Chapter 2
Equilibrium and Non-Equilibrium Bosonization

In this chapter we consider one of the most important theoretical methods in one-dimensional physics—the bosonization approach. This method allows one to exactly diagonalize the Hamiltonian of interacting fermions [1, 2]. The main result of this method is that the excitation spectrum of such a system in one-dimension consist of gapless boson modes, in striking contrast with the Fermi liquid picture. This is because the restricted dimensions enhance the scattering between the electrons and completely destroy the quasi-particle picture in one dimension. We start with the basic introduction to the bosonization approach in Sect. 2.1. Then we compute the equilibrium correlation functions of interacting fermions in the framework of this approach. Finally, in Sect. 2.3 we generalize the bosonization technique to the non-equilibrium situation and introduce a new method—the non-equilibrium bosonization, which will be used in several chapters of this thesis.

2.1 Bosonization of One-Dimensional Fermions

2.1.1 One-Dimensional Interacting Systems

Let us consider a system of one-dimensional fermions. These can be electrons in quantum wires, quantum Hall edge states, etc. Here we would like to introduce a general approach, where one considers n one-dimensional channels of either same or different chiralities. In the case of integer quantum Hall effect, ν edge states have the same chirality, while in fractional Hall states some one-dimensional channels might have opposite chiralities. Let us denote the operator of annihilation of an electron in a state with wave vector k in αth channel as $c_{k\alpha}$. These operators obey the fermionic

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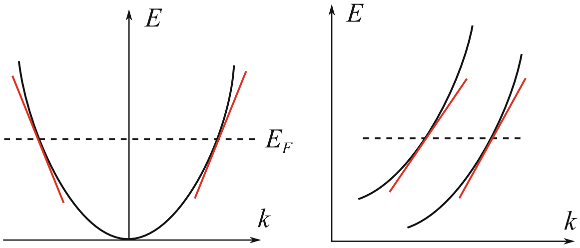

Fig. 2.1 Illustration of the linearization of spectrum of one-dimensional electrons. *Left panel* shows the spectrum of electrons in quantum wires. The electrons with energies near the Fermi energy E_F give a contribution to transport. Thus, one can linearize the actual spectrum (*black curve*) and replace it with two branches (*red*, *straight lines*) of left and right moving chiral electrons with the same Fermi velocity. *Right panel* shows the situation at the edge of a two-dimensional electron gas in the regime of integer quantum Hall effect. Here the Fermi surface is located near the actual edge of the sample. Different branches correspond to different Landau levels.The spectrum can be linearized in this case too, and one obtains several channels of the same chirality. Note that the Fermi velocities are generally different in the case of integer quantum Hall effect (Color figure online)

anti-commutation relations, i.e.,

$$\{c^\dagger_{k\alpha}, c_{k'\beta}\} = \delta_{kk'}\delta_{\alpha\beta}, \tag{2.1}$$

while all other anti-commutators are zero. The free single-particle Hamiltonian can be written as:

$$\mathcal{H}_0 = \sum_\alpha \sum_{k=-\Lambda}^{\Lambda} \epsilon_{k\alpha} c^\dagger_{k\alpha} c_{k\alpha}, \tag{2.2}$$

where $\epsilon_{k\alpha}$ are single-particle energies (see Fig. 2.1), and we introduced an ultraviolet cut-off Λ. If one is interested in low energy physics, such as a transport through the system, then one may choose $\Lambda \ll E_F$ and linearize the spectrum of fermions, so that $\epsilon_{k\alpha} = v_{F\alpha}k$, where the wave vector in each channel is defined with respect to the corresponding Fermi wave vector. For convenience, we also redefine the index α so that it enumerates the *Fermi points*, for example, the left and right movers in a quantum wire, referred as two different channels.

Next, we would like to introduce the set of fermion field operators in each channel as follows:

$$\psi_\alpha(x) = \frac{1}{\sqrt{W}} \sum_k c_{k\alpha} e^{ikx}, \tag{2.3}$$

where W is the length of the system. These operators obey the standard anti-commutation relations $\{\psi^\dagger_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y)$, and the free Hamiltonian (2.2) can be written in terms of these operators as following:

$$\mathcal{H}_0 = -i \sum_\alpha v_{F\alpha} \int dx \psi^\dagger_\alpha(x) \partial_x \psi_\alpha(x). \tag{2.4}$$

A naive definition of the charge density operator at a point x in terms of the fermion field is $\rho_\alpha(x) = \psi^\dagger_\alpha(x)\psi_\alpha(x)$. However, such definition leads to the divergent result for the ground state density:

$$\langle\rho_\alpha(x)\rangle = (1/W)\sum_{k,k'} e^{ix(k'-k)}\langle c^\dagger_{k\alpha}c_{k'\alpha}\rangle = (1/W)\sum_k \langle c^\dagger_{k\alpha}c_{k\alpha}\rangle. \tag{2.5}$$

This result is partly an artifact of the introduced linearization, but in the case of quantum Hall effect it has a physical meaning. Namely, there is no preferred choice of minimal momentum, starting from which we attribute a state to the edge and not to the bulk. Fortunately, this divergent contribution is constant. Therefore, we define the edge density operator where this divergent constant is subtracted:

$$\rho_\alpha(x) = :\psi^\dagger_\alpha(x)\psi_\alpha(x): \equiv \psi^\dagger_\alpha(x)\psi_\alpha(x) - \langle\psi^\dagger_\alpha(x)\psi_\alpha(x)\rangle. \tag{2.6}$$

One can check that the fermion operator annihilates a unit local charge:

$$[\psi_\alpha(x), \rho_\beta(y)] = \delta_{\alpha\beta}\psi_\alpha(x)\delta(x-y). \tag{2.7}$$

Finally, we would like to take into account the fact that electrons interact with each other. In general, the interaction Hamiltonian is a functional of the density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}[\rho_\alpha(x)]$. Here we will consider only the case of (screened) Coulomb interaction, so that:

$$\mathcal{H}_{\text{int}} = \iint dxdy U_{\alpha\beta}(x-y)\rho_\alpha(x)\rho_\beta(y), \tag{2.8}$$

where $U_{\alpha\beta}(x-y)$ is the Coulomb interaction potential.

2.1.2 Bosonic Fields and Hamiltonian

In this section we would like to introduce and investigate the Fourier modes of the density field which are defined as following,

$$\rho_{k\alpha} \equiv (1/W)\int dx e^{ikx}\rho_\alpha(x) = (1/W)\sum_{k'} c^\dagger_{k+k'\alpha}c_{k'\alpha}, \quad k \neq 0, \tag{2.9}$$

and the zero mode operator:

$$\pi_\alpha = (1/W)\int \rho_\alpha(x)dx, \tag{2.10}$$

representing the homogeneous part of the charge density. One can check the relation $\rho_{-k\alpha} = \rho^\dagger_{k\alpha}$, which follows from the fact that the density field is real. Importantly, the operators (2.9) have bosonic nature, namely, one can check that the commutators of two such operators are zero:

$$[\rho_{k\alpha}, \rho_{k'\beta}] = 0, \quad k \neq -k'. \tag{2.11}$$

However, for the case $k' = -k$ and $\alpha = \beta$ the above commutator is ill defined if the summation in the definition (2.9) goes up to infinity. The expression for this commutator can be written in terms of the occupation number operators $n_{k\alpha} \equiv c^{\dagger}_{k\alpha} c_{k\alpha}$:

$$[\rho_{k\alpha}, \rho_{-k\alpha}] = (1/W^2)\big[\sum_{k'<-\Lambda} (n_{k'-k,\alpha} - n_{k'\alpha}) + \sum_{k'\geq-\Lambda} (n_{k'-k,\alpha} - n_{k'\alpha})\big]. \tag{2.12}$$

If we assume, that all the states below the cut-off are always filled, then we can shift the summation variable in the second sum and get

$$[\rho_{k\alpha}, \rho_{-k\alpha}] = \pm\frac{k}{2\pi W}, \tag{2.13}$$

where we have taken into account the quantization of the wave vector $\Delta k = 2\pi/W$, and the sign stays for the chirality of the corresponding channel. This anomalous commutator will be important in Chap. 6 for the connection between the bulk and the edge effective theories in the classification of the edge model of fractional quantum Hall effect. Here we only note, that this commutation relation completes the canonical bosonization of the density operators.

The most important step in the bosonization procedure is to express the fermion fields in terms of the density fields [1, 3]. Namely, one can check that the expression:

$$\psi_\alpha(x) = \frac{1}{\sqrt{a}} \exp\left[i\varphi_\alpha + 2\pi i \cdot \pi_\alpha x + \sum_k \frac{2\pi i}{k} \rho_{k\alpha} e^{ikx} \right], \tag{2.14}$$

where φ_α is zero mode canonically conjugated to π_α and $a = 2\pi/\Lambda$, has all the properties of the electron operator. First of all, one can check that the commutation relation with charge density operators (2.7) holds for the operator (2.14). Second, the operator (2.14) has fermionic statistics:

$$\{\psi^{\dagger}_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y). \tag{2.15}$$

In fact, it is more convenient to introduce new fields $\phi_\alpha(x)$, related to the densities as

$$\rho_\alpha(x) = \frac{1}{2\pi}\partial_x \phi_\alpha(x). \tag{2.16}$$

Then, it follows from (2.16), (2.11) and (2.13) that these fields satisfy the commutation relations of the following form:

$$[\phi_\alpha(x), \phi_\beta(y)] = \pm i\pi \mathrm{sgn}(x-y)\delta_{\alpha\beta} \tag{2.17}$$

where the sign stays for the chirality of the corresponding channel. In terms of the boson fields ϕ_α, Eq. (2.14) becomes much simpler:

$$\psi_\alpha(x) = \frac{1}{\sqrt{a}} e^{i\phi_\alpha(x)}. \tag{2.18}$$

This expression is one of the main equations of the bosonization approach.

In the next step we represent the Hamiltonian (2.2) in terms of the boson fields as well. In order to do so, we note that the equation of motion for the density modes generated by this Hamiltonian is given by:

$$[\mathcal{H}_0, \rho_{k\alpha}] = v_{F\alpha} k \rho_{k\alpha}. \tag{2.19}$$

Taking into account the commutation relations (2.11), (2.13) for bosons, one concludes that the Hamiltonian could be expressed as follows:

$$\mathcal{H}_0 = \sum_\alpha \frac{v_{F\alpha}}{4\pi} \int dx (\partial_x \phi_\alpha(x))^2 \tag{2.20}$$

This result can be obtained alternatively substituting Eq. (2.18) into Eq. (2.4) and performing point splitting procedure [4]. Thus, we finally conclude, that in the case of interactions sensitive to the long wavelength part of the densities, the total Hamiltonian becomes *quadratic*:

$$\mathcal{H} \equiv \mathcal{H}_0 + \mathcal{H}_{\mathrm{int}} = \sum_{\alpha\beta} \iint \frac{dxdy}{8\pi^2} V_{\alpha\beta}(x-y) \partial_x \phi_\alpha(x) \partial_y \phi_\beta(y), \tag{2.21}$$

where the interaction potential is simply shifted by the Fermi velocity,

$$V_{\alpha\beta} = U_{\alpha\beta} + 2\pi v_{F\alpha} \delta_{\alpha\beta} \delta(x-y). \tag{2.22}$$

In fact, the effective edge theory action of fractional quantum Hall edge states (1.53) in absence of external fields coincides with the result of the bosonization approach. This means that the boson language is more universal than the fermion one, since the fermion single particle description exists only for integer quantum Hall effect, while the boson action of the same structure can be used to describe both integer and fractional quantum Hall effect.

2.1.3 Quantization of Boson Fields and Zero Modes

In this section we consider some important details of the above discussed construction, namely, the role of zero modes. The boson fields $\phi_\alpha(x)$ can be written in terms of boson creation and annihilation operators $a^\dagger_{k\alpha} \equiv \sqrt{W/2\pi k}\rho_{k\alpha}$ and $a_{k\alpha} \equiv \sqrt{W/2\pi k}\rho_{-k\alpha}$, which commutes as $[a^\dagger_{k\alpha}, a_{k'\beta}] = \delta_{\alpha\beta}\delta_{kk'}$, and in terms of the modes φ_α, π_α, as

$$\phi_\alpha(x) = \varphi_\alpha + 2\pi \cdot \pi_\alpha \, x + \sum_{k>0} \sqrt{\frac{2\pi}{Wk}} \left[a_{k\alpha} e^{ikx} + a^\dagger_{k\alpha} e^{-ikx} \right]. \tag{2.23}$$

We would like to recall that zero modes satisfy the canonical commutation relation $[\pi_\alpha, \varphi_\alpha] = i/W$, where W is the total size of the system. In the end of calculations we take the thermodynamic limit $W \to \infty$, so that W drops from the final results. The Hamiltonian acquires the following form in terms of the introduced operators:

$$\mathcal{H} = (1/2\pi) \sum_{k\alpha\beta} k \, V_{\alpha\beta}(k) a^\dagger_{k\alpha} a_{k\beta} + (W/2) \sum_{\alpha\beta} V_{\alpha\beta}(0) \pi_\alpha \pi_\beta, \tag{2.24}$$

where $V_{\alpha\beta}(k)$ is the Fourier transform of the potential (2.22).

The vacuum for collective excitations is defined as $a_{k\alpha}|0\rangle = 0$. The special care has to be taken about zero modes, because as we show in Sect. 3.4, zero modes determine charging effects and phase shifts, which are not small and important for the explanation of some experimental results in Chap. 3. From the definition (2.10) it is clear that the zero mode π_α has a meaning of a homogeneous density at the channel α. Therefore, we define "vacuum charges" Q_α

$$\pi_\alpha |0\rangle = Q_\alpha |0\rangle, \tag{2.25}$$

which are in fact charge densities at the one-dimensional channels accumulated in accordance with electrochemical potential in a given channel. The energy E_0 of the ground state, defined as $\mathcal{H}|0\rangle = E_0|0\rangle$, is then given by

$$E_0 = (W/2) \sum_{\alpha\beta} V_{\alpha\beta}(0) Q_\alpha Q_\beta. \tag{2.26}$$

Since the edge excitations in integer quantum Hall effect propagate along the equipotential lines, edge channels can be considered as metallic surfaces. We therefore can apply the well known electrostatic relation [5] for the potentials $\Delta\mu_\alpha$ of the edge channels:

$$\Delta\mu_\alpha \equiv (1/W)\delta E_0/\delta Q_\alpha = \sum_\beta V_{\alpha\beta}(0) Q_\beta. \tag{2.27}$$

Thus the quantity $V_{\alpha\beta}(0)$ is the inverse capacitance matrix.[1] Using now Eq. (2.24), (2.25), and the commutation relation for zero modes, we arrive at the following important result for the time evolution of zero modes

$$Q_\alpha(t) = \sum_\beta V^{-1}_{\alpha\beta}(0)\Delta\mu_\beta, \quad \varphi_\alpha(t) = \varphi_\alpha - \Delta\mu_\alpha t. \tag{2.28}$$

Sometimes, the zero modes discussed in this section are included in the fermion field in terms of the so called Klein factors. We think, however, that the formulation proposed here is more straightforward and clear, and thus can be easily generalized to different situations.

2.2 Correlation Function at Finite Temperature

After we have recast the Hamiltonian in a quadratic form, one can easily diagonalize it. Namely, one only needs to find a transformation matrix for boson fields, which preserves the commutator structure (2.17) and diagonalizes the interaction matrix (2.22). For the purely chiral case of integer quantum Hall effect such transformation is just an orthogonal transformation $q_{\alpha j}$ so that $qq^T = 1$ and

$$V_{\alpha\beta}(k) = \sum_j q_{\alpha j}\omega_j(k)q_{j\beta}. \tag{2.29}$$

Here $\omega_j(k)$ are the dispersion relations of the boson eigenmodes, which depend on the particular form of the interaction. Relation (2.16) indicates that these boson eigenmodes are in fact plasmons, the collective charge excitations. In the case of a short range interaction the dispersion is always linear $\omega_j(k) = v_j k$, and the transformation matrix is independent of k. Therefore, one can introduce diagonal fields defined as $\phi_j(x) = q_{j\alpha}\phi_\alpha(x)$. In terms of these fields, every fermion operator has a form $\psi_\alpha = \exp\left(i\sum_j p_j\phi_j\right)$, where $p_j = q_{j\alpha}$. In fact, similar operators with different values of the coefficients p_j appear in context of the effective theory of fractional quantum Hall edge. Thus, for further convenience we will do all the calculations in terms of these coefficients.

The main quantity which determines the physics of a fermion system is the time dependent two-point correlation function. In our case it can be written in the following form:

$$i\langle\psi^\dagger(x,t)\psi(0,0)\rangle = e^{i\varphi_0}K(x,t), \tag{2.30}$$

where the first factor is the zero-mode contribution and the second one is the contribution of the oscillator operators. The phase induced by zero modes is given by $\varphi_0 = \sum_j p_j\langle\pi_j\rangle(x - v_j t)$ and in the case of integer quantum Hall effect it can be

[1] Note that $\Delta\mu_{\alpha j}$ is the electrochemical potential, because our definition (2.22) contains a single-particle contribution.

written in a simple physical form:

$$\varphi_0 = \Delta\mu_\alpha t - 2\pi Q_\alpha x. \tag{2.31}$$

In the next step, we calculate the contribution of fluctuations. For the equilibrium state, this contribution can be rewritten in terms of the boson fields as:

$$\ln[K(x,t)] = \sum_{ij} p_i p_j \langle[\phi_i(x,t) - \phi_i(0,0)]\phi_j(0,0)\rangle. \tag{2.32}$$

Introducing the notation $X_j \equiv x + \sigma_j v_j t$, where σ_j denotes the chirality of the corresponding eigenmode, we express the fields in terms of creation and annihilation operators,

$$\phi_j(x,t) = i\sum_k \sqrt{\frac{2\pi}{Wk}}\Big[\tilde{a}_j(k)e^{ikX_j} + \tilde{a}_j^\dagger(k)e^{-ikX_j}\Big]. \tag{2.33}$$

Substituting this expression into Eq. (2.32), we obtain the following expression for the fluctuations contribution

$$\ln K = \sum_j p_j^2 \int\limits_0^\Lambda \frac{dk}{k}\left\{f_j(k)(e^{-ikX_j} - 1) + [1 + f_j(k)](e^{ikX_j} - 1)\right\}, \tag{2.34}$$

where $f_j(k) = [\exp(\beta v_j k) - 1]^{-1}$ are the equilibrium boson occupation numbers at inverse temperature β, and Λ is the ultraviolet cutoff.

The best way to proceed is to expand the occupation numbers in Boltzmann factors, $f_j(k) = \sum_{n=1}^\infty \exp(-\beta v_j n\, k)$, and integrate each term separately. This gives us

$$\ln K = -\sum_j p_j^2 \sum_{n=-\infty}^{\infty} \ln[\Lambda(i\beta v_j n - X_j)]. \tag{2.35}$$

Combining this expression with Eq. (2.30), we finally arrive at the following result for the fermion correlation function:

$$i\langle\psi^\dagger(x,t)\psi(0,0)\rangle \propto e^{i\varphi_0}\prod_i \left[\frac{v_i}{\pi T}\sinh\left(\pi\frac{TX_i}{v_i}\right)\right]^{-\delta_i}, \tag{2.36}$$

where $\delta_i = p_i^2$. The total scaling dimension of the correlation function $\Delta \equiv \sum_i \delta_i$, which determines the power law behavior or the correlation function at small times may be different in different cases. For example, in the situation of quantum wires it is expressed in terms of the Luttinger parameter [3]. Below in the thesis we calculate this dimension in several particular situations. However, it is important to note that the scaling dimension can be only equal to 1 in the situation of purely *chiral* system.

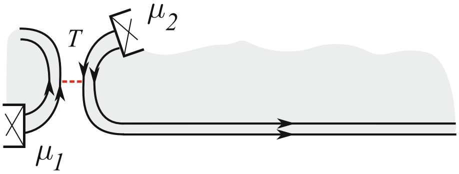

Fig. 2.2 Example of the situation where the equilibrium bosonization cannot be used. Typically, the distribution function of one-dimensional electrons is fixed by the reservoir from which they originate. But in the situation with tunneling (shown by the *red*, *dashed line*) between two edges with different chemical potentials, the distribution function of outgoing electrons in the bosonic language is strongly non-equilibrium, if the tunneling is not weak (Color figure online)

Indeed, in this case

$$\sum_i \delta_i = \sum_i p_i^2 = \sum_i q_{i\alpha} q_{\alpha i} = \delta_{\alpha\alpha} = 1, \tag{2.37}$$

because of the orthogonality of the transformation. Therefore, the correlation function in chiral systems behaves as $\langle \psi^\dagger(t)\psi(0)\rangle \sim 1/t$ even in the presence of interaction.

2.3 Non-Equilibrium Bosonization

In the previous section we have solved the interacting one-dimensional Hamiltonian and found the fermion correlation function. However, we did this only for an equilibrium state of electrons. In fact, the more general non-equilibrium conditions can not be included in the usual bosonization approach. For example, the problem appears in the situation where electron distribution function $f(\epsilon)$ at the boundary is not a Fermi function $f_F(\epsilon - \mu)$ but has a double step shape, i.e.,

$$f(\epsilon) = Rf_F(\epsilon - \mu_1) + Tf_F(\epsilon - \mu_2). \tag{2.38}$$

Of course in this situation one can bosonize the Hamiltonian and the fermion operators, but it is extremely difficult to express the distribution function (2.38) in terms of bosons. In order to treat the situations of such kind, as, e.g., illustrated in Fig. 2.2, we propose the non-equilibrium bosonization technique.

2.3.1 Non-Equilibrium Boundary Conditions and Full Counting Statistics

The Hamiltonian (2.21), together with the commutation relations (2.17), generates equations of motion for the fields ϕ_α. This equations have to be accompanied with boundary conditions. The key idea of the non-equilibrium bosonization approach is to describe the non-equilibrium state in terms of these boundary conditions. Namely, we write:

$$\partial_t \phi_\alpha(x,t) = -\frac{1}{2\pi}\sum_\beta \int dy V_{\alpha\beta}(x-y)\partial_y \phi_\beta(y,t), \tag{2.39a}$$

$$\partial_t \phi_\alpha(0,t) = 2\pi j_\alpha(t). \tag{2.39b}$$

Here $(1/2\pi)\partial_t \phi(x,t)$ is the expression for the edge currents in terms of the boson fields, and $j_\alpha(t)$ are the operators of current in terms of fermion degrees of freedom at the boundary. For example, in the situation depicted in Fig. 2.2, the $j_\alpha(t)$ are the tunneling currents.

The equations of motion (2.39a) are first order linear differential equations. Thus, these equations together with the boundary conditions (2.39b) can be easily solved with the help of the Green's functions method. The answer generally has the following form:

$$\phi_\alpha(x,t) = \sum_\beta \int dt' G_{\alpha\beta}(x,t-t')Q_\beta(t'), \tag{2.40}$$

where, the Green's function $G_{\alpha\beta}(x,t-t')$ depends on the particular form of the interaction potential $U_{\alpha\beta}(x-y)$, and we have introduced the charge operators defined as

$$Q_\alpha(t) = \int_{-\infty}^{t} dt' j_\alpha(t'). \tag{2.41}$$

Therefore, by solving Eq. (2.39), one may express the correlation functions of the fermion fields ψ_α in terms of the correlation function of the charge operator:

$$\begin{aligned}\langle \psi_\alpha^\dagger(x,t)\psi_\alpha(y,0)\rangle = \prod_\beta \Big\langle \exp\Big[-i\int dt' G_{\alpha\beta}(x,t-t')Q_\beta(t')\Big] \\ \times \exp\Big[i\int dt' G_{\alpha\beta}(y,-t')Q_\beta(t')\Big]\Big\rangle. \end{aligned}\tag{2.42}$$

Here, averaging on the right hand side is defined over the *free* electrons, e.g., over the distribution function (2.38). All the interaction effects are encoded in the Green's function. In general, the fields ϕ_α influence fluctuations of the currents j_α, leading to the dynamical Coulomb blockade in the quantum low-energy regime [6], and to cascade corrections to noise in the classical limit [7]. Importantly, simplification arises in such situations where such back-action effects are *absent*, e.g. for chiral edge states [8, 9]. As a consequence, the electron transport through the quantum point contact (see Fig. 2.2) is not affected by interactions, which has been recently confirmed in the experiment [10, 11].

In a chiral case with a short range interaction, the Green's function of Eq. (2.39a) can be found explicitly and it is given by a sum of delta functions. One can check by the direct substitution, that the solution for the boson fields is given by the following equation:

$$\phi_\alpha(x,t) = 2\pi \sum_{j\beta} q_{\alpha j} q_{j\beta} Q_\beta(t - x/v_j). \tag{2.43}$$

The above expression simplifies further in several cases considered in this thesis because of the special form of the transformation matrix q. In these cases one can express the correlation function (2.42) via the so called generator of full counting statistics of noise, defined as [12]

$$\chi_\alpha(\lambda, t) = \left\langle e^{i\lambda Q_\alpha(t)} e^{-i\lambda Q_\alpha(0)} \right\rangle. \tag{2.44}$$

This quantity is the quantum analog of the Fourier transform of the distribution function $P(Q)$ of charge $\chi(\lambda) = \langle e^{i\lambda Q} \rangle \equiv \int dQ P(Q) e^{i\lambda Q}$. Main property of the generator (2.44) is that it gives all the irreducible moments of the current in the long-time limit:

$$\partial^n_{i\lambda} \log(\chi_\alpha)/t = \langle\langle j^n_\alpha \rangle\rangle. \tag{2.45}$$

The full counting statistics generator (2.44) has been extensively studied recently [13], and it is known in several situations. In the next chapters we will study and use this generator for the statistics of electrons tunneling through a quantum point contact.

2.3.2 Equilibrium Boundary Conditions: A Simple Test

The simplest test of the non-equilibrium bosonization approach is to consider an infinite quantum Hall edge and formally split it in two parts at the point $x = 0$. Then, the boundary conditions at this point describe only the thermal equilibrium fluctuations of the charge density. Thus, one can calculate the electron correlation function within our approach, and compare it to the well known result for the finite temperature correlator in a chiral one-dimensional system (2.36). The solution of the equations of motion (2.39) in the system with single channel and short range interaction is simple:

$$\phi(x,t) = Q(t - x/v), \tag{2.46}$$

where $v = v_F + U/2\pi$. The non-equilibrium bosonization prescription gives the expression for the correlation function $\psi^\dagger(x,t)\psi(0,0) = \chi(2\pi, t - x/v)$.

The thermal fluctuations have Gaussian statistics, therefore one can write down the generating function as follows:

$$\log[\chi(\lambda, t)] = -(\lambda^2/2)\langle Q_\alpha^2(t) - 2Q_\alpha(t)Q_\alpha(0) + Q_\alpha^2(0)\rangle. \tag{2.47}$$

Taking into account the definition of charge operator (2.41) one can rewrite this result via the power spectrum of current fluctuations:

$$\log[\chi(\lambda, t)] = -(\lambda^2/2)\int \frac{d\omega}{2\pi\omega^2}(1 - e^{i\omega t})S(\omega), \tag{2.48}$$

where $S_\alpha(\omega) \equiv \int dt e^{i\omega t}\langle j_\alpha(t) j_\alpha(0)\rangle$.

In the next step we use the expression for the *fermion* current to calculate the equilibrium noise power spectrum. The current of non-interacting fermions at finite time can be written as:

$$j(t) = \iint dE d\omega c^\dagger_{\omega+E} c_E e^{i\omega t}. \tag{2.49}$$

Using this expression and the statistics of the Fermi distribution, one can find that the noise spectrum at inverse temperature β is given by the following equation [14]:

$$S_\alpha(\omega) = \frac{1}{2\pi}\frac{\omega}{1 - e^{-\beta\omega}}. \tag{2.50}$$

Note, that r.h.s. of this equation is proportional to the Bose equilibrium distribution function. Substituting this expression in Eq. (2.48) one gets:

$$\log[\chi(\lambda, t)] = \lambda^2/4\pi^2 \int \frac{d\omega}{\omega}\frac{1 - e^{-i\omega t}}{1 - e^{-\beta\omega}}. \tag{2.51}$$

This integral can be evaluated expanding in Boltzmann factors and integrating each term. Finally, we take into account that in our situation $\lambda = 2\pi$ and come to the conclusion that the correlation function equals to

$$\psi^\dagger(x, t)\psi(0, 0) = \frac{\pi/\beta}{\sinh[\pi(t - x/v)/\beta]}. \tag{2.52}$$

We see that the result is consistent with the predictions of the usual bosonization approach (2.36).

To summarize, here we have explicitly illustrated that the usual bosonization approach may be split in two steps. First one is the calculation of the statistics of the boundary conditions, and the second one is the solution of the equations of motion for the excitations at the edge. Below, in Chap. 4, we make another test of non-equilibrium bosonization by considering an example of a more complex system.

2.4 Conclusions

We have shown that the bosonization approach is very powerful tool for the theoretical description of one-dimensional systems. The main idea of the bosonization approach is to rewrite the fermion field operators in terms of new collective boson fields. This allows one to recast the Hamiltonian of the interacting one-dimensional fermions with linear spectrum in a quadratic form and thus diagonalize it and find correlation functions of electrons. This approach is suitable, however, only for finding the equilibrium correlation functions. The non-equilibrium generalization of the bosonization procedure, introduced in this section, allows one to reduce the problem of finding a non-equilibrium correlation function of an interacting system to the problem of finding some correlation functions of non-interacting electrons.

The key idea of this approach is to take advantage of the absence of back-action of the interaction effects on the statistics of processes which determine the boundary conditions. Such situation is realized in chiral systems with short range interaction and several other systems. In some of these cases, the non-equilibrium bosonization allows one to rewrite the correlation function in terms of the *full counting statistics* generator. As we will see below, this method gives a simple and efficient approach to several problems.

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