Preface

This book is a rigorous study of q-fractional calculus and q-fractional difference equations. Our study is developed starting from the work of Agarwal [17], Al-Salam [18, 19], and Al-Salam and Verma [20]. In [17], the q-fractional Riemann–Liouville calculus is defined formally and many properties are given as well. In comparison with the celebrated monographs on fractional calculus, for example, the first book on fractional calculus [227] written by Oldham and Spanier, the books of Samko et al. [269], Kilbas et al. [169], Podlubny [234], and Diethelm [82], we can see that the q-fractional theory is far from being a well-established q-counterpart of the existing fractional theory. However, we hope that the present work would be a takeoff point to establish a more comprehensive q-fractional theory. We would like to mention that our q-study is based on a q-difference operator and its associated right inverse. This q-difference operator goes back to Euler and may go back to Heine and is reintroduced by Jackson in [158]. Sometimes it is called Euler–Jackson q-difference operator or simply Jackson q-difference operator as we do through the entire book. The main objective of this book is to provide such an overview of the basic theory of fractional q-difference equations, methods of their solutions and applications, taking into account the audience of this book, namely the applied scientists interested in developing the q-theory, investigating and exploring its applications. Applications of the classical fractional calculus appeared in many publications. For example, Sneddon's book [276] on mixed boundary value problems from the mid-1960s included a survey of fractional calculus and how to use it to solve integral equation arising in elasticity. In addition, applications of fractional calculus in mathematical physics, probability, and modeling were introduced in the mid-1970s in [261]. Recently, more applications in classical mechanics, particle physics, diffusion systems, viscoelastic and disordered modern electrical systems, modeling and control are in [267]. Perhaps Leibniz [179] did not expect this number of applications when he sent a letter in 1695 to L'Hôpital asking about the meaning of the derivative of order half. From this point of view, we expect that in the long run, many applications of the fractional q-calculus will appear.

This book consists of nine chapters. Chapter 1 provides some basic definitions and properties of q-analysis as the q-difference operator, the q-integral operator, q-special functions, and q-integral transforms. Chapter 2 is a study of the existence and uniqueness of the solutions of first-order systems of q-difference equations and linear q-difference equations. This chapter also includes some results on zeros of q-trigonometric functions and q-Bessel functions. Chapter 3 includes the basic Sturm–Liouville problem formulated and studied in [30]. It also includes the reformulation introduced in [207] of the q^2-Fourier transform introduced by Rubin in [265, 266]. In Chap. 4, we survey the developments in the fractional q-theory since Al-Salam and Agarwal introduced their generalization to Jackson q-integral and derivatives to fractional orders. It contains the fractional q-calculus associated with Al-Salam and Agarwal fractional q-analogue of the Riemann–Liouville fractional derivatives. Chapter 5 is devoted to other approaches of extending the notion of q-integrals and q-derivatives to fractional orders like q-Caputo fractional derivatives and q-Weyl fractional derivatives. In this chapter we show that a generalization to Grünwald–Letnikov fractional derivatives in the q-settings leads to Al-Salam–Agarwal fractional q-derivatives. We outline the generalization of the Askey–Wilson q-difference operator to fractional orders introduced by Ismail and Rahman in [147]. We conclude this chapter with a generalization of the q-difference operator introduced by Rubin in [265, 266] to fractional orders. In Chap. 6, we give a rigorous proof of Al-Salam–Verma fractional q-Leibniz rule [20] and a generalization of the fractional q-Leibniz rule introduced by Agarwal in [15]. We also introduce a fractional q-Leibniz rule associated with Weyl fractional q-operator. This result is a generalization of the result introduced by Purohit in [248]. At the last section of this chapter, we derive some q-identities using the fractional q-Leibniz formulae represented in this chapter. Chapter 7 is fully devoted to q-Mittag-Leffler functions and their major properties. We explore the Mellin–Barnes contour representations and Hankel contour representations of the two q-analogues of the Mittag-Leffler functions considered in this book. Chapter 8 includes fundamental existence and uniqueness theorems for linear and nonlinear fractional q-difference equations as well as first-order systems of fractional q-difference equations, where the q-derivative is either the Riemann–Liouville fractional q-derivative or Caputo fractional q-derivative. Most of the results of this chapter are a generalization of the results mentioned in Chap. 2. In Chap. 9, the last chapter, we investigate the applications of the q-Laplace, q-Mellin, and q^2-Fourier integral transforms to constructing explicit solutions of certain classes of linear fractional q-difference equations. In the appendix, we include tables of fractional q-derivatives of q-special functions and generalized Rodrigues-type formulae for some q-special functions. The bibliography consists of 302 books and articles, including some recent preprints submitted for publications, up to 2011. However, it cannot be considered as a complete bibliography since this discipline is a fast-growing area. But, on the other hand, we believe that the references of the bibliography and references mentioned therein are enough to get a complete overview of the developments occurred in this subject up to the year 2011.

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