Chapter 2
Conservation of Momentum

Conservation of momentum provides the next basic differential equation of the stellar-structure problem. We will derive this in several steps of gradually increasing generality. The first assumes mechanical equilibrium (Sect. 2.1), the equation of motion for spherical symmetry follows in Sect. 2.4, while in Sect. 2.5 even the assumption of spherical symmetry is dropped. In Sect. 2.6 we briefly discuss general relativistic effects in the case of hydrostatic equilibrium.

2.1 Hydrostatic Equilibrium

Most stars are obviously in such long-lasting phases of their evolution that no changes can be observed at all. Then the stellar matter cannot be accelerated noticeably, which means that all forces acting on a given mass element of the star compensate each other. This mechanical equilibrium in a star is called "hydrostatic equilibrium", since the same condition also governs the pressure stratification, say, in a basin of water. With our assumptions (gaseous stars without rotation, magnetic fields, or close companions), the only forces are due to gravity and to the pressure gradient.

For a given moment of time, we consider a thin spherical mass shell with (an infinitesimal) thickness dr at a radius r inside the star. Per unit area of the shell, the mass is $\varrho\, dr$, and the weight of the shell is $-g\varrho\, dr$. This weight is the gravitational force acting towards the centre (as indicated by the minus sign).

In order to prevent the mass elements of the shell from being accelerated in this direction, they must experience a net force due to pressure of the same absolute value, but acting outwards. This means that the shell must feel a larger pressure P_i at its interior (lower) boundary than the pressure P_e at its outer (upper) boundary (see Fig. 2.1). The total net force per unit area acting on the shell due to this pressure difference is

$$P_\mathrm{i} - P_\mathrm{e} = -\frac{\partial P}{\partial r}\, dr. \tag{2.1}$$

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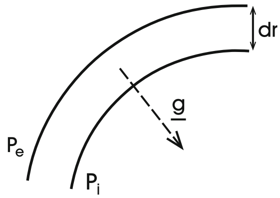

Fig. 2.1 Pressure at the upper and lower border of a mass shell of thickness *dr,* and the vector of gravitational acceleration (*dashed*) acting at one point on the shell

(The right-hand side of this equation is in fact a positive quantity, since P decreases with increasing r.) The sum of the forces arising from pressure and gravity has to be zero,

$$\frac{\partial P}{\partial r} + g\varrho = 0, \tag{2.2}$$

which gives the condition of hydrostatic equilibrium as

$$\frac{\partial P}{\partial r} = -g\varrho. \tag{2.3}$$

This shows the balance of the forces from pressure (left-hand side) and gravity (right-hand side), both per unit volume of the thin shell. Equation (1.8) gives $g = Gm/r^2$ so that (2.3) finally becomes

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\varrho. \tag{2.4}$$

This hydrostatic equation is the second basic equation describing the stellar-structure problem in the Eulerian form (r as an independent variable).

If we take m as the independent variable instead of r, we obtain the hydrostatic condition by multiplying (2.4) with $\partial r/\partial m = (4\pi r^2 \varrho)^{-1}$, according to (1.5) and (1.6):

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}. \tag{2.5}$$

This is the second of our basic equations in the Lagrangian form.

2.2 The Role of Density and Simple Solutions

We have dealt up to now with the distribution of matter, the gravitational field, and the pressure stratification in the star. This purely mechanical problem yielded two differential equations, for example, with m as independent variable (a choice not affecting the discussion),

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho}, \qquad \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}. \tag{2.6}$$

Let us see whether solutions can be obtained at this stage for the problem as stated so far.

We have only two differential equations for three unknown functions, namely r, P, and ϱ. Obviously we can solve this mechanical problem only if we can express one of them in terms of the others, for example, the density ϱ as a function of P. In general, this will not be the case. But there are some exceptional situations where ϱ is a well-known function of P and r or P and m. We can then treat the equations as ordinary differential equations, since they do not contain the time explicitly.

If such integrations are to be carried out starting from the centre, the difficulty occurs that (2.6) are singular there, since $r \to 0$ for $m \to 0$, though one can easily overcome this problem by the standard procedure of expansion in powers of m, as given later in (11.3) and (11.6).

A rather artificial example that can be solved by quadrature is $\varrho = \varrho(m)$, in particular ϱ = constant in the homogeneous gaseous sphere.

Physically more realistic are solutions obtained for the so-called *barotropic* case, for which the density is a function of the gas pressure only: $\varrho = \varrho(P)$. A simple example would be a perfect[1] gas at constant temperature. After assuming a value P_c for the central pressure, both equations (2.6) have to be solved simultaneously, since $\varrho(P)$ in the first of them is not known before P is evaluated.

As we will see later (for instance, in Sects. 19.3 and 19.8), there are also cases for which no choice of P_c yields a surface of zero pressure at finite values of r. In the theory of stellar structure there is even a use for these types of solution.

Among the barotropic solutions is a wide class of models for gaseous spheres called *polytropes.* These important solutions will later be discussed extensively (Chap. 19). Barotropic solutions also describe white dwarfs, i.e. stars that really exist (Sect. 37.1).

But in general the density is not a function of pressure only but depends also on the temperature T. For a given chemical composition of the gas, its thermodynamic behaviour yields an equation of state of the form $\varrho = \varrho(P, T)$. A well-known case is that of a perfect gas, where

$$\varrho = \frac{\mu}{\Re}\frac{P}{T} \tag{2.7}$$

with the gas constant $\Re = 8.315 \times 10^7\,\mathrm{erg\,K^{-1}\,g^{-1}}$ (which we define per g instead of per mole), while μ is the (dimensionless) mean molecular weight, i.e. the average number of atomic mass units per particle; in the case of ionized hydrogen, $\mu = 0.5$ (see Sect. 4.2).

[1]Throughout this book we will use the terms *perfect* and *ideal* gas synonymously, as they describe the same physical concept.

Once the temperature appears in the equation of state and cannot be eliminated by means of additional conditions, it then becomes much more difficult to determine the internal structure of a self-gravitating gaseous sphere. The mechanical structure is then also determined by the temperature distribution, which in turn is coupled to the transport and generation of energy in the star. This requires new equations, with which we shall deal in Chaps. 4 and 5.

2.3 Simple Estimates of Central Values P_c, T_c

The hydrostatic condition (2.5) together with an equation of state for a perfect gas (2.7) enables us to estimate the pressure and the temperature in the interior of a star of given mass and radius.

Let us replace the left-hand side of (2.5) by an average pressure gradient $(P_0 - P_c)/M$, where $P_0(= 0)$ and P_c are the pressures at the surface and at the centre. On the right-hand side of (2.5) we replace m and r by rough mean values $M/2$ and $R/2$, and we obtain

$$P_c \approx \frac{2GM^2}{\pi R^4}. \tag{2.8}$$

From the equation of state for a perfect gas, and with the mean density

$$\bar{\varrho} = \frac{3M}{4\pi R^3}, \tag{2.9}$$

we find with (2.8) that

$$\begin{aligned} T_c &= \frac{P_c}{\varrho_c}\frac{\mu}{\Re} = P_c \frac{\mu}{\Re}\frac{\bar{\varrho}}{\varrho_c}\frac{4\pi R^3}{3M} \\ &\approx \frac{8}{3}\frac{\mu}{\Re}\frac{Gm}{R}\frac{\bar{\varrho}}{\varrho_c}. \end{aligned} \tag{2.10}$$

Since in most stars the density increases monotonically from the surface to the centre, we have $\bar{\varrho}/\varrho_c < 1$ (Numerical solutions show that $\bar{\varrho}/\varrho_c \approx 0.03 \ldots 0.01.$). Therefore (2.10) yields

$$T_c \lesssim \frac{8}{3}\frac{G\mu}{\Re}\frac{M}{R}. \tag{2.11}$$

With the mass and the radius of the Sun ($M_\odot = 1.989 \times 10^{33}$ g, $R_\odot = 6.96 \times 10^{10}$ cm) and with $\mu = 0.5$, we find that

$$P_c \approx 7 \times 10^{15}\,\mathrm{dyn/cm^2}, \quad T_c < 3 \times 10^7\,\mathrm{K}. \tag{2.12}$$

Modern numerical solutions (Chap. 29) give $P_c = 2.4 \times 10^{17}$ dyn/cm^2, $T_c = 1.6 \times 10^7$ K.

So we can expect to encounter enormous pressures and very high temperatures in the central regions of the stars. Moreover, our assumption of a perfect gas turns out to be fully justified for these values of P and T.

2.4 The Equation of Motion for Spherical Symmetry

Our equation of hydrostatic equilibrium (2.5) is a special case of conservation of momentum. If the (spherical) star undergoes accelerated radial motions, we have to consider the inertia of the mass elements, which introduces an additional term. We confine ourselves here to the Lagrangian description (m, t as independent variables), which is especially convenient for spherical symmetry.

We go back to the derivation of the hydrostatic equation in Sect. 2.1 and again consider a thin shell of mass dm at the distance r from the centre (Fig. 1.1). Owing to the pressure gradient, this shell experiences a force per unit area f_P given by (2.1), the right-hand side of which is easily rewritten in terms of $\partial P/\partial m$ according to (1.7):

$$f_P = -\frac{\partial P}{\partial m}\, dm. \tag{2.13}$$

The gravitational force per unit area acting on the mass shell is, with the use of (1.8),

$$f_g = -\frac{g\, dm}{4\pi r^2} = -\frac{Gm}{r^2}\frac{dm}{4\pi r^2}. \tag{2.14}$$

If the sum of the two forces is not equal to zero, the mass shell will be accelerated according to

$$\frac{dm}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = f_P + f_g. \tag{2.15}$$

This gives with (2.13) and (2.14) the equation of motion as

$$\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{\partial P}{\partial m} - \frac{Gm}{4\pi r^4}. \tag{2.16}$$

The signs in (2.16) are such that the pressure gradient alone would produce an outward acceleration (since $\partial P/\partial m < 0$), while the gravity alone would produce an inward acceleration.

Equation (2.16) would give exactly the equation of hydrostatic equilibrium (2.5) if the second time derivative of r vanished, i.e. if all mass elements were at rest or moved radially at constant velocity. Moreover, the term on the left-hand side is certainly unimportant if its absolute value is small compared to the absolute values of any term on the right, i.e. if the two terms on the right-hand side cancel each other nearly to zero. Then the hydrostatic condition is a very good approximation, and the configuration moves through neighbouring near-equilibrium states. In this sense we are allowed to apply the simpler hydrostatic equation to a much wider class of solutions than those fulfilling the strict requirement $\partial^2 r/\partial t^2 = 0$. To illustrate this further we assume a deviation from hydrostatic equilibrium such that, for example, in (2.16), the pressure term suddenly "disappears". The inertial term on the left would then have to compensate the gravitational term on the right. We now define a characteristic time-scale τ_{ff} for the ensuing collapse of the star by setting $|\partial^2 r/\partial t^2| = R/\tau_{\mathrm{ff}}^2$. Then we obtain from (2.16) $R/\tau_{\mathrm{ff}}^2 \approx g$, or

$$\tau_{\mathrm{ff}} \approx \left(\frac{R}{g}\right)^{1/2} . \tag{2.17}$$

This is some kind of a mean value for the free-fall time over a distance of order R following the sudden disappearance of the pressure. We can correspondingly determine a timescale τ_{expl} for the explosion of our star for the case that gravity were suddenly to disappear: $R/\tau_{\mathrm{expl}}^2 = P/\varrho R$, where we have replaced $\partial P/\partial r$ by P/R after writing $4\pi r^2(\partial P/\partial m) = (\partial P/\partial r)/\varrho$ (P and ϱ are here average values over the entire star). We then find that

$$\tau_{\mathrm{expl}} \approx R\left(\frac{\varrho}{P}\right)^{1/2} . \tag{2.18}$$

Since $(P/\varrho)^{1/2}$ is of the order of the mean velocity of sound in the stellar interior, one can see that τ_{expl} is of the order of the time a sound wave needs to travel from the centre to the surface.

If our model is near hydrostatic equilibrium, then the two terms on the right side of (2.16) have about equal absolute value and $\tau_{\mathrm{ff}} \approx \tau_{\mathrm{expl}}$. We then call this timescale the *hydrostatic timescale* τ_{hydr}, since it gives the typical time in which a (dynamically stable) star reacts on a slight perturbation of hydrostatic equilibrium. With $g \approx GM/R^2$, we obtain from (2.17) up to factors of order 1 that

$$\tau_{\mathrm{hydr}} \approx \left(\frac{R^3}{GM}\right)^{1/2} \approx \frac{1}{2}(G\bar{\varrho})^{-1/2}. \tag{2.19}$$

In the case of the Sun we find the surprisingly small value $\tau_{\mathrm{hydr}} \approx 27$ min. Even in the case of a red giant ($M \approx M_\odot, R \approx 100 R_\odot$), one has only $\tau_{\mathrm{hydr}} \approx 18$ days, while for a white dwarf ($M \approx M_\odot, R \approx R_\odot/50$), the hydrostatic timescale is extremely short: $\tau_{\mathrm{hydr}} \approx 4.5$ s. In most phases of their life the stars change slowly on a timescale that is very long compared to τ_{hydr}. Then they are very close to hydrostatic equilibrium and the inertial terms in (2.16) can be ignored.

2.5 The Non-spherical Case

Up to now we have dealt with spherically symmetric configurations only. It is easy to see how the equations would have to be modified for more general cases without this symmetry.

After rewriting (2.16) for the independent variable r, we easily identify it as a special case of the Eulerian equation of motion of hydrodynamics

$$\varrho \frac{dv}{dt} = -\nabla P - \varrho \nabla \Phi, \tag{2.20}$$

where v is the velocity vector, and the substantial time derivative on the left is defined by the operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \tag{2.21}$$

The general form of (1.4) has already been shown to be the continuity equation of hydrodynamics

$$\frac{\partial \varrho}{\partial t} = -\nabla \cdot (\varrho \boldsymbol{v}), \tag{2.22}$$

and, as described in Sect. 1.3, the gravitational potential Φ is connected with an arbitrary distribution of the density by the Poisson equation (1.9):

$$\nabla^2 \Phi = 4\pi G \varrho. \tag{2.23}$$

We see in fact that the stellar-structure equations discussed up to now are just special cases of normal textbook hydrodynamics.

2.6 Hydrostatic Equilibrium in General Relativity

To help with subsequent work (Chap. 38), we briefly refer to the change of the equation of hydrostatic equilibrium due to effects of general relativity. For details see, for example, Zeldovich and Novikov (1971).

Very strong gravitational fields, as in the case of neutron stars, are described by the Einstein field equations

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{\kappa}{c^2} T_{ik}, \quad \kappa = \frac{8\pi G}{c^2}, \tag{2.24}$$

where R_{ik} is the Ricci tensor, g_{ik} is the metric tensor and the scalar R is the Riemann curvature. T_{ik} is the energy-momentum tensor, which for a perfect gas has as the only non-vanishing components $T_{00} = \varrho c^2$, $T_{11} = T_{22} = T_{33} = P$ (ϱ includes the energy density, P = pressure). We are interested in static (time-independent), spherically symmetric mass distributions. Then the line element *ds,* i.e. the distance between two neighbouring events, is given in spherical coordinates (r, ϑ, φ) by the general form

$$ds^2 = \mathrm{e}^{\nu} c^2 dt^2 - \mathrm{e}^{\lambda} dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \tag{2.25}$$

with $\nu = \nu(r), \lambda = \lambda(r)$. With these expressions for T_{ik} and *ds,* the field equations (2.24) can be reduced to three ordinary differential equations:

$$\frac{\kappa P}{c^2} = \mathrm{e}^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}, \tag{2.26}$$

$$\frac{\kappa P}{c^2} = \frac{1}{2} \mathrm{e}^{-\lambda} \left(\nu'' + \frac{1}{2} \nu'^2 + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right), \tag{2.27}$$

$$\kappa \varrho = \mathrm{e}^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \tag{2.28}$$

where primes denote derivatives with respect to r. After multiplication with $4\pi r^2$, (2.28) can be integrated giving

$$\kappa m = 4\pi r (1 - \mathrm{e}^{-\lambda}). \tag{2.29}$$

Here m denotes the *gravitational mass* inside r defined by

$$m = \int_0^r 4\pi r^2 \varrho \, dr. \tag{2.30}$$

For $r = R, m$ becomes the gravitational mass M of the star. It is the mass a distant observer would measure by its gravitational effects, for example, on orbiting planets. It is not, however, the mass which we naïvely identify with the baryon number times the atomic mass unit: M contains not only the rest mass, but the whole energy (divided by c^2). This includes the internal and the gravitational energy, the latter being negative and reducing the gravitational mass (just as the binding energy of a nucleus results in a mass defect; see Chap. 18). The seemingly familiar form of (2.30) is treacherous. First of all, $\varrho = \varrho_0 + U/c^2$ contains the whole energy density U as well as the rest-mass density ϱ_0, and the changed metric would give the spherical volume element as $\mathrm{e}^{\lambda/2} 4\pi r^2 \, dr$ instead of the usual form $4\pi r^2 \, dr$ [over which (2.30) is integrated].

Differentiation of (2.26) with respect to r gives $P' = P'(\lambda, \lambda', \nu', \nu'', r)$. When $\lambda, \lambda', \nu', \nu''$ are eliminated by (2.26), (2.27) and (2.29), one arrives at the

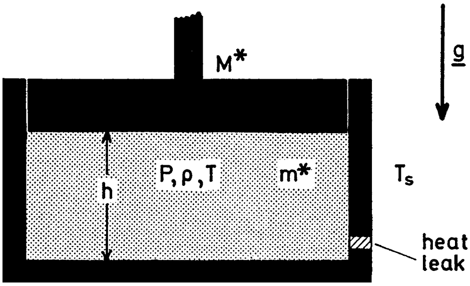

Fig. 2.2 The piston model. Gas of mass m^* (with pressure P, density ϱ, temperature T) is held in a container with a movable piston of mass M^*. The gravitational acceleration g acts on the piston. The container is embedded in a medium of temperature T_s; a possible heat leak is indicated (*dashed*) in the right wall of the container. In Chap. 2, only the mechanical properties of the model are discussed

Tolman-Oppenheimer-Volkoff (TOV) equation for hydrostatic equilibrium in general relativity:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\varrho\left(1+\frac{P}{\varrho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1}. \tag{2.31}$$

Obviously this reverts to the usual form (2.4) for $c^2 \to \infty$.

For gravitational fields that are not too large (small deviations from Newtonian mechanics), one can expand the product of the parentheses in (2.31) and retain only terms linear in $1/c^2$. This gives the so-called *post-Newtonian approximation:*

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\varrho\left(1+\frac{P}{\varrho c^2}+\frac{4\pi r^3 P}{mc^2}+\frac{2Gm}{rc^2}\right). \tag{2.32}$$

2.7 The Piston Model

From time to time we shall make use of a simple mechanical model which in some respects mimics the behaviour of stars, and which is shown in Fig. 2.2. A piston of mass M^* encloses a gas of mass m^* in a box. $G^* = gM^*$ is the weight of the piston in a gravitational field described by the gravitational acceleration g. A is the cross-sectional area of the piston and h its height above the bottom. Then $V = Ah$ is the volume of the gas, while its density is $\varrho = m^*/V$.

In the case of hydrostatic equilibrium, the gas pressure P adjusts in such a way that the weight per unit area is balanced by the pressure:

$$G^* = PA. \tag{2.33}$$

If the forces do not compensate each other, the piston is accelerated in the vertical direction according to the equation of motion

$$M^* \frac{d^2h}{dt^2} = -G^* + PA. \tag{2.34}$$

In a similar manner to our considerations of Sect. 2.4, we can define two timescales $\tau_{\rm ff}$ and $\tau_{\rm expl}$:

$$\tau_{\rm ff} \approx \left(\frac{h}{g}\right)^{1/2}, \tag{2.35}$$

$$\tau_{\rm expl} \approx h \left(\frac{\varrho}{P}\right)^{1/2} \left(\frac{M^*}{m^*}\right)^{1/2}. \tag{2.36}$$

In the limit of hydrostatic equilibrium both timescales are the same, and we then call $\tau_{\rm ff} = \tau_{\rm expl}$ the hydrostatic timescale $\tau_{\rm hydr}$.

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