

Mass Transport in Porous Media With Variable Mass

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Abstract We present a theoretical and numerical study of mass transport in a porous medium saturated with a fluid and characterised by an evolving internal structure. The dynamics of the porous medium and the fluid as well as their reciprocal interactions are described at a coarse scale, so that the fundamental tools of Mixture Theory and Continuum Mechanics can be used. The evolution of the internal structure of the porous medium, which is here primarily imputed either to growth or to mass exchange with the fluid, is investigated by enriching the space of kinematic variables of the mixture with a set of structural descriptors, each of which is power-conjugate to generalised forces satisfying a balance law. Establishing the influence of the structural change of the porous medium on the transport

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properties of the mixture and, thus, on the quantities characterising fluid flow is the crux of our contribution.

Keywords Mass transport • Porous media • Inelastic deformations • Variable mass • Dissipation

1 Introduction

The scope of this contribution is to study the transport of fluid in a deformable porous medium whose mass and internal structure may vary in time. Our level of investigation is coarse enough, so that we can formulate our problem with the aid of Mixture Theory. Thus, we consider a biphasic mixture consisting of a solid and a fluid phase. The solid phase represents the porous medium, whereas the fluid phase consists of a fluid that completely saturates the pores of the solid and may move throughout it. A system of this type is perhaps the most essential model of those soft biological tissues whose main constituents are a porous solid matrix and an interstitial fluid. An example of such tissues is articular cartilage, in which the solid consists of cells (chondrocytes), collagen fibres and matrix of proteoglycans, and the fluid comprises water, ions and various chemical compounds. The latter ones can be either nutrients for the cells or byproducts of the cellular metabolic activity. From here on, we call “fluid phase” the system made of water and all chemicals dissolved in it, and we refer to these as “constituents of the fluid phase”.

Among the factors that assess the health of a tissue, an important role is played by the constituents that supply nutriment to the cells and regulate their metabolism. Other relevant constituents are those that can either promote or hinder processes, which can damage the tissue (e.g., growth of tumours). In any of these cases, a mathematical model of a tissue should provide information about the concentration of constituents and the transport processes to which they are subjected. As long as continuum models are concerned (these models do not explicitly track molecular or sub-cellular processes), the evolution of constituents is put in the form of diffusion-advection-reaction equations. Therefore, if the validity of Fick’s and Darcy’s laws is accepted, it is essential to determine the diffusivity and permeability of the tissue in order to quantify its capability of transporting matter. This capability, on the other hand, depends on geometrical (i.e., geometry of the pore space) and mechanical factors (i.e., deformation and stress), and couplings between them. Indeed, when the tissue deforms, the geometry of the pore space changes and so do the transport properties (diffusivity and permeability). Moreover, different compositions of the fluid phase lead to different hydraulic conductivities. However, we shall neglect this effect in the rest of our study, and we shall focus on the much simpler case of a fluid phase comprising a single constituent only. We shall also assume that the fluid phase is incompressible.

The picture sketched above becomes more complex when the structural changes of the tissue are considered beside deformation. By “structural changes” we mean processes that contribute to modify the properties of the tissue (e.g., the stiffness, diffusivity and permeability) in response to growth, mass exchange between the fluid and the solid phase, and re-organisation of cells and fibrils. Even though these phenomena are all intermingled with each other, a conceptual classification is pointed out in the biomechanical literature (cf., for example, Fung [25] and Taber [57]), where growth and mass exchange are said to lead to the variation of mass of a body, while the re-organisation of cells and fibrils is referred to as remodelling.

From the point of view of Continuum Mechanics, the structural change of the solid phase of a tissue is modelled by means of a class of deformations that describe how the material particles are distributed in the tissue. The mathematical object used to define these deformations is a second-order tensor, which we denote by $\mathbf{F}_a$. With the terminology of [22], $\mathbf{F}_a$ measures the material inhomogeneities triggered by growth, mass exchange processes, and cellular re-organisation. There is, thus, a strong conceptual difference between the standard deformations and those described by $\mathbf{F}_a$: whereas the former ones are related to the gradient of displacement of the body, the latter deformations need not be the gradient of any vector field. Rather, they are primitive entities that define, together with displacements, the parameters that are necessary and sufficient for describing the kinematics of deformable bodies with variable internal structure.

In “classical” Continuum Mechanics¹, tensor $\mathbf{F}_a$ represents the anelastic part of the deformation gradient tensor, $\mathbf{F}_s$, which describes the overall change of shape of a solid. This anelastic deformation may be due to, for example, plastic deformations, thermal distortions, and damage [42]. Tensor $\mathbf{F}_a$ individuates an evolving relaxed configuration of body elements. The accommodating deformation, which determines the actual configuration of the body from the relaxed one, is denoted by $\mathbf{F}_e$, and obeys the multiplicative decomposition $\mathbf{F}_s = \mathbf{F}_e \mathbf{F}_a$ [36, 37, 39]. Usually, $\mathbf{F}_e$ is said to be the elastic part of $\mathbf{F}_s$.

Rodriguez et al. [53] used the decomposition of $\mathbf{F}_s$ to study growth mechanics, and identified $\mathbf{F}_a$ with the deformations due to growth. When $\mathbf{F}_a$ is not the gradient of any vector field, it is said to be incompatible. Physically, this represents the case in which grown material points lose their geometric compatibility (this situation usually leads to residual stresses). The rate of anelastic deformation, $\mathbf{L}_a = \dot{\mathbf{F}}_a(\mathbf{F}_a)^{-1}$, is related to the variation of body mass in such a way that the mass density of the body is constant when measured with respect to the relaxed configuration (cf., for example, [22]).

The kinematic entities $\mathbf{F}_s$ and $\mathbf{F}_a$ are the mathematical objects describing the physical processes that influence the transport properties of a tissue (i.e., diffusivity and permeability). To be more specific, $\mathbf{F}_a$ may be decomposed as the product of tensors, which distinguish the anelastic deformations associated with growth from those associated with the exchange of mass between the solid and the fluid phase.

¹ By “classical” we mean here the Continuum Mechanics that studies non-living matter.

The case in which both processes are modelled together has been studied in [29, 27]. In our present contribution, however, we do not consider mass exchange, so that $\mathbf{F}_a$ accounts for the anelastic deformations associated with growth only.

Under the hypothesis of negligible inertial terms and incompressible solid and fluid phases, the unknowns to be determined are given by the displacement field (whose material gradient is $\mathbf{F}_s$), pressure, and $\mathbf{F}_a$. We formulate a boundary value problem for the calculation of pressure and displacements, and discuss how to find an equation for $\mathbf{L}_a$ (tensor $\mathbf{F}_a$ is then found by solving the initial value problem $\dot{\mathbf{F}}_a = \mathbf{L}_a \mathbf{F}_a$, with $\mathbf{F}_a(t_0) = \mathbf{F}_{a0}$). Accepting Darcy's law amounts to say that fluid flow is determined by the hydraulic conductivity, $\mathbf{K}$, and pressure gradient. Therefore, the study of the transport of fluid in a deformable porous medium with variable mass and internal structure reduces to the determination of the influence of $\mathbf{F}_a$ on $\mathbf{K}$ and pressure. In particular, we show that different choices of the initial value $\mathbf{F}_{a0}$ lead to different pressure distributions and displacements. The latter ones, in turn, affect $\mathbf{K}$ and are thus able to influence the capability of the medium of conveying fluid. The alteration of pressure may be relevant for some biomechanical applications in which the health of the cells of a tissue depends on the pressure (for example, this is the case of chondrocytes in articular cartilage). Our study aims to put some of concepts presented in [30] in a more rigorous framework.

Our contribution is organised as follows. In Sect. 2, we review the kinematics of biphasic mixtures with consideration of the structural changes of the solid phase; we study the balance laws relevant for our purposes and discuss the dissipation inequality characterising the system under investigation. In Sect. 3, we formulate the mathematical model in weak form in order to supply the basis for finite element implementations. Since our equations are non-linear, we present the linearisation procedure and discuss the spatial discretization of the computational domain as well as some numerical issues related to the solution of the problem. In Sect. 4, we study a benchmark problem. Finally, we summarise our results in Sect. 5.

2 Theoretical Background

The equations governing the coarse-scale evolution of biphasic systems can be found by averaging the relations that describe the systems at a smaller scale, e.g., the pore scale. The averaging procedures are often based on volume- and mass-averages [33, 10]. These assume the existence of a representative volume element (RVE) $\Omega(\mathbf{x})$, which supplies information about the composition and structure of the mixture at the point $\mathbf{x} \in \mathbb{R}^3$, where $\mathbb{R}^3$ is the ambient space. The characteristic size of the RVE depends on the system under investigation. The subvolume $\Omega_\alpha(t, \mathbf{x})$, with $\alpha = \ell, s$, represents the subset of $\Omega(\mathbf{x})$ occupied by the α th phase at time t . The ratio $\phi_\alpha(t, \mathbf{x}) := |\Omega_\alpha(t, \mathbf{x})|/|\Omega(\mathbf{x})|$ is referred to as volumetric fraction.

Moreover, the abbreviations $\mathcal{P}_s$ and $\mathcal{P}_\ell$ will be sometimes used to denote the solid and the fluid phase, respectively.

If the void space of the porous medium is completely filled with the fluid, the medium is said to be “saturated”, and the condition $\phi_s + \phi_\ell = 1$ applies at all times and all points. The distribution of mass of $\mathcal{P}_\alpha$ in Ω_α is the “true” mass density of $\mathcal{P}_\alpha$, and is denoted here by $\hat{\rho}_\alpha$. The product $\rho_\alpha = \phi_\alpha \hat{\rho}_\alpha$ measures the distribution of mass of $\mathcal{P}_\alpha$ in Ω , and is called “apparent” mass density of $\mathcal{P}_\alpha$.

2.1 Kinematics

We formulate the kinematic description of the mixture in two steps. We consider first the velocities $\mathbf{v}_s$ and $\mathbf{v}_\ell$, which characterise the standard motion of $\mathcal{P}_s$ and $\mathcal{P}_\ell$. Subsequently, we introduce the second-order tensors $\mathbf{F}_a$ and $\mathbf{L}_a$, which describe the anelastic deformation and the rate of anelastic deformation associated with the change of internal structure of the solid phase.

Among the various ways of describing the motion of the phases $\mathcal{P}_s$ and $\mathcal{P}_\ell$, we choose that based on the set of “standard” velocities

$$\mathcal{V}_{\text{st}} := \{\mathbf{v}_s, \mathbf{w}_{\ell s}\}, \quad (1)$$

where $\mathbf{w}_{\ell s} := \mathbf{v}_\ell - \mathbf{v}_s$ is the relative velocity of $\mathcal{P}_\ell$ with respect to $\mathcal{P}_s$. Another relevant kinematic quantity is the velocity of the centre of mass of the mixture, which is defined by

$$\mathbf{v} := \frac{\rho_s \mathbf{v}_s + \rho_\ell \mathbf{v}_\ell}{\rho}, \quad \rho := \rho_\ell + \rho_s. \quad (2)$$

Here, ρ denotes the mass density of the mixture as a whole. The relative velocities $\tilde{\mathbf{v}}_\ell := \mathbf{v}_\ell - \mathbf{v}$ and $\tilde{\mathbf{v}}_s := \mathbf{v}_s - \mathbf{v}$ describe the relative motion of each phase with respect to the motion of the mixture as a whole. For each phase, we denote by $\mathbf{a}_\alpha$ (with $\alpha = \ell, s$) the acceleration of that phase, which is defined by the convective derivative of $\mathbf{v}_\alpha$ with respect to the motion of the phase $\mathcal{P}_\alpha$, i.e.

$$\mathbf{a}_\alpha := \partial_t \mathbf{v}_\alpha + \text{grad}(\mathbf{v}_\alpha) \mathbf{v}_\alpha, \quad \alpha = \ell, s. \quad (3)$$

The acceleration of the mixture is indicated by $\mathbf{a}$ and is defined as the convective derivative of $\mathbf{v}$ with respect to the motion of the mixture as a whole, i.e.

$$\mathbf{a} := \partial_t \mathbf{v} + \text{grad}(\mathbf{v}) \mathbf{v}. \quad (4)$$

The description of kinematics shown so far is based on the spatial, or Eulerian, formalism. In order to make it consistent with the Lagrangian description, we need to relate the velocities $\mathbf{v}_s$ and $\mathbf{v}_\ell$ with the motions of the solid and fluid phases. To this end, we present a formulation that follows, with some slight differences, the picture put forward by Quiligotti [48]. The starting point is that the biphasic

mixture represents a porous medium whose void space is saturated by a fluid. If we ideally remove the fluid, we are left with a solid skeleton, which we embed in the three-dimensional Euclidean space $\mathbb{R}^3$. We call $\mathcal{B}_r$ the subset of $\mathbb{R}^3$ in which the solid skeleton is embedded, and denote by $\mathbf{X}$ the coordinates of the centroid of the RVE associated with $\mathcal{B}_r$. The set $\mathcal{B}_r$ is also said to be the “reference configuration” of the solid phase $\mathcal{P}_s$, and the coordinates $\mathbf{X}$ may be called “material coordinates”. A smooth motion of $\mathcal{P}_s$, which is referred to as s -motion in [48], is a sequence of mappings $\chi_s(t, \cdot): \mathcal{B}_r \mapsto \mathbb{R}^3$, such that $\mathbf{x} = \chi_s(t, \mathbf{X}) \in \mathbb{R}^3$. The material gradient of the map χ_s equals the deformation gradient $\mathbf{F}_s := \text{Grad}(\chi_s)$, i.e. $(F_s)_{il} = (\chi_s)_{i,l}$ in Cartesian coordinates, where the comma means differentiation with respect to X_l . The time derivative of χ_s is defined by

$$\partial_t \chi_s(t, \mathbf{X}) = \mathbf{v}_s(t, \chi_s(t, \mathbf{X})) = \mathbf{v}_s(t, \mathbf{x}). \quad (5)$$

In order to describe the kinematics of the fluid phase $\mathcal{P}_\ell$, we introduce the material manifold $\mathcal{B}_\ell$, which consists of fluid particles. A fluid particle, labelled by X_ℓ , is placed in the Euclidean space by means of an embedding that locates the particle at $\mathbf{x}$. A smooth motion of $\mathcal{P}_\ell$, the ℓ -motion, is a sequence of mappings defined by $\chi_\ell(t, \cdot): \mathcal{B}_\ell \mapsto \mathbb{R}^3$, such that $\mathbf{x} = \chi_\ell(t, X_\ell)$. The velocity of the particle labelled by X_ℓ satisfies the identity

$$\partial_t \chi_\ell(t, X_\ell) = \mathbf{v}_\ell(t, \chi_\ell(t, X_\ell)) = \mathbf{v}_\ell(t, \mathbf{x}). \quad (6)$$

The definitions of s - and ℓ -motion imply the chain of identities

$$\mathbf{x} = \chi_s(t, \mathbf{X}) = \chi_\ell(t, X_\ell), \quad (7)$$

which means that both the solid and the fluid phase co-exist at the same point $\mathbf{x}$ of the Euclidean space. A quantity associated with the fluid phase, e.g., the velocity $\mathbf{v}_\ell$, can be expressed in terms of the material coordinates by using the following composition of maps:

$$\mathbf{v}_\ell(t, \mathbf{x}) = \mathbf{v}_\ell(t, \chi_s(t, \mathbf{X})) = [\mathbf{v}_\ell(t, \cdot) \circ \chi_s(t, \cdot)](\mathbf{X}). \quad (8)$$

An analogous result holds true for $\mathbf{v}_s$. The portion of $\mathbb{R}^3$ occupied by the mixture at time t is given by the intersection $\mathcal{B}_t = \chi_s(t, \mathcal{B}_r) \cap \chi_\ell(t, \mathcal{B}_\ell)$.

The gradient of $\mathbf{v}_s$ is denoted by $\mathbf{L}_s := \text{grad}(\mathbf{v}_s)$. The deformation process of $\mathcal{P}_s$ is determined by a tensor field $\mathbf{F}_s$, which satisfies the condition $\dot{\mathbf{F}}_s = \mathbf{L}_s \mathbf{F}_s$. The determinant of $\mathbf{F}_s$, $J_s := \det(\mathbf{F}_s)$, accounts for the change of volume associated with the change of configuration. The definitions introduced so far can be found in several treatises about the classical Theory of Mixtures (cf., for example, [7, 13, 58, 51]).

In order to complete the kinematic analysis of the considered biphasic mixture, we have to introduce a non-standard descriptor in addition to the standard velocities collected in $\mathcal{V}_{\text{st}}$. This descriptor has to model the structural change of the solid phase, $\mathcal{P}_s$, in response to interactions that lead to the variation and redistribution of its mass. On the basis of the motivations reported, for example, in

[53, 22, 20, 41, 3, 4], these types of structural evolution are viewed as anelastic processes. Accordingly, the kinematic descriptor of these processes is the tensor of anelastic deformation, $\mathbf{F}_a$, which is related to the rate of anelastic deformation, $\mathbf{L}_a$, through the differential equation $\dot{\mathbf{F}}_a := \mathbf{L}_a \mathbf{F}_a$. If the solid phase exhibits elastic behaviour, the tensor $\mathbf{F}_s$ is decomposed as $\mathbf{F}_s = \mathbf{F}_e \mathbf{F}_a$, where $\mathbf{F}_e$ represents the elastic contribution to the overall deformation. The variations of volume of the solid phase due to the elastic and the anelastic deformations are denoted by $J_e = \det(\mathbf{F}_e)$ and $J_a = \det(\mathbf{F}_a)$, respectively. The multiplicative decomposition of $\mathbf{F}_s$ implies $J_s = J_e J_a$. The determinants J_s , J_e and J_a are strictly positive.

The multiplicative decomposition $\mathbf{F}_s = \mathbf{F}_e \mathbf{F}_a$ was introduced by Kröner [36, 37], Lee [39] and other scientists working in Continuum Mechanics and, in particular, in the Theory of Plasticity. In Biomechanics, it was firstly used by Rodriguez et al. [53]. The tensor $\mathbf{F}_a$ maps vectors attached to $\mathcal{B}_r$ into vectors attached to a relaxed configuration, which is often referred to as “natural configuration” [49] and denoted by $\mathcal{B}_n$.

In the following, the index “s” associated with $\mathbf{F}_s$ and J_s will be dropped for the sake of simpler notation.

2.2 Balance Laws

The Eulerian, local forms of the balance of mass of the solid and fluid phase read

$$D_s \rho_s + \rho_s \operatorname{div}(\mathbf{v}_s) = \rho_s \gamma_s, \quad (9)$$

$$D_s \rho_\ell + \rho_\ell \operatorname{div}(\mathbf{v}_s) + \operatorname{div}(\rho_\ell \mathbf{w}_{\ell s}) = 0, \quad (10)$$

where the operator $D_s \mathbf{A} = \partial_t \mathbf{A} + \operatorname{grad}(\mathbf{A}) \cdot \mathbf{v}_s$ is the convective derivative of the generic tensor field $\mathbf{A}$ with respect to the motion of the solid phase, and γ_s is the rate at which the mass of the solid phase is produced or depleted. Multiplying (9) and (10) by J , and passing to the material description lead to the following form of the mass balance laws of the constituents of the mixture:

$$\overline{\dot{(J\rho_s)}} = J\rho_s \gamma_s, \quad (11)$$

$$\overline{\dot{(J\rho_\ell)}} + \operatorname{Div}(J\rho_\ell \mathbf{F}^{-1} \mathbf{w}_{\ell s}) = 0. \quad (12)$$

The operators “Div” and “Grad” are the divergence and gradient operators computed with respect to the material coordinates. They are related to “div” and “grad” by the formulae $\operatorname{grad}(\mathbf{A}) = \operatorname{Grad}(\mathbf{A}_\chi) \mathbf{F}^{-1}$ and $\operatorname{div}(\mathbf{A}) = \operatorname{Grad}(\mathbf{A}_\chi) : \mathbf{F}^{-T}$, where $\mathbf{A}$ is a given vector field, the symbol “ χ ” denotes the inner product between tensors, and the index “ χ ”, which will be dropped for here on, means $\mathbf{A}_\chi(t, \cdot) = \mathbf{A}(t, \cdot) \circ \chi_s(t, \cdot)$.

The product $\rho_{sr} := J\rho_s$ in (11) defines the mass of $\mathcal{P}_s$ measured per unit volume of $\mathcal{B}_r$. By using the definition of apparent mass density, and the fact that $J = J_e J_a$, the quantity ρ_{sr} can be rewritten as

$$\rho_{sr} = J_e J_a \rho_s = J_a \rho_{sn}, \quad (13)$$

where $\rho_{sn} := J_e \rho_s$ indicates the mass density of $\mathcal{P}_s$ computed with respect to the natural configuration $\mathcal{B}_n$. Furthermore, substituting (13) into (11) yields

$$\text{tr}(\mathbf{L}_a)\rho_{sn} + \dot{\rho}_{sn} = \rho_{sn}\gamma_s. \quad (14)$$

We enforce now the condition that the variation of body mass is compensated for by the rate $\text{tr}(\mathbf{L}_a)$, which implies that the mass density ρ_{sn} is constant in time. Thus, we arrive at the results

$$\gamma_s = \text{tr}(\mathbf{L}_a), \quad \rho_{sn} = \rho_{s0}, \quad (15)$$

where ρ_{s0} may be a function of material coordinates only. A consequence of (13)–(15) is that the solution to (11) is given by

$$\rho_s = \frac{\rho_{sn}}{J_e} = \rho_{sn} \frac{J_a}{J}. \quad (16)$$

This means that the apparent density of the solid phase, ρ_s , is determined if the constant mass distribution ρ_{sn} is assigned, and the volumetric deformations J and J_a are known.

A simplification may be obtained under the hypothesis that the true mass density $\hat{\rho}_s$ is a given constant, which implies that $\mathcal{P}_s$ is incompressible. This allows to reformulate (16) in terms of the stronger condition

$$\phi_s = \frac{\phi_{sn}}{J_e} = \phi_{sn} \frac{J_a}{J}, \quad (17)$$

which involves only different measures of the volumetric fraction of the solid phase. In (17), ϕ_{sn} denotes the volumetric fraction of $\mathcal{P}_s$ “seen” by $\mathcal{B}_n$. We remark that ϕ_{sn} is constant and should be regarded as a known quantity of the model.

Another simplification follows from requiring that the true mass density of the fluid phase, $\hat{\rho}_\ell$, is a given constant, so that $\mathcal{P}_\ell$ is incompressible too. Granted this condition and the saturation constraint, a consequence of (17) and (15)₁ is that (12) acquires the simpler form

$$\text{Div}[J\mathbf{F}^{-1}\mathbf{q}_{\ell s}] = -\dot{J} + J_a \phi_{sn} \text{tr}(\mathbf{L}_a), \quad (18)$$

with $\mathbf{q}_{\ell s} := \phi_\ell \mathbf{w}_{\ell s}$. In summary, (17) and (18) provide the balances of mass of the solid and the fluid phase, respectively.

Together with mass balance, also the balance of momentum of the phases $\mathcal{P}_s$ and $\mathcal{P}_\ell$ has to be studied. The Eulerian, local form of these balance laws is given by

$$\partial_t(\rho_s \mathbf{v}_s) + \operatorname{div}(\rho_s \mathbf{v}_s \otimes \mathbf{v}_s) - \operatorname{div}(\mathbf{T}_s) = \rho_s \mathbf{m}_s + (\rho_s \mathbf{p}_s + \rho_s \gamma_s \mathbf{v}_s), \quad (19)$$

$$\partial_t(\rho_\ell \mathbf{v}_\ell) + \operatorname{div}(\rho_\ell \mathbf{v}_\ell \otimes \mathbf{v}_\ell) - \operatorname{div}(\mathbf{T}_\ell) = \rho_\ell \mathbf{m}_\ell, \quad (20)$$

where $\mathbf{T}_s$ and $\mathbf{T}_\ell$ denote the Cauchy stress tensors of the solid and fluid phase, $\rho_s \mathbf{m}_s$ and $\rho_\ell \mathbf{m}_\ell$ are the rate of exchange of momentum between the two phases, and $\rho_s \mathbf{p}_s$ is rate of change of momentum due to growth. The forces $\rho_s \mathbf{m}_s$ and $\rho_\ell \mathbf{m}_\ell$ satisfy the condition

$$\rho_s \mathbf{m}_s + \rho_\ell \mathbf{m}_\ell = \mathbf{0}, \quad (21)$$

which states that, in the absence of growth (i.e., when $\gamma_s = 0$ and $\mathbf{p}_s = \mathbf{0}$), the mixture is closed with respect to momentum. An explanation of the physical meaning of (21) in terms of pore scale considerations can be found, for example, in [33]. The use of (21), and the definition of the relative velocities $\tilde{\mathbf{v}}_\ell$ and $\tilde{\mathbf{v}}_s$ as well as of the accelerations $\mathbf{a}$ and $\mathbf{a}_\ell$ allow for reformulating the balance laws (19) and (20) in the following way

$$\rho \mathbf{a} - \operatorname{div}(\mathbf{T}) = \rho_s \mathbf{p}_s + \rho_s \gamma_s \tilde{\mathbf{v}}_s, \quad (22)$$

$$\rho_\ell \mathbf{a}_\ell - \operatorname{div}(\mathbf{T}_\ell) = \rho_\ell \mathbf{m}_\ell, \quad (23)$$

where $\mathbf{T}$, which denotes the Cauchy stress tensor of the mixture, is defined by

$$\mathbf{T} := \mathbf{T}_s + \mathbf{T}_\ell - [\rho_s \tilde{\mathbf{v}}_s \otimes \tilde{\mathbf{v}}_s + \rho_\ell \tilde{\mathbf{v}}_\ell \otimes \tilde{\mathbf{v}}_\ell]. \quad (24)$$

Equation (22) represents the balance of momentum of the mixture as a whole, and is obtained by adding together (19) and (20) and applying the definitions (2)–(4) to the result. Furthermore, substituting the identity

$$\rho \mathbf{a} := \sum_{\alpha=\ell,s} \rho_\alpha \mathbf{a}_\alpha - \sum_{\alpha=\ell,s} \operatorname{div}(\rho_\alpha \tilde{\mathbf{v}}_\alpha \otimes \tilde{\mathbf{v}}_\alpha) + \rho_s \gamma_s \tilde{\mathbf{v}}_s \quad (25)$$

into (22) yields

$$\rho_s \mathbf{a}_s + \rho_\ell \mathbf{a}_\ell - \operatorname{div}(\mathbf{T}_s + \mathbf{T}_\ell) = \rho_s \mathbf{p}_s, \quad (26)$$

$$\rho_\ell \mathbf{a}_\ell - \operatorname{div}(\mathbf{T}_\ell) = \rho_\ell \mathbf{m}_\ell. \quad (27)$$

Finally, neglecting the inertial forces of both phases, the balances of momentum (19) and (20) become

$$\operatorname{div}(\mathbf{T}_s + \mathbf{T}_\ell) + \rho_s \mathbf{p}_s = \mathbf{0}, \quad (28)$$

$$\operatorname{div}(\mathbf{T}_\ell) + \rho_\ell \mathbf{m}_\ell = \mathbf{0}. \quad (29)$$

By means of the Piola transformations of (28) and (29), the momentum balance laws of the mixture can be written with respect to the reference placement $\mathcal{B}_r$, i.e.

$$\operatorname{Div}(\mathbf{P}_s + \mathbf{P}_\ell) + J \rho_s \mathbf{p}_s = \mathbf{0}, \quad (30)$$

$$\text{Div}(\mathbf{P}_\ell) + J\rho_\ell \mathbf{m}_\ell = \mathbf{0}, \quad (31)$$

where

$$\mathbf{P}_s := J\mathbf{T}_s \mathbf{F}^{-T}, \quad \mathbf{P}_\ell := J\mathbf{T}_\ell \mathbf{F}^{-T} \quad (32)$$

denote, respectively, the first Piola–Kirchhoff stress tensors of the solid and fluid phase. In order to close the mathematical problem resulting from (18), (30) and (31), it is necessary to provide information about the stresses $\mathbf{P}_s$ and $\mathbf{P}_\ell$, and the force densities $\rho_\ell \mathbf{m}_\ell$ and $\rho_s \mathbf{p}_s$.

The change of internal structure of the solid phase is a process whose kinematics are described by the tensor map $\mathbf{F}_a$ and the generalised velocity $\mathbf{L}_a = \dot{\mathbf{F}}_a(\mathbf{F}_a)^{-1}$. The set of generalised forces that perform working on $\mathbf{L}_a = \dot{\mathbf{F}}_a(\mathbf{F}_a)^{-1}$ comprises an internal force, $\mathbf{Z}_n$, which drives the structural evolution, and an external force, $\mathbf{Y}_n$, which models the interaction of the system with its surrounding environment. Both forces are second-order tensors. It is postulated that they obey the balance law

$$\mathbf{Z}_n = \mathbf{Y}_n. \quad (33)$$

The index “ n ” means that $\mathbf{Z}_n$ and $\mathbf{Y}_n$ are conceived as forces acting on the natural configuration $\mathcal{B}_n$, although they can be also written with respect to $\mathcal{B}_r$ and $\mathcal{B}_t$ by performing proper transformations. The reader is referred to [20] for an exhaustive explanation of the physical concepts leading to (33). Here, we simply state that, in analogy with the balance laws (28) and (29) (which describe a balance of forces that perform working on the set of standard velocities), also the forces that perform working on the non-standard descriptor $\mathbf{L}_a$ should satisfy a balance law. Some extensions of the results presented in [20] can be found, for example, in [1, 26, 2, 5, 29, 28, 27].

The internal force-like variables $\rho_\ell \mathbf{m}_\ell$ and $\mathbf{Z}_n$ are responsible for dissipation, and should thus comply with the dissipation inequality that characterises the system under investigation.

2.3 Study of the Residual Dissipation

We introduce the total internal working associated with $\mathcal{P}_t \subset \mathcal{B}_t$

$$\mathfrak{W}^{in}(\mathcal{P}_t) = \mathfrak{W}_{st}^{in}(\mathcal{P}_t) + \mathfrak{W}_{n-st}^{in}(\mathcal{P}_t), \quad (34)$$

where $\mathfrak{W}_{st}^{in}(\mathcal{P}_t)$ and $\mathfrak{W}_{n-st}^{in}(\mathcal{P}_t)$ describe, respectively, the working performed by the standard and non-standard forces acting on the system. These two contributions are defined by the following expressions

$$\mathfrak{W}_{\text{n-st}}^{\text{in}}(\mathcal{P}_t) := \int_{\mathcal{P}_t} (J_e)^{-1} \mathbf{Z}_n : \mathbf{L}_a, \quad (35)$$

$$\mathfrak{W}_{\text{st}}^{\text{in}}(\mathcal{P}_t) := \int_{\mathcal{P}_t} \left\{ -\rho_\ell \mathbf{m}_\ell \cdot \mathbf{w}_{\ell s} + \mathbf{T}_\ell : \text{grad}(\mathbf{w}_{\ell s}) + (\mathbf{T}_s + \mathbf{T}_\ell) : \text{grad}(\mathbf{v}_s) \right\}. \quad (36)$$

In a purely mechanical context, it can be proven that dissipation is given by

$$\int_{\mathcal{P}_t} \mathcal{D} = - \int_{\mathcal{P}_t} [\rho_s D_s \Psi_s + \rho_\ell D_\ell \Psi_\ell] + \mathfrak{W}_{\text{st}}^{\text{in}}(\mathcal{P}_t) + \mathfrak{W}_{\text{n-st}}^{\text{in}}(\mathcal{P}_t) \geq 0, \quad (37)$$

which is assumed to be non-negative. In terms of the overall Helmholtz free energy density of the system, $\rho\Psi$, the first term on the RHS of (37) can be written as

$$- \int_{\mathcal{P}_t} [\rho_s D_s \Psi_s + \rho_\ell D_\ell \Psi_\ell] = -d_t \int_{\mathcal{P}_t} \rho\Psi + \int_{\partial\mathcal{P}_t} \mathbf{q}_\Psi \cdot \mathbf{n} + \int_{\mathcal{P}_t} \rho_s \gamma_s \Psi_s, \quad (38)$$

where

$$\rho\Psi := \rho_s \Psi_s + \rho_\ell \Psi_\ell, \quad (39)$$

$$\mathbf{q}_\Psi := -[\rho_s \Psi_s \tilde{\mathbf{v}}_s + \rho_\ell \Psi_\ell \tilde{\mathbf{v}}_\ell]. \quad (40)$$

Under the hypotheses of hyperelastic solid phase and macroscopically inviscid fluid, the study of the inequality (37) yields the following results for the Cauchy stresses $\mathbf{T}_s$ and $\mathbf{T}_\ell$:

$$\mathbf{T}_s = -\phi_s \pi \mathbf{g}^{-1} + \mathbf{g}^{-1} \left(\rho_s \frac{\partial \Psi_s}{\partial \mathbf{F}_e} \right) (\mathbf{F}_a)^{-T} \mathbf{F}^T, \quad (41)$$

$$\mathbf{T}_\ell = -\phi_\ell \pi \mathbf{g}^{-1}, \quad (42)$$

where $\mathbf{g}$ is the metric tensor associated with $\mathcal{B}_t$

Requiring the invariance of constitutive laws under superimposed rigid motions places further restrictions on the results (41) and (42). If a rigid motion is impressed, the points $\mathbf{x} \in \mathcal{B}_t$ transform as $\mathbf{x} \mapsto \bar{\mathbf{x}} = \mathbf{R} \mathbf{x} + \mathbf{c}$, where $\mathbf{R}$ is a proper orthogonal tensor defining a pure rotation, and $\mathbf{c}$ is a vector defining a pure rigid translation [44, 32]. Consequently, $\mathbf{F}$, $\mathbf{F}_e$ and $\mathbf{F}_a$ transform as follows

$$\mathbf{F} \mapsto \bar{\mathbf{F}} = \mathbf{R} \mathbf{F}, \quad \mathbf{F}_e \mapsto \bar{\mathbf{F}}_e = \mathbf{R} \mathbf{F}_e, \quad \mathbf{F}_a \mapsto \bar{\mathbf{F}}_a = \mathbf{F}_a. \quad (43)$$

However, the Helmholtz free energy density Ψ_s has to remain invariant under these transformations. Therefore, Ψ_s may depend on $\mathbf{F}_e$ only through the Cauchy stretch tensor $\mathbf{C}_e = (\mathbf{F}_e)^T \mathbf{F}_e$, which is independent on $\mathbf{R}$. This yields the relation

$$\mathbf{g}^{-1} \left(\rho_s \frac{\partial \Psi_s}{\partial \mathbf{F}_e} \right) = \mathbf{F}_e \left(2\rho_s \frac{\partial \mathcal{W}_s}{\partial \mathbf{C}_e} \right), \quad (44)$$

where $\mathcal{W}_s$ is the Helmholtz free energy density of the solid phase written as function of $\mathbf{C}_e$. On the other hand, the Cauchy stresses $\mathbf{T}_s$ and $\mathbf{T}_\ell$ transform as $\bar{\mathbf{T}}_\alpha = \mathbf{R}\mathbf{T}_\alpha\mathbf{R}^T$, with $\alpha = s, \ell$.

Furthermore, by invoking the incompressibility of the solid phase and using the definitions (17) and (32), the first Piola–Kirchhoff stress tensors become

$$\mathbf{P}_s = -J_a \phi_{sn} \pi \mathbf{g}^{-1} \mathbf{F}^{-T} + \mathbf{F} \left[J_a (\mathbf{F}_a)^{-1} \left(2 \frac{\partial \mathcal{W}_{sn}}{\partial \mathbf{C}_e} \right) (\mathbf{F}_a)^{-T} \right], \quad (45)$$

$$\mathbf{P}_\ell = -(J - J_a \phi_{sn}) \pi \mathbf{g}^{-1} \mathbf{F}^{-T}, \quad (46)$$

where $\mathcal{W}_{sn} = \phi_{sn} \hat{\rho}_s \mathcal{W}_s$. Since $\mathbf{F}^{-T} = (\mathbf{F}_e)^{-T} (\mathbf{F}_a)^{-T}$, the stress $\mathbf{P}_s$ can be rewritten as

$$\mathbf{P}_s = J_a \mathbf{P}_{sn} (\mathbf{F}_a)^{-T}, \quad \mathbf{P}_{sn} := -\phi_{sn} \pi \mathbf{g}^{-1} (\mathbf{F}_e)^{-T} + \mathbf{F}_e \left(2 \frac{\partial \mathcal{W}_{sn}}{\partial \mathbf{C}_e} \right). \quad (47)$$

Finally, we call Mandel stress the tensor

$$\mathbf{M}_{sn} := (\mathbf{F}_e)^t \mathbf{P}_{sn} := -\phi_{sn} \pi (\mathbf{g}_n)^{-1} + \mathbf{C}_e \left(2 \frac{\partial \mathcal{W}_{sn}}{\partial \mathbf{C}_e} \right), \quad (48)$$

with $(\mathbf{F}_e)^t = (\mathbf{g}_n)^{-1} (\mathbf{F}_e)^T \mathbf{g}$.

The constitutive results (45) and (46) allow for a simplification of the expression of dissipation. After localisation, we obtain

$$\mathcal{D} = -\{ \rho_\ell \mathbf{m}_\ell - \pi \text{grad}(\phi_\ell) \} \cdot \mathbf{w}_{\ell s} + (J_e)^{-1} \{ \mathbf{M}_{sn} + \mathbf{Z}_n \} : \mathbf{L}_a \geq 0 \quad (49)$$

We introduce now the dissipative part of the force-like variables $\mathbf{m}_\ell$ and $\mathbf{Z}_n$, which are given by

$$\rho_\ell \check{\mathbf{m}}_\ell := \rho_\ell \mathbf{m}_\ell - \pi \text{grad}(\phi_\ell), \quad \check{\mathbf{Z}}_n := \mathbf{M}_{sn} + \mathbf{Z}_n, \quad (50)$$

and are constrained to satisfy the inequality

$$\mathcal{D} = -\rho_\ell \check{\mathbf{m}}_\ell \cdot \mathbf{w}_{\ell s} + (J_e)^{-1} \check{\mathbf{Z}}_n : \mathbf{L}_a \geq 0. \quad (51)$$

Dissipation has to be zero when $\mathbf{w}_{\ell s}$ and $\mathbf{L}_a$ vanish. Substitution of (50) into the balance laws (29) and (33), and use of the constitutive result (42) yield

$$\rho_\ell \check{\mathbf{m}}_\ell = \phi_\ell \text{grad}(\pi), \quad (52)$$

$$\check{\mathbf{Z}}_n = \mathbf{M}_{sn} + \mathbf{Y}_n. \quad (53)$$

Equations (52) and (53) represent the new form of the force balances that define, respectively, the dynamics of the fluid phase and the internal structure of the solid phase (the dynamics of the mixture as a whole are defined by (28) or (30)). Before proceeding with the determination of the forces $-\rho_\ell \check{\mathbf{m}}_\ell$ and $(J_e)^{-1} \check{\mathbf{Z}}_n$,

a discussion about the study of the dissipation inequality (51), and some of its implications, is mandatory.

Let us set $\mathbf{L}_a = \mathbf{0}$ and focus on the pair $(-\rho_\ell \check{\mathbf{m}}_\ell, \mathbf{w}_{\ell s})$. In Biomechanics, it is often assumed that fluid flow obeys Darcy's law (cf., for example, [6]). Darcy's model of flow can be retrieved consistently with the study of dissipation (cf., for example, [33]) by expressing $-\rho_\ell \check{\mathbf{m}}_\ell$ as a constitutive function of $\mathbf{w}_{\ell s}$ that vanishes when $\mathbf{w}_{\ell s} = \mathbf{0}$. This function is then expanded in Taylor series in a neighbourhood of $\mathbf{w}_{\ell s} = \mathbf{0}$ and, for small velocities, only the first-order term of the expansion is maintained. Therefore, one obtains

$$-\rho_\ell \check{\mathbf{m}}_\ell = \mathbf{A} \cdot \mathbf{q}_{\ell s}, \quad (54)$$

where $\mathbf{q}_{\ell s} := \phi_\ell \mathbf{w}_{\ell s}$ is called “specific discharge” and $\mathbf{A}$ is a positive-definite second-order tensor that represents the resistivity of the medium (in this discussion, ϕ_ℓ is assumed to be strictly different from zero). Substitution of this result into (52), and inverting $\mathbf{A}$ yield

$$\mathbf{q}_{\ell s} = -\mathbf{K} \cdot \text{grad}(\pi), \quad (55)$$

where $\mathbf{K} = \phi_\ell \mathbf{A}^{-1}$ is said to be the hydraulic conductivity of the medium. Equation (55) is a simplified form of Darcy's law, in which the buoyancy term is neglected. According to (55), $\mathbf{q}_{\ell s}$ vanishes in a permeable medium when the pressure gradient is zero. It is known, however, that Darcy's law may cease to be valid if, for example, the flow starts only when $-\rho_\ell \check{\mathbf{m}}_\ell$ exceeds a certain threshold, which is given by the pressure gradient [9]. In this case, the force $-\rho_\ell \check{\mathbf{m}}_\ell$ need not be smooth at $\mathbf{w}_{\ell s} = \mathbf{0}$.

Let us put now $\mathbf{w}_{\ell s} = \mathbf{0}$ and study the pair $((J_e)^{-1} \check{\mathbf{Z}}_n, \mathbf{L}_a)$. In some models of growth mechanics, constitutive laws of the type $\check{\mathbf{Z}}_n = \mathbb{H}_n : \mathbf{L}_a$ have been proposed (cf., for example, [1, 40, 2, 28, 5, 56, 27]), with $\mathbb{H}_n$ being a diagonally symmetric, positive-definite fourth-order tensor. The rate of anelastic deformation $\mathbf{L}_a$ was thus presented in the form $\mathbf{L}_a = \mathbb{G}_n : \check{\mathbf{Z}}_n$, with $\mathbb{G}_n = (\mathbb{H}_n)^{-1}$. Substitution of (53) in this relation yields

$$\mathbf{L}_a = \mathbb{G}_n : (\mathbf{M}_{sn} + \mathbf{Y}_n). \quad (56)$$

Equation (56) follows from the hypothesis that $\check{\mathbf{Z}}_n$ can be assigned as a constitutive function of $\mathbf{L}_a$ that vanishes when $\mathbf{L}_a = \mathbf{0}$. This function is then assumed to be smooth and linearised in a neighbourhood of $\mathbf{L}_a = \mathbf{0}$. For a positive-definite $\mathbb{G}_n$, the formula (56) admits the following interpretation: the rate of anelastic deformation, $\mathbf{L}_a$, becomes zero when the external force $\mathbf{Y}_n$ can be tuned in such a way that the sum $(\mathbf{M}_{sn} + \mathbf{Y}_n)$ vanishes. This situation implies that $\mathbf{F}_a$ (which always satisfies the kinematic relation $\dot{\mathbf{F}}_a = \mathbf{L}_a \mathbf{F}_a$) either ceases to evolve in time or remains equal to its initial value. In some biomechanical applications, $\mathbf{Y}_n$ is thought of as the “target stress” that regulates the process with which it is associated (when the target stress is reached, the process ceases). For example, in the model of arterial growth proposed in [45], $\mathbf{Y}_n$ is related to the homeostatic stress.

On the other hand, if the tensor $\mathbf{Y}_n$ is zero (or negligibly small), the equality $\mathbf{L}_a = \mathbf{0}$ cannot be recovered in general, since the Mandel stress, $\mathbf{M}_{sn}$, is not compensated by any external force. Furthermore, the relation (56) could be too restrictive for some applications. In fact, one may relax (56) and postulate an evolution law of the type [42]

$$\dot{\mathbf{V}}_a = \langle f \rangle \sum_{h=0}^2 \sum_{k=0}^2 \beta_{hk}(\mathbf{V}_a, \mathbf{S}_{sn}) [(\mathbf{V}_a)^h (\mathbf{S}_{sn})^k + (\mathbf{S}_{sn})^k (\mathbf{V}_a)^h]. \quad (57)$$

Here, $\mathbf{V}_a$ is the symmetric part of $\mathbf{F}_a$ (cf. the decomposition $\mathbf{F}_a = \mathbf{V}_a \mathbf{R}_a$ [44], where $\mathbf{R}_a$ –the tensor of anelastic rotation– is set equal to the identity), $\mathbf{S}_{sn} := (\mathbf{F}_e)^{-1} \mathbf{P}_{sn}$ is the second Piola–Kirchhoff stress tensor, β_{hk} is a given constitutive function of the invariants of $\mathbf{V}_a$, $\mathbf{S}_{sn}$ and compositions of these tensors, f is said to be dynamic yield function, and the symbol $\langle f \rangle$ equals unity when the anelastic deformation changes with time and equals zero otherwise. Constitutive restrictions on $\mathbf{L}_a$, $\mathbf{S}_{sn}$ and the dynamic yield function are then found by using (57) in the computation of the extrema of the anelastic working.

Another method for determining evolution laws is given in [17], where rate-independent plasticity is investigated. The dissipation is defined as a function of $\mathbf{L}_a$ and is assumed to be continuous, but generally non-differentiable, at $\mathbf{L}_a = \mathbf{0}$, while the tensor $\tilde{\mathbf{Z}}_n$ is constitutively indeterminate at $\mathbf{L}_a = \mathbf{0}$. Within this framework, a maximum-dissipation criterion is formulated and it is proven that the dissipation function is everywhere sub-differentiable and, thus, convex with respect to $\mathbf{L}_a$, and that $\tilde{\mathbf{Z}}_n$ must belong to the sub-differential of the dissipation function. The evolution of $\mathbf{L}_a$ is determined by introducing a scalar yield criterion through the yield function f and showing that $\mathbf{L}_a$ has to be an element of the sub-differential of f . In the case of a smooth yield function f , it is found that $\mathbf{L}_a$ follows the “normality rule”

$$\mathbf{L}_a = \lambda \frac{\partial f}{\partial \tilde{\mathbf{Z}}_n}, \quad (58)$$

where λ is a Kuhn-Tucker multiplier satisfying the conditions $\lambda \geq 0$, $f \leq 0$, $\lambda f = 0$, and determined by the consistency requirement $\dot{\lambda} f = 0$. Many of the mathematical tools for presenting this theory can be found in [52]. In the context of biological materials, an evolution law of the type (58) was proposed in [46] for modelling the reorganisation of cells (an anelastic deformation) of a tissue in the presence of growth (e.g., a tumour).

We remark that (56–58) are all plausible ways to determine the evolution of $\mathbf{L}_a$. They are, however, different from each other since they are conceived for modelling different physical situations. On the other hand, the common feature of all these models is that the relations linking the generalised forces $-\rho_\ell \tilde{\mathbf{m}}_\ell$ and $\tilde{\mathbf{Z}}_n$ with their power-conjugate generalised velocities $\mathbf{q}_{\ell s}$ and $\mathbf{L}_a$ satisfy a maximum-dissipation principle (cf. [50]).

2.4 Summary of the Model

Our purpose is to study how the structural change of the solid phase influences fluid flow through the modulation of the transport properties of the mixture. We accept the validity of Darcy's law, so that the fluid flow depends on hydraulic conductivity and pressure gradient. Therefore, to accomplish our task, we have to show how, for a given type of problem and assigned boundary conditions, different tensors $\mathbf{F}_a$ modulate the hydraulic conductivity of the medium and the pressure field inside it.

We remark that the medium is assumed to be isotropic with respect to both its elastic properties and permeability.

By substituting (55) into (18), using the constitutive results (45) and (46), neglecting $\mathbf{p}_s$ in (28), and writing the pressure gradient in material coordinates, i.e. $\text{grad}(\pi) = \mathbf{F}^{-T} \text{Grad}(\pi)$, the equations to solve are

$$-\text{Div} \left[-J\pi \mathbf{F}^{-T} + \mathbf{P} \right] = \mathbf{0}, \quad (59)$$

$$\text{Div} [\mathbf{K}_r \cdot \text{Grad}(\pi)] = \dot{J} - J_a \phi_{sn} \text{tr}(\mathbf{L}_a), \quad (60)$$

$$\dot{\mathbf{F}}_a = \mathbf{L}_a \mathbf{F}_a \quad (61)$$

where $\mathbf{P} = J_a \mathbf{F} \mathbf{F}_a^{-1} \mathbf{S}_n \mathbf{F}_a^{-T}$ denotes from here on the constitutive part of $\mathbf{P}_s$ (cf. (45)), $\mathbf{S}_n$ is the second Piola–Kirchhoff stress tensor measured with respect to $\mathcal{B}_n$, and $\mathbf{K}_r$ is the material form of the tensor of hydraulic conductivity, i.e., $\mathbf{K}_r = J \mathbf{F}^{-1} \mathbf{K} \mathbf{F}^{-T}$. The material is assumed to be hyperelastic and isotropic, and is modelled by the Neo-Hookean elastic energy given below [12], which leads to the following expressions of $\mathbf{S}_n$ and elasticity tensor $\mathbb{C}_n$:

$$\mathcal{W}_{sn}(\mathbf{C}_e) = \frac{\mu_n}{2} [\text{tr}(\mathbf{C}_e) - 3] - \mu_n \ln(J_e) + \frac{\lambda_n}{2} [\ln(J_e)]^2, \quad (62)$$

$$\mathbf{S}_n = 2 \frac{\partial \mathcal{W}_{sn}}{\partial \mathbf{C}_e} = \mu_n [(\mathbf{g}_n)^{-1} - (\mathbf{C}_e)^{-1}] + \lambda_n [\ln(J_e)] (\mathbf{C}_e)^{-1}, \quad (63)$$

$$\mathbb{C}_n = 4 \frac{\partial^2 \mathcal{W}_{sn}}{\partial \mathbf{C}_e \otimes \partial \mathbf{C}_e} = \lambda_n (\mathbf{C}_e)^{-1} \otimes (\mathbf{C}_e)^{-1} + 2[\mu_n - \lambda_n \ln(J_e)] \mathbb{I}_n, \quad (64)$$

where $\mathbb{I}_n$ is defined in the Appendix. Tensor $\mathbf{K}$ is taken from [34] and adapted to our framework, i.e.

$$\mathbf{K} = k_0 \left[\frac{\phi_{s0}}{1 - \phi_{s0}} \frac{J - \phi_{sn} J_a}{J_a \phi_{sn}} \right]^{m_0} \exp \left\{ \frac{m_1}{2} \left[\frac{J^2 - J_a^2}{J_a^2} \right] \right\} \mathbf{g}^{-1}. \quad (65)$$

The numbers m_0 and m_1 featuring in (65) are material parameters. To close the problem, $\mathbf{L}_a$ should be supplied by one of the formulae (56)–(58). The formulae of the elasticity tensor and hydraulic conductivity given in (64) and (65),

respectively, define isotropic tensors. However, other forms of hydraulic conductivity, which account for tissue anisotropy, have been recently proposed in [6, 24].

In the absence of growth, the field equations (59) and (60) were studied in [21] in the case of a linear viscoelastic biphasic model for soft tissues.

Equations (59) and (60), which hold in the internal points of $\mathcal{B}_r$, are completed with conditions prescribed on the boundary $\partial\mathcal{B}_r$. The unknowns of the problem are displacements, $\mathbf{u}(t, \mathbf{X}) := \chi(t, \mathbf{X}) - \mathbf{X}$, and pressure, π . For each unknown, the boundary $\partial\mathcal{B}_r$ is split into a Dirichlet- and a Neumann-type subset. This means that $\partial\mathcal{B}_r$ admits the representations $\partial\mathcal{B}_r = \Gamma_N^u \cup \Gamma_D^u$ and $\partial\mathcal{B}_r = \Gamma_N^\pi \cup \Gamma_D^\pi$, where Γ_N^u and Γ_N^π are the subsets of $\partial\mathcal{B}_r$ on which Neumann boundary conditions for the displacements and pressure are prescribed, while Γ_D^u and Γ_D^π are the subsets on which Dirichlet conditions are supplied. Formally, boundary conditions are written as

$$\left\{ \begin{array}{ll} \mathbf{u} = \mathbf{u}_b, & \text{on } \Gamma_D^u, \\ [-J\pi\mathbf{F}^{-T} + \mathbf{P}]N = \mathbf{f}_{rb}, & \text{on } \Gamma_N^u, \\ \pi = \pi_b, & \text{on } \Gamma_D^\pi, \\ [-\mathbf{K}_r \text{Grad}(\pi)] \cdot N = Q_{rb}, & \text{on } \Gamma_N^\pi, \end{array} \right. \quad (66)$$

where N is the unit vector normal to $\partial\mathcal{B}_r$. The surface force $\mathbf{f}_{rb}$ is defined per unit area of the reference boundary Γ_N^u and is, thus, generally different from the force $\mathbf{f}_b$ associated with the actual configuration $\mathcal{B}_t$. An analogous argument holds true for the quantities Q_{rb} and Q_b , the latter being the flux prescribed per unit area of the boundary $\partial\mathcal{B}_t$ of the actual configuration. The pairs $\mathbf{f}_{rb} - \mathbf{f}_b$ and $Q_{rb} - Q_b$ are related to each other by the formulae [12]

$$\mathbf{f}_{rb} = \mathbf{f}_b J \sqrt{N \otimes N : \mathbf{C}^{-1}}, \quad Q_{rb} = Q_b J \sqrt{N \otimes N : \mathbf{C}^{-1}}. \quad (67)$$

In order to reduce the number of equations to solve numerically, we consider the very particular case in which the tensor of anelastic deformation is kept constant (i.e. $\mathbf{F}_a$ is constant and known from the outset), so that no anelastic evolution occurs. This implies that $\mathbf{L}_a$ is zero. In other words, we assume that anelastic deformations have already taken place, which means that the tissue has already grown and remodelled. Physically, this can be rephrased by saying that the tissue grows and remodels over a time scale much larger than the scale over which fluid flows and elastic deformations take place.

3 Weak Formulation of the Field Equations

Throughout this section, we adopt Cartesian coordinates. To obtain the weak form of the field equations (59) and (60), we multiply (59) by the virtual velocity $\mathbf{v}_v$, and (60) by the virtual pressure π_v , and apply Leibniz's rule of differentiation. By

integrating the resulting expressions over the reference configuration $\mathcal{B}_r$, and using Gauss' Theorem, we obtain

$$- \int_{\mathcal{B}_r} J \pi \mathbf{F}^{-T} : \dot{\mathbf{F}}_v \, d\Omega + \int_{\mathcal{B}_r} \mathbf{P} : \dot{\mathbf{F}}_v \, d\Omega - \int_{\Gamma_N^u} \mathbf{v}_v \cdot \mathbf{f}_{rb} \, dA = 0, \quad (68)$$

$$- \int_{\mathcal{B}_r} \text{Grad}(\pi_v) \cdot \mathbf{K}_r \text{Grad}(\pi) \, d\Omega - \int_{\Gamma_N^\pi} \pi_v Q_{rb} \, dA - \int_{\mathcal{B}_r} \pi_v \dot{J} \, d\Omega = 0, \quad (69)$$

where $\dot{\mathbf{F}}_v = \text{Grad}(\mathbf{v}_v)$. In the jargon of Numerical Analysis, the virtual fields $\mathbf{v}_v$ and π_v are referred to as test functions. These functions are chosen in such a way that they vanish identically on Γ_D^u and Γ_D^π , respectively.

Apart from boundary conditions, which might be time-dependent in general, the only place in which time features explicitly in (68) and (69) is in the time derivative $\dot{J}$. We approximate this derivative with a finite difference, and use an implicit Euler method [47]. In order to do that, we consider the time interval (t_i, t_f) and discretize it in N sub-intervals (t^{m-1}, t^m) . Thus, we write $\dot{J} \approx (J^m - J^{m-1})/(\Delta t)^m$, where $(\Delta t)^m$ is the size of the m th time step, and reformulate the system of equations (68) and (69) in the following way [12]

$$- \int_{\mathcal{B}_r} J^m \pi^m (\mathbf{C}^m)^{-1} : \dot{\mathbf{E}}_v^m \, d\Omega + \int_{\mathcal{B}_r} \mathbf{S}^m : \dot{\mathbf{E}}_v^m \, d\Omega - \int_{\Gamma_N^u} \mathbf{v}_v \cdot \mathbf{f}_{rb}^m \, dA = 0, \quad (70)$$

$$\begin{aligned} & - \int_{\mathcal{B}_r} \text{Grad}(\pi_v) \cdot \mathbf{K}_r^m \text{Grad}(\pi^m) \, d\Omega - \int_{\Gamma_N^\pi} \pi_v Q_{rb}^m \, dA \\ & - \int_{\mathcal{B}_r} \pi_v \frac{J^m}{(\Delta t)^m} \, d\Omega = - \int_{\mathcal{B}_r} \pi_v \frac{J^{m-1}}{(\Delta t)^m} \, d\Omega, \end{aligned} \quad (71)$$

where $\mathbf{C}^m = (\mathbf{F}^m)^T \mathbf{F}^m$, $\mathbf{S}^m := (\mathbf{F}^m)^{-1} \mathbf{P}^m$, and $\dot{\mathbf{E}}_v^m$ is given by

$$\dot{\mathbf{E}}_v^m := \frac{1}{2} [(\mathbf{F}^m)^T \dot{\mathbf{F}}_v + (\dot{\mathbf{F}}_v)^T \mathbf{F}^m]. \quad (72)$$

In this notation, the Green-Lagrange strain tensor at time t^m reads $\mathbf{E}^m = \frac{1}{2} (\mathbf{C}^m - \mathbf{I}_r)$, with $\mathbf{I}_r$ being the identity tensor in the reference configuration. Given a generic function $\mathbf{A}$, the symbol $\mathbf{A}^m$ means $\mathbf{A}^m = \mathbf{A}(t^m, \cdot)$.

3.1 Linearisation

Formally, equations (70) and (71) can be rewritten in compact form as

$$\mathfrak{F}_u(\mathbf{u}^m, \pi^m, \mathbf{v}_v) = 0, \quad (73)$$

$$\mathfrak{F}_\pi(\mathbf{u}^m, \pi^m, \pi_v) = 0, \quad (74)$$

where $\mathfrak{F}_u$ and $\mathfrak{F}_\pi$ are functionals of displacement and pressure, and their virtual counterparts $\mathbf{v}_v$ and π_v . In the sequel, we drop the dependence of these functionals on $\mathbf{v}_v$ and π_v for the sake of a simpler notation. The non-linear system of equations (73) and (74) is solved by Newton's method [12, 47]. At the k th iteration, the linearised system reads

$$\mathfrak{F}_u(\mathbf{u}^{m,k-1}, \pi^{m,k-1}) + \mathfrak{D}\mathfrak{F}_u(\mathbf{u}^{m,k-1}, \pi^{m,k-1})[\mathbf{h}^{m,k}, \theta^{m,k}] = \mathbf{0}, \quad (75)$$

$$\mathfrak{F}_\pi(\mathbf{u}^{m,k-1}, \pi^{m,k-1}) + \mathfrak{D}\mathfrak{F}_\pi(\mathbf{u}^{m,k-1}, \pi^{m,k-1})[\mathbf{h}^{m,k}, \theta^{m,k}] = 0. \quad (76)$$

where $\mathfrak{D}\mathfrak{G}(\mathbf{u}^{m,k-1}, \pi^{m,k-1})[\mathbf{h}^{m,k}, \theta^{m,k}]$ denotes the Gateaux-derivative of a generic functional $\mathfrak{G}$, evaluated at $(\mathbf{u}^{m,k-1}, \pi^{m,k-1})$, and along the direction $[\mathbf{h}^{m,k}, \theta^{m,k}]$.

Given the pair $(\mathbf{u}^{m,k-1}, \pi^{m,k-1})$, in the neighbourhood of which the functionals (73) and (74) are linearised, the increments $\mathbf{h}^{m,k}$ and $\theta^{m,k}$ are computed by solving (75) and (76), and the updated pair of fields $(\mathbf{u}^{m,k}, \pi^{m,k})$ is determined as follows

$$\mathbf{u}^{m,k} = \mathbf{u}^{m,k-1} + \mathbf{h}^{m,k}, \quad (77)$$

$$\pi^{m,k} = \pi^{m,k-1} + \theta^{m,k}. \quad (78)$$

We notice that $\dot{\mathbf{E}}_v^m$, as defined in (72), is a bilinear functional of $\mathbf{u}^m$ and $\mathbf{v}_v$. Thus, by introducing the notation

$$\dot{\mathbf{E}}_v^m = \dot{\mathbf{E}}_v(\mathbf{u}^m) = \frac{1}{2}[(\mathbf{F}^m)^T \dot{\mathbf{F}}_v + (\dot{\mathbf{F}}_v)^T \mathbf{F}^m], \quad (79)$$

replacing $\mathbf{u}^m$ with $\mathbf{u}^{m,k}$, and invoking (77), we conclude that

$$\dot{\mathbf{E}}_v^{m,k} = \dot{\mathbf{E}}_v(\mathbf{u}^{m,k}) = \dot{\mathbf{E}}_v^{m,k-1} + \text{sym}[(\mathbf{H}^{m,k})^T \dot{\mathbf{F}}_v], \quad (80)$$

with $\mathbf{H}^{m,k} := \text{Grad}(\mathbf{h}^{m,k})$, and that the following identity holds true:

$$\dot{\mathbf{E}}_v^{m,k-1} := \frac{1}{2}[(\mathbf{F}^{m,k-1})^T \dot{\mathbf{F}}_v + (\dot{\mathbf{F}}_v)^T \mathbf{F}^{m,k-1}] = \mathfrak{D}\mathbf{E}(\mathbf{u}^{m,k-1})[\mathbf{v}_v]. \quad (81)$$

Furthermore, we recall that [12]

$$\mathfrak{D}\mathbf{F}(\mathbf{u}^{m,k-1})[\mathbf{h}^{m,k}] = \text{Grad}(\mathbf{h}^{m,k}) = \mathbf{H}^{m,k}. \quad (82)$$

With the formalism of this section, the boundary terms $\mathbf{f}_{rb}^m$ and Q_{rb}^m should be written as

$$\mathbf{f}_{rb}^m = \mathbf{f}_{rb}(t^m, \mathbf{u}^m), \quad Q_{rb}^m = Q_{rb}(t^m, \mathbf{u}^m). \quad (83)$$

Since $\mathbf{f}_{rb}^m$ and Q_{rb}^m depend on displacements in a non-linear way, they should be linearised too. In the computational praxis, however, this linearisation is sometimes avoided, and the approximations $\mathbf{f}_{rb}^m \approx \mathbf{f}_{rb}^m(t^m, \mathbf{u}^{m,k-1})$ and $Q_{rb}^m \approx Q_{rb}^m(t^m, \mathbf{u}^{m,k-1})$ are made. This is done because it is often too difficult to connect the boundary finite element over which $\mathbf{f}_{rb}^m$ and Q_{rb}^m are defined with the volume

element to which it belongs. This approximation does not usually destroy the convergence of Newton's method.

When $\mathbf{L}_a$ is different from zero (which means that anelastic deformations evolve in time), the rate of anelastic deformation could be either given by one of (56) and (58) or computed as $\mathbf{L}_a = \dot{\mathbf{V}}_a(\mathbf{V}_a)^{-1}$, with $\dot{\mathbf{V}}_a$ specified in (57). In any case, the resulting expression is a function of deformation and should, thus, be linearised. If a plasticity-like flow rule is taken (cf. (58)), dedicated numerical procedures (e.g., the Return Mapping Algorithm [54] or the Linearised Projection Algorithm [59]) have to be implemented for the computation of the admissible stresses. However, the use of such numerical schemes may rise stability issues. In the case of associative multiplicative elasto-plasticity, these stability problems were studied, for example, by Miehe et al. [43]. On the other hand, if $\mathbf{L}_a$ is determined by (57), the linearisation procedure may become too demanding. Finally, also in the case in which the validity of (56) is assumed, a stable numerical procedure should be tested in order to obtain reliable solutions of (70), (71) and the evolution equation $\dot{\mathbf{F}}_a = \mathbf{L}_a \mathbf{F}_a$.

3.2 Spatial Discretization

The variational problem associated with (56) and (58) can be written in abstract form as:

$$\boxed{\begin{array}{l} \text{Find } \mathbf{h}^{m,k} \text{ and } \theta^{m,k} \in Q, \text{ such that :} \\ \left\{ \begin{array}{ll} a(\mathbf{h}^{m,k}, \mathbf{v}_v) - b(\mathbf{v}_v, \theta^{m,k}) = f(\mathbf{v}_v) & \forall \mathbf{v}_v \in V \\ -c(\mathbf{h}^{m,k}, \pi_v) - d(\theta^{m,k}, \pi_v) = g(\pi_v) & \forall \pi_v \in Q \end{array} \right. \end{array}} \quad (84)$$

where $V := (H_{\Gamma_b}^1(\mathcal{B}_r))^3$ and $Q := H_{\Gamma_p}^1(\mathcal{B}_r)$ are Hilbert spaces. The bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, $c(\cdot, \cdot)$, $d(\cdot, \cdot)$ are defined as follows

$$\begin{aligned} a(\mathbf{h}^{m,k}, \mathbf{v}_v) := & \int_{\mathcal{B}_t} \{ \mathfrak{D} \mathbf{E}^{m,k-1}[\mathbf{v}_v] : [\mathbb{C}_c^{m,k-1} + \mathbb{C}_r^{m,k-1}] : \mathfrak{D} \mathbf{E}^{m,k-1}[\mathbf{h}^{m,k}] \} d\Omega \\ & + \int_{\mathcal{B}_r} [-J^{m,k-1} \pi^{m,k-1} (\mathbf{C}^{m,k-1})^{-1} + \mathbf{S}^{m,k-1}] : \mathfrak{D} \dot{\mathbf{E}}_v^{m,k-1}[\mathbf{h}^{m,k}] d\Omega, \end{aligned} \quad (85)$$

$$b(\mathbf{v}_v, \theta^{m,k}) := \int_{\mathcal{B}_r} J^{m,k-1} \theta^{m,k} (\mathbf{C}^{m,k-1})^{-1} : \mathfrak{D} \mathbf{E}^{m,k-1}[\mathbf{v}_v] d\Omega, \quad (86)$$

$$c(\mathbf{h}^{m,k}, \pi_v) := \frac{b(\mathbf{h}^{m,k}, \pi_v)}{(\Delta t)^m} + \int_{\mathcal{B}_r} \text{Grad}(\pi_v) \cdot \mathfrak{D} \mathbf{K}_r^{m,k-1}[\mathbf{h}^{m,k}] \cdot \text{Grad}(\pi^{m,k-1}) d\Omega, \quad (87)$$

$$d(\theta^{m,k}, \pi_v) := \int_{\mathcal{B}_r} \text{Grad}(\pi_v) \cdot \mathbf{K}_r^{m,k-1} \text{Grad}(\theta^{m,k}), d\Omega, \quad (88)$$

and the right-hand sides $f(\cdot)$, $g(\cdot)$ are given by

$$f(\mathbf{v}_v) := -\mathfrak{F}_u(\mathbf{u}^{m,k-1}, \pi^{m,k-1}, \mathbf{v}_v), \quad (89)$$

$$g(\pi_v) := -\mathfrak{F}_\pi(\mathbf{u}^{m,k-1}, \pi^{m,k-1}, \pi_v). \quad (90)$$

We notice that two “elasticity” tensors feature in (85). They are defined by:

$$\mathbb{C}_r^{m,k-1} := J_a(\mathbf{F}_a)^{-1} \underline{\otimes} (\mathbf{F}_a)^{-1} : \mathbb{C}_n^{m,k-1} : (\mathbf{F}_a)^{-T} \underline{\otimes} (\mathbf{F}_a)^{-T}, \quad (91)$$

$$\mathbb{C}_c^{m,k-1} := 2J^{m,k-1} \pi^{m,k-1} \left[\mathbb{I}^{m,k-1} - \frac{1}{2} (\mathbf{C}^{m,k-1})^{-1} \otimes (\mathbf{C}^{m,k-1})^{-1} \right]. \quad (92)$$

The tensor $\mathbb{C}_r^{m,k-1}$ is the true elasticity tensor, which can be found by the constitutive prescriptions. The tensor $\mathbb{C}_c^{m,k-1}$, on the other hand, is due to the incompressibility constraint, which, however, does not imply the restriction $J(\mathbf{C}) = 1$ to the motion of the solid phase of a biphasic material.

All quantities with indices “ $(m, k-1)$ ” (e.g., $J^{m,k-1}$) are computed at the “point” $(\mathbf{u}^{m,k-1}, \pi^{m,k-1})$, i.e. $J^{m,k-1} \equiv J(\mathbf{u}^{m,k-1})$. Moreover, for sake of shorter notation, the Gauteaux derivatives have been rewritten as

$$\mathfrak{D}\mathbf{E}(\mathbf{u}^{m,k-1})[\mathbf{v}_v] \equiv \mathfrak{D}\mathbf{E}^{m,k-1}[\mathbf{v}_v], \quad (93)$$

$$\mathfrak{D}\mathbf{E}(\mathbf{u}^{m,k-1})[\mathbf{h}^{m,k}] \equiv \mathfrak{D}\mathbf{E}^{m,k-1}[\mathbf{h}^{m,k}], \quad (94)$$

$$\mathfrak{D}\dot{\mathbf{E}}_v(\mathbf{u}^{m,k-1}, \mathbf{v}_v)[\mathbf{h}^{m,k}] \equiv \mathfrak{D}\dot{\mathbf{E}}_v^{m,k-1}[\mathbf{h}^{m,k}], \quad (95)$$

$$\mathfrak{D}\mathbf{K}_r(\mathbf{u}^{m,k-1})[\mathbf{h}^{m,k}] \equiv \mathfrak{D}\mathbf{K}_r^{m,k-1}[\mathbf{h}^{m,k}]. \quad (96)$$

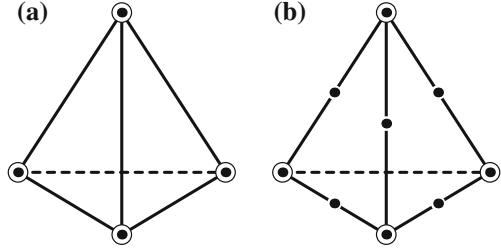
These results (85)–(96) can be obtained by adopting the formulae in Appendix, and using (64) and (65).

In order to write the finite element formulation of the problem, we introduce a conforming shape-regular mesh $\mathcal{T}_h$ consisting of N_h non-overlapping elements $\{K_i\}_{i=1}^{N_h}$. Moreover, we define the following finite dimensional spaces:

$$\begin{aligned} V_h^l &= \left\{ \mathbf{v}_h \in V : \mathbf{v}_h|_{K_i} \in (\mathbb{P}_l)^3, \text{ for } i = 1, \dots, N_h \right\} \\ Q_h^\sigma &= \left\{ \pi_h \in Q : \pi_h|_{K_i} \in \mathbb{P}_\sigma, \text{ for } i = 1, \dots, N_h \right\}, \end{aligned} \quad (97)$$

where $(\mathbb{P}_l)^3$ and $\mathbb{P}_\sigma$ are spaces of polynomials of degree l and σ , respectively. In particular, the notation $(\mathbb{P}_l)^3$ means that all components of the three-dimensional vector $\mathbf{v}_h|_{K_i}$ are polynomials of degree l . Denoting by $\{\boldsymbol{\varphi}_q\}_{q=1}^M$ and $\{\phi_s\}_{s=1}^N$, with $M = \dim(V_h^l)$ and $N = \dim(Q_h^\sigma)$, the Lagrangian basis functions of V_h^l and Q_h^σ , respectively, the discrete solutions can be written as

Fig. 1 DOFs for the employed P1-P1 couple (a) and for Taylor-Hood P2-P1 elements (b). The full circles indicate DOFs for displacement, while the empty circles indicate the DOFs for pressure



$$\mathbf{h}_h^{m,k} = \sum_{q=1}^M h_q^{m,k} \boldsymbol{\varphi}_q, \quad \boldsymbol{\theta}_h^{m,k} = \sum_{s=1}^N \theta_s^{m,k} \boldsymbol{\phi}_s, \quad (98)$$

Employing these definitions, the finite element formulation of the problem reads

Find $\mathbf{h}_h^{m,k} \in V_h^l$ and $\boldsymbol{\theta}_h^{m,k} \in Q_h^\sigma$, such that :

$$\begin{cases} a(\mathbf{h}_h^{m,k}, \boldsymbol{\varphi}_p) - b(\boldsymbol{\varphi}_p, \boldsymbol{\theta}_h^{m,k}) = f(\boldsymbol{\varphi}_p), & p = 1 \dots M, \\ -c(\mathbf{h}_h^{m,k}, \boldsymbol{\phi}_r) - d(\boldsymbol{\theta}_h^{m,k}, \boldsymbol{\phi}_r) = g(\boldsymbol{\phi}_r), & r = 1 \dots N, \end{cases} \quad (99)$$

or in an algebraic form:

$$\begin{pmatrix} A & -B^T \\ -C & -D \end{pmatrix} \begin{pmatrix} \mathbf{h} \\ \boldsymbol{\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}. \quad (100)$$

The matrices $A \in \mathbb{R}^{M \times M}$, $B, C \in \mathbb{R}^{M \times N}$, and $D \in \mathbb{R}^{N \times N}$ are the algebraic representations of the weak forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, $c(\cdot, \cdot)$, and $d(\cdot, \cdot)$, respectively, and their entries are given by

$$\begin{aligned} A_{pq} &= a(\boldsymbol{\varphi}_q, \boldsymbol{\varphi}_p) & B_{rq} &= b(\boldsymbol{\varphi}_q, \boldsymbol{\phi}_r) \\ C_{rq} &= c(\boldsymbol{\varphi}_q, \boldsymbol{\phi}_r) & D_{rs} &= d(\boldsymbol{\phi}_s, \boldsymbol{\phi}_r). \end{aligned} \quad (101)$$

The entries of vectors $\mathbf{f}$ and $\mathbf{g}$ are given by

$$\mathbf{f}_p = f(\boldsymbol{\varphi}_p), \quad \mathbf{g}_r = g(\boldsymbol{\phi}_r), \quad (102)$$

while the vectors $\mathbf{h}$ and $\boldsymbol{\theta}$ are defined such that $\mathbf{h}_q = h_q$ and $\boldsymbol{\theta}_s = \theta_s$.

The discrete problem (100) has the same form as the saddle-point problem arising from the discretization of the incompressible Navier-Stokes equations or the almost incompressible elasticity equations. However, there are some important differences. On the one hand, the matrix A is symmetric but not positive definite since, in general, ellipticity does not hold for the terms defining the bilinear form $a(\cdot, \cdot)$ (cf. (85)). On the other hand, the second row of the system does not arise from the constraint $\text{div}(\mathbf{v}) = 0$, rather it comes from the discretization of (69). In fact, we have $C \neq B$ and

the matrix D is the algebraic representation of a diffusion operator. Problems of this form are known as “generalised saddle-point problems” [11].

Note that our choice of finite element spaces V_h^1 and Q_h^1 (see Fig. 1a) does not comply with the Ladyženskaja-Babuška-Brezzi (LBB) condition [15]. This results in a matrix B^T with a non-trivial kernel. Still, since D is non-zero, spurious pressure modes are avoided. This effect of the Laplacian is the same as obtained by the Brezzi-Pitktäranta stabilization for Stokes and Navier-Stokes problems [16].

Let us point out that in the case of very small permeability the stabilizing effect of the diffusion matrix D becomes less effective, which can be the source of numerical instabilities. In this case, the choice of discrete spaces that satisfy the LBB condition is necessary, e.g. Taylor-Hood elements (Fig. 1b).

In our simulation, no occurrence of volume locking was observed. However, despite the formulation of the problem in a saddle-point form, volume locking can occur in the case of large bulk moduli. To overcome this issue, different variational formulations involving a mixed problem [55, 14] or second order finite elements can be applied.

3.3 Numerical Solution

The discretization method described above has been implemented in the software toolbox UG/Obslib++ [8, 31].

For small number of degrees of freedom, direct solvers are advantageous due to their robustness and fast solution phase. In our numerical experiments (see Sect. 3) we used UMFPACK [19].

For large system sizes, direct solvers are prohibitively expensive in terms of computing time and memory consumption. However, standard efficient solution strategies like domain-decomposition and multigrid methods are not readily applicable to the type of *coupled* problems discussed here. For example, in the case of an additive Schwarz preconditioned, it can be shown that the iterations of a GMRES solver grows proportional to the square root of the number of the processors [35]. In the case of multigrid methods, the challenge is to find suitable smoothers for the indefinite global matrix.

An alternative solution method is based on the Schur complement of the system (100). This method is based on the elimination of the displacement degrees of freedom which results in a reduced system of size N :

$$(CA^{-1}B^T + D)\theta = -CA^{-1}f - g. \quad (103)$$

Note that if $f = 0$, forming the right-hand side does not require any solution of a linear system. A well-known solution method for the system (103) is the Uzawa algorithm. This method is a two-step method and reads:

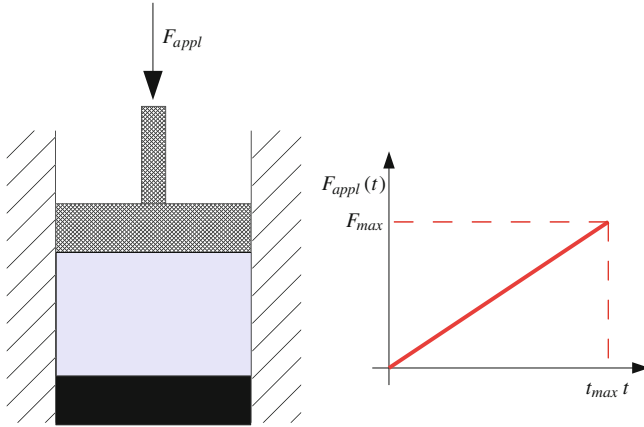


Fig. 2 Schematic representation of the parallel plate apparatus used in the benchmark problem: the lower plate is impermeable, whereas liquid can flow through the upper plate. Results are reported in the case of an applied load linearly increasing in time

Given the initial guess θ_0 , at every step k solve :

$$A\mathbf{h}_k = B^T\theta_k + \mathbf{f},$$

and update pressure :

$$\theta_{k+1} = \theta_k + \omega(-\mathbf{g} - C\mathbf{h}_k - D\theta_k).$$

The index k used in this box should not be confused with that denoting the k th Newton step in the procedure sketched above.

This method is equivalent to the Richardson method for the Schur complement system and requires the solution of one linear system $A\mathbf{h}_k = B^T\theta_k + \mathbf{f}$ at every step. The matrix A obtained from the discretization of a non-linear elasticity problem can be efficiently solved with a multigrid method. An efficient preconditioning strategy for the Schur complement arising in the linear problem has been presented in [38, 23].

4 Solution of a Benchmark Problem

We apply the model presented in Sect. 2.4 to describe a confined compression test under given loading conditions. We consider the case in which the biphasic material is positioned inside a rigid cylinder and left free to grow. The cylindrical sample is then compressed between two plates: the lower plate is impermeable, whereas the upper plate allows fluid exudation, so that the liquid embedded in the material can escape from the specimen due to compression (see Fig. 2).

The formulation of the confined compression is based on the assumption that the matrix representation of the deformation gradient is given a very simple diagonal form. Indeed, since the cylindrical wall of the parallel-plate-apparatus is supposed to be rigid and impervious, it is reasonable to assume deformations and velocities of all constituents to be along the Z-axis. Therefore, using a cylindrical coordinate system, the deformation generated by a uniaxial force applied along the Z-axis is

$$\chi^r(t, \mathbf{X}) = R, \quad \chi^\theta(t, \mathbf{X}) = \Theta, \quad \chi^z(t, \mathbf{X}) = z, \quad (104)$$

where $\mathbf{X} = (R, \Theta, Z)$. We restrict our investigation to the case in which χ^z depends on Z and t only, so that the matrix representation of the deformation gradient tensor is diagonal and given by

$$\mathbf{F} = \text{diag}\{1, 1, \partial_Z \chi^z\}. \quad (105)$$

We remark that, due to the particular form of $\mathbf{F}$, the identity $J = \partial_Z \chi^z$ holds true. Then, we assume that $\mathbf{F}_a$, the tensor of anelastic deformation that maps the tangent space of the reference configuration, $T\mathcal{B}_r$, onto the tangent space of the natural configuration, $T\mathcal{B}_n$, has the diagonal form

$$\mathbf{F}_a = \text{diag}\{g_1, g_1, g_3\}. \quad (106)$$

We choose a non-spherical growth in order to study the influence of non-isotropic growth on the distribution of load and pressure throughout compression. For the problem under investigation, g_1 and g_3 are assumed to be constant in time and given from the outset. From (105) and (106), it follows that

$$\mathbf{F}_e = \text{diag}\left\{\frac{1}{g_1}, \frac{1}{g_1}, \frac{J}{g_3}\right\}. \quad (107)$$

The next step is to re-write (59) and (60) in cylindrical coordinates. For ease of notation, we introduce the symbol

$$\mathbf{Q} := -\mathbf{K}_r \cdot \text{Grad}(\pi). \quad (108)$$

It follows from (60) that

$$\dot{J} = -\text{Div}(\mathbf{Q}) = -\frac{1}{R} \frac{\partial}{\partial R}(RQ^R) - \frac{1}{R} \frac{\partial Q^\Theta}{\partial \Theta} - \frac{\partial Q^Z}{\partial Z} = -\frac{\partial Q^Z}{\partial Z}, \quad (109)$$

where the last identity holds true by requiring that the derivatives with respect to both the radial and tangential directions vanish identically, and that Q^R is zero. The latter condition amounts to say that there is no fluid flow along the radial direction. It is thus sufficient to determine Q^Z , which is given by

$$Q^Z = -(K_r)^{ZZ} \frac{\partial \pi}{\partial Z} = -JK \frac{1}{J^2} \frac{\partial \pi}{\partial Z} = -\frac{K}{J} \frac{\partial \pi}{\partial Z}. \quad (110)$$

By substituting (110) into (109), we obtain

$$\dot{J} = \frac{\partial}{\partial Z} \left(\frac{K}{J} \frac{\partial \pi}{\partial Z} \right). \quad (111)$$

Taking into account that the Piola–Kirchhoff stress tensor $\mathbf{P}$ is diagonal, i.e.

$$\begin{aligned} \mathbf{P} &= \text{diag}\{P^{rR}, P^{\vartheta\Theta}, P^{zZ}\} \\ &= \mu_n J_a \text{diag}\left\{\frac{1}{g_1^2}, \frac{1}{g_1^2}, \frac{J}{g_3^2}\right\} + \left[\lambda_n \log\left(\frac{J}{J_a}\right) - \mu_n\right] \text{diag}\left\{J_a, J_a, \frac{J_a}{J}\right\}, \end{aligned} \quad (112)$$

and that the liquid and the solid phases move only along the z -direction, the balance of momentum (30) reduces to

$$\frac{\partial \pi}{\partial Z} = \frac{\partial P^{zZ}}{\partial Z}. \quad (113)$$

By coupling (113) with (111), we obtain

$$\begin{aligned} \dot{J} &= \frac{\partial}{\partial Z} \left(\frac{K}{J} \frac{\partial P^{zZ}}{\partial Z} \right) = \\ &= \frac{\partial}{\partial Z} \left[\frac{K}{J} \frac{\partial}{\partial Z} \left(\mu_n \frac{g_1^2}{g_3} J + \left(\lambda_n \log\left(\frac{J}{g_1^2 g_3}\right) - \mu_n \right) \frac{g_1^2 g_3}{J} \right) \right]. \end{aligned} \quad (114)$$

It can be proven that the partial derivative of P^{zZ} with respect to the axial direction reads

$$\frac{\partial P^{zZ}}{\partial Z} = \left[\frac{J_a \mu_n}{g_3^2} + \frac{J_a \mu_n}{J^2} + \frac{J_a \lambda_n}{J^2} - \frac{J_a \lambda_n}{J^2} \ln\left(\frac{J}{J_a}\right) \right] \frac{\partial J}{\partial Z}. \quad (115)$$

Therefore, the mass balance law acquires the form of a nonlinear diffusion equation, in which the “transported” quantity is the volumetric deformation J . Indeed, by substituting (115) into (114), we obtain

$$\dot{J} = \frac{\partial}{\partial Z} \left[D(J) \frac{\partial J}{\partial Z} \right] \quad (116)$$

where $D(J)$ represents a fictitious diffusion coefficient defined by

$$D(J) := K \left[\frac{J_a \mu_n}{J g_3^2} + \frac{J_a \mu_n}{J^3} + \frac{J_a \lambda_n}{J^3} - \frac{J_a \lambda_n}{J^3} \ln\left(\frac{J}{J_a}\right) \right]. \quad (117)$$

It is important to notice that $D(J)$ is always positive, being $J < J(0, Z) = J_a$ because of compression, which leads to $J_e < 1$. Therefore, all the terms on the right-hand-side of (117) are positive. This is consistent with the diffusive nature of the problem and it will be important also in defining boundary conditions.

Finally, the diffusion equation (116) has to be solved together with the auxiliary equations

$$\frac{\partial \chi^z}{\partial Z} = J, \quad (118)$$

$$\frac{\partial \pi}{\partial Z} = \left[\frac{J_a \mu_n}{g_3^2} + \frac{J_a \mu_n}{J^2} + \frac{J_a \lambda_n}{J^2} - \frac{J_a \lambda_n}{J^2} \ln \left(\frac{J}{J_a} \right) \right] \frac{\partial J}{\partial Z}. \quad (119)$$

We remark, however, that (118) and (119) are decoupled from (116), and can thus be solved *a posteriori* once J is known, provided proper boundary conditions are supplied.

4.1 Boundary and Initial Conditions

In order to solve (114), (118) and (119), we have to supply boundary conditions (BCs) and an initial condition (IC). In particular, the mass balance (114) requires two BCs and one IC, whereas both (118) and (119) require one BC only. Boundary conditions have to be provided at the boundary points $Z = 0$ and $Z = L$, which identify the lower and upper boundary of the specimen, respectively.

The boundary conditions have to be consistent with the following requirements: (i) the axial stress at the upper boundary of the specimen has to be equal to the applied load, $P_{app}(t)$; (ii) the velocity of the fluid and of the solid phase have to be zero at the bottom because the lower plate is impermeable and fixed; and (iii) the pressure π has to be zero at $Z = L$ since the liquid is in equilibrium with the atmosphere. These observations are translated in the following set of boundary conditions

$$\chi^z(t, 0) = 0, \quad (120)$$

$$Q^Z(t, 0) = 0 \quad \Rightarrow \quad \frac{\partial \pi}{\partial Z}(t, 0) = 0, \quad (121)$$

$$\tilde{P}^{zZ}(t, L) + P^{zZ}(t, L) = P_{app}(t), \quad (122)$$

$$\pi(t, L) = 0, \quad (123)$$

where $\tilde{\mathbf{P}} = -J\pi\mathbf{g}^{-1}\mathbf{F}^{-T}$. We remark that, by virtue of the identity (113), we may rephrase (121) as follows

$$\frac{\partial P^{zZ}}{\partial Z}(t, 0) = \left[\frac{J_a \mu_n}{g_3^2} + \frac{J_a \mu_n}{J^2} + \frac{J_a \lambda_n}{J^2} - \frac{J_a \lambda_n}{J^2} \ln \left(\frac{J}{J_a} \right) \right]_{(t,0)} \frac{\partial J}{\partial Z}(t, 0) = 0. \quad (124)$$

We recall that the argument in the square brackets is always positive. Therefore, this condition leads to a zero-Neumann BC for J at the lower boundary:

$$\frac{\partial J}{\partial Z}(t, 0) = 0. \quad (125)$$

On the other hand, (122) leads to a Dirichlet condition on J at the upper boundary:

$$\mu_n \frac{J_a J(t, L)}{g_3^2} - \mu_n \frac{J_a}{J(t, L)} + \lambda_n \frac{J_a}{J(t, L)} \ln \left(\frac{J(t, L)}{J_a} \right) = P_{appl}(t). \quad (126)$$

Since this equation is nonlinear with respect to $J(t, L)$, solutions can be found by applying Newton's method or other techniques.

As initial condition, we take $J(0, Z) = J_a(0, Z)J_e(0, Z) = J_a$. Indeed, at the initial time, there is no elastic deformation, although the anelastic deformation has already occurred. We remark that, for consistency, the condition $\mathbf{F}_e(0, \mathbf{X}) = \delta$ entails that $g_1 = 1$.

4.2 Discretization

Equation (114) can be solved using central differences for space derivatives and then a proper ODE solver to obtain the temporal evolution. In the following we depict the main steps of this procedure.

The 1D-domain, represented by the interval $[0, L]$, is divided into $N - 1$ sub-intervals of the same width ΔZ through the introduction of N equispaced nodes

$$0 = Z_1 < Z_2 = Z_1 + \Delta Z < \dots < Z_j < \dots < Z_{N-1} < Z_N = L.$$

Spaces derivatives are then approximated by finite differences, so that the following system of $N - 2$ equations is obtained:

$$j_j = \frac{1}{(\Delta Z)^2} \left(\frac{K_{j+1}}{J_{j+1}} P_{j+1}^{zZ} - \left(\frac{K_{j+1}}{J_{j+1}} + \frac{K_j}{J_j} \right) P_j^{zZ} + \frac{K_j}{J_j} P_{j-1}^{zZ} \right). \quad (127)$$

Here, j enumerates the nodes of the grid, i.e. $J_j = J(t, Z_j)$, $K_j = K(t, Z_j)$ and $P_j^{zZ} = P^{zZ}(t, Z_j)$, with $j = 2, \dots, N - 1$. The boundary values J_1 and J_N are given by (125) and (126). A special treatment is performed for the initial node: in order to preserve the second-order-accuracy of the discretization method, a fictitious node Z_0 is introduced, and the Neumann boundary condition (125) is approximated by the central difference

$$\frac{J_2 - J_0}{2(\Delta Z)} = 0,$$

which implies $J_0 = J_2$. This allows to prolong the validity of the discretization used in (127) to the node $j = 1$.

At the upper boundary, we solve (126) numerically in order to determine J_N . For this purpose, we implement a standard Newton-Raphson method. According to this procedure, the initial partial differential equation (114) is approximated by a system of ordinary differential equations that can be integrated by choosing a stable ODE solver, with the initial condition $J_j(0) = J(0, Z_j) = g_1^2 g_3$, for $j = 1, \dots, N$.

Table 1 Parameters of the benchmark problem

Parameter	Description	Value
L	Height of the specimen	10 mm
$2R$	Diameter of the specimen	5 mm
$F_{\max}$	Maximum applied force	$-0.2 \cdot 9.81\text{N}$
$t_{\max}$	Time of load application	30 s
k_0	Hydraulic conductivity	$3.6454 \cdot 10^{-12} \text{ m}^4/(\text{N} \cdot \text{s})$
m_0	Material parameter	0.0848
m_1	Material parameter	4.638
ϕ_{s0}	Referential value of solidity.	0.6
ϕ_{sn}	Solidity in the relaxed configuration	0.6
λ_n	Lamé's first modulus	0.3137 MPa
μ_n	Shear modulus	0.3566 MPa

After computing J , the function χ^z is calculated by invoking (118) coupled with (120), and using a standard forward Euler method. The variation of pressure inside the specimen can be calculated a posteriori, once P^{zZ} is known. Indeed, integrating the balance of momentum (113), with the boundary condition $\pi(t, L) = 0$ MPa, yields

$$\pi(t, Z) = P^{zZ}(t, Z) - P^{zZ}(t, L). \quad (128)$$

4.3 Results

The model presented in the previous sections is applied to describe the compression of a cylindric specimen of soft biological tissue, which is positioned in a chamber delimited by a rigid cylindric wall and two parallel plates. The wall and the lower plate are impermeable to liquid, whereas the upper plate allows for fluid exudation. An external compressive force is applied at the upper plate, parallel to the symmetry axis of the specimen. The experimental apparatus is schematically shown in Fig. 2.

We restrict our analysis to the case in which the external force increases linearly in time until $t_{\max} = 30$ s when the maximum force, $F_{\max} = -0.2 \cdot 9.81$ N, is reached.

In (106), we consider

$$\mathbf{F}_a = \text{diag}\{g_1, g_1, g_3\} = \text{diag}\{g, g, g + \epsilon\}, \quad (129)$$

where ϵ measures the deviation of $\mathbf{F}_a$ from a spherical anelastic deformation. The numerical results shown in this section are obtained by implementing in Matlab[®] the procedure described in Sect. 4.2. All the parameters are listed in Table 1.

We recall that, for the considered problem, the only possible value of g is unity. This implies that the deviations of $\mathbf{F}_a$ from a referential spherical tensor are actually the deviations from the identity tensor. We start the simulations with $\epsilon = 0.1$.

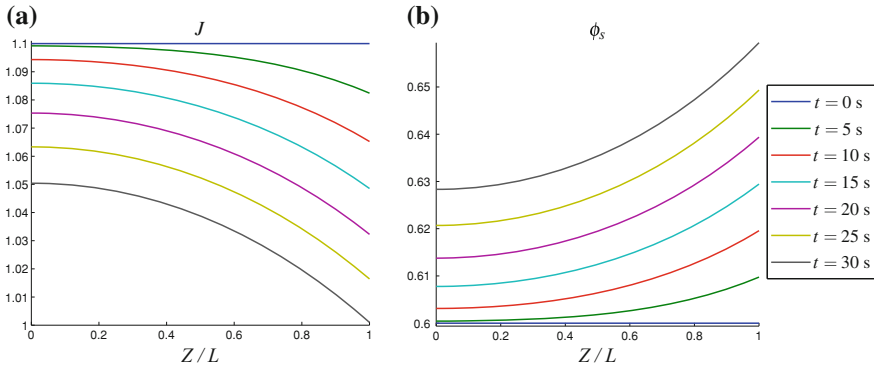


Fig. 3 Evolution in time and space of $J(t, Z)$ (a) and solid volumetric fraction (b), when $\epsilon = 0.1$, starting from the initial values $J(0, Z) = J_a$ and $\phi_s = \phi_{sn}$. Solutions are reported every 5 s for 30 s, which correspond to the maximum time of load application. All the parameters used in the simulation are listed in Table 1

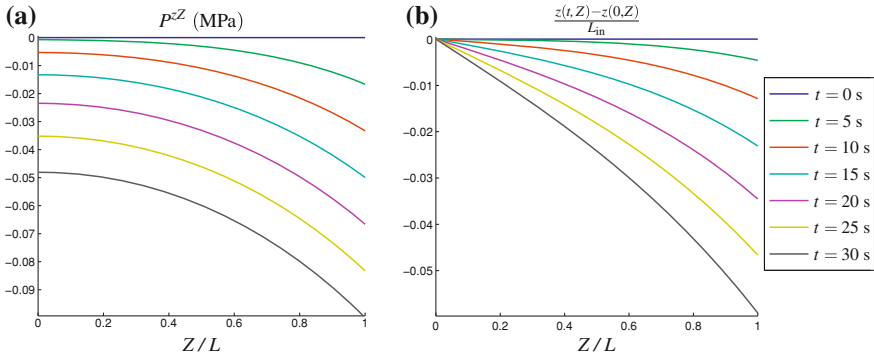


Fig. 4 Evolution in time and space of $P^Z(t, Z)$ (a) and relative displacement, $[z(t, Z) - z(0, Z)]/L_{in}$ (b), with $P^Z(0, Z) = 0$ MPa and $z(0, Z) = J_a Z$. Results are plotted every 5 s up to 30 s, in the case of the parameters listed in Table 1 and $\epsilon = 0.1$

Equation (114), integrated with the initial condition $J(0, Z) = J_a$ and BCs (125) and (126), gives origin to the curves in Fig. 3a, which represent $J(t, Z)$ plotted over space, at different instants of time (every 5 s for 30 s). The corresponding volumetric fraction of the solid, $\phi_s(t, Z) = \phi_{sn} J_a [J(t, Z)]^{-1}$, is reported in Fig. 3b. The characteristic time of the diffusive process described in (114) is defined by $t_d := L^2 [D(J)]^{-1}$, which is a function of time and material coordinates through J . For the considered load, $D(J)$ is an increasing function of J . Thus, the maximum characteristic time, t_d^M , corresponds to the minimum value of J , which is reached at the end of the simulation in $Z = L$ (cf. Fig. 3a). On the other hand, the minimum characteristic time, t_d^m , is reached at the beginning of the simulation, for $J(0, Z) = J_a$ and $D(J(0, Z)) = k_0(\mu_n g_3^{-2} + \mu_n J_a^{-2} + \lambda_n J_a^{-2}) = k_0(2\mu_n + \lambda_n) J_a^{-2}$. We remark

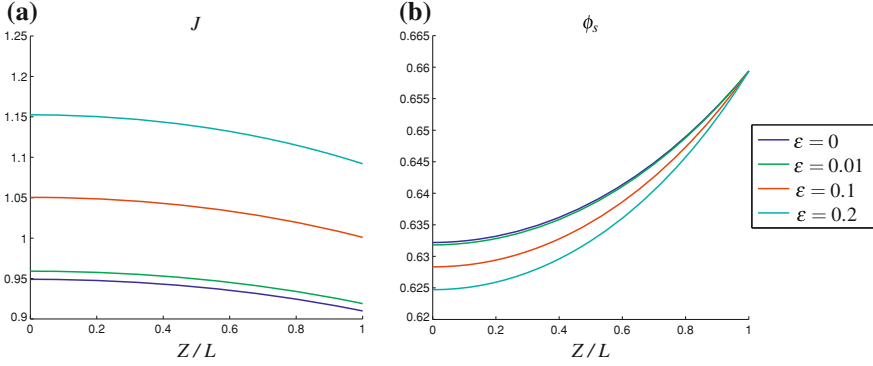


Fig. 5 Distribution of J (a) and solid volumetric fraction (b) at the final time of compression $t_{\max} = 30$ s, for different values of ϵ

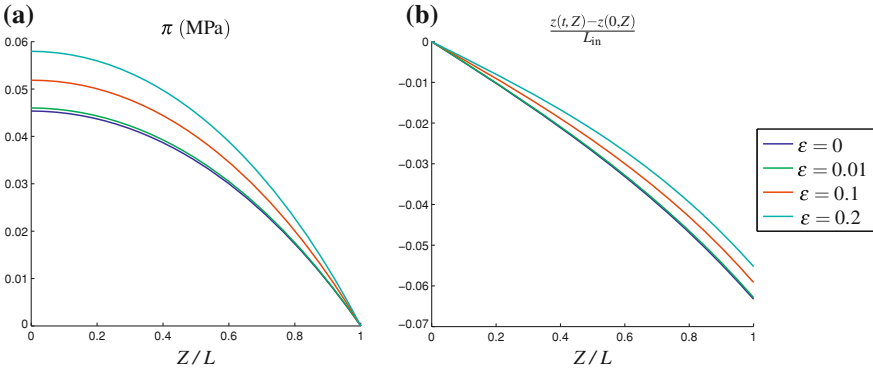


Fig. 6 Evolution of the pressure over space (a) and relative displacement (b) at the final time of compression $t_{\max} = 30$ s, for different values of ϵ

that the factor $(2\mu_n + \lambda_n)$ coincides with the P-wave modulus of the material. For the parameters listed in Table 1, we find $t_d^M \approx 38.2625$ s and $t_d^m \approx 32.3206$ s. We notice that t_d^M and t_d^m are of the same order as $t_{\max}$.

The axial component of the constitutive part of the first Piola–Kirchhoff stress tensor, P^{zZ} , and the relative displacement, $[z(t,Z) - z(0,Z)]/L_{\text{in}}$ are plotted in Fig. 4a and b, respectively. Here, L_{in} denotes the length of the specimen at time $t = 0$ s, which is defined by $L_{\text{in}} := \int_0^L \partial_Z \chi^z(0, Z) dZ = J_a L$ consistently with (118).

The value of P^{zZ} at the upper boundary is given by $P^{zZ}(t, L) = (F_{\text{appl}}(t, L))/S$, where $F_{\text{appl}}(t, L) = F_{\max}[t/t_{\max}]$ and $S = \pi R^2$, the area of the surface over which the applied load is distributed, coincides with the cross section of the specimen in the reference configuration. The amplitude of the displacement increases in time with the applied load (cf. Fig. 4b). This behaviour is qualitatively the same also for

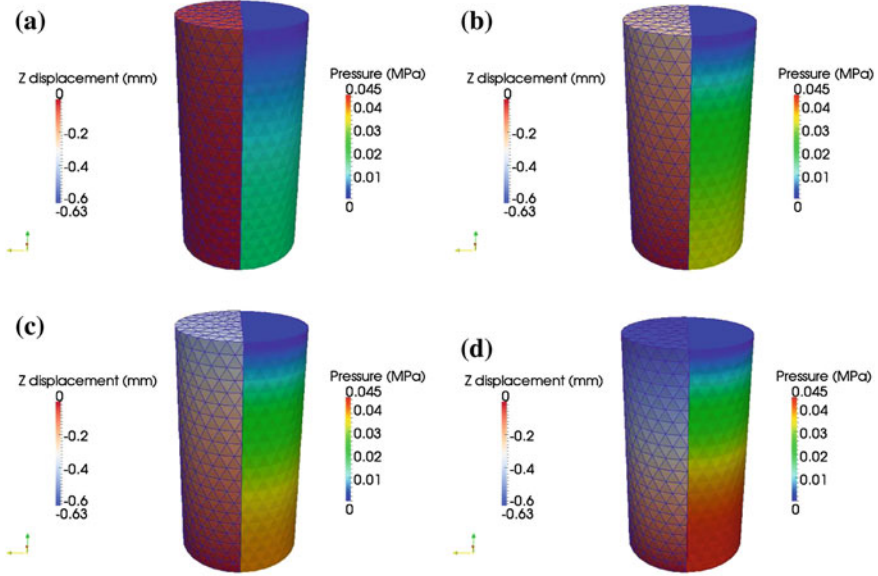


Fig. 7 Time evolution of the displacement and the pressure without growth ($\epsilon = 0$)

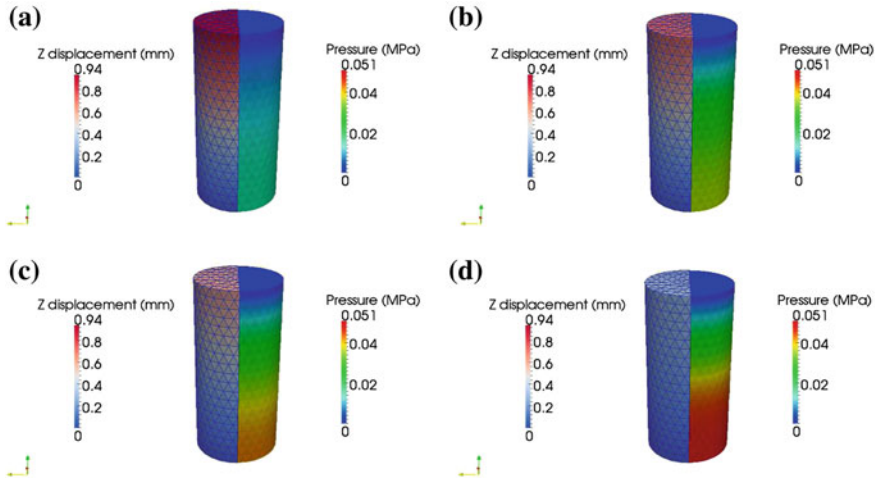


Fig. 8 Time evolution of the displacement and the pressure with $\epsilon = 0.1$

the other values of ϵ considered in the following. However, the diffusive process tends to become slower as ϵ gets bigger.

In the following, we run a set of simulations with varying ϵ in order to highlight the influence of this parameter on the response of the material (e.g., distribution of stress and deformation inside the specimen). Results are presented in Figs. 5 and 6 for $\epsilon = \{0, 0.01, 0.1, 0.2\}$. In particular, the volumetric deformation J , which solves (114), and the volumetric fraction of the solid phase at $t_{\max} = 30$ s are reported in Fig. 5a and b, respectively. We remark that the value of the solid volumetric fraction at the upper boundary, $\phi_s(t, L)$, is the same for every value of ϵ because $J_a[J(t, L)]^{-1}$ is constrained to satisfy (126) independently of ϵ . Pressure and relative displacements are plotted in Fig. 6 at time $t = t_{\max}$. Pressure is obtained by solving (128) in consistency with the condition (123). The value of the pressure at the lower boundary rises as ϵ increases, and the pressure distribution tends to become more inhomogeneous for larger deviations of $\mathbf{F}_a$ from sphericity. For the considered load, the normalised final displacement, $[z(t_{\max}, Z) - z(0, Z)]/L_{\text{in}}$, which is zero at the bottom of the specimen, diminishes with increasing ϵ (cf. Fig. 6b).

The results of the simulations obtained by means of the computational methods outlined in Sects. 3.2 and 3.3 are shown in Figs. 7 and 8, for two values of ϵ , for comparison with the results obtained in Matlab[®].

5 Conclusions and Outlook

We reviewed some fundamental aspects of the theory of biphasic materials with variable mass and internal structure. The structural change, described by the second-order tensor $\mathbf{F}_a$, and the variation of mass (which is assumed to be due exclusively to growth) are connected with each other since the rate at which mass increases (or decreases), γ_s , is related to the rate of anelastic deformation $\mathbf{L}_a$ through $\gamma_s = \text{tr}(\mathbf{L}_a)$. For our purposes, however, we considered a simplified framework in which $\mathbf{L}_a$ is set equal to zero. Consequently, $\mathbf{F}_a$ is taken to be constant. This amounts to study situations in which the biphasic medium evolves under external actions after growth has already occurred. Physically, this means that we are hypothesising that the time scale over which the medium grows is much slower than the scale over which it deforms. Based on this approximation, we study how different choices of $\mathbf{F}_a$ (which correspond to different possible ways of changing the internal structure of the solid phase) influence the deformation and, thus, the displacement field, as well as the distribution of pressure inside the medium. Our next goal is to consider fully coupled equations, in which the value of $\mathbf{F}_a$ changes in time according one of the evolution laws given in (56), (57), or (58). Furthermore, on the basis of a model presented in [27], we would like to study the influence of the structural change of the porous medium on the transport mechanisms governing the evolution of chemical agents.

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A.1 Appendix

The following relations are used in the computations above:

$$\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2} J \mathbf{C}^{-1}, \quad \frac{\partial \mathbf{C}}{\partial \mathbf{F}} = \delta \otimes \mathbf{F}^T + \mathbf{F}^T \otimes \delta, \quad (130)$$

$$\frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} = -\mathbb{I} = -\frac{1}{2} [\mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{C}^{-1}]. \quad (131)$$

The tensor $\mathbb{I}_n$ is obtained from (131) by substituting $\mathbf{C}$ with $\mathbf{C}_e$.

Given two second-order tensors $\mathbf{A}$ and $\mathbf{B}$, the products $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A} \underline{\otimes} \mathbf{B}$ have the following index representation [18]

$$[\mathbf{A} \otimes \mathbf{B}]_{IJMN} = A_{IN} B_{JM}, \quad [\mathbf{A} \underline{\otimes} \mathbf{B}]_{IJMN} = A_{IM} B_{JN}. \quad (132)$$

To perform the linearisation procedure, we set $\mathbf{u} = \mathbf{u}_0 + \mathbf{h}$, and use the following Gateaux-derivatives:

$$\begin{aligned} \mathfrak{D}(J\pi\mathbf{C}^{-1})(\mathbf{u}_0, \pi_0)[\mathbf{h}, \theta] &= (J_0(\mathbf{C}_0)^{-T} : \mathfrak{D}\mathbf{E}(\mathbf{u}_0)[\mathbf{h}])\pi_0(\mathbf{C}_0)^{-1} \\ &\quad + J_0\theta(\mathbf{C}_0)^{-1} - J_0\pi_0 2(\mathbf{C}_0)^{-1} \{ \mathfrak{D}\mathbf{E}(\mathbf{u}_0)[\mathbf{h}] \} (\mathbf{C}_0)^{-1}, \end{aligned} \quad (133)$$

$$\mathfrak{D}\dot{\mathbf{E}}_v(\mathbf{u}_0, \mathbf{v}_v)[\mathbf{h}] = \frac{1}{2} [(\mathbf{H})^T \dot{\mathbf{F}}_v + (\dot{\mathbf{F}}_v)^T \mathbf{H}], \quad (134)$$

$$\mathfrak{D}\mathbf{S}(\mathbf{u}_0)[\mathbf{h}] = \mathbb{C}_r(\mathbf{u}_0) : \mathfrak{D}\mathbf{E}(\mathbf{u}_0)[\mathbf{h}], \quad (135)$$

$$\mathbb{C}_r = J_a(\mathbf{F}_a)^{-1} \underline{\otimes} (\mathbf{F}_a)^{-1} : \mathbb{C}_n : (\mathbf{F}_a)^{-T} \underline{\otimes} (\mathbf{F}_a)^{-T}. \quad (136)$$

The formulae in Sect. 3.1 are retrieved by setting $\mathbf{u} \equiv \mathbf{u}^{m,k}$, $\mathbf{u}_0 \equiv \mathbf{u}^{m,k-1}$ and $\mathbf{h} \equiv \mathbf{h}^{m,k}$.

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