Chapter 2
Control of Inventories with Markov Demand

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Abstract We consider inventory control problems in discrete time. The horizon is infinite, and we consider discounted payoffs as well nondiscounted payoffs (ergodic control). We may have backlog or not. We may have set-up costs or not. In the traditional framework, the demand is modeled as a sequence of i.i.d. random variables. The ordering strategy is given by a base stock policy or an s, S policy, whether or not there is a set-up cost. We consider here the situation when the demand is modeled by a Markov chain. We show how the base stock policy and the s, S policy can be extended.

2.1 Introduction

We consider inventory control problems, with nonindependent demands. In real problems, the successive demands are linked for a lot of reasons, and the assumption of independence is too limited. The simplest way to model the linkage is to assume that the demands form a Markov process as such or are derived from a Markov process. Our objective is to show how the methods used in the case of independent demands can be adapted to this situation. In a recent book by Beyer et al. [1] a comprehensive presentation of these problems is given. We refer also to the references of this book for the related literature [2, 3]. In this work, the authors consider that the demand comes from an underlying state of demand, which is modeled as a Markov chain with a finite number of states. The fact that the number of states is finite simplifies mathematical arguments. We will here discuss the

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situation in which the demand itself is a Markov process. We will consider backlog and non-backlog situations (but develop the no backlog case only) and set-up and non-set-up situations. The model is on an infinite horizon, and we study discounted as well as non-discounted (ergodic control) situations.

2.2 No Backlog and No Set-Up Cost

2.2.1 The Model

Let $\Omega, \mathscr{A}, P$ be a probability space, on which is defined a Markov chain z_n. This Markov chain represents the demand. Its state space is R^+ and its transition probability is $f(\zeta|z)$. We shall assume that

$$f(\zeta|z) \text{ is uniformly continuous in both variables and bounded} \tag{2.1}$$

$$\int_0^{+\infty} \zeta f(\zeta|z) d\zeta \leq c_0 z + c_1 \tag{2.2}$$

We can assume that $z_1 = z$, a fixed constant or more generally a random varible with given probability distribution. We define the filtration

$$\mathscr{F}^n = \sigma(z_1, \cdots, z_n)$$

A control policy, denoted by V, is a sequence of random variables v_n such that v_n is $\mathscr{F}^n$ measurable. When $z_1 = z$, then v_1 is deterministic. Also, we assume as usual that $v_n \geq 0$. We next define the inventory as the sequence

$$y_{n+1} = (y_n + v_n - z_{n+1})^+, \quad y_1 = x \tag{2.3}$$

The process y_n is adapted to the filtration $\mathscr{F}^n$. The joint process y_n, z_n is also a Markov chain. We can write, for a test function $\varphi(x,z)$ (bounded continuous on $R^+ \times R^+$)

$$E[\varphi(y_2, z_2)|y_1 = x, z_1 = z, v_1 = v] = E[\varphi((x + v - z_2)^+, z_2)|z_1 = z]$$

This defines a Markov chain to which is associated the operator

$$\Phi^v \varphi(x,z) = \int_{x+v}^{+\infty} \varphi(0,\zeta) f(\zeta|z) d\zeta + \int_0^{x+v} \varphi(x + v - \zeta, \zeta) f(\zeta|z) d\varsigma \tag{2.4}$$

We notice the property

$$\begin{aligned}&\Phi^v \varphi_v(x,z) \text{ is uniformly continuous in } v, x, z \text{ if}\\&\varphi_v(x,z) = \phi(x,z,v) \text{ is uniformly continuous and bounded in } x, z, v\end{aligned} \tag{2.5}$$

We next define the function

$$l(x,z,v) = cv + hx + pE[(x+v-z_2)^-|z_1 = z] \tag{2.6}$$

which will model the one period cost. The cost function is then

$$J_{x,z}(V) = E\sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, z_n, v_n) \tag{2.7}$$

We are interested in the value function

$$u(x,z) = \inf_V J_{x,z}(V) \tag{2.8}$$

Set $l_v(x,z) = l(x,z,v)$. Note the inequalities

$$cv + hx \leq l_v(x,z) \leq cv + hx + p(c_0 z + c_1) \tag{2.9}$$

and

$$cv \leq \Phi^v l_v(x,z) \leq cv + h(x+v) + p(c_0^2 z + c_0 c_1 + c_1) \tag{2.10}$$

Consider then the function

$$w_0(x,z) = \sum_{n=1}^{\infty} \alpha^{n-1} (\Phi^0)^{n-1} l_0(x,z) \tag{2.11}$$

which corresponds to the payoff, when the control is identically 0.

Lemma 2.1. *We assume that*

$$\alpha c_0 < 1 \tag{2.12}$$

then the series $w_0(x,z) < \infty$ *and more precisely*

$$w_0(x,z) \leq \frac{hx}{1-\alpha} + \frac{pc_0 z}{1-c_0\alpha} + \frac{pc_1}{(1-\alpha)(1-c_0\alpha)} \tag{2.13}$$

Proof. It is an immediate consequence of formulas (2.11) and (2.10). □

We can now write the Bellman equation

$$u(x,z) = \inf_{v \geq 0}[l(x,z,v) + \alpha \Phi^v u(x,z)] \tag{2.14}$$

We state the following:

Theorem 2.1. *We assume* (2.1)*,* (2.2)*,* (2.6)*,* (2.12)*. Then there exists one and only one solution of equation* (2.14)*, such that* $0 \leq u \leq w_0$*. It is continuous and coincides with the value function* (2.8)*. There exists an optimal feedback* $\hat{v}(x,z)$ *and an optimal control policy* $\hat{V}$*.*

Proof. We use a standard monotonicity argument to prove that on the interval $(0, w_0)$, the set of solutions of (2.14) is not empty. It has a minimum and a maximum solution. The minimum solution is l.s.c. and the maximum solution is u.s.c. The minimum solution coincides with the value function. There exists an optimal feedback $\hat{v}(x,z)$and an optimal control policy $\hat{V}$. To prove uniqueness, we have to prove that the minimum and maximum solution coincide. We begin by proving a bound on $\hat{v}(x,z)$. It will be first convenient to mention a slightly better estimate for w_0. Indeed, we can write

$$w_0(x,z) \leq h \sum_{n=1}^{\infty} (\alpha \Phi^0)^{n-1} x(x,z) + \frac{pc_0 z}{1 - c_0 \alpha} + \frac{pc_1}{(1-\alpha)(1-c_0\alpha)} \tag{2.15}$$

and (2.13) was simply derived from (2.15) by using $\Phi^0 x(x,z) \leq x$. Next, from (2.14), considering $\underline{u}$, the minimum solution, we can state that

$$\underline{u}(x,z) \geq h \sum_{n=1}^{\infty} (\alpha \Phi^0)^{n-1} x(x.z)$$

which follows from $l(x,z,v) \geq hx$ and

$$\Phi^v \varphi(x,z) \geq \Phi^0 \varphi(x,z), \forall v \geq 0, \forall \varphi \text{ increasing in } x \tag{2.16}$$

Therefore, we can write

$$l(x,z,v) + \alpha \Phi^v \underline{u}(x,z) \geq cv + h \sum_{n=1}^{\infty} (\alpha \Phi^0)^{n-1} x(x,z)$$

Therefore, in minimizing in v, we can bound from above the range of v. More precisely, we get

$$\hat{v}(x,z) \leq \frac{pc_0 z}{c(1-c_0\alpha)} + \frac{pc_1}{c(1-\alpha)(1-c_0\alpha)} \tag{2.17}$$

We consider the optimal trajectory $\hat{y}_n, z_n$ obtained from the optimal feedback namely

$$\hat{y}_{n+1} = (\hat{y}_n + \hat{v}_n - z_{n+1})^+, \; \hat{v}_n = \hat{v}(\hat{y}_n, z_n)$$

with $\hat{y}_1 = x, z_1 = z$. It can be shown that the maximum solution will coincide with the minimum solution, if we can check that $\hat{V}$ satisfies the property

$$\alpha^n E\bar{u}(\hat{y}_{n+1}, z_{n+1}) \to 0, \text{ as } n \to \infty$$

It is sufficient to replace $\bar{u}$ by w_0 and by estimate (2.13), it is sufficient to show that

$$\alpha^n E\hat{y}_{n+1}, \quad \alpha^n E z_{n+1} \to 0, \text{ as } n \to \infty \tag{2.18}$$

However, by standard Markov arguments, $\alpha^n E z_{n+1} \leq (\alpha c_0)^n z$. From the Assumption (2.12), the second part of (2.18) follows immediately. We next use

$$\begin{aligned}
\hat{y}_{n+1} &\leq \hat{y}_n + \hat{v}_n \\
&\leq \hat{y}_n + \frac{pc_0 z_n}{c(1-c_0\alpha)} + \frac{pc_1}{c(1-\alpha)(1-c_0\alpha)}
\end{aligned}$$

Therefore,

$$E\hat{y}_{n+1} \leq E\hat{y}_n + \frac{pc_0^n z}{c(1-c_0\alpha)} + \frac{pc_1}{c(1-\alpha)(1-c_0\alpha)}$$

and we deduce the estimate

$$E\hat{y}_{n+1} \leq x + \frac{pzc_0}{c(1-\alpha c_0)}\frac{1-c_0^n}{1-c_0} + \frac{npc_1}{c(1-\alpha)(1-c_0\alpha)}$$

Using again the Assumption (2.12), we deduce the first part of (2.18). This completes the proof. □

2.2.2 *Base-Stock Policy*

We want now to check that the optimal feedback $\hat{v}(x,z)$ can be obtained by a base stock policy, with a base stock depending on the value of z. We have

Theorem 2.2. *We make the assumptions of Theorem 2.1 and $p > c$. We assume also*

$$f(x|z) \geq a_0(M) > 0, \ \forall x, z \leq M \tag{2.19}$$

Then the function $u(x,z)$ is convex and C^1in the argument x. Moreover the optimal feedback is given by

$$\hat{v}(x,z) = \left| \begin{array}{ll} S(z) - x & \text{if } x \leq S(z) \\ 0 & \text{if } x \geq S(z) \end{array} \right. \tag{2.20}$$

The function $S(z)$ is uniformly continuous and the derivative in x, $u'(x,z)$ is uniformly continuous.

Proof. We consider the increasing process

$$u_{n+1}(x,z) = \inf_{v \geq 0}[l(x,z,v) + \alpha\Phi^v u_n(x,z)], \qquad u_0(x) = 0$$

which we write also as

$$u_{n+1}(x,z) = (h-c)x + \inf_{\eta \geq x}\{c\eta + pE[(\eta - z_2)^- | z_1 = z] + \alpha E[u_n((\eta - z_2)^+, z_2) | z_1 = z]\}$$

and we are going to show recursively that the sequence $u_n(x,z)$ is convex and C^1 in x. Define

$$G_n(x,z) = cx + pE[(x - z_2)^- | z_1 = z] + \alpha E[u_n((x - z_2)^+, z_2) | z_1 = z]$$

then the function $G_n(x,z)$ attains its minimum in $S_n(z)$, which can be uniquely defined by taking the smallest minimum. We prove by induction that $u_n(x,z)$, $G_n(x,z)$ are convex and C^1in x. Moreover

$$G_n'(x,z) = c - p\bar{F}(x|z) + \alpha E[u_n'(x - z_2, z_2)\mathbf{1}_{x \geq z_2} | z_1 = z]$$

We see that $G_n'(0,z) = c - p$ and we can check that

$$G_n'(+\infty, z) = c + h\alpha \frac{1-\alpha^n}{1-\alpha}$$

Therefore, there exists a point $S_n(z)$ such that $G_n'(S_n(z),z) = 0$. Note that $S_n(z) > 0$, since $G_n'(0,z) = c - p < 0$. We have

$$u_{n+1}(x,z) = \left| \begin{array}{ll} (h-c)x + G_n(S_n(z),z) & \text{if } x \leq S_n(z) \\ (h-c)x + G_n(x,z) & \text{if } x \geq S_n(z) \end{array} \right.$$

It follows that the limit $u(x,z)$ is convex in x. Clearly $u(x,z) \to +\infty$, as $x \to +\infty$. Also

$$G(x,z) = cx + pE[(x - z_2)^- | z_1 = z] + \alpha E[u((x - z_2)^+, z_2) | z_1 = z]$$

is convex and $\to +\infty$ as $x \to +\infty$. So the minimum is attained in $S(z)$, which can be defined in a unique way, by taking the smallest minimum.

Since $h - c \leq u_n'(x,z) \leq \dfrac{h}{1-\alpha}$, we can assert that $u(x,z)$ is Lipschitz continuous in x. The same is true for $G(x,z)$. But

$$G'(x,z) = c - p\bar{F}(x|z) + \alpha E[u'(x - z_2, z_2)\mathbf{1}_{x > z_2} | z_1 = z] \to c - p \text{ as } x \to 0$$

therefore also $S(z) > 0$. Then, from convexity

$$u(x,z) = \left| \begin{array}{ll} (h-c)x + G(S(z),z) & \text{if } x \leq S(z) \\ (h-c)x + G(x,z) & \text{if } x \geq S(z) \end{array} \right.$$

Next, from

$$c - p\bar{F}(S_n(z)|z) + \alpha E[u_n'(S_n(z) - z_2, z_2)\mathbf{1}_{S_n(z)>z_2}|z_1 = z] = 0$$

we deduce, using the estimate on $u_n'$and the property

$$\bar{F}(S_n(z)|z) \leq \frac{c_0 z + c_1}{S_n(z)}$$

that

$$S_n(z) \leq \frac{(c_0 z + c_1)(p + \alpha(h - c))}{c + \alpha(h - c)} \tag{2.21}$$

The same estimate holds for $S(z)$. It is easy to check that $S_n(z) \to S(z)$, and $S(z)$ is the smallest minimum of $G(x, z)$ in x. Furthermore, from the continuity properties of $G(x, z)$ in both arguments, we can check that $S(z)$ is a continuous function. The feedback $\hat{v}(x, z)$ defined by (2.20) is also continuous in both arguments. Define

$$\chi(x, z) = u'(x, z) - h + c$$

as an element of B (space of measurable bounded functions on $R^+ \times R^+$), then χ is the unique solution in B of the equation

$$\chi(x, z) = g(x+\hat{v}(x, z), z)+\alpha E[\chi(x+\hat{v}(x, z)-z_2, z_2)\mathbf{1}_{x+\hat{v}(x,z)>z_2}|z_1 = z], \;\; x, z \in R^+ \tag{2.22}$$

where

$$g(x, z) = c + \alpha(h - c) - (p + \alpha(h - c))\bar{F}(x|z)$$

Since the function $g(x + \hat{v}(x, z), z)$ is continuous in the pair x, z, the solution $\chi(x, z)$ is also continuous. Let us check that $S(z)$ is uniformly continuous. Let us first check that

$$(G'(x_1, z)-G'(x_2, z))(x_1-x_2) \geq (p-\alpha(c-h))(F(x_1|z)-F(x_2|z))(x_1-x_2) \tag{2.23}$$

Assume to fix the ideas that $x_1 > x_2$. We have

$$(G'(x_1, z) - G'(x_2, z))(x_1 - x_2) = p(F(x_1|z) - F(x_2|z))(x_1 - x_2)+$$

$$\alpha E[(u'(x_1 - z_2, z_2) - u'(x_2 - z_2, z_2))(x_1 - x_2)\mathbf{1}_{z_2<x_2}|z_1 = z]+$$
$$+\alpha E[u'(x_1 - z_2, z_2)\mathbf{1}_{x_2<z_2<x_1}(x_1 - x_2)|z_1 = z]$$

The second term is positive, from the convexity of u. Using the left estimate on u' for the last term, we deduce (2.23). We then obtain

$$(S(z) - S(z'))(G'(S(z), z') - G'(S(z'), z)) \geq$$

$$(p - \alpha(c - h))(S(z) - S(z'))\left[\int_{S(z')}^{S(z)} (f(\xi|z) + f(\xi|z'))d\xi\right]$$

If $z, z' < m$, then from (2.21) we can find $M_m > m$ such that $S(z), S(z') < M_m$. From the Assumption (2.19), we deduce

$$(S(z)-S(z'))(G'(S(z),z')-G'(S(z'),z)) \geq 2a_0(M_m)(p-\alpha(c-h))(S(z)-S(z'))^2$$

Next we have

$$G'(x,z') - G'(x,z) = \int_0^x (p+\alpha u'(x-\zeta,\zeta))(f(\zeta|z') - f(\zeta|z))d\zeta$$

hence

$$|G'(x,z') - G(x,z)| \leq xC \sup_{0<\zeta<x} |f(\zeta|z') - f(\zeta|z)|$$

Applying this estimate with $x = S(z)$ and $x = S(z')$ and combining estimates, we obtain easily that $S(z)$ is uniformly continuous. It follows that the feedback $\hat{v}(x,z)$ is uniformly continuous. Then from (2.22), we obtain that $\chi(x,z)$ is uniformly continuous. The proof has been completed. □

Remark 2.1. We have $\chi(x,z) = 0, \forall x \leq S(z)$, and

$$\chi(x,z) = g(x,z) + \alpha E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1 = z], \ \forall x \geq S(z) \tag{2.24}$$

Also

$$0 = g(S(z),z) + \alpha E[\chi(S(z)-z_2,z_2)\mathbf{1}_{S(z)>z_2}|z_1 = z] \tag{2.25}$$

So $S(z)$ is not the solution of $g(S(z),z) = 0$. If we consider the function

$$G^*(x,z) = (p+\alpha(h-c))E[(x-z_2)^+|z_1 = z] - (p-c)x + pE[z_2|z_1 = z]$$

then the solution of $g(S,z) = 0$ is denoted by $S^*(z)$ and minimizes $G^*(x,z)$. We have $G(x,z) \geq G^*(x,z)$.

2.2.3 Ergodic Theory

We turn now to the case when $\alpha \to 1$. We write $u_\alpha(x,z)$ to satisfy

$$u_\alpha(x,z) = \left| \begin{array}{ll} (h-c)x + cS_\alpha(z) + pE[(S_\alpha(z)-z_2)^-|z_1 = z]+ & \\ \alpha E[u_\alpha((S_\alpha(z)-z_2)^+,z_2)|z_1 = z] & \text{if } x \leq S_\alpha(z) \\ hx + pE[(x-z_2)^-|z_1 = z] + \alpha E[u_\alpha((x-z_2)^+,z_2)|z_1 = z] & \text{if } x \geq S_\alpha(z) \end{array} \right. \tag{2.26}$$

We shall make the assumptions

$$c_0 = 0 \tag{2.27}$$

$$\inf_{\substack{0 \leq \zeta \leq a \\ z}} f(\zeta|z) \geq \gamma(a) > 0, \forall a$$

$$f(\zeta|z) \text{ is ergodic} \tag{2.28}$$

$$\int |f(\zeta|z) - f(\zeta|z')|d\zeta \leq \delta|z - z'| \tag{2.29}$$

$$\sup_z F(x|z) = \delta_0(x) < 1, \forall x$$

We denote by $\varpi(z)$ the invariant probability density corresponding to the Markov chain $f(\zeta|z)$. We state

Theorem 2.3. *We assume* (2.1), (2.2) *with* $c_0 = 0$, (2.6) *with* $p > c$ *and* (2.27)–(2.29). *Then, for a subsequence (still denoted* α*) converging to* 1*, we have, for any compact* K *of* R^+

$$\sup_{z \in K} |S_\alpha(z) - S(z)| \leq \epsilon(\alpha, K), \quad \epsilon(\alpha, K) \to 0, \text{ as } \alpha \uparrow 1 \tag{2.30}$$

$$\sup_{\substack{x \leq M \\ z \leq N}} |u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)| \to 0, \forall M, N \tag{2.31}$$

with $\rho_\alpha \to \rho$ *and*

$$u(x,z) + \rho = \left| \begin{array}{ll} (h-c)x + cS(z) + pE[(S(z) - z_2)^-|z_1 = z] + & \\ E[u((S(z) - z_2)^+, z_2)|z_1 = z] & \text{if } x \leq S(z) \\ hx + pE[(x - z_2)^-|z_1 = z] + E[u((x - z_2)^+, z_2)|z_1 = z] & \text{if } x \geq S(z) \end{array} \right. \tag{2.32}$$

The function $u(x,z)$ *satisfies the growth condition*

$$\sup_{\substack{\xi \leq x \\ z}} |u(\xi, z)| \leq C_0 + \frac{C_1 x}{1 - \delta_0(x)} \tag{2.33}$$

It is C^1 *in* x *and Lipschitz continuous in* z. *The following estimates hold:*

$$\sup_{\substack{\xi \leq x \\ z}} |u_x(\xi, z)| \leq \frac{C}{1 - \delta_0(x)}, \tag{2.34}$$

$$|u(x,z) - u(x,z')| \leq [C_0(m_0) + \frac{C_1(m_0)x}{1 - \delta_0(x)}]\delta|z - z'| \tag{2.35}$$

$$(1 - \delta_0(x)) \sup_{\substack{\xi \leq x \\ z}} |u_{xx}(\xi, z)| \leq C, \ (1 - \delta_0(x)) \sup_{\substack{\xi \leq x \\ z}} |u_{xz}(\xi, z)| \leq C, \ a.e. \tag{2.36}$$

Given $S(z)$, the pair $u(x,z),\rho$ satisfying the above conditions and $\int u(0,z)\varpi(z)dz = 0$ is uniquely defined. Also

$$u(x,z) + \rho = \inf_{v\geq 0}[l(x,z,v) + \Phi^v u(x,z)] \tag{2.37}$$

Proof. We begin with (2.30). We first note that, thanks to $c_0 = 0$, we have

$$S_\alpha(z) \leq \frac{c_1(p+h)}{\min(h,c)} = m_0 \tag{2.38}$$

We pursue the estimates obtained in Theorem 2.2. We have first

$$(S_\alpha(z)-S_\alpha(z'))(G'_\alpha(S_\alpha(z),z')-G'_\alpha(S_\alpha(z'),z)) \geq 2\min(h,c)\gamma(m_0)(S_\alpha(z)-S_\alpha(z'))^2$$

We know that $u_\alpha(x,z)$ is C^1 in x. Then, from (2.26), we have (denoting $u'_\alpha(x,z) = u'_{\alpha x}(x,z)$

$$\begin{aligned} u'_\alpha(x,z) &= h - c \quad \text{if} \quad x < S_\alpha(z) \\ &= h - p\bar{F}(x|z) + \alpha E[u'_\alpha(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1 = z], \text{ if } x > S_\alpha(z) \end{aligned}$$

from which we can assert that

$$\sup_{\substack{\xi \leq x \\ z}} |u'_\alpha(\xi,z)| \leq \frac{\max(h,p)}{1-\delta_0(x)} \tag{2.39}$$

Therefore,

$$|G'_\alpha(S_\alpha(z),z') - G'_\alpha(S_\alpha(z),z)| = (p + \frac{\max(h,p)}{1-\sup_z F(m_0|z)})\int |f(\zeta|z') - f(\zeta|z)|d\zeta$$

$$|G'_\alpha(S_\alpha(z'),z') - G'_\alpha(S_\alpha(z'),z)| = (p + \frac{\max(h,p)}{1-\sup_z F(m_0|z)})\int |f(\zeta|z') - f(\zeta|z)|d\zeta$$

Collecting results we obtain the estimate

$$|S_\alpha(z) - S_\alpha(z')| \leq \frac{p + \dfrac{\max(h,p)}{1-\delta_0(m_0)}}{\min(h,c)\gamma(m_0)}\int |f(\zeta|z') - f(\zeta|z)|d\zeta \tag{2.40}$$

and from the first Assumption (2.29), we finally obtain

$$|S_\alpha(z) - S_\alpha(z')| \le \frac{p + \dfrac{\max(h,p)}{1-\delta_0(m_0)}}{\min(h,c)\gamma(m_0)} \delta|z - z'| \tag{2.41}$$

Therefore, the sequence $S_\alpha(z)$ is uniformly Lipschitz continuous. It is standard that one can extract a subsequence, which converges in the space of continuous functions on compact sets, for any compact set K, towards a function $S(z)$. Therefore, (2.30) is satisfied.

Define $\chi_\alpha(x,z) = u'_\alpha(x,z) - h + c$. We have

$$\chi_\alpha(x,z) = g_\alpha(x,z) + \tag{2.42}$$

$$+ \alpha E[\chi_\alpha(x - z_2, z_2)\mathbf{1}_{x>z_2}|z_1 = z], \; x > S_\alpha(z) \tag{2.43}$$

$$= 0, \qquad x \le S_\alpha(z)$$

with

$$g_\alpha(x,z) = (c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x|z) \tag{2.44}$$

As it has been done for $u'_\alpha(x,z)$, we can state

$$\sup_{\substack{0 \le \xi \le x \\ z}} |\chi_\alpha(\xi,z)| \le \frac{\max(h,p)}{1-\delta_0(x)} \tag{2.45}$$

If we consider $\psi_\alpha(x,z) = \chi'_\alpha(x,z) = u''_\alpha(x,z)$, then using the fact that $\chi_\alpha(0,z) = 0$, we see that

$$\psi_\alpha(x,z) = (p + \alpha(h-c)) f(x|z) + \tag{2.46}$$

$$+ \alpha E[\psi_\alpha(x - z_2, z_2)\mathbf{1}_{x>z_2}|z_1 = z], \quad x > S_\alpha(z)$$

$$\psi_\alpha(x,z) = 0, \quad x < S_\alpha(z)$$

The function $\psi_\alpha(x,z)$ is not continuous, however it is measurable and bounded. We have

$$\sup_{\substack{0 \le \xi \le x \\ z}} |\psi_\alpha(\xi,z)| \le \frac{((h-c)^+ + p)\|f\|}{1-\delta_0(x)} \tag{2.47}$$

where $\|f\| = \sup_{x,z} f(x|z)$. We next obtain an estimate on $\chi_\alpha(x,z) - \chi_\alpha(x,z')$. Assume first $x > S_\alpha(z), x > S_\alpha(z')$. Then, from (2.42), we have

$$\chi_\alpha(x,z) - \chi_\alpha(x,z') = g_\alpha(x,z) - g_\alpha(x,z') +$$

$$+ \alpha \int_0^x \chi_\alpha(x-\zeta,\zeta)(f(\zeta|z) - f(\zeta|z'))d\zeta$$

From estimate (2.45) and the first Assumption (2.29), we deduce

$$|\chi_\alpha(x,z) - \chi_\alpha(x,z')| \leq \left(p + (h-c)^+ + \frac{\max(h,p)}{1-\delta_0(x)}\right)\delta|z-z'|$$
$$x > S_\alpha(z), x > S_\alpha(z')$$

Assume now to fix ideas that $S_\alpha(z') > x > S_\alpha(z)$. Then $\chi_\alpha(x,z') = 0$ and

$$0 = g_\alpha(S_\alpha(z), z) + \alpha E[\chi_\alpha(S_\alpha(z) - z_2, z_2)\mathbf{1}_{S_\alpha(z)>z_2}|z_1 = z]$$

Therefore,

$$\begin{aligned}
\chi_\alpha(x,z) - \chi_\alpha(x,z') &= g_\alpha(x,z) - g_\alpha(S_\alpha(z),z) + \alpha E[\chi_\alpha(x - z_2, z_2)\mathbf{1}_{x>z_2}|z_1 = z] - \\
&\quad - \alpha E[\chi_\alpha(S_\alpha(z) - z_2, z_2)\mathbf{1}_{S_\alpha(z)>z_2}|z_1 = z] \\
&= (p + \alpha(h-c))(F(x|z) - F(S_\alpha(z)|z)) + \\
&\quad + \alpha E[\chi_\alpha(x - z_2, z_2)\mathbf{1}_{x>z_2>S_\alpha(z)}|z_1 = z] + \\
&\quad + \alpha E[(\chi_\alpha(x - z_2, z_2) - \chi_\alpha(S_\alpha(z) - z_2, z_2))\mathbf{1}_{S_\alpha(z)>z_2}|z_1 = z]
\end{aligned}$$

It follows

$$|\chi_\alpha(x,z) - \chi_\alpha(x,z')| \leq \left[p + (h-c)^+ + \frac{\max(h,p)}{1-\delta_0(x)} + \frac{(h-c)^+ + p}{1-\delta_0(x)}\right]\|f\|(x - S_\alpha(z))$$

Finally, we can state the estimate

$$|\chi_\alpha(x,z) - \chi_\alpha(x,z')| \leq \frac{C(m_0)}{1-\delta_0(x)}\delta|z-z'| \tag{2.48}$$

where $C(m_0)$ depends only on constants and on m_0. Therefore, considering the gradient of χ_α in both variables, we have obtained the estimate

$$|D\chi_\alpha(x,z)| \leq \frac{C}{1-\delta_0(x)} \tag{2.49}$$

From this estimate, we can assert that, for a subsequence (still denoted α),

$$\sup_{\substack{x \leq M \\ z \leq N}} |\chi_\alpha(x,z) - \chi(x,z)| \to 0, \text{ as } \alpha \to 0, \forall M, N \tag{2.50}$$

Therefore, also

$$\sup_{\substack{x \le M \\ z}} |\alpha E[\chi_\alpha(x-z_2,z_2)\mathbf{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1=z]| \le$$

$$\sup_{\substack{x \le M \\ z}} |E[\chi_\alpha(x-z_2,z_2)\mathbf{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1=z]| + (1-\alpha)\frac{\max(h,p)}{1-\delta_0(M)}$$

and

$$\sup_{\substack{x \le M \\ z}} |E[\chi_\alpha(x-z_2,z_2)\mathbf{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1=z]| \le$$

$$\sup_{\substack{x \le M \\ z \le N}} |\chi_\alpha(x,z)-\chi(x,z)| + 2\frac{\max(h,p)}{1-\delta_0(M)}\frac{c_1}{N}$$

from which we deduce easily that

$$\sup_{\substack{x \le M \\ z}} |\alpha E[\chi_\alpha(x-z_2,z_2)\mathbf{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1=z]| \to 0, \forall M$$

From (2.42), it follows that

$$\chi(x,z) = g(x,z) + E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1=z],\ \forall x > S(z)$$

where

$$g(x,z) = h - (p+h-c)\bar{F}(x|z) \tag{2.51}$$

Also χ(x,z) $=0$, if $x < S(z)$. Moreover from

$$0 = g_\alpha(S_\alpha(z),z) + \alpha E[\chi_\alpha(S_\alpha(z)-z_2,z_2)\mathbf{1}_{S_\alpha(z)>z_2}|z_1=z]$$

and

$$|\alpha E[\chi_\alpha(S_\alpha(z)-z_2,z_2)\mathbf{1}_{S_\alpha(z)>z_2}|z_1=z] - E[\chi(S_\alpha(z)-z_2,z_2)\mathbf{1}_{S_\alpha(z)>z_2}|z_1=z]| \le$$

$$\sup_{\substack{x \le m_0 \\ z}} |\alpha E[\chi_\alpha(x-z_2,z_2)\mathbf{1}_{x>z_2} - E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1=z]|$$

$$|E[\chi(S_\alpha(z)-z_2,z_2)\mathbf{1}_{S_\alpha(z)>z_2}|z_1=z]-E[\chi(S(z)-z_2,z_2)\mathbf{1}_{S(z)>z_2}|z_1=z]|\leq$$
$$\frac{((h-c)^++p)\|f\|}{1-\delta_0(m_0)}|S_\alpha(z)-S(z)|+\frac{2\max(h,p)}{1-\delta_0(m_0)}|F(S_\alpha(z)|z)-F(S(z)|z)|$$

we obtain easily

$$0=g(S(z),z)+E[\chi((S(z)-z_2)^+,z_2)|z_1=z]$$

and the function $\chi(x,z)$ is continuous in x. Let us next set

$$\Gamma_\alpha(z)=u_\alpha(0,z)$$
$$G_\alpha(z)=E[(S_\alpha(z)-z_2)^-|z_1=z]$$

then from the first equation (2.26) one can check

$$\begin{aligned}\Gamma_\alpha(z)&=cS_\alpha(z)+pG_\alpha(z)+\alpha E[u_\alpha((S_\alpha(z)-z_2)^+,z_2)|z_1=z]\\&=\Psi_\alpha(z)+\alpha E[\Gamma_\alpha(z_2)|z_1=z]\end{aligned}$$

with

$$\Psi_\alpha(z)=cS_\alpha(z)+pG_\alpha(z)+\alpha E\left[\int_0^{(S_\alpha(z)-z_2)^+}(h-c+\chi_\alpha(\xi,z_2))d\xi|z_1=z\right]$$

and

$$0\leq\Psi_\alpha(z)\leq\left[\max(h,c)+\frac{\max(h,p)}{1-\delta(m_0)}\right]m_0+pc_1$$

This estimate also holds for the limit

$$\Psi(z)=cS(z)+pG(z)+E\left[\int_0^{(S(z)-z_2)^+}(h-c+\chi(\xi,z_2))d\xi|z_1=z\right]$$

with

$$G(z)=E[(S(z)-z_2)^-|z_1=z]$$

Define

$$\rho=\int\Psi(z)\varpi(z)dz$$

Consider now the equation

$$\Gamma(z)+\rho=\Psi(z)+E[\Gamma(z_2)|z_1=z],\quad\int\Gamma(z)\varpi(z)dz=0$$

From ergodic theory, we can assert that

$$\sup_z |\Gamma(z)| \le \sup_z |\Psi(z) - \rho| \frac{3-\beta}{1-\beta}$$

where $0 < \beta < 1$ depends only on the Markov chain. Similarly, if we set

$$\rho_\alpha = \int \Psi_\alpha(z)\varpi(z)dz, \qquad \tilde{\Gamma}_\alpha(z) = \Gamma_\alpha(z) - \frac{\rho_\alpha}{1-\alpha}$$

we can write

$$\tilde{\Gamma}_\alpha(z) + \rho_\alpha = \Psi_\alpha(z) + \alpha E[\tilde{\Gamma}_\alpha(z_2)|z_1 = z]$$

we can also assert that

$$\begin{aligned}\sup_z |\tilde{\Gamma}_\alpha(z)| &\le \sup_z |\Psi_\alpha(z) - \rho_\alpha| \frac{3-\beta}{1-\beta} \\ &\le 2\left[(\max(h,c) + \frac{\max(h,p)}{1-\delta(m_0)})m_0 + pc_1\right]\frac{3-\beta}{1-\beta}\end{aligned}$$

Moreover

$$\tilde{\Gamma}_\alpha(z) - \tilde{\Gamma}_\alpha(z') = \Psi_\alpha(z) - \Psi_\alpha(z') + \alpha \int \tilde{\Gamma}_\alpha(\zeta)(f(\zeta|z) - f(\zeta|z'))d\zeta$$

Using properties (2.41), (2.45) and the Assumption (2.29), we can check that

$$|\Psi_\alpha(z) - \Psi_\alpha(z')| \le C(m_0)\delta|z - z'|$$

and thus also

$$|\tilde{\Gamma}_\alpha(z) - \tilde{\Gamma}_\alpha(z')| \le C_1(m_0)\delta|z - z'|$$

Therefore, the functions $\tilde{\Gamma}_\alpha(z)$ are uniformly Lipschitz continuous and bounded. It follows that for a subsequence we obtain

$$\sup_{0\le z\le N} \left|\Gamma_\alpha(z) - \frac{\rho_\alpha}{1-\alpha} - \Gamma(z)\right| \to 0, \ \forall M \tag{2.52}$$

Therefore, also

$$\sup_{\substack{0 \le x \le M \\ 0 \le z \le N}} \left|u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)\right| \to 0, \forall x, \forall M, N \tag{2.53}$$

with

$$u(x,z) = (h-c)x + + \int_0^x \chi(\xi,z)d\xi + \Gamma(z) \tag{2.54}$$

We deduce

$$u(x,z) = (h-c)x + \Gamma(z), \forall x \leq S(z) \tag{2.55}$$

From (2.52), (2.53) it is clear that

$$\Gamma(z) = u(0,z) \tag{2.56}$$

However

$$E[u((S(z)-z_2)^+, z_2|z_1 = z] = E[u(0,z_2)|z_1 = z] + E\left[\int_0^{(S(z)-z_2)^+} (h-c+\chi(\xi,z_2))d\xi|z_1 = z\right]$$

hence

$$\Gamma(z) = -\rho + cS(z) + pG(z) + E[u((S(z)-z_2)^+, z_2|z_1 = z]$$

and thus the first relation (2.32) is proven. Consider now the situation with $x \geq S(z)$. Define the function

$$\tilde{u}(x,z) = hx + pE[(x-z_2)^-|z_1 = z] + E[u((x-z_2)^+, z_2)|z_1 = z]$$

We obtain

$$\begin{aligned} \tilde{u}'(x,z) &= h - p\bar{F}(x|z) + E[u'(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1 = z] \\ &= h - c + \chi(x,z), \quad x \geq S(z) \end{aligned}$$

Also

$$\begin{aligned} \tilde{u}(S(z),z) &= hS(z) + pE[(S(z)-z_2)^-|z_1 = z] + \Gamma(z) \\ &= u(S(z),z) + \rho \end{aligned}$$

From these two relations we get $\tilde{u}(x,z) = u(x,z) + \rho, \forall x \geq S(z)$. This concludes the second part of (2.32). Note also that

$$u_\alpha(x,z) - u_\alpha(x,z') = \Gamma_\alpha(z) - \Gamma_\alpha(z') + \int_0^x (\chi_\alpha(\xi,z) - \chi_\alpha(\xi,z')d\xi$$

Using already proven estimates we obtain

$$|u_\alpha(x,z) - u_\alpha(x,z')| \leq \left[C_0(m_0) + \frac{C_1(m_0)x}{1-\delta_0(x)}\right]\delta|z-z'|$$

The limit $u(x,z)$ satisfies all estimates (2.34), (2.36). To prove (2.37), we first check that

$$u_\alpha(x,z) \leq l(x,z,v) + \alpha E[u_\alpha((x+v-z_2)^+, z_2)|z_1 = z], \forall x,z,v$$

Therefore, it easily follows that

$$u(x,z) + \rho \leq l(x,z,v) + E[u((x+v-z_2)^+, z_2)|z_1 = z], \forall x,z,v$$

However (2.37) can be read as

$$u(x,z) + \rho = l(x,z,\hat{v}(x,z)) + E[u((x+\hat{v}(x,z)-z_2)^+, z_2)|z_1 = z], \forall x,z$$

where

$$\hat{v}(x,z) = \left| \begin{array}{ll} S(z)-x & \text{if } x \leq S(z) \\ \quad 0 & \text{if } x \geq S(z) \end{array} \right.$$

Combining we get equation (2.37). Let us prove uniqueness, for $S(z)$ given. We first prove that $\chi(x,z)$ is uniquely defined. To prove this it is sufficient to prove that if

$$\begin{aligned} \chi(x,z) &= E[\chi(x-z_2,z_2)\mathbf{1}_{x>z_2}|z_1 = z], \ \forall x > S(z) \\ &= 0, \ \forall x \leq S(z) \end{aligned}$$

and

$$(1-\delta_0(x)) \sup_{\substack{0 \leq \xi \leq x \\ z}} |\chi(\xi,z)| < \infty$$

then $\chi(x,z) = 0$. This is clear. The function $\Psi(z)$ is thus uniquely defined. It follows that the pair ρ, $\Gamma(z) = u(0,z)$ is also uniquely defined, with the condition $\int \Gamma(z)\varpi(z)dz = 0$. Therefore, $u(x,z)$ is also uniquely defined. The proof of the theorem has been completed. □

Example 2.1. Consider the situation of independent demands, then $f(x|z) = f(x)$. In that case $\varpi(x) = f(x)$. Then $S(z) = S$, and

$$\rho = (p+h-c)(S-D)^+ - (p-c)S + p\bar{D}$$

We see also that $\Psi(z) = \rho$, and thus $\Gamma(z) = 0$. □

Consider next the situation

$$f(x|z) = \lambda(z) \exp -\lambda(z)x$$

with the assumption $0 < \lambda_0 \leq \lambda(z) \leq \lambda_1$. We also assume that $\lambda(z)$ is Lipschitz continuous. Then all assumptions of Theorem 2.3 are satisfied. □

We turn to the interpretation of the number ρ. Consider the feedback $\hat{v}(x,z)$ associated with the base stock $S(z)$. Define the controlled process

$$\hat{y}_{n+1} = (\hat{y}_n + \hat{v}_n - z_{n+1})^+, \ \hat{v}_n = \hat{v}(\hat{y}_n, z_n)$$

with $\hat{y}_1 = x, \ z_1 = z$. We define the policy $\hat{V} = (\hat{v}_1, \cdots, \hat{v}_n, \cdots)$. We consider the averaged cost

$$J^n_{x,z}(\hat{V}) = \frac{\sum_{j=1}^{n} El(\hat{y}_j, z_j, \hat{v}_j)}{n}$$

Similarly, for any policy $V = (v_1, \cdots, v_n \cdots)$ we define the averaged cost

$$J^n_{x,z}(V) = \frac{\sum_{j=1}^{n} El(y_j, z_j, v_j)}{n}$$

with

$$y_{n+1} = (y_n + v_n - z_{n+1})^+, \quad y_1 = x, \ z_1 = z$$

We state

Proposition 2.1. *We have the property*

$$\rho = \lim_{n\to\infty} J^n_{x,z}(\hat{V}) \tag{2.57}$$

Furthermore, consider the set of policies

$$\mathscr{U} = \{V \mid |u(y_n, z_n)| \leq C_x\}$$

then we have

$$\rho = \inf_{V\in\mathscr{U}} \lim_{n\to\infty} J^n_{x,z}(V) \tag{2.58}$$

Proof. We first notice that

$$\hat{y}_n \leq \max(x, m_0)$$

Therefore, from estimate (2.33), we get

$$|u(\hat{y}_n, z_n)| \leq C_0 + \frac{C_1 \max(x, m_0)}{1 - \delta_0(\max(x, m_0))}$$

Therefore, $\hat{V}$ belongs to $\mathscr{U}$. From (2.37), we can write

$$u(\hat{y}_n, z_n) + \rho = l(\hat{y}_n, z_n, \hat{v}_n) + E[u(\hat{y}_{n+1}, z_{n+1}) | \hat{y}_n, z_n]$$

Taking the expectation and adding up we obtain

$$\rho = J^n_{x,z}(\hat{V}) + \frac{Eu(\hat{y}_{n+1}, z_{n+1}) - u(x,z)}{n}$$

and thus the property (2.57) follows immediately. Similarly, for any policy we can write

$$\rho \leq J^n_{x,z}(V) + \frac{Eu(y_{n+1}, z_{n+1}) - u(x,z)}{n}$$

Therefore, if $V \in \mathscr{U}$, we have $\rho \leq J^n_{x,z}(V)$. This implies (2.58). The proof has been completed. □

Remark 2.2. We cannot state that the process $\hat{y}_n, z_n$ is ergodic. Consequently, we cannot give an interpretation of the function $u(x,z)$ itself.

2.3 No Backlog and Set-Up Cost

2.3.1 Model

We now study the situation of set-up cost, and we consider the no shortage model. We have to study the Bellman equation

$$u(x,z) = \inf_{v \geq 0}[K\mathbf{1}_{v>0} + l(x,z,v) + \alpha\Phi^v u(x,z)] \tag{2.59}$$

where

$$\Phi^v\varphi(x,z) = E[\varphi((x+v-z_2)^+, z_2)|z_1 = z] \tag{2.60}$$

and

$$f(\zeta|z) \text{ is uniformly continuous in both variables and bounded} \tag{2.61}$$

$$\int_0^{+\infty} \zeta f(\zeta|z)d\zeta \leq c_0 z + c_1 \tag{2.62}$$

$$l(x,z,v) = cv + hx + pE[(x+v-z_2)^-|z_1 = z] \tag{2.63}$$

We look for solutions of (2.59) in the interval $[0, w_0(x,z)]$ with

$$w_0(x,z) = l_0(x,z) + \alpha E[w_0(x-z_2, z_2)|z_1 = z] \tag{2.64}$$

In Lemma 2.1, we have proven the estimate

$$w_0(x,z) \leq \frac{hx}{1-\alpha} + \frac{pc_0 z}{1-c_0\alpha} + \frac{pc_1}{(1-\alpha)(1-c_0\alpha)} \tag{2.65}$$

with the assumption

$$c_0\alpha < 1 \tag{2.66}$$

As usual we consider the payoff function

$$J_{x,z}(V) = E\sum_{n=1}^{\infty} \alpha^{n-1}[K\mathbf{1}_{v_n>0} + l(y_n, z_n, v_n)] \tag{2.67}$$

with $V = (v_1, \cdots, v_n, \cdots)$ adapted process with positive values and

$$y_{n+1} = (y_n + v_n - z_{n+1})^+, \;\; y_1 = x \tag{2.68}$$

We define the value function

$$u(x,z) = \inf_V J_{x,z}(V) \tag{2.69}$$

We state

Theorem 2.4. *We assume* (2.60)–(2.63)*,* (2.66)*. The value function defined in* (2.69) *is the unique l.s.c. solution of the Bellman equation* (2.59) *in the interval* $[0, w_0]$*. There exists an optimal feedback* $\hat{v}(x,z)$*.*

2.3.2 $s(z), S(z)$ *Policy*

We now prove the following result.

Theorem 2.5. *We make the assumptions of Theorem 2.4 and* $p > c$*. Then the function* $u(x,z)$ *is* K*-convex and continuous. It tends to* $+\infty$ *as* $x \to +\infty$*. Considering the numbers* $s(z), S(z)$ *associated to* $u(x,z)$*, the optimal feedback is given by*

$$\hat{v}(x,z) = \left| \begin{array}{ll} S(z) - x & \text{if } x \leq s(z) \\ \quad 0 & \text{if } x > s(z) \end{array} \right. \tag{2.70}$$

The functions $s(z)$ $S(z)$ *are continuous.*

Proof. We consider the increasing sequence

$$u_{n+1}(x,z) = \inf_{v\geq 0}\{K\mathbf{1}_{v>0}+cv+hx+pE[(x+v-z_2)^-|z_1 = z]+\alpha E[u_n((x+v-z_2)^+, z_2)|z_1 = z]\} \tag{2.71}$$

with $u_0(x,z) = 0$. Define

$$G_n(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha E[u_n((x-z_2)^+, z_2)|z_1 = z] \quad (2.72)$$

We can write

$$u_{n+1}(x,z) = (h-c)x + \inf_{\eta \geq x}[K\mathbf{1}_{\eta \geq x} + G_n(\eta,z)] \quad (2.73)$$

We are going to show, by induction, that both $u_n(x,z), G_n(x,z)$ are $K-$convex in x, continuous, and $\to +\infty$ as $x \to +\infty$, for $n \geq 1$. The properties are clear for $n = 1$. We assume they are verified for n, we prove them for $n+1$. Since $G_n(x,z)$ is $K-$convex in x, continuous, and $\to +\infty$ as $x \to +\infty$, we can define $s_n(z)$, $S_n(z)$ with

$$G_n(S_n(z),z) = \inf_{\eta} G_n(\eta,z) \quad (2.74)$$

$$s_n(z) = \left| \begin{array}{ll} 0 \quad \text{if} & G_n(0,z) \leq K + \inf_\eta G_n(\eta,z) \\ G_n(s_n(z),z) = K + \inf_\eta G_n(\eta,z) \text{ if} & G_n(0,z) > K + \inf_\eta G_n(\eta,z) \end{array} \right. \quad (2.75)$$

As usual we take the smallest minimum to define $S_n(z)$in a unique way. Since $G_n(\eta,z)$ is continuous, it is easy to check that $S_n(z)$ is continuous. Also $s_n(z)$ is continuous. We can write

$$u_{n+1}(x,z) = (h-c)x + G_n(\max(x,s_n(z)),z)$$

which shows immediately that $u_{n+1}(x,z)$ is K-convex and continuous. Furthermore $u_{n+1}(x,z) \to +\infty$, as $x \to +\infty$. We then have

$$\begin{aligned} G_{n+1}(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha(h-c)E[(x-z_2)^+|z_1 = z] + \\ + \alpha E[G_n(\max(x-z_2,s_n(z_2)),z_2)|z_1 = z] \end{aligned}$$

It is the sum of a convex function and a K-convex function, hence $K-$convex. It is continuous and $\to +\infty$ as $x \to +\infty$. So the recurrence is proven. If we write (formally)

$$u_n'(x,z) = h - c + \chi_n(x,z)$$

then one has the recurrence

$$\begin{aligned} \chi_{n+1}(x,z) &= 0 \quad \text{if} \quad x < s_n(z) \\ &= \mu(x,z) + \alpha E[\chi_n(x-z_2,z_2)|z_1 = z] \end{aligned}$$

with

$$\mu(x,z) = c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x|z) \quad (2.76)$$

The function $\chi_{n+1}(x,z)$ is discontinuous in $s_n(z)$. However one has the bound

$$-\frac{(p-c)}{1-\alpha} \leq \chi_{n+1}(x,z) \leq \frac{c+\alpha(h-c)}{1-\alpha} \quad \text{a.e.}$$

Therefore, the limit $u(x,z)$ is K-convex and satisfies

$$u'(x,z) = h - c + \chi(x,z)$$

with

$$-\frac{(p-c)}{1-\alpha} \leq \chi(x,z) \leq \frac{c+\alpha(h-c)}{1-\alpha} \quad \text{a.e.}$$

Hence $u(x,z)$ is continuous in x and $\to +\infty$ as $x \to +\infty$. Therefore, one defines uniquely $s(z), S(z)$ where $S(z)$ minimizes in x the function

$$G(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha E[u((x-z_2)^+, z_2)|z_1 = z]$$

From this formula and the Lipschitz continuity of u in x, using the Assumption (2.61) one can see that $G(x,z)$ is continuous in both arguments. Hence $S(z)$ and $s(z)$ are continuous. The proof has been completed. □

2.3.3 *Ergodic Theory*

We now study the behavior of $u(x,z)$ as $\alpha \to +\infty$. We denote it by $u_\alpha(x,z)$ and we write the relations

$$u_\alpha(x,z) = (h-c)x + G_\alpha(\max(x, s_\alpha(z)), z) \tag{2.77}$$

$$G_\alpha(x,z) = g_\alpha(x,z) + \alpha E[G_\alpha(\max((x-z_2)^+, s_\alpha(z_2)), z_2)|z_1 = z] \tag{2.78}$$

with

$$g_\alpha(x,z) = cx + pE[(x-z_2)^-|z_1 = z] + \alpha(h-c)E[(x-z_2)^+|z_1 = z] \tag{2.79}$$

We will use an approach different from that of the base stock case, since we cannot prove uniform Lipschitz properties for the function $s_\alpha(z)$. The present method will use less assumptions. We shall assume $c_0 = 0$ and

$$z_n \quad \text{is ergodic} \tag{2.80}$$

We denote by $\varpi(z)$ the invariant measure. We also assume

$$\int |f(\zeta|z) - f(\zeta|z')|d\zeta \leq \delta|z-z'|, \tag{2.81}$$

$$\int \zeta |f(\zeta|z) - f(\zeta|z')| d\zeta \ \le \ \delta |z - z'|$$

$$\sup_z F(x|z) \ = \ \delta_0(x) < 1, \forall x \tag{2.82}$$

$$\sup_z \bar{F}(x|z) \to 0 \text{ as } x \to +\infty \tag{2.83}$$

We begin with

Lemma 2.2. *If* $s(z) > 0$*, then*

$$S(z) \le \frac{p + \alpha(h-c)}{c + \alpha(h-c)} E[z_2|z]$$

Proof. If $s(z) > 0$ then we have

$$u(0,z) = K + G(S(z),z)$$

Set

$$\hat{y}_2 = (S(z) - z_2)^+, \quad \hat{v}_2 = \hat{v}(\hat{y}_2, z_2)$$

then

$$G(S(z),z) = cS(z) + pE[(S(z) - z_2)^-|z_1 = z] + \alpha E[u(\hat{y}_2, z_2)|z_1 = z]$$

Also we can write

$$u(\hat{y}_2, z_2) = (h-c)\hat{y}_2 + K\mathbf{1}_{\hat{v}_2 > 0} + G(\hat{y}_2 + \hat{v}_2, z_2)$$

$$= h\hat{y}_2 + K\mathbf{1}_{\hat{v}_2 > 0} + c\hat{v}_2 + pE[(\hat{y}_2 + \hat{v}_2 - z_3)^-|z_2] + \alpha E[u((\hat{y}_2 + \hat{v}_2 - z_3)^+, z_3)|z_2]$$

Therefore, we can write

$$u(0,z) = K + cS(z) + pE[(S(z) - z_2)^-|z_1 = z] +$$

$$+ \alpha E[h\hat{y}_2 + K\mathbf{1}_{\hat{v}_2 > 0} + c\hat{v}_2 + p(\hat{y}_2 + \hat{v}_2 - z_3)^- + +\alpha u((\hat{y}_2 + \hat{v}_2 - z_3)^+, z_3)|z_1 = z] \tag{2.84}$$

We next have

$$\begin{aligned} u(0,z) &\le G(0,z) \\ &= E[pz_2 + \alpha u(0, z_2)|z_1 = z] \end{aligned}$$

Furthermore

$$u(0, z_2) \le K + G(\hat{y}_2 + \hat{v}_2, z_2)$$

Replacing G and combining the two inequalities we get

$$u(0,z) \leq pE[z_2|z_1 = z] + \alpha K + \\ + \alpha E[c(\hat{y}_2 + \hat{v}_2) + p(\hat{y}_2 + \hat{v}_2 - z_3)^- + \alpha u((\hat{y}_2 + \hat{v}_2 - z_3)^+, z_3)|z_1 = z]$$

Comparing with (2.84) we obtain easily the desired inequality. □

We deduce from the lemma that, whenever $c_0 = 0$

$$s_\alpha(z) \leq \frac{p + (h-c)^+}{\min(c,h)} c_1 \tag{2.85}$$

We now state

Theorem 2.6. *We assume* (2.61), (2.62), *with* $c_0 = 0$, (2.80)–(2.82)*. Then there exists a number* ρ_α *such that, for a subsequence, still denoted* $\alpha \to 1$

$$\sup_{\substack{x \leq M \\ z}} |u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha} - u(x,z)| \to 0, \forall M \tag{2.86}$$

and $\rho_\alpha \to \rho$*. The function* $u(x,z)$ *is Lipschitz continuous,* K*-convex and satisfies*

$$|u(x,z)| \leq C_0 + \frac{C_1 x}{1 - \delta_0(x)} \tag{2.87}$$

$$|u_x(x,z)| \leq \frac{C}{1-\delta_0(x)}, \quad |u_z(x,z)| \leq C_0 + \frac{C_1 x}{1-\delta_0(x)} \; a.e. \tag{2.88}$$

The pair $u(x,z), \rho$ *is the solution of*

$$u(x,z) + \rho = \inf_{v \geq 0} [K\mathbf{1}_{v>0} + l(x,z,v) + \Phi^v u(x,z)] \tag{2.89}$$

Proof. We set

$$\chi_\alpha(x,z) = u'_\alpha(x,z) - h + c$$

then we can write

$$\chi_\alpha(x,z) = \mathbf{1}_{x > s_\alpha(z)} \{\mu_\alpha(x,z) + \alpha E[\chi_\alpha(x - z_2, z_2)\mathbf{1}_{x - z_2 > 0}|z_1 = z] \tag{2.90}$$

with

$$\mu_\alpha(x,z) = g'_\alpha(x,z) \\ = c + \alpha(h-c) - (p + \alpha(h-c))\bar{F}(x|z)$$

The function $\chi_\alpha(x,z)$ is not continuous, but satisfies

$$\sup_{\substack{0 \le \xi \le x \\ z}} |\chi_\alpha(\xi,z)| \le \frac{\max(h,p)}{1-\delta_0(x)} \tag{2.91}$$

Let us next define $\Gamma_\alpha(z) = u_\alpha(0,z)$. We have

$$\begin{aligned}
\Gamma_\alpha(z) &= G_\alpha(s_\alpha(z),z) \\
&= cs_\alpha(z) + pE[(s_\alpha(z)-z_2)^-|z_1 = z] + \alpha E[u_\alpha((s_\alpha(z)-z_2)^+, z_2)|z_1 = z] \\
&= g_\alpha(s_\alpha(z),z) + \alpha E\left[\int_0^{(s_\alpha(z)-z_2)^+} \chi_\alpha(\xi,z_2)d\xi|z_1 = z\right] + \alpha E[\Gamma_\alpha(z_2)|z_1 = z]
\end{aligned}$$

Therefore, we can write

$$\Gamma_\alpha(z) = \Psi_\alpha(z) + \alpha E[\Gamma_\alpha(z_2)|z_1 = z] \tag{2.92}$$

with

$$\Psi_\alpha(z) = g_\alpha(s_\alpha(z),z) + \alpha E\left[\int_0^{(s_\alpha(z)-z_2)^+} \chi_\alpha(\xi,z_2)d\xi|z_1 = z\right]$$

Thanks to Lemma 2.2, we have estimate (2.85), so $s_\alpha(z) \le m_0$. Using estimate (2.91), we see that $|\Psi_\alpha(z)| \le C$. Let us then define

$$\rho_\alpha = \int \Psi_\alpha(z)\varpi(z)dz$$

and $\tilde{\Gamma}_\alpha(z) = \Gamma_\alpha(z) - \dfrac{\rho_\alpha}{1-\alpha}$. From ergodic theory, we can assert that

$$\sup_z |\tilde{\Gamma}_\alpha(z)| \le \sup_z |\Psi_\alpha(z) - \rho_\alpha| \frac{3-\beta}{1-\beta} \le C$$

and we can state the estimate

$$|u_\alpha(x,z) - \frac{\rho_\alpha}{1-\alpha}| \le C + \frac{Cx}{1-\delta_0(x)}$$

Define $\tilde{u}_\alpha(x,z) = u_\alpha(x,z) - \dfrac{\rho_\alpha}{1-\alpha}$. We write equation (2.59) as

$$\tilde{u}_\alpha(x,z) + \rho_\alpha = \inf_{v \ge 0}[K\mathbf{1}_{v>0} + l(x,z,v) + \alpha\Phi^v\tilde{u}_\alpha(x,z)] \tag{2.93}$$

From Lemma 2.2 we can assert that the optimal feedback satisfies $x+\hat{v}_\alpha(x,z) \leq \max(x,m_0)$. Therefore, we can replace (2.93) by

$$\begin{aligned}\tilde{u}_\alpha(x,z) + \rho_\alpha &= \inf_{x+v\leq\max(x,m_0)} [K\mathbf{1}_{v>0} + l(x,z,v) + \alpha\Phi^v\tilde{u}_\alpha(x,z)] \\ &= \inf_{0\leq v\leq\max(x,m_0)-x} L_\alpha(x,z,v)\end{aligned}$$

Next

$$L_\alpha(x,z,v) - L_\alpha(x,z',v) = \int [p(x+v-\zeta)^- + \alpha\tilde{u}_\alpha((x+v-\zeta)^+,\zeta)](f(\zeta|z) - f(\zeta|z'))d\zeta$$

For $v \leq \max(x,m_0) - x$ we can write

$$\begin{aligned}|\tilde{u}_\alpha((x+v-\zeta)^+,\zeta)| &\leq C + \frac{C\max(x,m_0)}{1-\delta_0(\max(x,m_0))} \\ &\leq C_1' + \frac{C_2 x}{1-\delta_0(x)}\end{aligned}$$

Using the Assumption (2.81) we get easily

$$|\tilde{u}_\alpha(x,z) - \tilde{u}_\alpha(x,z')| \leq \left(C_1 + \frac{C_2 x}{1-\delta_0(x)}\right)\delta|z-z'|$$

From the estimates obtained we can assert that $\tilde{u}_\alpha(x,z)$ has a converging subsequence (still denoted α) in the sense

$$\sup_{\substack{x\leq M\\ z\leq N}} |\tilde{u}_\alpha(x,z) - u(x,z)| \to 0, \text{as } \alpha \to 0 \tag{2.94}$$

Since ρ_α is bounded, we can always assume that $\rho_\alpha \to \rho$. Denote

$$L(x,z,v) = K\mathbf{1}_{v>0} + l(x,z,v) + \Phi^v u(x,z)$$

then

$$L_\alpha(x,z,v) - L(x,z,v) = \alpha\Phi^v(\tilde{u}_\alpha - u)(x,z) - (1-\alpha)\Phi^v u(x,z)$$

For $v \leq \max(x,m_0) - x$, we have assuming $M > m_0$

$$\sup_{\substack{x\leq M\\ z}} |\Phi^v u(x,z)| \leq C + \frac{CM}{1-\delta_0(M)}$$

We also have

$$\sup_{\substack{x \leq M \\ z}} |\Phi^v(\tilde{u}_\alpha - u)(x,z)| \leq \sup_{\substack{x \leq M \\ z}} |\tilde{u}_\alpha - u|(x,z) \sup_z \bar{F}(N,z) + \\ + \sup_{\substack{x \leq M \\ z \leq N}} |\tilde{u}_\alpha - u|(x,z)$$

Using the Assumption (2.83) and (2.94), letting first $\alpha \rightarrow 1$, then $N \rightarrow \infty$, we deduce

$$\sup_{\substack{x \leq M \\ v \leq \max(x, m_0) - x \\ z}} |L_\alpha(x,z,v) - L(x,z,v)| \rightarrow 0, \text{ as } \alpha \rightarrow 1$$

Therefore we deduce easily (2.86) and also that the pair $u(x,z), \rho$ satisfies (2.89). Estimates (2.87), (2.88) follow immediately from the corresponding ones on $\tilde{u}_\alpha(x,z)$. The $K-$convexity of u follows from the $K-$convexity of $\tilde{u}_\alpha$. The proof has been completed. □

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