Chapter 2
Elliptic Systems

Abstract This is an intermediate chapter which first introduces into the theory of non-linear elliptic systems with quadratic growth in the gradient, and which presents secondly some results concerning curvature estimates and theorems of Bernstein-type for surfaces in Euclidean spaces of arbitrary dimensions.

A famous result of S. Bernstein states that a smooth minimal graph in $\mathbb{R}^3$, defined on the whole plane $\mathbb{R}^2$, must necessarily be a plane. Today we know various strategies to prove this result, and the idea goes back to E. Heinz to establish first a curvature estimate and to deduce Bernstein's result in a second step. However, minimal surfaces with higher codimensions do not share this Bernstein property, as one of our main examples $X(w) = (w, w^2) \in \mathbb{R}^4$ with $w = u + iv$ convincingly shows. It is still a great challenge to find geometrical criteria, preferably in terms of the curvature quantities of the surfaces' normal bundles, which guarantee the validity of Bernstein's theorem.

We must admit that we can only discuss briefly some points where we would wish to employ our tools we develop in this book, but up to now we can not continue to drive further developments.

2.1 The Mean Curvature Vector

2.1.1 Mean Curvature and Mean Curvature Vector

Elliptic systems with quadratic growth in the gradient of the form

$$|\Delta Z| \le a_0 |\nabla Z|^2$$

with the Euclidean Laplace operator Δ and the Euclidean gradient ∇ will play an important role in our analysis. It particularly turns out that the Euler–Lagrange equations for normal Coulomb frames satisfy such non-linear elliptic systems.

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The construction of normal Coulomb frames thus requires a profound knowledge of analytical properties of the underlying geometrical objects. For this reason we devote this intermediate chapter to present some basic facts of conformally parametrized immersions with prescribed mean curvature vector $\mathfrak{H}$ as the standard example of a non-linear elliptic system of the type from above.

Definition 2.1. Let the immersion X together with an ONF $\mathfrak{N}$ be given. Then the *mean curvature* H_{N_σ} of an immersion X *w.r.t. an unit normal vector* $N_\sigma \in \mathfrak{N}$ is defined as

$$H_{N_\sigma} := \frac{1}{2} \sum_{i,j=1}^{2} g^{ij} L_{N_\sigma,ij} = \frac{L_{N_\sigma,11} g_{22} - 2L_{N_\sigma,12} g_{12} + L_{N_\sigma,22} g_{11}}{2W^2} .$$

Consider an ONF $\mathfrak{N} = (N_1, \ldots, N_n)$, and set $H_\sigma := H_{N_\sigma}$ for abbreviation.

Definition 2.2. The *mean curvature vector* $\mathfrak{H} \in \mathbb{R}^n$ of the immersion X is given by

$$\mathfrak{H} := \sum_{\sigma=1}^{n} H_\sigma N_\sigma .$$

For surfaces in $\mathbb{R}^3$ there is, up to orientation, exactly one unit normal vector N and thus exactly one *mean curvature*

$$H = \frac{L_{11} g_{22} - 2L_{12} g_{12} + L_{22} g_{11}}{2W^2} .$$

Nevertheless, sometimes ones speaks of the mean curvature vector $\mathfrak{H} = HN$ even in this case of one codimension.

It misleads to believe that the mean curvature vector $\mathfrak{H}$ could *replace* this special unit normal vector N for surfaces in $\mathbb{R}^3$. This is not the case since, for example, for minimal surfaces it always holds $\mathfrak{H} \equiv 0$ while, of course, N does not vanish.

Definition 2.3. The immersion X is called a *minimal surface* if and only if

$$\mathfrak{H} \equiv 0 \quad \text{in } \overline{B} .$$

The property $\mathfrak{H} \equiv 0$ does neither depend on the choice of the normal frame $\mathfrak{N}$ nor on the choice of the parametrization.

In fact, in general it holds: *The mean curvature vector $\mathfrak{H}$ neither depends on the parametrization (if we only admit regular parameter transformations which do not affect the orientation of the unit normal vectors $N_\sigma \in \mathfrak{N}$) nor on the choice of the ONF (if we only admit transformations of class $SO(n)$ between those frames).*

Minimal surfaces are the topic of a huge amount of literature: Courant [28], Nitsche [92], Osserman [94], Dierkes et al. [34], Colding and Minicozzi [27], Eschenburg and Jost [44] to enumerate only some few significant contributions and to illustrate the importance of this surface class in the fields of geometric analysis.

2.1.2 Parallel Mean Curvature Vector

Surfaces with constant mean curvature vector generalize the minimal surface concept. They actually play a central role in modern geometric analysis of surfaces with one codimension $n = 1$ which are immersed in Riemannian or Lorentzian spaces. We refer the reader e.g. to the classical textbook Kenmotsu [78], or to the extensive works of Große-Brauckmann, Heinz, Hildebrandt, Karcher, Korevaar, Kusner, Lawson, Meeks, Sauvigny, Sullivan, Wente, and many others; see for example [54] and the references therein.

So assume now that the mean curvature vector of the immersion X satisfies

$$|\mathfrak{H}| \equiv \text{const} \quad \text{in } \overline{B},$$

where, of course, $\mathfrak{H} \equiv 0$ is allowed. Differentiation yields

$$\sum_{\sigma=1}^{n} H_\sigma H_{\sigma,u^i} = 0,$$

and therefore

$$\sum_{\sigma=1}^{n} H_\sigma H_{\sigma,u^i} - \sum_{\sigma=1}^{n}\sum_{\vartheta=1}^{n} H_\sigma H_\vartheta T_{\sigma,i}^{\vartheta} = 0$$

since the double sum is zero due to $T_{\sigma,i}^{\vartheta} = -T_{\vartheta,i}^{\sigma}$.

We want to associate this property with the following concept.

Definition 2.4. The mean curvature vector $\mathfrak{H}$ of the immersion X is called *parallel in the normal bundle* if the normal parts of the partial derivatives $\partial_{u^i}\mathfrak{H}$ vanish identically, i.e. if there hold

$$\partial_{u^i}^{\perp}\mathfrak{H} \equiv 0 \quad \text{in } B \quad \text{for } i = 1,2.$$

For reasons of simplicity we want to concentrate on the case $n = 2$ of two codimensions. Then the following interesting result holds true (see e.g. Chen [21], or Kenmotsu and Zhou [79] and the references therein).

Proposition 2.1. *If the mean curvature vector $\mathfrak{H}$ of the immersion $X\colon\overline{B} \to \mathbb{R}^4$ is parallel in the normal bundle then it has constant length. If additionally $\mathfrak{H} \neq 0$, then it holds $S \equiv 0$ for the scalar curvature of the normal bundle.*

Proof. The identities

$$\partial_{u^i}^{\perp}\mathfrak{H} = \sum_{\sigma=1}^{n} H_{\sigma,u^i} N_\sigma + \sum_{\vartheta=1}^{n} H_\vartheta N_{\vartheta,u^i}^{\perp} = \sum_{\sigma=1}^{2} H_{\sigma,u^i} N_\sigma + \sum_{\vartheta=1}^{2}\sum_{\sigma=1}^{2} H_\vartheta T_{\vartheta,i}^{\sigma} N_\sigma = 0$$

for $i = 1, 2$ can be written in the form

$$H_{1,u} = H_2 T^2_{1,1}\,, \quad H_{1,v} = H_2 T^2_{1,2}\,, \quad H_{2,u} = -H_1 T^2_{1,1}\,, \quad H_{2,v} = -H_1 T^2_{1,2}\,.$$

Thus, we compute

$$\begin{aligned}
\frac{1}{2}\,\partial_u |\mathfrak{H}|^2 &= H_1 H_{1,u} + H_2 H_{2,u} \;=\; H_1 H_2 T^2_{1,1} - H_1 H_2 T^2_{1,1} \;=\; 0, \\
\frac{1}{2}\,\partial_v |\mathfrak{H}|^2 &= H_1 H_{1,v} + H_2 H_{2,v} \;=\; H_1 H_2 T^2_{1,2} - H_1 H_2 T^2_{1,2} \;=\; 0
\end{aligned}$$

and infer $|\mathfrak{H}|^2 \equiv \text{const}$. Moreover, it holds

$$\begin{aligned}
0 = \partial_{uv} H_1 - \partial_{vu} H_1 \;&=\; H_{2,u} T^2_{1,2} + H_2 \partial_u T^2_{1,2} - H_{2,v} T^2_{1,1} - H_2 \partial_v T^2_{1,1} \\
&= -H_1 T^2_{1,1} T^2_{1,2} + H_1 T^2_{1,1} T^2_{1,2} + H_2 \partial_u T^2_{1,2} - H_2 \partial_v T^2_{1,1} \\
&= -\,H_2 (\partial_v T^2_{1,1} - \partial_u T^2_{1,2}) \\
&= -\,H_2 S W
\end{aligned}$$

and analogously $0 = -H_1 S W$. Therefore, either X is a minimal immersion with $\mathfrak{H} \equiv 0$, or if not then it is a surface with mean curvature vector of constant length greater than zero and with flat normal bundle. The statement is proved. □

2.1.3 The Mean Curvature System

From the Gauß equations in connection with the conformal representation of the Christoffel symbols from Sect. 1.4.2 we now derive an elliptic system for conformally parametrized immersions X with prescribed mean curvature vector $\mathfrak{H}$.

Proposition 2.2. *Let the conformally parametrized immersion X of prescribed mean curvature vector $\mathfrak{H}$ together with an ONF $\mathfrak{N}$ be given. Then it holds*

$$\Delta X = 2 \sum_{\vartheta=1}^{n} H_\vartheta W N_\vartheta = 2 \mathfrak{H} W \quad \textit{in } B.$$

Proof. From the Gauß equations we infer

$$\begin{aligned}
\Delta X &= (\Gamma^1_{11} + \Gamma^1_{22}) X_u + (\Gamma^2_{11} + \Gamma^2_{22}) X_v + \sum_{\vartheta=1}^{n} (L_{\vartheta,11} + L_{\vartheta,22}) N_\vartheta \\
&= \sum_{\vartheta=1}^{n} (L_{\vartheta,11} + L_{\vartheta,22}) N_\vartheta\,.
\end{aligned}$$

Here we take into account that

$$\Gamma_{11}^1 + \Gamma_{22}^1 = \frac{W_u}{2W} - \frac{W_u}{2W} = 0,$$
$$\Gamma_{11}^2 + \Gamma_{22}^2 = -\frac{W_v}{2W} + \frac{W_v}{2W} = 0$$

as well as

$$L_{\vartheta,11} + L_{\vartheta,22} = 2H_\vartheta W$$

from the definition of H_ϑ. The statement follows. □

This system generalizes the *classical mean curvature system*

$$\triangle X = 2HWN \quad \text{in } B$$

from Hopf [72] in case $n = 1$ of one codimension with the scalar mean curvature $H \in \mathbb{R}$ and the unit normal vector N of the surface X.

In particular, we infer that conformally parametrized minimal surfaces represent harmonic vectors, i.e. it then holds

$$\Delta X = 0 \quad \text{in } B$$

which offers the possibility to apply the powerful tools of complex analysis to the differential geometry of minimal surfaces. We will discuss this fact later.

2.1.4 *Quadratic Growth in the Gradient: A Maximum Principle*

Now we want to give a geometric application of the classical maximum principle for subharmonic functions. Namely, assume there is an upper bound $|\mathfrak{H}| \le h_0$ in $\overline{B}$ be given such that for the conformally parametrized immersion X it holds

$$|\triangle X| \le 2h_0 W \le h_0|\nabla X|^2 \quad \text{in } B$$

on account of

$$\begin{aligned} W &= \sqrt{(X_u \cdot X_u)(X_v \cdot X_v) - (X_u \cdot X_v)^2} = \sqrt{(X_u \cdot X_u)^2} \\ &= |X_u||X_u| \le \frac{1}{2}\left(X_u^2 + X_u^2\right) = \frac{1}{2}\left(X_u^2 + X_v^2\right) = \frac{1}{2}|\nabla X|^2 . \end{aligned}$$

Thus, the surface vector X is *solution of a non-linear elliptic system with quadratic growth in the gradient.*

Proposition 2.3. *Let $X\colon \overline{B} \to \mathbb{R}^{n+2}$ be a conformally parametrized immersion with prescribed mean curvature vector $\mathfrak{H}$. Let $|\mathfrak{H}| \le h_0$ in $\overline{B}$, and suppose that*

$$h_0 \sup_{(u,v)\in B} |X(u,v)| \le 1.$$

Then it holds the geometric maximum principle

$$\max_{(u,v)\in\overline{B}} |X(u,v)|^2 = \max_{(u,v)\in\partial B} |X(u,v)|^2 .$$

Proof. We remark that the statement is obviously true without the assumption on the conformal parametrization since introducing a conformal parameter system $(u,v) \in \overline{B}$ does not affect the maximum norm of the representation $X(u,v)$. Nevertheless, using conformal parameters we compute

$$\begin{aligned}\Delta|X|^2 &= 2\big(|\nabla X|^2 + X \cdot \triangle X\big) \ \ge\ 2\big(|\nabla X|^2 - h_0|X||\nabla X|^2\big)\\ &= 2|\nabla X|^2(1-h_0|X|) \ \ge\ 0.\end{aligned}$$

Therefore, the vector $|X(u,v)|^2$ is subharmonic, and the statement follows from the classical maximum principle. □

Surfaces X with the property

$$h_0 \sup_{(u,v)\in B} |X(u,v)| \le 1$$

are also called *small solutions* of the mean curvature system in contrast to *large solutions* which do not necessarily obey the maximum principle. We will encounter this fact later again. Minimal surfaces are always small in this sense.

The method of proof we presented here goes already back to Heinz (see also Sauvigny [107], vol. 2, Chap. XII). For further considerations we refer e.g. to Dierkes [33] and the references therein.

2.2 Curvature Estimates

2.2.1 Problem Statement

With this intermediate chapter we also want to draw the reader's attention to the problem of curvature estimates and Bernstein-type theorems for minimal surfaces in higher-dimensional Euclidean spaces. In particular, we have in mind to confront some of the methods and results from this field of geometric analysis with the concepts of extrinsic differential geometry which we developed in the first chapter.

This plan must be left incomplete due to its complexity. We will therefore concentrate on some "light" versions of curvature estimates and their immediate consequences, and we will only discuss briefly more profound approaches and methods.

2.2.2 Estimate of the $S^{\vartheta}_{\sigma,12}$

Our first observation is based upon the representation formula

$$S^{\omega}_{\sigma,12} = \frac{1}{W}\,(L_{\sigma,11} - L_{\sigma,22})L_{\omega,12} - \frac{1}{W}\,(L_{\omega,11} - L_{\omega,22})L_{\sigma,12}$$

of the normal curvature tensor from Sect. 1.6.2. Applying the Cauchy–Schwarz inequality gives us

$$|S^{\omega}_{\sigma,12}| \le \frac{1}{2W}\,(L^2_{\sigma,11} + 2L^2_{\sigma,12} + L^2_{\sigma,22}) + \frac{1}{2W}\,(L^2_{\omega,11} + 2L^2_{\omega,12} + L^2_{\omega,22}).$$

On the other hand we verify

$$\begin{aligned} 2H^2_\sigma - K_\sigma &= \frac{L^2_{\sigma,11} + 2L_{\sigma,11}L_{\sigma,22} + L^2_{\sigma,22}}{2W^2} - \frac{L_{\sigma,11}L_{\sigma,22} - L^2_{\sigma,12}}{W^2} \\ &= \frac{L^2_{\sigma,11} + 2L^2_{\sigma,12} + L^2_{\sigma,22}}{2W^2} \end{aligned}$$

so that we arrive at the

Proposition 2.4. *Let the immersion X together with an ONF $\mathfrak{N}$ be given. Then the components $S^{\omega}_{\sigma,12}$ of the curvature vector of its normal bundle can be estimated as follows*

$$|S^{\omega}_{\sigma,12}| \le (2H^2_\sigma - K_\sigma)W + (2H^2_\omega - K_\omega)W \quad \textit{for all } \sigma, \omega = 1, \ldots, n.$$

In particular, immersions with the property

$$2H^2_\sigma - K_\sigma \equiv 0 \quad \text{for all } \sigma = 1, \ldots, n$$

have flat normal bundle: $S^{\omega}_{\sigma,12} = 0$. But, in general, bounds for $|S^{\omega}_{\sigma,12}|$ can only be achieved by establishing bounds for the curvatures and the area element W.

The special case of two codimensions $n = 2$ leads us to

$$|S|W \le (2H^2_1 - K_1)W + (2H^2_2 - K_2)W = 2|\mathfrak{H}|^2W - KW$$

due to $S = \frac{1}{W} S_{1,12}^2$. Integration then yields the estimate

$$2 \iint_B |\mathfrak{H}|^2 W\, du dv \geq \iint_B |S| W\, du dv + \iint_B K W\, du dv$$

which we will employ at the end of Sect. 2.2.11. Guadalupe and Rodriguez in [55] derive this integral inequality in case of compact surfaces without boundary.

The *Willmore functional*

$$\iint_B |\mathfrak{H}|^2 W\, du dv$$

on the left hand side enjoys a special attention of the geometric analysis due to its complexity of its non-linear, fourth-order Euler–Lagrange equations, but also due to its wide range of applications in mathematical biology, chemistry, or physics, see e.g. the pioneering work of Helfrich [65] who discusses the significant role of higher-order geometric functionals in $\mathbb{R}^3$ of the general form

$$\iint_B \left\{ \alpha + \beta (H - H_0)^2 + \gamma K \right\} W\, du dv$$

in the theory of so-called elastic bilayers, α, β, γ and H_0 being material constants.

We want to refer the reader to the classical monograph [125] for Willmore's own introduction into the fascinating problem of determining immersions which are critical or even minimal for this functional named after him.

In e.g. Palmer [95] and the recent work Dall'Acqua [31] we find uniqueness results for the Willmore problem for special boundary data. Dall'Acqua et al. [32] prove existence and classical regularity of Willmore surfaces of catenoid-type which were observed phenomenologically e.g. by Fröhlich and Große-Brauckmann using Ken Brakke's surface evolver, see [47]. Concerning the general boundary value problem we want to refer to Schätzle's paper [108].

Moreover, Rivière [98, 99] extends techniques and results e.g. from Helein [64] to derive a non-linear differential equation in a divergence-type form for critical points of the Willmore functional—the basis for further existence and regularity investigations.

Some of Helein's results, on the other hand, will play an important role in our considerations in the fourth chapter.

Let us finally remark that the integral over the Gaussian curvature on the right hand side of the above inequality can be expressed by the Gauß–Bonnet formula in terms of the geodesic curvature κ_g of the immersion X along the boundary curve ∂B,

$$\iint_B K W\, du dv = \int_{\partial B} \kappa_g\, ds - 2\pi,$$

see e.g. Blaschke and Leichtweiß [12] for more details on this famous identity connecting analysis, topology and differential geometry.

And the conformally invariant functional

$$\iint_B |S|W\,du\,dv$$

measures the *total normal curvature* of the surface. In Sakamoto [101] we find the probably first investigations on critical points of this functional, and this should open new fields in classical differential geometry.

2.2.3 The Special Case of Holomorphic Minimal Graphs

We want to specify the foregoing estimate

$$|S|W \le 2|\mathfrak{H}|^2 W - KW$$

in case of holomorphic minimal graphs.

Proposition 2.5. *Let the minimal graph $X(w) = (x, \Phi(w))$ on $\overline{B}$ with a holomorphic function*

$$\Phi(w) = \varphi(w) + i\psi(w)$$

be given. Then it holds

$$S(w) = -K(w) \quad \textit{for all } w \in \overline{B}\,.$$

Proof. Making use of the special ONF (see Sect. 1.2.2)

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,(-\varphi_u, -\varphi_v, 1, 0), \\ N_2 &= \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,(\varphi_v, -\varphi_u, 0, 1) \end{aligned}$$

we will compute the Gaussian curvature K and the normal curvature scalar S. Since X is minimal we already know $|S| \le (-K)$, and we will verify $S = -K$.

For this purpose, we first note

$$L_{1,11} = \frac{\varphi_{uu}}{\sqrt{1+|\nabla\varphi|^2}}\,, \quad L_{1,12} = \frac{\varphi_{uv}}{\sqrt{1+|\nabla\varphi|^2}}\,, \quad L_{1,22} = \frac{\varphi_{vv}}{\sqrt{1+|\nabla\varphi|^2}}$$

as well as

$$L_{2,11} = -\frac{\varphi_{uv}}{\sqrt{1+|\nabla\varphi|^2}}, \quad L_{2,12} = \frac{\varphi_{uu}}{\sqrt{1+|\nabla\varphi|^2}} = -\frac{\varphi_{vv}}{\sqrt{1+|\nabla\varphi|^2}},$$
$$L_{2,22} = \frac{\varphi_{uv}}{\sqrt{1+|\nabla\varphi|^2}}$$

what leads us to (recall $W = 1 + |\nabla\varphi|^2$)

$$K_1 = \frac{L_{1,11}L_{1,22} - L_{1,12}^2}{W^2} = \frac{\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2}{(1+|\nabla\varphi|^2)^3},$$
$$K_2 = \frac{L_{2,11}L_{2,22} - L_{2,12}^2}{W^2} = \frac{-\varphi_{uv}^2 + \varphi_{uu}\varphi_{vv}}{(1+|\nabla\varphi|^2)^3}.$$

Thus, the Gaussian curvature of the holomorphic graph turns out to be

$$K = 2\,\frac{\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2}{(1+|\nabla\varphi|^2)^3}.$$

Now let us come to the calculation of S : We have

$$\begin{aligned} T_{1,1}^2 &= \big[\partial_u(1+|\nabla\varphi|^2)^{-\frac{1}{2}}\big](-\varphi_u, -\varphi_v, 1, 0)\cdot N_2 \\ &\quad + \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\,(-\varphi_{uu}, -\varphi_{uv}, 0, 0)\cdot N_2 \\ &= \frac{1}{1+|\nabla\varphi|^2}\,(-\varphi_{uu}, -\varphi_{uv}, 0, 0)\cdot(\varphi_v, -\varphi_u, 0, 1) \\ &= \frac{\varphi_u\varphi_{uv} - \varphi_v\varphi_{uu}}{1+|\nabla\varphi|^2}, \end{aligned}$$

and analogously

$$T_{1,2}^2 = \frac{\varphi_u\varphi_{vv} - \varphi_v\varphi_{uv}}{1+|\nabla\varphi|^2}.$$

Compute now the derivatives

$$\partial_u T_{1,2}^2 = \frac{\varphi_{uu}\varphi_{vv} + \varphi_u\varphi_{uvv} - \varphi_{uv}^2 - \varphi_v\varphi_{uuv}}{1+|\nabla\varphi|^2} - \frac{2(\varphi_u\varphi_{vv} - \varphi_v\varphi_{uv})(\varphi_u\varphi_{uu} + \varphi_v\varphi_{uv})}{(1+|\nabla\varphi|^2)^2},$$

$$\partial_v T_{1,1}^2 = \frac{\varphi_{uv}^2 + \varphi_u\varphi_{uvv} - \varphi_{vv}\varphi_{uu} - \varphi_v\varphi_{uuv}}{1+|\nabla\varphi|^2} - \frac{2(\varphi_u\varphi_{uv} - \varphi_v\varphi_{uu})(\varphi_u\varphi_{uv} + \varphi_v\varphi_{vv})}{(1+|\nabla\varphi|^2)^2}.$$

A final calculation of

$$S = \frac{1}{W}\,(\partial_v T^2_{1,1} - \partial_u T^2_{1,2})$$

would then show the stated identity.

□

2.2.4 Minimal Surfaces in $\mathbb{R}^3$

Bernstein in 1914 proved the following result (see [10] and Hopf [70, 71]).

Proposition 2.6. *A minimal graph $X(x,y) = (x,y,\zeta(x,y))$ satisfying the minimal surface equation*

$$(1+\zeta_y^2)\zeta_{xx} - 2\zeta_x\zeta_y\zeta_{xy} + (1+\zeta_x^2)\zeta_{yy} = 0,$$

defined on the whole plane $\mathbb{R}^2$ and with continuous partial derivatives of first and second order, is necessarily a plane.

This result characterizes insistently the non-linear character of the minimal surface equation in contrast to its linearization

$$\Delta\zeta = \zeta_{xx} + \zeta_{yy} = 0,$$

the Laplace equation, which actually possesses non-flat solutions over $\mathbb{R}^2$.

Bernstein's proof relies essentially on his

Lemma 2.1. *Let $\zeta = \zeta(x,y)$ be bounded and twice continuously differentiable, and suppose it solves*

$$A\zeta_{xx} + 2B\zeta_{xy} + C\zeta_{yy} = 0$$

with coefficients A, B and C which depend on $(x,y,\zeta,\zeta_x,\zeta_y,\zeta_{xx},\zeta_{xy},\zeta_{yy})$ and fulfill $AC - B^2 > 0$. Then it necessarily holds $\zeta \equiv const.$

Bernstein verifies that $u = \arctan\zeta_x$ is a solution of such a differential equation, and the boundedness of u implies his proposition.

While Bernstein's method was topological in its nature, Heinz [61] in 1952 gave a completely new proof of Bernstein's principle for minimal graphs by establishing a curvature estimate first, what requires deep analytical estimates of the derivatives of the conformally parametrized minimal surface vector from above and an estimate for its area element from below.

For a comprehensive presentation of the theory of plane harmonic mappings together with this estimate of the area element we also want to refer to Duren [40]. For complete treatments of the theory of non-linear elliptic systems of second order with quadratic growth in the gradient we refer the reader to Heinz [62], Sauvigny [107], vol. 2, or Schulz [111].

The point we want to stress is that Bernstein's principle fails for minimal graphs with higher codimensions, for (w, w^2) is obviously a counter-example. One should find geometric conditions which make this principle hold again.

2.2.5 *How a Curvature Estimate Could Work*

Let the minimal graph on the closed disc $\overline{B}_R$ of radius $R > 0$ together with an ONF $\mathfrak{N}$ be given. We introduce conformal parameters and obtain a harmonic vector-valued mapping $X\colon \overline{B} \to \mathbb{R}^{n+2}$.

The Gaussian curvature $K_\sigma(0,0)$ in the origin $(0,0) \in B$ w.r.t. an arbitrary $N_\sigma \in \mathfrak{N}$ can be estimated by

$$-K_\sigma(0,0) \le \frac{|L_{\sigma,11}(0,0)||L_{\sigma,22}(0,0)| + |L_{\sigma,12}(0,0)|^2}{W(0,0)^2}$$

where in the enumerator

$$|L_{\sigma,ij}(0,0)| \le |N_\sigma(0,0)||X_{u^i u^j}(0,0)| \le |X_{u^i u^j}(0,0)|$$

or

$$|L_{\sigma,ij}(0,0)| \le |N_{\sigma,u^i}(0,0)||X_{u^j}(0,0)|.$$

Thus, the problem we are faced with is to find (a) upper bounds for the second derivatives of X, or for its first derivatives and the first derivatives of N_σ in the origin, and (b) to establish a lower bound for the area element $W(0,0)$.

2.2.6 *Estimate of the Area Element from Below: The Heinz Lemma*

Let the minimal graph $X(x, y) = (x, y, \zeta_1(x, y), \ldots, \zeta_n(x, y))$ on the closed disc $\overline{B}_R$ of radius $R > 0$ be given. Introduce conformal parameters $(u, v) \in \overline{B}$ such that it holds

$$\Delta X(u, v) = 0 \quad \text{in } B.$$

We now consider the *harmonic plane mapping*

$$f(u, v) := \big(x^1(u, v), x^2(u, v)\big), \quad (u, v) \in \overline{B}.$$

Since $X(x, y)$ is a graph, *this mapping represents the reparametrization of the graph into the new form* $X(u, v)$, and therefore the *scaled plane mapping*

$$F\colon \overline{B} \longrightarrow \overline{B} \quad \text{via } F(u, v) := \frac{1}{R}\, f(u, v)$$

can be chosen with the properties (see e.g. Sauvigny [107], vol. 2):

- F is one-to-one and satisfies $F(0,0) = (0,0)$.
- F maps the boundary ∂B positively oriented and topologically onto ∂B.
- $J_F(u,v) > 0$ in B for the Jacobian of F.

The Heinz lemma on harmonic plane mappings with all these properties states now the following universal estimate.

Proposition 2.7. *With the Heinz constant* $C_H = \frac{27}{4\pi^2} \approx 0.6839\ldots$ *it always holds*

$$|F_w(0,0)|^2 + |F_{\overline{w}}(0,0)|^2 \geq C_H\,.$$

This estimate immediately implies a lower bound for the area element, namely

$$\begin{aligned} W(0,0) &= \frac{1}{2}\,|\nabla X(0,0)|^2 \;\geq\; \frac{1}{2}\,|\nabla x^1(0,0)|^2 + \frac{1}{2}\,|\nabla x^2(0,0)|^2 \\ &= R^2|F_w(0,0)|^2 + R^2|F_{\overline{w}}(0,0)|^2 \;\geq\; R^2 C_H\,. \end{aligned}$$

Actually, Heinz first proved

$$|F_w(0,0)|^2 + |F_{\overline{w}}(0,0)|^2 \geq 1 - \frac{2\pi}{3} + \frac{4}{\pi} \approx 0.1788\ldots$$

while the sharp form given in the proposition above goes back to Hall [57], see e.g. Duren's monograph [40] for more details.

Thus, for a complete curvature estimate it remains to estimate the derivatives of X and/or the derivatives of the unit normal vectors N_σ.

2.2.7 Minimal Surfaces with Controlled Growth

Let the minimal graph be conformally parametrized via $X\colon \overline{B} \to \mathbb{R}^{n+2}$. We have

$$|L_{\sigma,11}||L_{\sigma,22}| + |L_{\sigma,12}|^2 \leq |X_{uu}||X_{vv}| + |X_{uv}|^2\,.$$

Due to $\Delta X = 0$, potential theory yields a universal constant $C_1 \in (0,\infty)$ such that

$$|X_{u^i u^j}(0,0)| \leq C_1 \sup_{(u,v)\in B} |X(u,v)| = C_1 \sup_{(x,y)\in B_R} |X(x,y)|,$$

see e.g. Gilbarg and Trudinger [53], Theorem 4.6. Now we arrive at the following curvature estimate and theorem of Bernstein-type from Fröhlich [48].

Theorem 2.1. *Let there exist a constant* $\Omega \in (0,\infty)$ *such that the minimal graph* $X\colon \overline{B}_R \to \mathbb{R}^{n+2}$ *satisfies the following growth condition*

$$|X(x,y)| \leq \Omega R^\varepsilon$$

with some $\varepsilon \in [0,2)$. Then it holds the curvature estimate

$$|K_\sigma(0,0)| \le \frac{2C_1^2\Omega^2}{C_H^2} \cdot \frac{R^{2\varepsilon}}{R^4}\,.$$

Thus, if the minimal graph is defined over the whole $\mathbb{R}^2$ then it is a plane.

The last statement in this theorem follows after performing the limit $R \to \infty$.

This result is sharp in the following sense: $X(w) = (w, w^2)$, defined on the whole plane $\mathbb{R}^2$, has quadratic growth, i.e. $\varepsilon = 2$ in the terminology of our theorem, and it is obviously not a plane!

We also want to mention that our theorem generalizes the classical Liouville theorem from complex analysis.

It arises the question whether the critical growth $\varepsilon = 2$ has something to do with the non-vanishing of the scalar curvature of the normal bundle. This question must be left open.

2.2.8 The First and Second Variation of the Area Functional

Now we want to draw the reader's attention to curvature estimates for *stable minimal surfaces.* For this purpose we first consider immersions $X\colon \overline{B} \to \mathbb{R}^{n+2}$ which (a) are critical for the area functional

$$\mathscr{A}[X] := \iint\limits_B W\,du\,dv \quad \text{with } W = \sqrt{g_{11}g_{22} - g_{12}^2}\,,$$

and (b) for which its second variation is always positive.

For the next two results we especially refer to Sauvigny [102].

Proposition 2.8. *The immersion $X\colon \overline{B} \to \mathbb{R}^{n+2}$ is critical for $\mathscr{A}[X]$ if its mean curvature vector vanishes identically, i.e. if*

$$\mathfrak{H} \equiv 0 \quad \textit{in } B.$$

In other words, minimal surfaces are stationary for the area functional.

Proposition 2.9. *The second variation of $\mathscr{A}[X]$ w.r.t. an unit normal vector $N_\sigma \in \mathfrak{N}$ for a conformally parametrized minimal immersion $X\colon \overline{B} \to \mathbb{R}^{n+2}$ reads*

$$\begin{aligned}\delta_\sigma^2\mathscr{A}[X] &= \iint\limits_B \left(|\nabla\varphi|^2 + 2K_\sigma W\varphi^2\right) du\,dv\\ &\quad + \sum_{\vartheta=1}^n \iint\limits_B \left\{(T_{\sigma,1}^\vartheta)^2 + (T_{\sigma,2}^\vartheta)^2\right\}\varphi^2\,du\,dv\end{aligned}$$

for arbitrary $\varphi \in C_0^\infty(B,\mathbb{R})$.

In case $n = 1$ of one codimension there is only one unit normal vector w.r.t. which we can evaluate the second variation. Thus, we would then arrive at

$$\delta^2\mathscr{A}[X] = \iint\limits_B \left(|\nabla\varphi|^2 + 2KW\varphi^2\right) du dv$$

since the integral over the squared torsion coefficients drops out.

2.2.9 Stable Minimal Surfaces

The second variation leads us directly to the

Definition 2.5. The minimal surface X is called *stable* if it holds

$$\delta_N^2\mathscr{A}[X] \ge 0 \quad \text{for all } \varphi \in C_0^\infty(B,\mathbb{R})$$

and all unit normal vectors N.

For fixed ONF $\mathfrak{N}$ and fixed test function $\varphi \in C_0^\infty(B,\mathbb{R})$ we could sum up all the n stability inequalities $\delta_\sigma^2\mathscr{A}[X] \ge 0$ for $\sigma = 1,\ldots,n$ to get

$$\begin{aligned}\iint\limits_B |\nabla\varphi|^2 du dv \ge \frac{2}{n}\iint\limits_B (-K)W\varphi^2 du dv \\ - \frac{1}{n}\sum_{\sigma,\omega=1}^{n}\iint\limits_B \left\{(T_{\sigma,1}^{\omega})^2 + (T_{\sigma,2}^{\omega})^2\right\}\varphi^2 du dv,\end{aligned}$$

again for all test functions φ. Note that $\mathfrak{H} \equiv 0$ and $K \le 0$ for the minimal surface.

It must be remarked that *the right hand side of these inequalities depends on the choice of the ONF $\mathfrak{N}$ while the left hand side does not.* Thus, it arises the question whether there exists an ONF $\mathfrak{N}$ with controlled torsion coefficients such that the difference at the right hand side stays positive for all φ.

In the next two chapters we will construct special Coulomb-gauged ONF's for which we can in fact control the torsion by means of the curvature of the normal bundle (and certain smallness conditions in case $n > 2$).

In particular, we will show that *if the normal bundle is flat then there exist an ONF $\mathfrak{N}$ which is free of torsion,* and then the minimal surface is stable if

$$\iint\limits_B |\nabla\varphi|^2 du dv \ge 2\iint\limits_B (-K_N)W\varphi^2 du dv$$

for all test functions $\varphi \in C_0^\infty(B,\mathbb{R})$ and all unit normal vectors N. It will turn out that the curvature of the normal bundle acts as a barrier for the existence of orthogonal unit normal frames with vanishing torsion coefficients. So if we set

$$T := \sum_{1\le\sigma<\omega\le n} \left\{(T_{\sigma,1}^{\omega})^2 + (T_{\sigma,2}^{\omega})^2\right\}$$

in the general case $T > 0$, when we can not expect existence of torsion-free ONF's, we obtain from the above stability inequality after partial integration

$$0\le \iint\limits_B \left\{|\nabla\varphi|^2 + \frac{2}{n}(K+T)W\varphi^2\right\} du\,dv = \iint\limits_B \left\{-\Delta\varphi+\frac{2}{n}(K+T)W\varphi\right\} \varphi\, du\,dv.$$

The Schwarzian eigenvalue problem which arises from here,

$$-\Delta\varphi + \lambda(K+T)\varphi = 0 \quad \text{in } B, \quad \varphi = 0 \quad \text{on } \partial B,$$

was first considered in Barbosa and do Carmo [5], later in Sauvigny [102, 104] in his studies of minimal surfaces with polygonal boundaries, but, however, always without taken the curvature of the normal bundle into particular account.

Thus, also here it remains the question whether we can characterize stability of minimal surfaces in terms of the eigenvalues of that Schwarzian eigenvalue problem, and how these eigenvalues depend on the curvature of the normal bundle.

Sauvigny applied his results to prove uniqueness for minimal surfaces spanning so-called extreme polygonal boundary curves, see e.g. [103]. Moreover, in [106] he establishes compactness and finiteness results for stable and unstable small immersions with constant mean curvature spanning regular, extreme Jordan curves.

Concerning new results on finiteness for minimal surfaces with polygonal boundaries we want to draw the reader's attention to the papers Jakob [73–75]. In this context we would also like to refer to a recent result of Bergner and Jakob [9] on the non-existence of branch points for minimal surfaces in $\mathbb{R}^{n+2}$.

Finally, we want to remark that already Wirtinger in [126] proved the absolutely area minimizing property of holomorphic minimal surfaces w.r.t. compactly supported variations which implies stability in the sense of our definition from the beginning. This minimizing character is also discussed in Eschenburg and Jost [44] by means of modern calibration methods.

2.2.10 Osserman's Curvature Estimate and a Generalization

In 1964, Osserman [93] proved the following

Proposition 2.10. *Assume that at each point of a minimal immersion X in $\mathbb{R}^{n+2}$ all unit normal vectors make an angle of at least $\omega > 0$ with a fixed axis in space. Then for the Gauß curvature $K(P)$ at some point $P = X(w_0)$, $w_0 \in B$, with interior distance $d > 0$ to the boundary, it holds*

$$|K(P)| \leq \frac{1}{d^2} \cdot \frac{16(n+1)}{\sin^4 \omega} \,.$$

In particular, if the surface is defined over the whole $\mathbb{R}^2$, *then it is a plane.*

A connection of Osserman's ω-condition with the curvature of the normal bundle of complete minimal graphs is not known to us. However, his result is sharp in the sense that the holomorphic graph (w, w^2), $w \in \mathbb{R}^2$, does not obey the ω-condition.

A refinement of Osserman's proof together with applications of potential theoretic methods enabled us in Bergner and Fröhlich [8] to prove the following curvature estimate for graphs with prescribed Hölder continuous mean curvature vector.

Proposition 2.11. *Let the graph*

$$X(x, y) = \big(x, y, \zeta_1(x, y), \ldots, \zeta_n(x, y)\big)$$

of prescribed mean curvature vector

$$\mathfrak{H} = \mathfrak{H}(X, Z)$$

be given, $(X, Z) \in \mathbb{R}^{n+2} \times S^{n+1}$ *with* $S^{n+1} \subset \mathbb{R}^{n+2}$ *being the* $(n+1)$*-dimensional unit sphere. Suppose that* $X(u, v)$ *represents a conformal reparametrization of this graph. Assume furthermore*

1. *The mean curvature vector* $\mathfrak{H} = \mathfrak{H}(X, Z)$ *satisfies*

$$|\mathfrak{H}(X, Z)| \leq h_0 \quad \text{for all } X \in \mathbb{R}^{n+2} \text{ and } Z \in S^{n+1}$$

   and

$$|\mathfrak{H}(X_1, Z_1) - \mathfrak{H}(X_2, Z_2)| \leq h_1 |X_1 - X_2|^\alpha + h_2 |Z_1 - Z_2|$$

   for all $X_1, X_2 \in \mathbb{R}^{n+2}$ *and* $Z_1, Z_2 \in S^{n+1}$, *with real constants* $h_0, h_1, h_2 \in [0, \infty)$ *and with some* $\alpha \in (0, 1)$.
2. *The surface represents (or contains) a geodesic disc* $\mathfrak{B}_r(X_0)$ *of radius* $r > 0$ *and center* $X_0 \in \mathbb{R}^{n+2}$ *(see the next paragraph for details).*
3. *With a real constant* $d_0 > 0$, *the area of this geodesic disc* $\mathfrak{B}_r(X_0)$ *can be estimated by*

$$\mathscr{A}[\mathfrak{B}_r(X_0)] \leq d_0 r^2 \,.$$

4. *At every point* $w \in B$, *each normal vector of* X *makes an angle of at least* $\omega > 0$ *with the* x_1*-axis.*

Then, for an arbitrarily chosen ONF $\mathfrak{N}$ *there exists a constant*

$$\Theta = \Theta(h_0 r, h_1 r^{1+\alpha}, h_2 r, d_0, \sin\omega, \alpha) \in (0, \infty)$$

such that it holds the curvature estimate

$$|K_\sigma(0,0)| \le \frac{1}{r^2}\left\{(h_0 r)^2 + \Theta\right\} \quad \textit{for all}\ \sigma = 1,\ldots,n.$$

In particular, if $\mathfrak{H} \equiv 0$, *and therefore* $h_0, h_1, h_2 = 0$, *and if the minimal graph is defined over the whole plane* $\mathbb{R}^2$, *then* $X(x,y)$ *must be affine linear.*

A few words to the assumptions in this theorem: Since X is a conformally parametrized graph with prescribed mean curvature vector $\mathfrak{H}$, it is a solution of

$$\Delta X = 2\sum_{\sigma=1}^{n} H_\sigma W N_\sigma = 2\mathfrak{H}W \quad \text{in } B.$$

Together with the first assumption we arrive at the following non-linear elliptic system with quadratic growth in the gradient (see Sect. 2.1.4)

$$|\Delta X| \le h_0 |\nabla X|^2 \quad \text{in } B.$$

The first and second derivatives (in the interior) of such a system can be controlled if X is either a small solution in the sense of

$$h_0 \cdot \sup_{(u,v)\in B} |X(u,v)| < 1,$$

or if a growth condition for the area as required is known such that a smallness condition can eventually be forced by means of the Courant–Lebesgue lemma in connection with a geometric maximum principle.

On the other hand, the assumption on the universal angle ω as well as the required graph property are needed (a) to ensure that also the plane mapping

$$f(u,v) = \big(x^1(u,v), x^2(u,v)\big), \quad (u,v) \in \overline{B}\,,$$

solves a non-linear elliptic system with quadratic growth in the gradient, and (b) that it is one-to-one with $F(0,0) = (0,0)$, positively oriented and topologically on the boundary, and possesses a positive Jacobian in B, see Sect. 2.2.6.

Most of our inputs were already discussed in the foregoing paragraphs. But also here the question remains open whether these assumptions can be connected to the inner geometry of the normal bundle.

2.2.11 On the Growth of Geodesic Discs

At least we can give a partial answer to this question regarding the growth condition for geodesic discs. Is it valid to require such a condition at all? In Bergner and

Fröhlich [8] we computed directly

$$\mathscr{A}[X] \le 192\pi r^2$$

for geodesic discs of the holomorphic graph (w, w^2), where $r > 0$ is chosen sufficiently large. Note that in this special case the scalar curvature S of the normal bundle vanishes asymptotically.

If S is otherwise everywhere strictly larger than a positive constant for some immersion $X\colon \overline{B} \to \mathbb{R}^4$ (or smaller than a negative constant), we can show

Proposition 2.12. *Let a minimal surface $X\colon \overline{B} \to \mathbb{R}^4$ be given such that the scalar curvature S of its normal bundle satisfies*

$$S(u,v) \ge S_0 > 0 \quad \textit{for all } (u,v) \in \overline{B}$$

with a fixed real number $S_0 > 0$.
Suppose furthermore that X represents (or contains) a geodesic disc $\mathfrak{B}_r(X_0)$ with geodesic radius $r > 0$ and with center X_0. Then for the area of this disc it holds the estimate

$$\mathscr{A}[\mathfrak{B}_r(X_0)] \ge \pi r^2 + \frac{S_0\pi}{12}\, r^4$$

Proof. Let the geodesic disc $\mathfrak{B}_r(X_0)$ be given parametrically as $X(\rho,\varphi)$ with geodesic polar coordinates $(\rho,\varphi) \in [0,r]\times[0,2\pi]$. With the area element $\sqrt{P(\rho,\tau)}$, the line element ds_P^2 w.r.t. this coordinate system takes the form

$$ds_P^2 = d\rho^2 + P(\rho,\varphi)\, d\varphi,$$

with smooth $P(\rho,\varphi) > 0$ for all $(\rho,\varphi) \in (0,r] \times [0,2\pi)$ satisfying

$$\lim_{\rho\to 0+} P(\rho,\varphi) = 0, \qquad \lim_{\rho\to 0+} \frac{\partial}{\partial\rho}\sqrt{P(\rho,\varphi)} = 1$$

for all $\varphi \in [0,2\pi)$. For these results and for the following identities we refer the reader e.g. to Blaschke and Leichtweiß [12]. In particular, with the geodesic curvature κ_g of the surface, the integral formula of Gauß–Bonnet gives

$$\int\limits_0^r \kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)}\, d\varphi + \int\limits_0^\rho \int\limits_0^{2\pi} K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\, d\tau d\varphi = 2\pi.$$

For curves with $\rho = \text{const}$ it holds

$$\kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)} = \frac{\partial}{\partial\rho}\sqrt{P(\rho,\varphi)} \quad \text{for all } (\rho,\varphi) \in (0,r] \times [0,2\pi),$$

and therefore, together with $-K \geq |S| \geq S_0 > 0$ (see Sect. 2.2.2) we can estimate

$$\begin{aligned}
\frac{\partial}{\partial\rho}\int\limits_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi &= \int\limits_0^{2\pi}\kappa_g(\rho,\varphi)\sqrt{P(\rho,\varphi)}\,d\varphi \\
&= 2\pi - \int\limits_0^{\rho}\int\limits_0^{2\pi}K(\tau,\varphi)\sqrt{P(\tau,\varphi)}\,d\tau d\varphi \\
&\geq 2\pi + \int\limits_0^{\rho}\int\limits_0^{2\pi}S_0\sqrt{P(\tau,\varphi)}\,d\tau d\varphi \\
&= 2\pi + S_0\mathscr{A}[\mathfrak{B}_\rho(X_0)] \\
&\geq 2\pi + S_0\pi\rho^2
\end{aligned}$$

for all $\rho \in (0, r]$. Integration over the radius coordinate yields

$$\int\limits_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi \geq 2\pi\rho + \frac{S_0\pi}{3}\,\rho^3\,,$$

and a further integration over $\rho = 0\ldots r$ shows

$$\mathscr{A}[\mathfrak{B}_r(X_0)] = \int\limits_0^{r}\int\limits_0^{2\pi}\sqrt{P(\rho,\varphi)}\,d\varphi d\rho \geq \pi r^2 + \frac{S_0\pi}{12}\,r^4$$

proving the statement. □

At least for certain stable minimal geodesic discs we can show that their areas grow quadratically in the radius r. Namely, let us start again from the stability inequality

$$\begin{aligned}
\delta^2_\omega\mathscr{A}[X] = &\iint\limits_B |\nabla\varphi|^2\,dudv + 2\iint\limits_B K_\omega W\varphi^2\,dudv \\
&+ \sum_{\sigma=1}^{n}\iint\limits_B \left\{(T^\sigma_{\omega,1})^2 + (T^\sigma_{\omega,2})^2\right\}\varphi\,dudv \\
\geq 0 &
\end{aligned}$$

for all $\varphi \in C^\infty_0(B,\mathbb{R})$ and using conformal parameters. Since for the non-positive Gauß curvature K we know

$$K_\omega \geq \sum_{\sigma=1}^{n} K_\sigma = K,$$

we have the estimate

$$\delta_\omega^2 \mathscr{A}[X] \geq D^2 \mathscr{A}[X] := \iint\limits_B |\nabla \varphi|^2 \, du dv + 2 \iint\limits_B K W \varphi^2 \, du dv$$

for all $\varphi \in C_0^\infty(B, \mathbb{R})$. The integral $D^2 \mathscr{A}[X]$ agrees with the functional of the second variation for minimal surfaces in $\mathbb{R}^3$ such that in this situation the condition

$$D^2 \mathscr{A}[X] \geq 0 \quad \text{for all } \varphi \in C_0^\infty(B, \mathbb{R})$$

actually defines *stability for minimal surfaces* $X : \overline{B} \to \mathbb{R}^3$.

Sauvigny in [102] revives this condition to define (strict) stability for minimal immersions $X : \overline{B} \to \mathbb{R}^{n+2}$. It holds the

Proposition 2.13. *Let the geodesic disc* $\mathfrak{B}_r(X_0)$ *be stable in the sense of*

$$D^2 \mathscr{A}[\mathfrak{B}_r(X_0)] \geq 0 \quad \textit{for all } \varphi \in C_0^\infty(B, \mathbb{R}).$$

Then its area can be estimates by

$$\mathscr{A}[\mathfrak{B}_r(X_0)] \leq \frac{4\pi}{3} r^2 .$$

For the proof we refer the reader to Gulliver [56] and Sauvigny [105].

Two concluding remarks are due: First, and this follows from Sect. 2.2.2, we estimate

$$S_0 \iint\limits_B W \, du dv \leq \iint\limits_B |S_0| W \, du dv \leq \iint\limits_B (-K) W \, du dv,$$

that is, *if* $S_0 \neq 0$ *then the curvatura integra is not finite for complete minimal graphs.*

And secondly, and this is to round out the beginning of this paragraph, Micallef in [88] showed that if a minimal graph is complete and stable, and if its area growths quadratically, then it is holomorphic. Wirtinger in [126] proved that holomorphic minimal surfaces area absolutely area minimizing w.r.t. compactly supported variations, see our discussion in Sect. 2.2.9 above.

2.2.12 Curvature Estimates for Higher-Dimensional Minimal Graphs

The next result, which goes back to Hildebrandt, Jost and Widman [68], states a Bernstein-type result for higher-dimensional minimal surface graphs in Euclidean spaces of arbitrary dimensions.

In particular, it contains a gradient bound of the graph a priori which essentially reflects the fact that a generalized Gauß map of the surface (concerning this, see also Hoffman and Osserman [69]) must be contained in a geodesic ball of radius $\frac{\sqrt{2}\pi}{4}$ in the so-called Grassmannian manifold $G_{r,s}$.

Proposition 2.14. *Let $z^\alpha = f^\alpha(x)$ with $\alpha = 1, \ldots, s$ and $x = (x^1, \ldots, x^r) \in \mathbb{R}^r$ be C^2-regular, and let it generate a r-dimensional minimal graph*

$$X(x, y) = \big(x^1, \ldots, x^r, f^1(x^1, \ldots, x^r), \ldots, f^s(x^1, \ldots, x^r)\big).$$

Let there furthermore exist a real number $\beta > 0$ such that

$$\beta < \cos^{-t}\left(\frac{\pi}{2\sqrt{t}\,K}\right)$$

with

$$K := \begin{cases} 1 & \textit{if } t = 1 \\ 2 & \textit{if } t \geq 2 \end{cases} \quad \textit{and} \quad t := \min\{r, s\}.$$

Assume finally that

$$\sqrt{\det\left(\delta_{ij} + \sum_{\alpha=1}^{s} \frac{\partial f^\alpha}{\partial x^i}\,\frac{\partial f^\alpha}{\partial x^j}\right)_{i,j=1,\ldots,r}} < \beta.$$

Then the functions $f^1, \ldots, f^s$ are affine linear on $\mathbb{R}^r$, and the minimal graph is an affine linear r-dimensional plane.

In subsequent works, e.g. Jost and Xin [76], Wang [119], or Xin [127], one finds various improvements of this result as well as generalizations to surfaces with prescribed mean curvature vector.

Also here the question must be left open how the assumption on the gradient bound stands in connection with our geometrical and analytical concepts of the normal bundle. We expect that it can be weakened at least in the special case $r = 2$ of surfaces in view of other results where less restrictive assumptions are required, see e.g. Fröhlich [46] and the references therein, and combine them for instance with the discussions from Barbosa and do Carmo [5, 6], and Ruchert [100].

Finally we want to mention the curvature estimates and theorems of Bernstein-type for minimal submanifolds with flat normal bundle from Smoczyk et al. [113], where e.g. classical methods from Schoen et al. [109] and Ecker, Huisken [42] were employed. In Fröhlich and Winklmann [52] we succeeded in proving similar results for graphs of dimension $m \in [2, 5]$ but with prescribed mean curvature vector. Can one find methods and techniques comparable to the ones presented in this book to establish more general results for submanifolds with *arbitrary normal bundles?*

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