

# Preface

An operad is an algebraic device which encodes a type of algebras. Instead of studying the properties of a particular algebra, we focus on the universal operations that can be performed on the elements of any algebra of a given type. The information contained in an operad consists in these operations and all the ways of composing them. The classical types of algebras, that is associative algebras, commutative algebras and Lie algebras, give the first examples of algebraic operads. Recently, there has been much interest in other types of algebras, to name a few: Poisson algebras, Gerstenhaber algebras, Jordan algebras, pre-Lie algebras, Batalin–Vilkovisky algebras, Leibniz algebras, dendriform algebras and the various types of algebras up to homotopy. The notion of operad permits us to study them conceptually and to compare them.

The operadic point of view has several advantages. First, many results known for classical types of algebras, when written in the operadic language, can be applied to other types of algebras. Second, the operadic language simplifies both the statements and the proofs. So, it clarifies the global understanding and allows one to go further. Third, even for classical algebras, the operad theory provides new results that had not been unraveled before. Operadic theorems have been applied to prove results in other fields, like the deformation-quantization of Poisson manifolds by Maxim Kontsevich and Dmitry Tamarkin for instance. Nowadays, operads appear in many different themes: algebraic topology, differential geometry, noncommutative geometry,  $C^*$ -algebras, symplectic geometry, deformation theory, quantum field theory, string topology, renormalization theory, combinatorial algebra, category theory, universal algebra and computer science.

Historically, the theoretical study of compositions of operations appeared in the 1950s in the work of Michel Lazard as “analyseurs”. Operad theory emerged as an efficient tool in algebraic topology in the 1960s in the work of Frank Adams, J. Michael Boardmann, André Joyal, Gregory Kelly, Peter May, Saunders McLane, Jim Stasheff, Rainer Vogt and other topologists and category theorists. In the 1990s, there was a “renaissance” of the theory in the development of deformation theory and quantum field theory, with a shift from topology to algebra, that can be found in the work of Ezra Getzler, Victor Ginzburg, Vladimir Hinich, John Jones, Mikhail

Kapranov, Maxim Kontsevich, Yuri I. Manin, Martin Markl, Vadim Schechtman, Vladimir Smirnov and Dmitry Tamarkin for instance. Ten years later, a first monograph [MSS02] on this subject was written by Martin Markl, Steve Shnider and Jim Stasheff in which one can find more details on the history of operad theory.

Now, 20 years after the renaissance of the operad theory, most of the basic aspects of it have been settled and it seems to be the right time to provide a comprehensive account of algebraic operad theory. This is the purpose of this book.

One of the main fruitful problems in the study of a given type of algebras is its relationship with algebraic homotopy theory. For instance, starting with a chain complex equipped with some compatible algebraic structure, can this structure be transferred to any homotopy equivalent chain complex? In general, the answer is negative. However, one can prove the existence of higher operations on the homotopy equivalent chain complex, which endow it with a richer algebraic structure. In the particular case of associative algebras, this higher structure is encoded into the notion of associative algebra up to homotopy, alias  $A$ -infinity algebra, unearthed by Stasheff in the 1960s. In the particular case of Lie algebras, it gives rise to the notion of  $L$ -infinity algebras, which was successfully used in the proof of the Kontsevich formality theorem. It is exactly the problem of governing these higher structures that prompted the introduction of the notion of operad.

Operad theory provides an explicit answer to this transfer problem for a large family of types of algebras, for instance those encoded by Koszul operads. Koszul duality was first developed at the level of associative algebras by Stewart Priddy in the 1970s. It was then extended to algebraic operads by Ginzburg and Kapranov, and also Getzler and Jones in the 1990s (part of the renaissance period). The duality between Lie algebras and commutative algebras in rational homotopy theory was recognized to coincide with the Koszul duality theory between the operad encoding Lie algebras and the operad encoding commutative algebras. The application of Koszul duality theory for operads to homotopical algebra is a far-reaching generalization of the ideas of Dan Quillen and Dennis Sullivan.

The aim of this book is, first, to provide an introduction to algebraic operads, second, to give a conceptual treatment of Koszul duality, and, third, to give applications to homotopical algebra.

We begin by developing the general theory of twisting morphisms, whose main application here is the Koszul duality theory for associative algebras. We do it in such a way that this pattern can be adapted to the operad setting. After giving the definition and the main properties of the notion of operad, we develop the operadic homological algebra. Finally, Koszul duality theory of operads permits us to study the homotopy properties of algebras over an operad.

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