilde{\mathcal{F}}_k$ -data. Our method is inspired by Rost [68] but avoids the use of transfers. The results of this section will be used in Sect. 3.2 below to construct the sheaves of unramified Milnor-Witt K-theory and unramified Milnor K-theory, etc., without using any transfers as it is usually done. As a consequence, our construction of transfers in Chap. 4 gives indeed a new construction of the transfers on the previous sheaves.

Let M_* be a functor $\mathcal{F}_k \rightarrow \mathcal{A}b_*$ to the category of \mathbb{Z} -graded abelian groups. We assume throughout this section that M_* is endowed with the following extra structures.

(D4) (i) For any $F \in \mathcal{F}_k$ a structure of $\mathbb{Z}[F^\times/(F^{\times 2})]$ -module on $M_*(F)$, which we denote by $(u, \alpha) \mapsto \langle u \rangle \alpha \in M_n(F)$ for $u \in F^\times$ and for $\alpha \in M_n(F)$. This structure should be functorial in the obvious sense in \mathcal{F}_k . \square

(D4) (ii) For any $F \in \mathcal{F}_k$ and any $n \in \mathbb{Z}$, a map $F^\times \times M_{n-1}(F) \rightarrow M_n(F)$, $(u, \alpha) \mapsto [u] \cdot \alpha$, functorial (in the obvious sense) in \mathcal{F}_k . \square

(D4) (iii) For any discrete valuation v on $F \in \mathcal{F}_k$ and uniformizing element π a graded epimorphism of degree -1

$$\partial_v^\pi : M_*(F) \rightarrow M_{*-1}(\kappa(v))$$

which is functorial, in the obvious sense, with respect to extensions $E \rightarrow F$ such that v restricts to a discrete valuation on E , with ramification index 1, if we choose as uniformizing element an element π in E . \square

We assume furthermore that the following axioms hold:

(B0) For $(u, v) \in (F^\times)^2$ and $\alpha \in M_n(F)$, one has

$$[uv]\alpha = [u]\alpha + \langle u \rangle [v]\alpha$$

and moreover $[u][v]\alpha = - \langle -1 \rangle [v][u]\alpha$.

- (B1) For a k -smooth integral domain A with field of fractions F , for any $\alpha \in M_n(F)$, then for all but only finitely many point $x \in \text{Spec}(A)^{(1)}$, one has that for any uniformizing element π for x , $\partial_x^\pi(\alpha) \neq 0$. \square
- (B2) For any discrete valuation v on $F \in \mathcal{F}_k$ with uniformizing element π one has $\partial_v^\pi([u]\alpha) = [\bar{u}]\partial_v^\pi(\alpha) \in M_n(\kappa(v))$ and $\partial_v^\pi(< u > \alpha) = < \bar{u} > \partial_v^\pi(\alpha) \in M_{(n-1)}(\kappa(v))$, for any unit u in $(\mathcal{O}_v)^\times$ and any $\alpha \in M_n(F)$. \square
- (B3) For any field extension $E \subset F \in \mathcal{F}_k$ and for any discrete valuation v on $F \in \mathcal{F}_k$ which restricts to a discrete valuation w on E , with ramification index e , let $\pi \in \mathcal{O}_v$ be a uniformizing element for v and $\rho \in \mathcal{O}_w$ be a uniformizing element for w . Write $\rho = u\pi^e$, with u a unit in \mathcal{O}_v . Then one has for $\alpha \in M_*(E)$, $\partial_v^\pi(\alpha|_F) = e_\epsilon < \bar{u} > (\partial_w^\rho(\alpha))|_{\kappa(v)} \in M_*(\kappa(v))$. \square

Here we set for any integer n ,

$$n_\epsilon = \sum_{i=1}^n < (-1)^{(i-1)} >$$

We observe that as a particular case of (B3) we may choose $E = F$ so that $e = 1$ and we get that for any discrete valuation v on $F \in \mathcal{F}_k$, any uniformizing element π , and any unit $u \in \mathcal{O}_v^\times$, then one has $\partial_v^{u\pi}(\alpha) = < \bar{u} > \partial_v^\pi(\alpha) \in M_{(n-1)}(\kappa(v))$ for any $\alpha \in M_n(F)$.

Thus in case Axiom (B3) holds, the kernel of the surjective homomorphism ∂_v^π only depends on the valuation v , not on any choice of π . In that case we then simply denote by

$$M_*(\mathcal{O}_v) \subset M_*(F)$$

this kernel. Axiom (B1) is then exactly equivalent to Axiom (A2) for unramified $\tilde{\mathcal{F}}_k$ -sets. The following is easy:

Lemma 2.36. *Assume M_* satisfies Axioms (B1), (B2) and (B3). Then it satisfies (in each degree) the axioms for a unramified $\tilde{\mathcal{F}}_k$ -abelian group datum. Moreover, it satisfies Axiom (A5) (i).*

We assume from now on (in this section) that M_* satisfies Axioms (B0), (B1), (B2) and (B3). Thus we may (and will) consider each M_n as a sheaf of abelian groups on $\mathcal{S}m_k$.

We recall that we denote, for any discrete valuation v on $F \in \mathcal{F}_k$, by $H_v^1(\mathcal{O}_v, M_n)$ the quotient group $M_n(F)/M_n(\mathcal{O}_v)$ and by $\partial_v : M_n(F) \rightarrow H_v^1(\mathcal{O}_v, M_n)$ the projection. Of course, if one chooses a uniformizing element π , one gets an isomorphism $\theta_\pi : M_{(n-1)}(\kappa(v)) \cong H_v^1(\mathcal{O}_v, M_n)$ with $\partial_v = \theta_\pi \circ \partial_v^\pi$.

For each discrete valuation v on $F \in \mathcal{F}_k$, and any uniformizing element π set

$$s_v^\pi : M_*(F) \rightarrow M_*(\kappa(v)) , \alpha \mapsto \partial_v^\pi([\pi]\alpha)$$

Lemma 2.37. *Assume M_* satisfies Axioms (B0), (B1), (B2) and (B3). Then for each discrete valuation v the homomorphism $s_v^\pi : M_*(\mathcal{O}_v) \subset M_*(F)$ doesn't depend on the choice of a uniformizing element π .*

Proof. From Axiom (B0) we get for any unit $u \in \mathcal{O}^\times$, any uniformizing element π and any $\alpha \in M_n(F)$: $[u\pi]\alpha = [u]\alpha + \langle u \rangle [\pi]\alpha$. Thus if moreover $\alpha \in M(\mathcal{O}_v)$, one has $s_v^{u\pi}(\alpha) = \partial_v^{u\pi}([u\pi]\alpha) = \partial_v^{u\pi}([u]\alpha) + \partial_v^{u\pi}(\langle u \rangle [\pi]\alpha) = \partial_v^{u\pi}(\langle u \rangle [\pi]\alpha)$, as by Axiom (B2) $\partial_v^{u\pi}([u]\alpha) = [\bar{u}]\partial_v^{u\pi}(\alpha) = [\bar{u}]0 = 0$. But by the same Axiom (B2), $\partial_v^{u\pi}(\langle u \rangle [\pi]\alpha) = \langle \bar{u} \rangle \partial_v^{u\pi}([\pi]\alpha)$, which by Axiom (B3) is equal to $\langle \bar{u} \rangle \partial_v^\pi([\pi]\alpha) = \partial_v^\pi([\pi]\alpha)$. This proves the claim. \square

We will denote by

$$s_v : M_*(\mathcal{O}_v) \rightarrow M_n(\kappa(v))$$

the common value of all the s_v^π 's. In this way M_* is endowed with a datum (D3).

We introduce the following Axiom:

(HA) (i) For any $F \in \mathcal{F}_k$, the following diagram

$$0 \rightarrow M_*(F) \rightarrow M_*(F(T)) \xrightarrow{\sum \partial_{(P)}^P} \bigoplus_{P \in \mathbb{A}_F^1} M_{*-1}(F[T]/P) \rightarrow 0$$

is a short exact sequence. Here P runs over the set of irreducible monic polynomials, and (P) means the associated discrete valuation. \square

(HA) (ii) For any $\alpha \in M(F)$, one has $\partial_{(T)}^T([T]\alpha|_{F(T)}) = \alpha$. \square

This axiom is obviously related to the Axiom (A6), as it immediately implies that for any $F \in \mathcal{F}_k$, $M(F) \rightarrow M(\mathbb{A}_F^1)$ is an isomorphism and $H_{Zar}^1(\mathbb{A}_F^1; M) = 0$.

We next claim:

Lemma 2.38. *Let M_* be as in Lemma 2.37, and suppose it additionally satisfies Axioms (HA) (i) and (HA) (ii). Then Axioms (A1) (ii), (A3) (i) and (A3) (ii) hold.*

Proof. The first part of Axiom (A1) (ii) follows from Axiom (B4). For the second part we choose a uniformizing element π in \mathcal{O}_w , which is still a uniformizing element for \mathcal{O}_v and the square

$$\begin{array}{ccc} M_*(F) & \xrightarrow{\partial_v^\pi} & M_{(*-1)}(\kappa(v)) \\ \uparrow & & \uparrow \\ M_*(E) & \xrightarrow{\partial_w^\pi} & M_{(*-1)}(\kappa(w)) \end{array}$$

is commutative by our definition (D4) (iii). Moreover the morphism $M_*(E) \rightarrow M_*(F)$ preserve the product by π by (D4) (i).

To prove Axiom (A3) we proceed as follows. By assumption we have $E \subset \mathcal{O}_v \subset F$. Choose a uniformizing element π of v . We consider the extension

$E(T) \subset F$ induced by $T \mapsto \pi$. The restriction of v is the valuation defined by T on $E[T]$. The ramification index is 1. Using the previous point, we see that we can reduce to the case $E \subset F$ is $E \subset E(T)$ and $v = (T)$. In that case, the claim follows from our Axioms **(HA)** (i) and **(HA)** (ii). \square

From now on, we assume that M_* satisfies all the Axioms previously met in this subsection. We observe that by construction the Axiom **(A5)** (i) is clear.

Fix a discrete valuation v on $F \in \mathcal{F}_k$. We denote by $v[T]$ the discrete valuation on $F(T)$ defined by the divisor $\mathbb{G}_m|_{\kappa(v)} \subset \mathbb{G}_m|_{\mathcal{O}_v}$ whose open complement is $\mathbb{G}_m|_F$. Choose a uniformizing element π for v . Observe that $\pi \in F(T)$ is still a uniformizing element for $v[T]$.

We want to analyze the following commutative diagram in which the horizontal rows are short exact sequences (given by Axiom **(HA)**):

$$\begin{array}{ccccccc}
 0 \rightarrow & M_*(F) & \rightarrow & M_*(F(T)) & \xrightarrow{\Sigma_P \partial_{(P)}^P} & \oplus_{P \in (\mathbb{A}_F^1)^{(1)}} M_{*-1}(F[T]/P) & \rightarrow 0 \\
 & \downarrow \partial_v^\pi & & \downarrow \partial_{v[T]}^\pi & & \downarrow \Sigma_{P,Q} \partial_Q^{\pi,P} & \\
 0 \rightarrow & M_{*-1}(\kappa(v)) & \rightarrow & M_{*-1}(\kappa(v)(T)) & \xrightarrow{\Sigma_Q \partial_{(Q)}^Q} & \oplus_{Q \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}} M_{*-2}(\kappa(v)[T]/Q) & \rightarrow 0
 \end{array} \tag{2.3}$$

and where the morphisms $\partial_Q^{\pi,P} : M_*(F[T]/P) \rightarrow M_{*-1}(\kappa(v)[T]/Q)$ are defined by the diagram.

For this we need the following Axiom:

(B4) Let v be discrete valuation on $F \in \mathcal{F}_k$ and let π be a uniformizing element. Let $P \in (\mathbb{A}_F^1)^{(1)}$ and $Q \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}$ be fixed.

- (i) If the closed point $Q \in \mathbb{A}_{\kappa(v)}^1 \subset \mathbb{A}_{\mathcal{O}_v}^1$ is not in the divisor $D_P \subset \mathbb{A}_{\mathcal{O}_v}^1$ with generic point $P \in \mathbb{A}_F^1 \subset \mathbb{A}_{\mathcal{O}_v}^1$ then the morphism $\partial_Q^{\pi,P}$ is zero.
- (ii) If Q is in $D_P \subset \mathbb{A}_{\mathcal{O}_v}^1$ and if the local ring $\mathcal{O}_{D_P,Q}$ is a discrete valuation ring with π as uniformizing element then

$$\partial_Q^{\pi,P} = - < -\frac{\overline{P'}}{Q'} > \partial_Q^Q : M_*(F[T]/P) \rightarrow M_{*-1}(\kappa(v)[T]/Q) \quad \square$$

We will set $U = \text{Spec}(\mathcal{O}_v)$ in the sequel. We first observe that $(\mathbb{A}_U^1)^{(1)} = (\mathbb{A}_F^1)^{(1)} \amalg \{v[T]\}$, where as usual $v[T]$ means the generic point of $\mathbb{A}_{\kappa(v)}^1 \subset \mathbb{A}_U^1$. For each $P \in (\mathbb{A}_F^1)^{(1)}$, there is a canonical isomorphism $M_{*-1}(F[T]/P) \cong H_P^1(\mathbb{A}_U^1; M_*)$, as P itself is a uniformizing element for the discrete valuation (P) on $F(T)$. For $v[T]$, there is also a canonical isomorphism $M_{*-1}(\kappa(v)[T]) \cong H_{v[T]}^1(\mathbb{A}_U^1; M_*)$ as π is also a uniformizing element for the discrete valuation $v[T]$ on $F(T)$.

Using the previous isomorphisms, we see that the beginning of the complex $C^*(\mathbb{A}_U^1; M_*)$ (see Sect. 2.2) is isomorphic to

$$0 \rightarrow M_*(\mathbb{A}_U^1) \rightarrow M_*(F(T)) \xrightarrow{\partial_{v[T]}^\pi + \sum_P \partial_{(P)}^P} M_{*-1}(\kappa(v)(T)) \oplus \left(\oplus_{P \in (\mathbb{A}_F^1)^{(1)}} M_{*-1}(F[T]/P) \right)$$

The diagram (2.3) can be used to compute the cokernel of the previous morphism $\partial : M_*(F(T)) \rightarrow M_{*-1}(\kappa(v)(T)) \oplus \left(\oplus_{P \in (\mathbb{A}_F^1)^{(1)}} M_{*-1}(F[T]/P) \right)$. Indeed the epimorphism ∂'

$$M_{*-1}(\kappa(v)(T)) \oplus (\oplus_P M_{*-1}(F[T]/P)) \xrightarrow{\sum_Q \partial_{(Q)}^Q - \sum_{P,Q} \partial_{Q,P}^{\pi,P}} \oplus_{Q \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}} M_{*-2}(\kappa(v)[T]/Q)$$

composed with ∂ is trivial, and the diagram

$$\begin{aligned} M_*(F(T)) &\xrightarrow{\partial} M_{*-1}(\kappa(v)(T)) \oplus (\oplus_P M_{*-1}(F[T]/P)) \\ &\xrightarrow{\partial'} \oplus_Q M_{*-2}(\kappa(v)[T]/Q) \rightarrow 0 \end{aligned} \quad (2.4)$$

is an exact sequence: this is just an obvious reformulation of the properties of (2.3).

Now fix $Q_0 \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}$. Let $(\mathbb{A}_F^1)_0^{(1)}$ be the set of P 's such that Q_0 lies in the divisor D_P of \mathbb{A}_U^1 defined by P .

Lemma 2.39. *Assume M_* satisfies all the previous Axioms (including (B4)). The obvious quotient*

$$\begin{aligned} M_*(F(T)) &\xrightarrow{\partial} M_{*-1}(\kappa(v)(T)) \oplus \left(\oplus_{P \in (\mathbb{A}_F^1)_0^{(1)}} M_{*-1}(F[T]/P) \right) \\ &\xrightarrow{\partial'_Q} M_{*-2}(\kappa(v)[T]/Q_0) \rightarrow 0 \end{aligned}$$

of the previous diagram is also an exact sequence.

Proof. Using the snake Lemma, it is sufficient to prove that the image of the composition $\oplus_{P \notin (\mathbb{A}_U^1)_0^{(1)}} M_{*-1}(F[T]/P) \subset \oplus_{P \in (\mathbb{A}_U^1)^{(1)}} M_{*-1}(F[T]/P) \rightarrow \oplus_{Q \in (\mathbb{A}_{\kappa(v)}^1)^{(1)}} M_{*-2}(\kappa(v)[T]/Q$ is exactly $\oplus_{Q \in (\mathbb{A}_{\kappa(v)}^1)^{(1)} - \{Q_0\}} M_{*-2}(\kappa(v)[T]/Q$. Axiom (B4)(i) readily implies that this image is contained in

$$\oplus_{Q \in (\mathbb{A}_{\kappa(v)}^1)^{(1)} - \{Q_0\}} M_{*-2}(\kappa(v)[T]/Q).$$

Now we want to show that the image entirely reaches each $M_{*-2}(\kappa(v)[T]/Q$, $Q \neq Q_0$. For any such Q , there is a P , irreducible, such that Q is αP , for some unit $\alpha \in \kappa(v)^\times$. Thus Q lies over D_P , but not Q_0 . Moreover, (π, P) is a system of generators of the maximal ideal of the local dimension 2 regular ring $(\mathcal{O}_v[T])_{(Q)}$, thus $(\mathcal{O}_v[T]/P)_{(Q)}$ is a discrete valuation ring with

uniformizing element the image of π . By Axiom **(B4)(ii)** now, we conclude that $\partial_Q^{\pi, P}$ is onto, proving the claim. \square

Now let X be a local smooth k -scheme of dimension 2 with closed point z and function field E . Recall from the beginning of Sect. 2.2 that we denote by $H_z^2(X; M)$ the cokernel of the sum of the residues $M_*(E) \xrightarrow{\sum_{y \in X^{(1)}} \partial_y} \oplus_{y \in X^{(1)}} H_y^1(X; M_*)$. We thus have a canonical exact sequence of the form:

$$\begin{aligned} 0 \rightarrow M_*(X) \rightarrow M_*(E) \xrightarrow{\sum_{y \in X^{(1)}} \partial_y} \oplus_{y \in X^{(1)}} H_y^1(X; M_*) \\ \xrightarrow{\sum_{y \in X^{(1)}} \partial_y^z} H_z^2(X; M_*) \rightarrow 0 \end{aligned} \quad (2.5)$$

where the homomorphisms denoted ∂_z^y are defined by the diagram. This diagram is the complex $C^*((\mathbb{A}_U^1)_0; M_*)$.

For X the localization $(\mathbb{A}_U^1)_0$ of \mathbb{A}_U^1 at some closed point $Q_0 \in \mathbb{A}_{\kappa(v)}^1$, with $U = \text{Spec}(\mathcal{O}_v)$ where v is a discrete valuation on some $F \in \mathcal{F}_k$, we thus get immediately:

Corollary 2.40. *Assume M_* satisfies all the previous Axioms. The complex $C^*((\mathbb{A}_U^1)_0; M_*)$ is canonically isomorphic to exact sequence:*

$$\begin{aligned} 0 \rightarrow M_*((\mathbb{A}_U^1)_Q) \rightarrow M_*(F(T)) \rightarrow M_{*-1}(\kappa(v)(T)) \\ \oplus \left(\oplus_{P \in (\mathbb{A}_F^1)_0^{(1)}} M_{*-1}(F[T]/P) \right) \\ \rightarrow M_{*-2}(\kappa(v)[T]/Q) \rightarrow 0 \end{aligned}$$

This isomorphism provides in particular a canonical isomorphism

$$M_{*-2}(\kappa(v)[T]/Q_0) \cong H_{Q_0}^2(\mathbb{A}_U^1; M_*)$$

Corollary 2.41. *Assume M_* satisfies all the previous Axioms. For each n , the unramified sheaves of abelian groups (on $\tilde{S}m_k$) M_n satisfies Axiom **(A2')**.*

Proof. From Remark 2.19, it suffices to check this when k is infinite.

Now assume X is a smooth k -scheme. Let $y \in X^{(1)}$ be a point of codimension 1. We wish to prove that given $\alpha \in H_y^1(X; M_*)$, there are only finitely many $z \in X^{(2)}$ such that $\partial_z^y(\alpha)$ is non trivial. By Gabber's Lemma, there is an open neighborhood $\Omega \subset X$ of y and an étale morphism $\Omega \rightarrow \mathbb{A}_V^1$, for V some open subset of an affine space over k , such that the morphism $\bar{y} \cap \Omega \rightarrow \mathbb{A}_V^1$ is a closed immersion.

The complement $\bar{y} - \bar{y} \cap \Omega$ is a closed subset everywhere of > 0 -dimension and thus contains only finitely many points of codimension 1 in \bar{y} .

For any $z \in (\bar{y} \cap \Omega)^{(1)}$, the étale morphism $\Omega \rightarrow \mathbb{A}_V^1$ obviously induces a commutative square

$$\begin{array}{ccc}
H_y^1(X; M_*) & \xrightarrow{\partial_y^y} & H_z^2(X; M_*) \\
\uparrow \wr & & \uparrow \wr \\
H_y^1(\mathbb{A}_V^1; M_*) & \xrightarrow{\partial_y^y} & H_z^2(\mathbb{A}_V^1; M_*)
\end{array}$$

(because $\bar{y} \cap \Omega \rightarrow \mathbb{A}_V^1$ is a closed immersion), we reduce to proving the claim for the image of y in \mathbb{A}_V^1 , which follows from our previous results. \square

Now that we know that M_* satisfies Axiom **(A2')**, for X a smooth k -scheme with function field E we may define as in Sect. 2.2 a (whole) complex $C^*(X; M_*)$ of the form

$$\begin{aligned}
0 \rightarrow M_*(X) \rightarrow M_*(E) &\xrightarrow{\Sigma_{y \in X^{(1)}} \partial_y} \oplus_{y \in X^{(1)}} H_y^1(X; M_*) \\
&\xrightarrow{\Sigma_{y,z} \partial_z^y} \oplus_{z \in X^{(2)}} H_z^2(X; M_*)
\end{aligned} \tag{2.6}$$

We thus get as an immediate consequence:

Corollary 2.42. *Assume M_* satisfies all the previous Axioms. For any discrete valuation v on $F \in \mathcal{F}_k$, setting $U = \text{Spec}(\mathcal{O}_v)$, the complex $C^*(\mathbb{A}_U^1; M_*)$ is canonically isomorphic to the exact sequence (2.4):*

$$\begin{aligned}
0 \rightarrow M_*(\mathbb{A}_U^1) \rightarrow M_*(F(T)) \rightarrow M_{*-1}(\kappa(v)(T)) \oplus \left(\oplus_{P \in (\mathbb{A}_F^1)^{(1)}} M_{*-1}(F[T]/P) \right) \\
\rightarrow \oplus_{Q \in (\mathbb{A}_U^1)^{(1)}} M_{*-2}(\kappa(v)[T]/Q) \rightarrow 0
\end{aligned}$$

Consequently, the complex $C^*(\mathbb{A}_U^1; M_*)$ is an exact complex, and in particular, for each n , the unramified sheaves of abelian groups (on $\tilde{S}m_k$) M_n satisfies Axiom **(A6)**.

Proof. Only the statement concerning Axiom **(A6)** is not completely clear: we need to prove that $M_n(U) \rightarrow M_n(\mathbb{A}_U^1)$ is an isomorphism for U a smooth local k -scheme of dimension ≤ 1 . The rest of the Axiom is clear. This claim is clear by Axiom **(HA)** for U of dimension 0. We need to prove it for U of the form $\text{Spec}(\mathcal{O}_v)$ for some discrete valuation v on some $F \in \mathcal{F}_k$ (observe that for the moment M_* only defines an unramified sheaf on $\tilde{S}m_k$, and we can only apply point (1) of Lemma 2.16. But this statement follows rather easily by contemplating the diagram (2.3). \square

We next prepare the statement of our last Axiom. Let X be a local smooth k -scheme of dimension 2, with field of functions F and closed point z . Consider the complex $C^*(X; M_*)$ associated to X in (2.5). By definition we have a short exact sequence:

$$0 \rightarrow M_*(F)/M_*(X) \rightarrow \oplus_{y \in X^{(1)}} H_y^1(X; M_*) \rightarrow H_z^2(X; M_*) \rightarrow 0$$

Let $y_0 \in X^{(1)}$ be such that \bar{y}_0 is smooth over k .

The properties of the induced morphism

$$M_*(F)/M_*(X) \rightarrow \bigoplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*) \quad (2.7)$$

will play a very important role. We first observe:

Lemma 2.43. *Assume M_* satisfies all the previous Axioms (including (B4)). Let X be a local smooth k -scheme of dimension 2, with field of functions F and closed point z , let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k . Then the homomorphism (2.7) is onto.*

Proof. We first observe that this property is true for any localization of a scheme of the form \mathbb{A}_U^1 at a point z of codimension 2, with $U = \text{Spec}(\mathcal{O}_v)$, for some discrete valuation v on F . If $\overline{y_0}$ is $\mathbb{A}_{\kappa(v)}^1$ this is just Axiom (HA). If $\overline{y_0}$ is not $\mathbb{A}_{\kappa(v)}^1$ we observe that the complex $C^*((\mathbb{A}_U^1)_z; M_*)$:

$$M(F(T)) \xrightarrow{\Sigma_{y \in ((\mathbb{A}_U^1)_z)^{(1)} \partial_y}} \bigoplus_{y \in ((\mathbb{A}_U^1)_z)^{(1)}} H_y^1(X; M) \rightarrow H_z^2(\mathbb{A}_{\kappa(v)}^1; M_*) \rightarrow 0$$

is isomorphic to the one of Corollary 2.40. By Axiom (B4)(ii) we deduce that the map $\partial_z^y : H_{y_0}^1(X; M) \rightarrow H_z^2(\mathbb{A}_{\kappa(v)}^1; M_*)$ is surjective. This implies the statement.

To prove the general case we use Gabber's Lemma. Let α be an element in $\bigoplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M)$. Let y_1, \dots, y_r be the points in the support of α . There exists an étale morphism $X \rightarrow \mathbb{A}_U^1$, for some local smooth scheme U of dimension 1, and with function field K , such that $\overline{y_i} \rightarrow \mathbb{A}_U^1$ is a closed immersion for each i . But then use the commutative square

$$\begin{array}{ccc} M_*(F) & \xrightarrow{\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_y} & \bigoplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*) \\ \uparrow & & \uparrow \\ M_*(K(T)) & \xrightarrow{\Sigma_{y \in ((\mathbb{A}_U^1)_z)^{(1)} - \{y_0\}} \partial_y} & \bigoplus_{y \in ((\mathbb{A}_U^1)_z)^{(1)} - \{y_0\}} H_y^1(\mathbb{A}_U^1; M_*) \end{array}$$

We now conclude that $\alpha = \sum_i \alpha_i$, with $\alpha_i \in H_{y_i}^1(X; M_*) \cong H_{y_i}^1(\mathbb{A}_U^1; M_*)$, $i \in \{1, \dots, r\}$ comes from an element from the bottom right corner. The isomorphism $H_{y_i}^1(X; M_*) \cong H_{y_i}^1(\mathbb{A}_U^1; M_*)$ is a consequence of our definition of $H_y^1(-; M_*)$ and (D4)(iii). The bottom horizontal morphism is onto by the first case we treated. Thus α lies in the image of our morphism. \square

Now for our X local smooth k -scheme of dimension 2, with field of functions F and closed point z , with $y_0 \in X^{(1)}$ such that $\overline{y_0}$ is smooth over k , choose a uniformizing element π of y_0 (in \mathcal{O}_{X, y_0}). This produces by definition an isomorphism $M_{*-1}(\kappa(y_0)) \cong H_{y_0}^1(X; M_*)$. Now the kernel of the morphism (2.7) is contained in $M_{*-1}(\kappa(y_0)) \cong H_{y_0}^1(X; M_*)$. We may now state our last Axiom:

(B5) Let X be a local smooth k -scheme of dimension 2, with field of functions F and closed point z , let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k . Choose a uniformizing element π of y_0 (in \mathcal{O}_{X,y_0}). Then the kernel of the morphism (2.7) is (identified to a subgroup of $M_{*-1}(\kappa(y_0))$) equal to $M_{*-1}(\mathcal{O}_{y_0,z}) \subset M_{*-1}(\kappa(y_0))$. \square

Remark 2.44. Thus if M_* satisfies Axiom **(B5)** one gets an exact sequence

$$0 \rightarrow M_{*-1}(\mathcal{O}_{y_0,z}) \rightarrow M_*(F)/M_*(X) \rightarrow \bigoplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*)$$

Lemma 2.43 shows that it is in fact a short exact sequence. \square

Lemma 2.45. *Assume that M_* satisfies all the previous Axioms of this section, including **(B4)**, **(B5)**.*

- 1) *Let X be a local smooth k -scheme of dimension 2, with field of functions F and closed point z , let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k . Choose a uniformizing element π of \mathcal{O}_{X,y_0} . Then the homomorphism $M_{*-1}(\kappa(y_0)) \cong H_{y_0}^1(X; M) \xrightarrow{\partial_z^{y_0}} H_z^2(X; M)$ induces an isomorphism*

$$\Theta_{y_0,\pi} : M_{*-1}(\kappa(y_0))/M_{*-1}(\mathcal{O}_{y_0,z}) = H_z^1(\overline{y_0}; M_{*-1}) \cong H_z^2(X; M)$$

- 2) *Assume $f : X' \rightarrow X$ is an étale morphism between smooth local k -schemes of dimension 2, with closed points respectively z' and z and with the same residue field $\kappa(z) = \kappa(z')$. Then the induced morphism $H_z^2(X; M_*) \rightarrow H_{z'}^2(X'; M_*)$ is an isomorphism. In particular, M_* satisfies Axiom **(A5)** (ii).*

Proof. 1) We know from the previous Remark that the sequence $0 \rightarrow M_{*-1}(\mathcal{O}_{y_0}) \rightarrow M_*(F)/M_*(X) \rightarrow \bigoplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*) \rightarrow 0$ is a short exact sequence. By the definition of $H_z^2(X; M)$ given by the short exact sequence (2.5), this provides a short exact sequence of the form

$$0 \rightarrow M_{*-1}(\mathcal{O}_{y_0,z}) \rightarrow M_{*-1}(\kappa(y_0)) \rightarrow H_z^2(X; M) \rightarrow 0$$

and produces the required isomorphism $\Theta_{y_0,\pi}$.

- 2) Choose $y_0 \in X^{(1)}$ such that $\overline{y_0}$ is smooth over k and a uniformizing element $\pi \in \mathcal{O}_{X,y_0}$. Clearly the pull back of y_0 to X' is still a smooth divisor denoted by y'_0 , and the image of π is a uniformizing element for $\mathcal{O}_{y'_0}$. Then the following diagram commutes

$$\begin{array}{ccc} H_{z'}^1(\overline{y'_0}; M_{*-1}) & \xrightarrow{\Theta_{y'_0,\pi'}} & H_{z'}^2(X'; M) \\ \uparrow & & \uparrow \\ H_z^1(\overline{y_0}; M_{*-1}) & \xrightarrow{\Theta_{y_0,\pi}} & H_z^2(X; M) \end{array}$$

Thus all the morphisms in this diagram are isomorphisms. \square

Theorem 2.46. *Let M_* be a functor $\mathcal{F}_k \rightarrow \mathcal{A}b_*$ endowed with data (D4) (i), (D4) (ii) and (D4) (iii) and satisfying the Axioms (B0), (B1), (B2), (B3), (HA), (B4) and (B5).*

Then for each n , endowed with the s_v 's constructed in Lemma 2.37, M_n is an unramified \mathcal{F}_k -abelian group datum in the sense of Definition 2.9. By Lemma 2.12 it thus defines an unramified sheaf of abelian groups on Sm_k that we still denote by M_n .

Moreover M_n is strongly \mathbb{A}^1 -invariant.

Proof of Theorem 2.46. The previous results (Lemmas 2.36 and 2.38) have already established that M_n is an unramified sheaf of abelian groups on \tilde{Sm}_k , satisfying all the Axioms for unramified sheaves on Sm_k except Axiom (A4) that we establish below.

Axiom (A2') is proven in Corollary 2.41. Axiom (A5)(i) is clear and Axiom (A5)(ii) holds by Lemma 2.45. Axiom (A6) holds by Corollary 2.42. Theorem 2.27 then establishes that each M_n is strongly \mathbb{A}^1 -invariant.

The only remaining point is thus to check Axiom (A4). By Remark 2.17 to prove (A4) in general it is sufficient to treat the case where the residue fields are infinite. We will freely use this remark in the proof below.

We start by checking the first part of Axiom (A4). Let $X = \text{Spec}(A)$ be a local smooth k -scheme of dimension 2 with closed point z and function field F . Let $y_0 \in X^{(1)}$ be such that $\overline{y_0}$ is smooth over k . Choose a pair (π_0, π_1) of generators for the maximal ideal of A , such that π_0 defines y_0 . Clearly $\pi_1 \in \mathcal{O}(\overline{y_0})$ is a uniformizing element for $z \in \mathcal{O}(\overline{y_0})$.

We consider the complex (2.5) of X with coefficients in M_* and the induced commutative square:

$$\begin{array}{ccc} M_*(F) & \xrightarrow{\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_y} & \oplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*) \\ \downarrow \partial_{y_0} & & \downarrow -\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_z^y \\ H_{y_0}^1(X; M_*) & \xrightarrow{\partial_z^{y_0}} & H_z^2(X; M_*) \end{array}$$

We put this square at the top of the commutative square

$$\begin{array}{ccc} H_{y_0}^1(X; M_*) & \xrightarrow{\partial_z^{y_0}} & H_z^2(X; M_*) \\ \downarrow \wr & & \downarrow \wr \\ M_{*-1}(\kappa(y_0)) & \xrightarrow{\partial_z^{\pi_1}} & M_{*-2}(\kappa(z)) \end{array}$$

where $H_{y_0}^1(X; M_*) \xrightarrow{\sim} M_{*-1}(\kappa(y_0))$ is the inverse to the canonical isomorphism θ_{π_0} induced by π_0 , and where $H_z^2(X; M_*) \xrightarrow{\sim} M_{*-2}(\kappa(z))$ is obtained by composing the inverse to the isomorphism θ_{y_0, π_0} obtained by the previous lemma and θ_{π_1} .

Now we add on the left top corner the morphism $M_{*-1}(\mathcal{O}_{X,y_0}) \rightarrow M_*(F)$, $\alpha \mapsto [\pi_0] \alpha$. We thus get a commutative square of the form:

$$\begin{array}{ccc}
 M_{*-1}(\mathcal{O}_{\pi_0}) \xrightarrow{[\pi_0]_-} & M_*(F) & \xrightarrow{\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_y} \oplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*) \\
 \downarrow \partial_{y_0}^{\pi_0} & & \downarrow \\
 M_{*-1}(\kappa(y_0)) & \xrightarrow{\partial_z^{\pi_1}} & M_{*-2}(\kappa(z))
 \end{array} \tag{2.8}$$

As for $y \neq y_0$, π_0 is unit in $\mathcal{O}_{X,y}$ we see that if $\alpha \in \cap_{y \in X^{(1)}} M_*(\mathcal{O}_y)$ the image of α through the composition $M_{*-1}(\mathcal{O}_{y_0}) \xrightarrow{[\pi_0]_-} M_*(F) \xrightarrow{\Sigma_{y \in X^{(1)} - \{y_0\}} \partial_y} \oplus_{y \in X^{(1)} - \{y_0\}} H_y^1(X; M_*)$ is zero. By the commutativity of the above diagram this shows that the image of such an α through $s_{y_0} = \partial_{y_0}^{\pi_0}([y_0]_-)$ lies in the kernel of $\partial_z^{\pi_1}$. But this kernel is $M_{*-1}(\mathcal{O}_{\overline{y_0}, z})$ and this proves the first part of Axiom **(A4)** (for M_{*-1} thus) for M_* .

Now we prove the second part of Axiom **(A4)**. Let $y_1 \in X^{(1)}$ be such that $\overline{y_1}$ is smooth over k and different from $\overline{y_0}$. The intersection $\overline{y_0} \cap \overline{y_1}$ is the point z as a closed subset. If $\overline{y_0}$ and $\overline{y_1}$ do not intersect transversally, we may choose (at least when $\kappa(z)$ is infinite which we may assume by Remark 2.17) a $y_2 \in X^{(1)}$ which intersects transversally both $\overline{y_0}$ and $\overline{y_1}$. Thus we may reduce to the case, that $\overline{y_0}$ and $\overline{y_1}$ do intersect transversally.

Choose $\pi_1 \in A$ which defines $\overline{y_1}$; (π_0, π_1) generate the maximal ideal of A . Now we want to prove that the two morphisms $\cap_{y \in X^{(1)}} M_*(\mathcal{O}_y) \rightarrow M_{*-2}(\kappa(z))$ obtained by using y_0 is the same as the one obtained by using y_1 .

We contemplate the complex (2.5) for X and expand the equation $\partial \circ \partial = 0$ for the elements of the form $[\pi_0][\pi_1]\alpha$ with $\alpha \in \cap_{y \in X^{(1)}} M_*(\mathcal{O}_y)$. From our axioms it follows that if $y \neq y_0$ and $y \neq y_1$ then $\partial_y([\pi_0][\pi_1]\alpha) = 0$. Now $\partial_{y_1}^{\pi_1}([\pi_0][\pi_1]\alpha)$ is $[\overline{\pi_0}]s_{y_1}(\alpha) \in M_{*-1}(\kappa(y_1)) \xrightarrow{\theta_{y_1}} H_{y_1}^1(X; M_*)$ and $\partial_{y_0}^{\pi_0}([\pi_0][\pi_1]\alpha)$ is (using Axiom **(B0)**) $- < -1 > [\overline{\pi_1}]s_{y_0}(\alpha) \in M_{*-1}(\kappa(y_0)) \xrightarrow{\theta_{y_0}} H_{y_0}^1(X; M_*)$. Now we compute the last boundary morphism and find that the sum

$$\Theta_{y_1, \pi_1} \circ \theta_{\pi_0}(s_z^{\overline{\pi_0}} \circ s_{y_1}(\alpha)) + \Theta_{y_0, \pi_0} \circ \theta_{\pi_1}(- < -1 > s_z^{\overline{\pi_1}} \circ s_{y_0}(\alpha)) = 0$$

vanishes in $H_z^2(X; M)$ (as $\partial \circ \partial = 0$). Lemma 2.47 below exactly yields, from this, the required equality $s_z \circ s_{y_1}(\alpha) = s_z \circ s_{y_0}(\alpha)$. \square

Lemma 2.47. *Assume that M_* is as above. Let $X = \text{Spec}(A)$ be a local smooth k -scheme of dimension 2, with field of functions F and closed point z . Let (π_0, π_1) be elements of A generating the maximal ideal of A and let $y_0 \in X^{(1)}$ the divisor of X corresponding to π_0 and $y_1 \in X^{(1)}$ that corresponding to π_1 . Assume both are smooth over k . Then the composed isomorphism*

$$M_{*-2}(\kappa(v)) \xrightarrow{\theta_{\pi_1}^-} H_z^1(\overline{y_0}; M_{*-1}) \xrightarrow{\Theta_{y_0, \pi_0}} H_z^2(X; M)$$

is equal to $< -1 >$ times the isomorphism

$$M_{*-2}(\kappa(v)) \xrightarrow{\theta_{\pi_0}^-} H_z^1(\overline{y_1}; M_{*-1}) \xrightarrow{\Theta_{y_1, \pi_1}} H_z^2(X; M)$$

Proof. We first observe that if $f : X' \rightarrow X$ is an étale morphism, with X' smooth local of dimension two, with closed point z' having the same residue field as z , and if y'_0 and y'_1 denote respectively the pull-back of y_0 and y_1 , then the elements (π_0, π_1) of $A' = \mathcal{O}(X')$ satisfy the same conditions. Clearly, by the previous Lemma, the assertion is true for X if and only if it is true for X' , because the θ_π 's and $\Theta_{y, \pi}$'s are compatible. Now there is a Nisnevich neighborhood of z : $\Omega \rightarrow X$ and an étale morphism $\Omega \rightarrow (\mathbb{A}_{\kappa(z)}^2)_{(0,0)}$ which is also an étale neighborhood and such that (π_0, π_1) corresponds to the coordinates (T_0, T_1) . In this way we reduce to the case $X = (\mathbb{A}_{\kappa(z)}^2)_{(0,0)}$ and $(\pi_0, \pi_1) = (T_0, T_1)$.

Now one reapplies exactly the same computation as in the proof of the Theorem to elements of the form $[T_0][T_1](\alpha|_{F(T_0, T_1)}) \in M_*(F(T_0, T_1))$ with $\alpha \in M_{*-2}(F)$. Now the point is that using our axioms $s_{(0,0)}^{\overline{T_0}} \circ s_{Y_1}(\alpha|_{F(T_0, T_1)}) = s_{(0,0)}^{\overline{T_0}}(\alpha|_{F(T_0)}) = \alpha$ and the same holds for the other term. We thus get from the proof the equality, for each $\alpha \in M_{*-2}(F)$

$$\Theta_{Y_1, T_1} \circ \theta_{T_0}^-(\alpha) = \Theta_{Y_0, T_0} \circ \theta_{T_1}^-(< -1 > \alpha)$$

which proves our claim. \square

Let M_* be as above. For any discrete valuation v on $F \in \mathcal{F}_k$ the image of $(\mathcal{O}_v)^\times \times M_{(*-1)}(\mathcal{O}_v) \rightarrow M_*(F)$, $(u, \alpha) \mapsto [u]\alpha$ lies in $M_*(\mathcal{O}_v)$. This produces for each $n \in \mathbb{Z}$ a morphism of sheaves on Sm_k : $\mathbb{G}_m \times M_{(*-1)} \rightarrow M_*$.

Lemma 2.48. *The previous morphism of sheaves induces for any n , an isomorphism $(M_n)_{-1} \cong M_{(n-1)}$.*

Proof. This follows from the short exact sequence

$$0 \rightarrow M_n(F) = M_n(\mathbb{A}_F^1) \rightarrow M_n(\mathbb{G}_m|_F) \xrightarrow{\partial_{D_0}^T} M_{n-1}(F) \rightarrow 0$$

given by Axiom **(HA)** (i). \square

Remark 2.49. 1) Conversely given a \mathbb{Z} -graded abelian sheaf M_* on Sm_k , consisting of strongly \mathbb{A}^1 -invariant sheaves, together with isomorphisms $(M_n)_{-1} \cong M_{(n-1)}$, then one may show that evaluation on fields yields a functor $\mathcal{F}_k \rightarrow \mathcal{A}b_*$ to \mathbb{Z} -graded abelian groups together with Data **(D4)**

- (i), **(D4) (ii)** and **(D4) (iii)** satisfying Axioms **(B0)**, **(B1)**, **(B2)**, **(B3)**, **(HA)**, **(B4)** and **(B5)**. This is an equivalence of categories.
- 2) We will prove in Chap. 5 that any strongly \mathbb{A}^1 -invariant sheaf is strictly \mathbb{A}^1 -invariant. Thus the previous category is also equivalent to that of homotopy modules over k consisting of \mathbb{Z} -graded strictly \mathbb{A}^1 -invariant abelian sheaves M_* on Sm_k , together with isomorphisms $(M_n)_{-1} \cong M_{(n-1)}$; see also [21]. This category is known to be the heart of the *homotopy t -structure* on the stable \mathbb{A}^1 -homotopy category of \mathbb{P}^1 -spectra over k , see [50–52]. \square

Remark 2.50. Our approach can be used also to analyze Rost cycle modules [68] over a perfect field k . Then Rost’s Axioms imply the existence of a obvious forgetful functor from his category of cycle modules over k to the category of M_* as above in the Theorem, with trivial $\mathbb{Z}[F^\times]$ -module structure, that is to say $\langle u \rangle = 1$ for each $u \in F^\times$. This can be shown to be an equivalence of categories (using for instance [21] or by direct inspection using our construction of transfers in Sect. 4.2). In particular, in the concept of cycle module, one might forget the transfers but should keep track of some consequences like Axioms **(B4)** and **(B5)** to get an equivalent notion. \square



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