

# 1 The Golden Number

THIS LECTURE is about the so-called Golden Number. This number defines a ratio which turns up in various guises in an amazing number of contexts. These include the ideal shapes of architectural objects such as temples, the arrangement of many botanical systems, and purely geometrical objects such as the pentagon. We shall start, however, by considering the sizes of pieces of paper.

## Pieces of paper

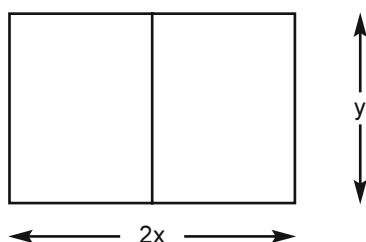
ONCE UPON A TIME paper came in all sorts of bizarre sizes with wonderful names such as Elephant, Foolscap and Crown. Books were printed in sizes such as Crown octavo which had eight pages to a sheet of Crown. However, in the 1960s, as part of the metrication process, the A series of paper sizes was introduced and all these names faded (the A series was originally a German DIN standard). But note that in this respect the US is stuck in a time warp and continues to use historical sizes known as Letter and Legal.

So now we use paper with boring names such as A3 and A4. A sheet of A3 is simply double a sheet of A4 and similarly A5 is half a sheet of A4. Thus, as we add one to the number, the area is divided by two. The diagram shows a sheet of A3 divided into two sheets of A4.

An important property of the A series is that all the sizes have the same proportions. Suppose the width of a sheet of A4 is  $x$  and its length is  $y$ , then a sheet of A3 will have width  $y$  and length  $2x$ . If the ratios of the length to width for both sizes are the same, it follows that

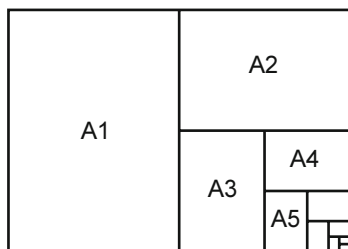
$$2x/y = y/x, \text{ so that } (y/x)^2 = 2 \text{ giving a ratio of } \sqrt{2} = 1.414\dots$$

So each sheet has sides  $\sqrt{2}$  smaller than the one before and area one-half the one before. Well this is well known. But what is less well known is that the area of A0, the biggest of the series, is defined to be exactly one square metre.



A sheet of A3 can be divided into two sheets of A4.

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A sheet of A0 subdivided into an infinite sequence of smaller sheets.

This means that the sides of a sheet of A0 have to be  $\sqrt{\sqrt{2}}$  and  $1/\sqrt{\sqrt{2}}$  respectively which give a size of about 1189 by 841 mm. A4 has sides one-quarter of this giving 297 by 210 which is the right answer.

Note that a sheet of A0 can be subdivided into one each of the (infinite number of) smaller sizes as above.

This shows quite clearly that the sum of the areas of the smaller sheets equals the area of the big sheet of A0. And so it follows that

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \dots = 1$$

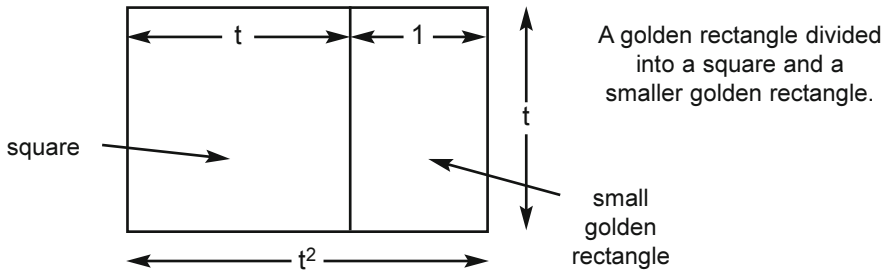
This is therefore a very simple illustration of the fact that the sum of an infinite series of numbers can be finite.

This, of course, is the essence of Zeno's paradox about Achilles and the tortoise. Remember that they have a race and Achilles gives the tortoise 100 metres start. Achilles runs 10 times as fast as the tortoise so by the time Achilles reaches the point from whence the tortoise started, the tortoise is now 10 metres ahead; when Achilles reaches that point the tortoise is now 1 metre ahead and so on. The argument is that the tortoise is always ahead and so Achilles never catches up. But of course Achilles catches up at the point  $100+10+1+0.1+0.01+\dots = 111.111\dots = 111\frac{1}{9}$  metres from the starting point. In the paradox the ratio of the speeds is 10 rather than 2 as in the paper sizes but if Achilles had arthritis and the tortoise was on drugs then maybe Achilles might only have twice the speed of the tortoise.

## The golden ratio

**A**NOTHER INTERESTING WAY to subdivide a sheet of paper is to remove a square from one end. If the remaining piece has the same shape as the original then we say that it is a golden rectangle and the ratio of the sides is the golden ratio. In other words a large golden rectangle can be subdivided into a square plus a small golden rectangle.

Suppose that the piece left after the square is removed has width 1 and length  $t$  so that the ratio of the sides is  $t$ . Then the square cut off has side  $t$  so that the length of the original piece must have been  $t+1$  as shown in the diagram opposite.



However, since the original piece has the same shape and its width is  $t$  it follows that its length must also have the value  $t^2$ . So we have

$$t^2 = t + 1 \quad \text{from which} \quad t^2 - t - 1 = 0$$

Using the familiar formula for the quadratic equation  $ax^2 + bx + c = 0$  which is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives  $t = (1 + \sqrt{5})/2 = 1.6180\dots$  This is the so-called golden number and is often denoted by the Greek letter  $\tau$  (*tau*). The equation  $\tau^2 = \tau + 1$  can be rearranged to give the following relations which should also be noted

$$\tau = 1 + 1/\tau \quad \text{and} \quad 1 = 1/\tau + 1/\tau^2 \quad \text{and so}$$

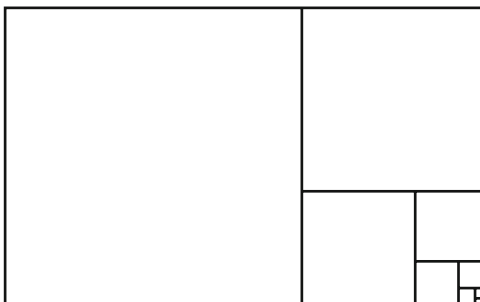
$$1/\tau = 0.6180\dots$$

$$\tau = 1.6180\dots$$

$$\tau^2 = 2.6180\dots$$

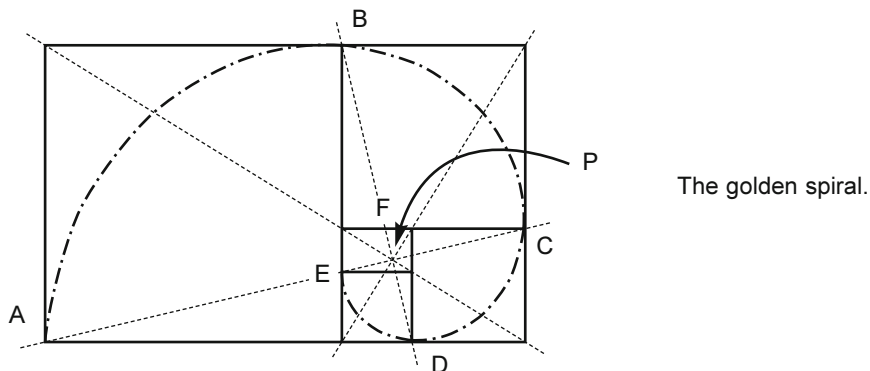
A piece of paper in the golden ratio is rather longer and narrower than A4. For comparison, the old Foolscap size is 13" by 8" which gives a ratio of 1.625 and so is very close to the golden ratio.

We can now see that a golden rectangle can be divided into an infinite number of squares of diminishing size. The arrangement shown below mirrors that of the division of the sheet of A0. Later in this lecture we shall see ways of



A golden rectangle subdivided into an infinite number of squares.

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dividing a rectangle into a finite number of squares all of which have a different size.

An alternative way to do the subdivision is in a spiral form where the squares converge onto a point  $P$  inside the original golden rectangle as shown above. Note how the lines which join the corner points all pass through the point  $P$ ; these four lines are at  $90^\circ$  to each other in pairs. Thus  $AEC$  is at right angles to  $BFD$ . In fact the whole picture simply consists of the basic square repeatedly rotated about the point  $P$  by  $90^\circ$  and shrunk by a factor of  $\tau$ .

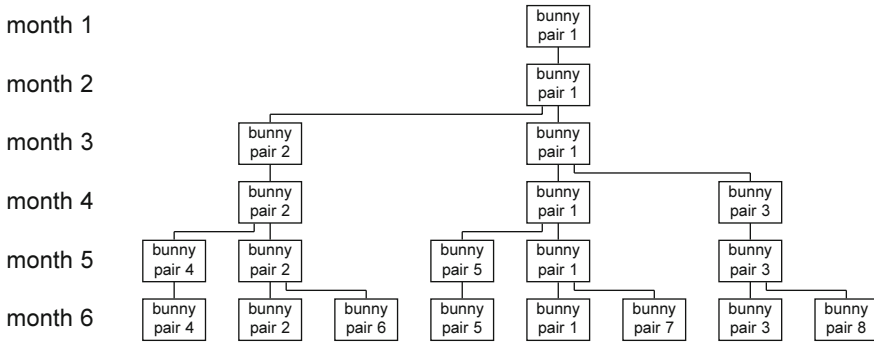
A spiral curve can be drawn through the points  $A, B, C, D, E, F$ , etc. In a single rotation it changes size by a factor of  $\tau^4$  which is about 6.85. This spiral is often called the golden spiral. At first glance it looks as if it is tangential to the squares but in fact it cuts the sides at about  $4.75^\circ$ . In general shape it approximates to that exhibited by many shells although it spirals too quickly.

It has been claimed that the golden ratio is the ideal shape for a room. Most modern houses have rooms that are too thin for their length thus giving an awkward end space. Pictures are often of the golden ratio as well. It is interesting that artists rarely paint a square picture. It is either landscape or portrait in format and typically around the golden ratio.

## Fibonacci's rabbits

IN 1202, LEONARDO OF PISA, who was nicknamed Fibonacci (son of good Nature), completed a book entitled *Liber Abbaci* (roughly, a Book of Counting). One of the topics concerns the breeding of rabbits. Suppose each pair of rabbits produces another pair of rabbits every month except in their first month of life and that we start with a single pair of new-born rabbits.

In month 1 we have just the original pair and at the start of month 2 we still have just the original pair since they only breed in that month. But at the start of the third month we have 2 pairs and by the fourth month 3 pairs. By the fifth



month not only do the original rabbits breed but their first born pair also breed so then we have 5 pairs. By the sixth month the second litter also breed giving 8 pairs. It then really gets going. This is shown in the tree of rabbits above.

A little thought shows that in each month the number of rabbits equals the sum of the numbers in the previous two months. (That's because all the rabbits of the previous month will still be around and all the rabbits of the month before will have bred a new pair.) So we get the famous series of Fibonacci numbers in which each one is the sum of the previous two:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

As the series progresses the ratio between successive numbers rapidly approaches a limit thus

1, 2, 1.5, 1.666..., 1.600, 1.625, 1.615..., 1.619..., 1.617...

The number the ratio approaches is, surprise, surprise, the golden number,  $\tau$ . That this should be the case is easily seen since in the limit if each number is  $x$  times its predecessor then any group of three successive numbers will have values  $n$ ,  $xn$ ,  $x^2n$  and since moreover each number is the sum of its two predecessors it follows that

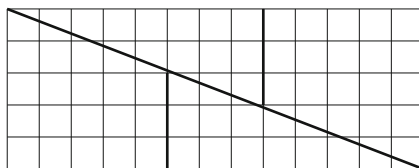
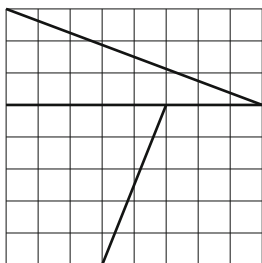
$$x^2n = xn + n \text{ and so } x^2 = x + 1 \text{ which is one of the formulae for } \tau$$

An interesting property of Fibonacci's numbers is that the square of any one of them differs from the product of the two adjacent numbers by exactly 1, for example

$$\begin{aligned} 3 \times 3 &= 9 = 2 \times 5 - 1 \\ 5 \times 5 &= 25 = 3 \times 8 + 1 \\ 8 \times 8 &= 64 = 5 \times 13 - 1 \\ 13 \times 13 &= 169 = 8 \times 21 + 1 \end{aligned}$$

Note how the squares are alternately one more than and one less than the product of the adjacent numbers.

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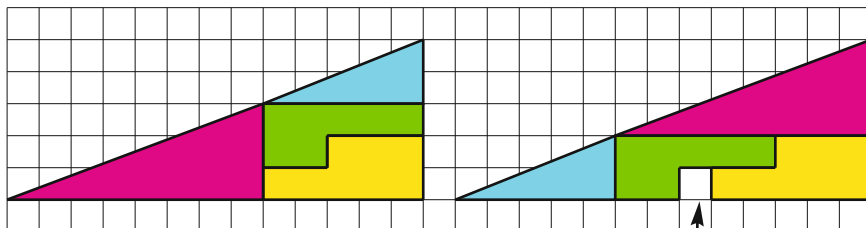
Dividing a chessboard of 64 squares into a rectangle with 65 squares.

This property is the basis of a very old puzzle. Consider a chessboard and cut it as shown above. It then seems as if the pieces can be rearranged to form a rectangle of size 5 by 13 which has 65 squares whereas the original chessboard had 64 squares.

The question is where did the extra unit of area come from? The answer is that the pieces don't quite fit together and in fact there is a thin parallelogram gap in the rearranged shape. But it is not easy to spot this unless the cutting is done carefully. This puzzle appeared in the Boy's Own Annual for 1917–1918 but surprisingly there was no explanation of where the extra square came from; the discussion implied that it truly appeared from nowhere. Similar puzzles can be made up with the smaller numbers but the gap is then much more obvious.

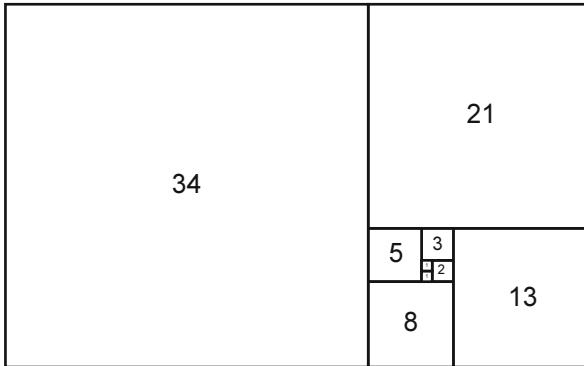
A related puzzle which is perhaps more surprising is shown below. The various shapes making up the left diagram can be rearranged to give the right diagram. This seems to produce a hole from nowhere!

The reason is that the angles of the two triangles are slightly different. The tangent of the angle at the base of the larger one is  $3/8$  whereas that at the base of the smaller one is  $2/5$ . These are both the ratios of successive Fibonacci numbers. The angles are about 20.56 and 21.80 degrees respectively and the difference of just over one degree is easily overlooked by the human eye. The top of the left diagram is slightly concave whereas that of the right is slightly convex. This same small angle is in the thin parallelogram in the puzzle above.



The magic hole puzzle.

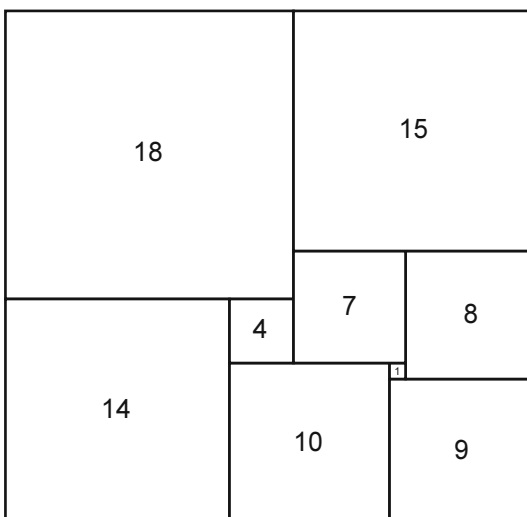
From whence  
cometh this  
hole?



A rectangle of sides  $34 \times 55$  divided into squares whose sides are the first nine Fibonacci numbers.

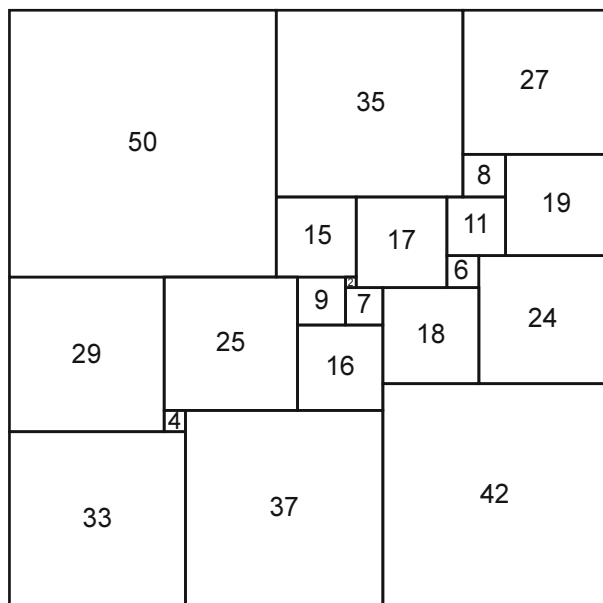
The Fibonacci numbers provide a way of subdividing an appropriate rectangle into a finite number of squares with one square corresponding to each number. The case of subdividing a rectangle 55 by 34 into 9 squares is shown above. But this doesn't quite solve the problem of dividing a rectangle into a number of *different* squares because the first two Fibonacci numbers are both equal to 1.

However, it is possible to subdivide a rectangle into 9 different squares of sizes 1, 4, 7, 8, 9, 10, 14, 15 and 18. They form a rectangle of sides 32 and 33 as shown below. It makes quite an amusing puzzle to assemble a set of such squares into a rectangle. A set of squares made of aluminium alloy or brass using a unit of perhaps  $\frac{1}{8}$ " or  $\frac{1}{2}$  cm is compact enough to keep in one's pocket and produce at parties to amuse (or bore) your friends. This is not the only subdivision into 9 squares (see Exercise 1 at the end of this lecture for another example) but there are no subdivisions of a rectangle into fewer.



A rectangle of sides  $32 \times 33$  divided into nine different squares.

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A square of side 112  
divided into 21  
different squares.

It is also possible to subdivide a *square* into different squares as shown above. The simplest case has 21 different squares of sides 2, 4, 6, 7, 8, 9, 11, 15, 16, 17, 18, 19, 24, 25, 27, 29, 33, 35, 37, 42 and 50. Note that the smallest has side 2 rather than 1. These squares can be arranged to form a square of side 112. I am sure that they would not make a sensible puzzle and would cause one to lose friends quite quickly.

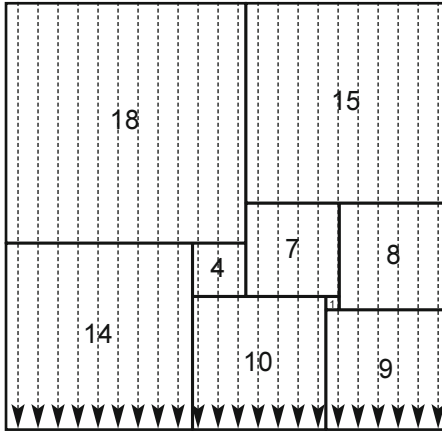
There is a curious relationship between electrical circuits and the subdivision of squares and rectangles into different squares. As a simple example, consider the case of the  $32 \times 33$  rectangle.

Imagine a current flowing from one edge to the opposite edge of a rectangle. The resistance increases as the rectangle gets longer and decreases as it gets wider. These effects cancel out so that all squares have the same resistance irrespective of their size; for simplicity we will assume that the resistance of a square is 1 ohm.

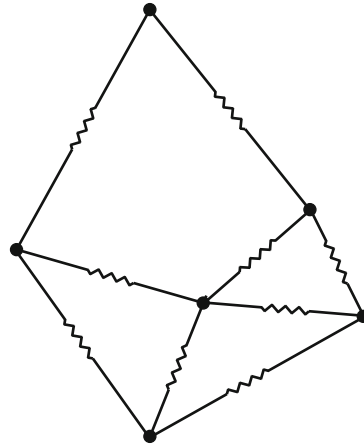
Now suppose that the  $32 \times 33$  rectangle be subdivided into squares and that a uniform current flows from the top edge to the bottom edge as shown opposite. Since the flow is uniform it follows that current never flows from a square to one alongside it but only from a square to one below it. As a consequence the collection of conducting squares is equivalent to a network of 1 ohm resistors connected in this case as shown alongside the rectangle.

Each resistance corresponds to a square and each junction point corresponds to a horizontal division between squares. The current flowing in each resistance is clearly proportional to the width of the corresponding square. Assuming that the current in the resistance corresponding to the square of side 1 is 1 amp it





Current flowing from one edge to the other of the  $32 \times 33$  rectangle.



The equivalent electrical circuit of nine equal resistors.

follows that the total current flowing through the network is 33 amps. It is easy to see that the total resistance of the rectangle is  $32/33$  ohms and that the voltage across the rectangle is 32 volts.

The problem of subdividing a rectangle into different squares thus becomes equivalent to finding an electrical network of 1 ohm resistors in which the current in each resistor is different. A network which exhibits symmetry somewhere (either as a whole or in part of it) will have equal currents somewhere. It is easy to convince oneself by just trying that any network joining two points with less than 9 lines has symmetry somewhere. It follows that 9 different squares is the minimum number for subdividing a rectangle.

The golden ratio and Fibonacci's numbers also turn up in the simplest form of so-called continued fractions which we now briefly explore before returning to the geometrical theme.

## Continued fractions

I AM SURE THAT Victorian schoolboys knew all about continued fractions – or maybe they didn't and got beaten as a consequence. Continued fractions are a somewhat neglected topic these days – probably considered too hard to learn at school (high school) and too obscure to learn at university (college). Nevertheless they have intriguing properties which are worth knowing.

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Consider any fraction less than 1 such as  $7/16$ . We can write this as

$$\frac{1}{16/7} = \frac{1}{2^2/7} = \frac{1}{2 + \frac{1}{7/2}} = \frac{1}{2 + \frac{1}{3^{1/2}}} = \frac{1}{2 + \frac{1}{3 + \frac{1}{2}}}$$

This is usually written in one of various shorthand notations such as

$$\frac{1}{2 +} \frac{1}{3 +} \frac{1}{2}$$

or even more succinctly as  $[2, 3, 2]$ . If a number is greater than 1 such as  $47/16$  then we can write it as  $[4; 2, 3, 2]$ .

We recall that some common fractions such as  $1/7$  expressed in decimal notation do not terminate but exhibit a recurring pattern thus

$$1/7 = 0.142857142857... \text{ sometimes written as } 0.\dot{1}4285\dot{7}$$

All common fractions expressed as continued fractions naturally terminate and of course  $1/7$  is simply expressed as  $[7]$ .

A surprising thing about continued fractions is that some numbers which are chaotic as a decimal fraction have a recurring form as a continued fraction. Consider for example the recurring fraction

$$x = [1; 2, 2, 2, ...]$$

This can be written as  $1 + y$  where

$$y = \frac{1}{2 +} \frac{1}{2 +} \frac{1}{2 +} \dots$$

A moment's thought shows that the fraction following the first 2 is itself  $y$  so we have

$$y = \frac{1}{2 + y} \quad \text{from which we get}$$

$$y^2 + 2y - 1 = 0 \quad \text{so that } y = -1 + \sqrt{2}$$

and so finally  $x = 1 + y = \sqrt{2} = 1.4142....$

In a similar way we can show that  $\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, ...]$ . Clearly the most beautiful recurring continued fraction of all has to be  $[1; 1, 1, 1, ...]$  and it will come as no surprise that this has the value  $\tau$ , the golden number.

Some so-called transcendental numbers such as  $e = 2.71828\dots$ , the base of natural logarithms, which are a complete mess as a decimal expansion, have a recognizable pattern as a continued fraction

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

Others such as  $\pi$  have no recognizable pattern;  $\pi = [3; 7, 15, 1, 292, \dots]$ .

An important property of continued fractions is that if we chop off the sequence then we obtain an approximation to the value. That is not so surprising because if we chop off a recurring decimal such as  $0.142857142857\dots$  then clearly we get an approximation. In the case of continued fractions the chopped off values are known as convergents. The approximations given by a chopped off decimal fraction are always too low but amazingly the convergents of a continued fraction are alternately too low and too high. For example if we consider some successive approximations to  $\tau = [1; 1, 1, 1, 1, \dots]$ , we find

$$\begin{aligned} [1; 1, 1] &= 3/2 \\ [1; 1, 1, 1] &= 5/3 \\ [1; 1, 1, 1, 1] &= 8/5 \end{aligned}$$

and Lo and Behold these are the ratios of successive pairs of Fibonacci numbers.

An important property of the convergents is that they provide excellent approximations to the original number. In fact it can be shown that a convergent provides the best approximation for the size of its terms. Thus  $13/8$  is the best approximation to  $\tau$  using only ratios involving integers of 13 or less.

The approximation of  $22/7$  for  $\pi$  is the first convergent from the continued fraction  $[3; 7, 15, 1, 292, \dots]$ . A convergent is a particularly good approximation if the first term ignored is large. So  $[3; 7] = 22/7$  is a good approximation to  $\pi$  because the first term ignored is 15. And  $[3; 7, 15, 1]$  is extremely good because the first term ignored is 292.

It is straightforward to compute one convergent from the previous two. Suppose the continued fraction is  $[a_0; a_1, a_2, a_3, \dots]$  and the convergents are  $c_0 = n_0/d_0$ ,  $c_1 = n_1/d_1$ ,  $c_2 = n_2/d_2$  and so on. Then the first two are easy

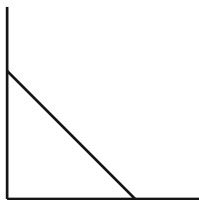
$$\begin{aligned} c_0 &= a_0/1 \text{ so } n_0 = a_0 \text{ and } d_0 = 1 \\ c_1 &= a_0 + 1/a_1 = (a_1 a_0 + 1)/a_1 \text{ so } n_1 = a_1 n_0 + 1 \text{ and } d_1 = a_1 \end{aligned}$$

The general pattern continues thus

$$\begin{aligned} n_2 &= a_2 n_1 + n_0; \quad d_2 = a_2 d_1 + d_0 \\ n_3 &= a_3 n_2 + n_1; \quad d_3 = a_3 d_2 + d_1 \end{aligned}$$

The rule is that to get the next convergent you multiply both top and bottom of the previous one by the next term and add them to the top and bottom of the one before that. Let's try that with  $\sqrt{2}$  which is  $[1; 2, 2, 2, \dots]$ . The first two

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The hypotenuse is  $\sqrt{2}$  times the sides.

convergents are  $[1] = 1/1$  and  $[1; 2] = 3/2$ . The next convergent is  $[1; 2, 2]$  so the new term added is 2 and then we follow the rule giving the sequence

$$\frac{1}{1}; \frac{3}{2}; \frac{2 \times 3 + 1}{2 \times 2 + 1} = \frac{7}{5}; \frac{2 \times 7 + 3}{2 \times 5 + 2} = \frac{17}{12}; \frac{2 \times 17 + 7}{2 \times 12 + 5} = \frac{41}{29}$$

So if you want to find a good approximation to a number first convert it to a continued fraction and then choose a convergent using these rules. As an example suppose we are trying to make a right angled corner out of Meccano (or Erector or Marklin) and wish to brace it with an isosceles triangle (that is one with two sides the same). Meccano strips only have holes every  $1/2''$  so we have to choose our bracing with care.

From the right angled triangle shown above (and using Pythagoras' theorem that the square on the hypotenuse is equal to the sum of the squares on the other two sides) it is clear that the length of the brace is going to be  $\sqrt{2}$  times the length of the sides. Now we have just computed the convergents for  $\sqrt{2}$ ; they are

$$1/1, 3/2, 7/5, 17/12, 41/29, \dots$$

We can quickly compute the decimal values of these convergents which are

$$1, 1.5, 1.4, 1.4166\dots, 1.4137\dots$$

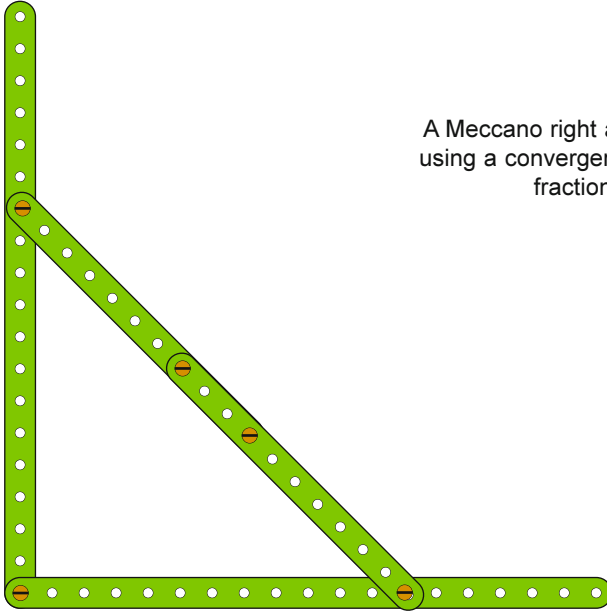
whereas the correct value is of course 1.4142...

It's pretty clear that  $3/2$  is not going to be very good but even  $7/5$  is only 1% out and  $17/12$  is more than four times better. In fact the slack in the bolts means that  $7/5$  will work but of course the slack in the bolts also means it isn't very rigid. So the best choice is  $17/12$ . Remember that the measurements are from the centres of the holes so we need 18 holes = 9" on the diagonal and 13 holes =  $6\frac{1}{2}''$  along the sides. Any Meccano person will tell you that there is no such thing as a 9" strip so we have to overlap two strips together as shown opposite.

Well that is quite enough about continued fractions so let's get back to some geometry. But before doing so consider the following curious expression

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}}}$$

A little thought reveals that  $x$  must satisfy  $x = \sqrt{1 + x}$  so that  $x^2 = 1 + x$  and hence  $x$  is again the golden number.

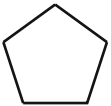


A Meccano right angle with bracing  
using a convergent of the continued  
fraction for  $\sqrt{2}$ .

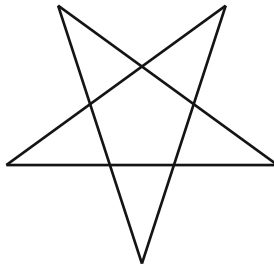
## Pentagons

THE MOST BEAUTIFUL two-dimensional shape in the world is undoubtedly the regular pentagon. It has long been used as a mystic symbol. If we extend the sides of a pentagon then we get the star-shaped pentagon. If we take a large pentagon and draw its five diagonals then these form a pentagram around a smaller pentagon inside the original pentagon.

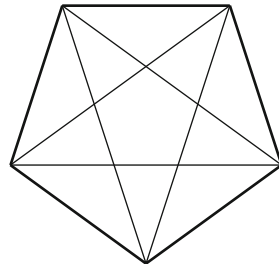
The pentagon with its diagonals reveals many instances of the golden number. Suppose for simplicity that the length of the sides of the inner pentagon is 1 unit. Each diagonal of the outer pentagon contains a side of the inner pentagon and two other sections which are sides of an isosceles triangle such as



pentagon

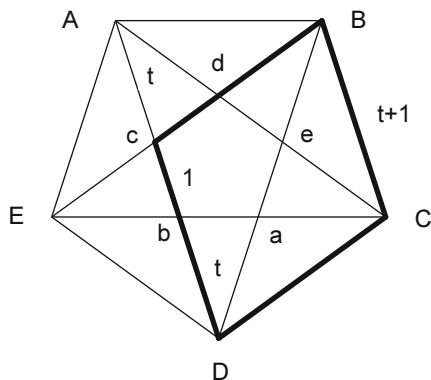


pentagram



pentagon with diagonals

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Pentagon with  
diagonals  
highlighting  
internal rhombus.

$abd$  in the diagram above. Suppose the length of these two other sections is  $t$ . Then the total length of a diagonal is  $2t + 1$ .

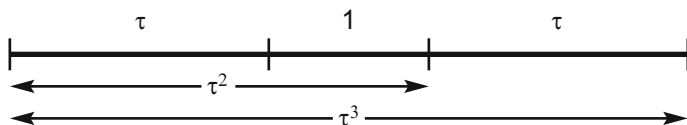
It is important to note that each diagonal of the outer pentagon (such as  $EbaC$ ) is parallel to the opposite side (in this case  $AB$ ). As a consequence the figure  $BcDC$  is a rhombus which is a parallelogram all of whose sides are equal. The rhombus is shown in bold in the diagram. Now the length of  $cD$  is clearly  $t+1$  and so it follows that the length of a side of the outer pentagon is also  $t+1$ .

Now notice that the large triangle  $ABD$  has the same shape as the small triangle  $baD$ . So the ratios of their sides must be the same. The small triangle has sides in the ratio  $t:1$  and the large triangle has sides in the ratio  $2t+1:t+1$ . And so

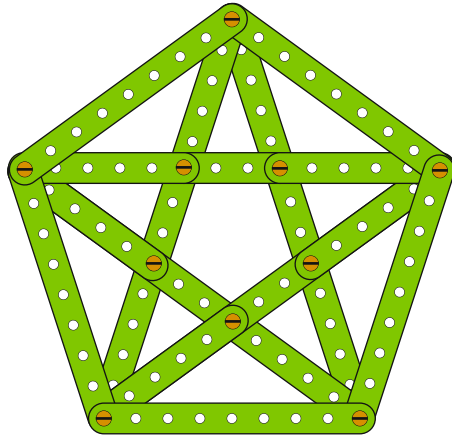
$$2t + 1 = t(t + 1) \text{ giving } t + 1 = t^2 \text{ which is the familiar equation for } \tau.$$

So  $t$  is actually the golden number  $\tau$ . The pentagon provides yet another example of the appearance of the golden ratio.

Indeed, the pentagon is absolutely riddled with golden ratios. The inner pentagon has side 1; the larger bits of the diagonal have length  $\tau$ , the outer pentagon has sides of length  $\tau^2$  and the diagonals have overall length  $\tau^3$ ! It is truly a golden figure. These powers of  $\tau$  are all revealed in the three parts of a diagonal as shown below. Note that since  $\tau^2 = \tau + 1$  it follows that  $\tau^3 = \tau^2 + \tau = (\tau + 1) + \tau = 2\tau + 1$ , the length of the whole diagonal.



The diagonal is riddled with golden ratios.



The Meccano pentagon with side 8 and diagonal 13.

It is interesting to compute further powers of  $\tau$  in the same way. We find

$$\tau^4 = 3\tau + 2$$

$$\tau^5 = 5\tau + 3$$

$$\tau^6 = 8\tau + 5$$

which shows those Fibonacci numbers once more!

Because the ratios of successive Fibonacci numbers are close approximations to  $\tau$  it follows that a sequence of four Fibonacci numbers forms a good approximation to the series of powers of  $\tau$  exhibited by the diagonal. We can make a Meccano model of the pentagonal figure using a series such as 5, 8, 13, 21 and the slack in the bolt holes takes up the small errors. We can even do it with 3, 5, 8, 13 as illustrated above but the next series down won't work.

The ancients knew all about the pentagon and the golden ratio. Euclid refers to the fact that the diagonals of a pentagon divide each other in “extreme and mean ratio”. The definition in Book VI says “Ἀκρον και μεσον λογον ευθεια τετμησθαι, οταν η ως η ολη προς το μειζον τμημα, ουτως το μειζον προς το ελαττον.” which obviously means “A straight line is divided in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.” Thus in the diagram the length of the whole line is  $\tau^3$ , the greater segment has length  $\tau^2$ , and the lesser segment has length  $\tau$  and therefore clearly both ratios are  $\tau$ .



## Phyllotaxis

**P**HYLLOTAXIS MEANS the arrangement of leaves and similar botanical items. It derives from the Greek word φύλον (*phulon*), meaning a leaf.

It has been known for ages that nature seems to prefer the Fibonacci numbers when it comes to the arrangement of branches, leaves and petals. Thus a pansy has five petals and some daisies have 21, 34 or even 55 petals. Flowers that seem to break this rule by having say 6 petals are seen on closer examination to have 2 groups of 3 petals.

However, these favoured numbers are sometimes violated. Thus clover usually has three leaves but just occasionally we find a four-leaved clover. Daisies which mostly have 21 petals will have some flowers with 20 or 22 petals. Poppy heads have around 13 seed pods but individuals may have 11, 12, 14 or even 15.

Arrangements involving Fibonacci numbers also occur as double interlocking spirals in objects such as pineapples and sunflower heads. A pineapple has its sections arranged so that they appear to form 5 spirals in one direction and 8 spirals in the other. Sunflower heads often have 55 and 89 spirals. Typical pinecones have three and five spirals.

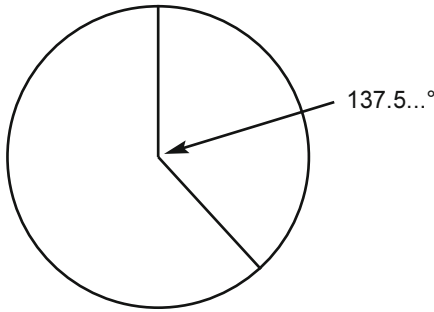
The cactus shown below looks somewhat like a pineapple from the side and shows spiky protuberances arranged in spirals of 21 and 34. On top it shows the new buds growing rather like the centre of a sunflower.

Until recently this was all thought to be “in the genes” but now it appears that maybe it is a phenomenon caused by natural processes where the emerging



Two views of the cactus, *Mammillaria tlalocii*. The side shows buds in spirals of 21 and 34. The top is similar to the centre of a sunflower.





A circle divided in extreme and mean ratio showing the golden angle.

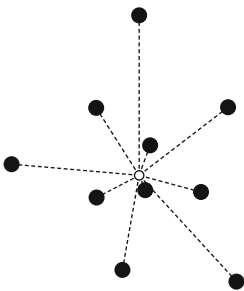
units appear to spread out in a way which minimizes interference and gives each maximum space for growth.

Growth in any plant commences at a conical tip from which new material emerges. The cone might be very sharp as in the tip of a bamboo shoot or quite flat as in a growing sunflower head. It seems however that there is a natural tendency for each new primordial growing bud to come out at an angle of about  $137.5^\circ$  from the previous one. Why this strange angle? Well this angle divides a circle into “extreme and mean ratio” in the words of Euclid. Now we might remember that

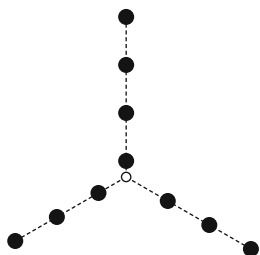
$$1 = 1/\tau + 1/\tau^2$$

so that the two angles which divide  $360^\circ$  in extreme and mean ratio are  $360/\tau$  and  $360/\tau^2$ . These angles are  $222.492\dots$  and  $137.508\dots$  degrees respectively. The latter is often called the golden angle.

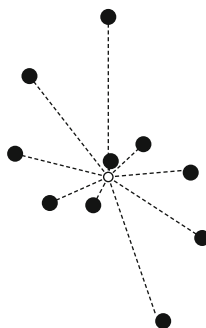
Now if successive buds grow out from the centre each  $137.5^\circ$  from the previous one it turns out that they each have maximum space around them. The diagram below shows ten buds having grown out in sequence at the same rate. Clearly if they were at an angle such as  $120^\circ$  from each other then the fourth bud would find itself up against the first one and the pattern would consist of three lines of buds with wasted spaces between. Indeed we have to avoid any angle which is a rational fraction of  $360^\circ$  (that is a factor of some multiple of  $360^\circ$ ). Numbers such as square roots are satisfactory so we might try  $360/\sqrt{5}$  which



Pattern for  $\tau$ .

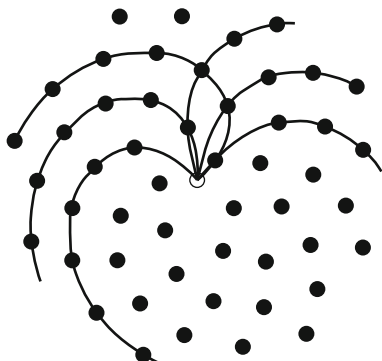


Pattern for 3.

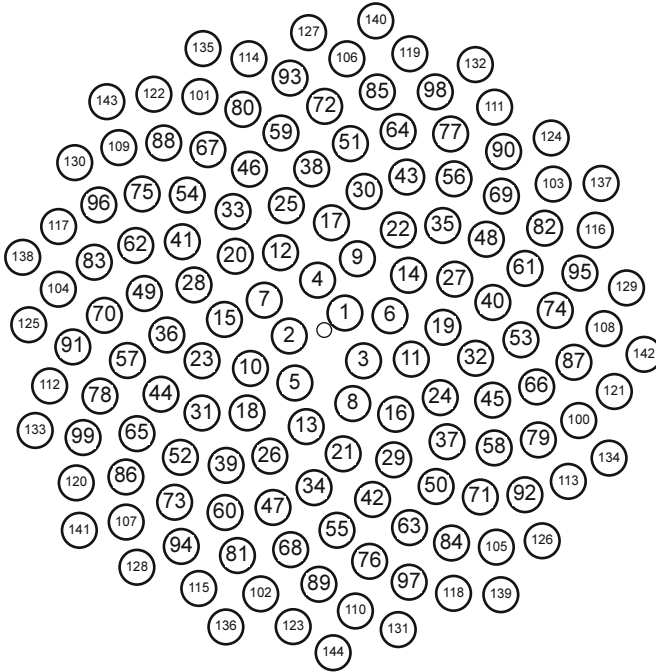
Pattern for  $\sqrt{5}$ .

gives 160.996... degrees. But, as seen in the diagram above, it is not that good since a pattern of spiral arms emerges and space is wasted between the arms. A better choice is  $360/\sqrt{2}$  which gives an angle of 105.442 degrees. But this is not quite so good as  $\tau$ . The reason that  $\tau$  is best is related to the fact that its continued fraction has every term equal to one thus  $[1; 1, 1, 1, \dots]$ . The smaller the terms in the continued fraction the more uniformly distributed is the pattern. The continued fraction for  $\sqrt{5}$  is  $[2; 4, 4, 4, \dots]$  and that for  $\sqrt{2}$  is  $[1; 2, 2, 2, \dots]$ . The diagrams above show the distribution for the three cases corresponding to dividing the circle by  $\tau$ , 3 and  $\sqrt{5}$ .

Assuming that the buds remain the same size, it is more realistic to slow their radial progress as they move away from the centre so that they continue to have the same space. Accordingly, a reasonable model is one where the distance out from the centre varies as the square root of the age of the bud. The resulting distribution for the first fifty buds shows that they align themselves as spirals with 13 in one direction and 8 in the other. However, as more buds are added it becomes clear that spirals corresponding to all the Fibonacci numbers can be traced. Adding more buds concentrates the eye on the outer regions where the spirals corresponding to the higher Fibonacci numbers are evident.



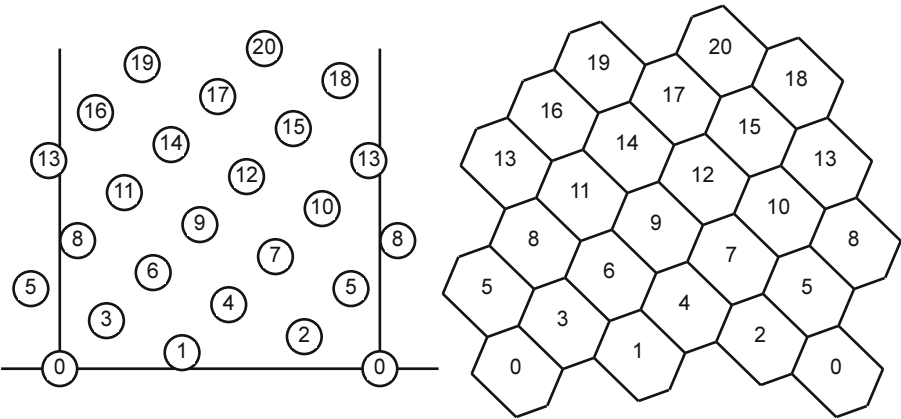
The first fifty buds and the spirals of 8 and 13.



The first 144 buds showing the various spirals.

Thus in the diagram above where the first 144 buds are shown the spirals most evident are those for 13 and 21 although the 34 spiral is beginning to appear. Note that the 21 spirals are formed by the numbers that are 21 apart and so on. The spirals for adjacent Fibonacci numbers always go in opposite directions. We can still trace the lower spirals but the buds on them become further and further apart. Note how the buds corresponding to the Fibonacci numbers themselves are aligned about the vertical. The pattern is clearly the sort of thing we see in a Sunflower head where typically the spirals are in the region of 55 and 89.

We will now consider a rather different model more appropriate to pineapples and fircones which have a general cylindrical appearance. We will suppose that buds are created along the base of a cylinder and migrate upwards at a constant rate. Again we will suppose that the buds emerge at the golden angle of  $137.5$  degrees apart. A big difference from the flat sunflower is that we can assume that the buds continue to travel at a uniform rate since they will then continue to have the same space about them. The general effect for the first twenty buds will be as in the diagram overleaf where the cylinder has been unrolled. This shows clearly the three spirals up to the right and five to the left and is a good representation of the pattern on a fircone. The spiral of two is also

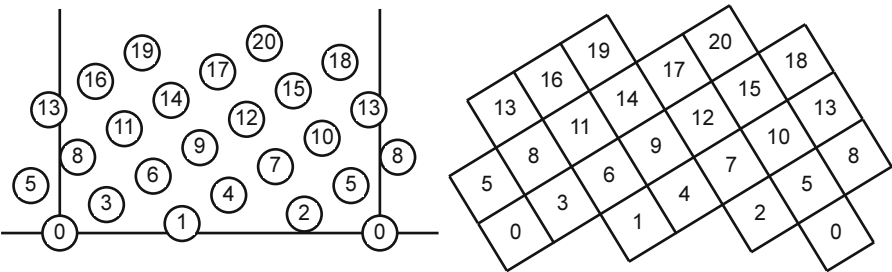


The first twenty points on a cylinder showing spirals of 2, 3 and 5.

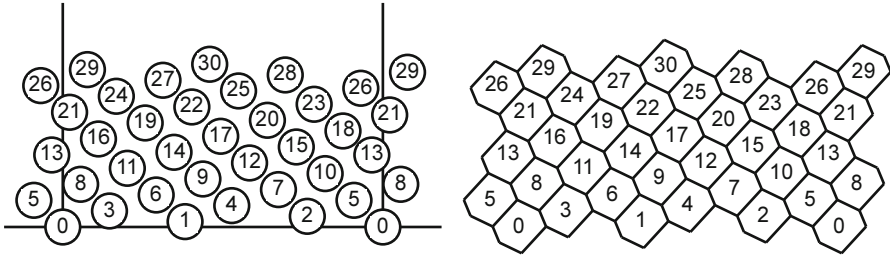
clearly visible on the flat picture but not so obvious on a fircone because it swiftly disappears from view.

In this rectangular pattern the spirals which dominate are determined by the ratio of vertical to horizontal separation. A neat presentation is to show the region consisting of the area containing those points closest to each point of the pattern. This is known as the Dirichlet region after the Prussian mathematician Peter Dirichlet (1805–1859). The region is usually hexagonal and is shown in the pattern on the right above which has the same spacing as that on the left and explicitly reveals that the regions which touch correspond to the spirals of 2, 3 and 5.

If we change the ratio of horizontal to vertical separation then the hexagonal regions change shape and the neighbouring regions eventually become different. At the transition positions the regions become rectangular as shown below which illustrates spirals of just 3 and 5. Note that we can still see the spiral of 2 and the nascent spiral of 8 lurking in the pattern.



The transition point where the regions are rectangular.

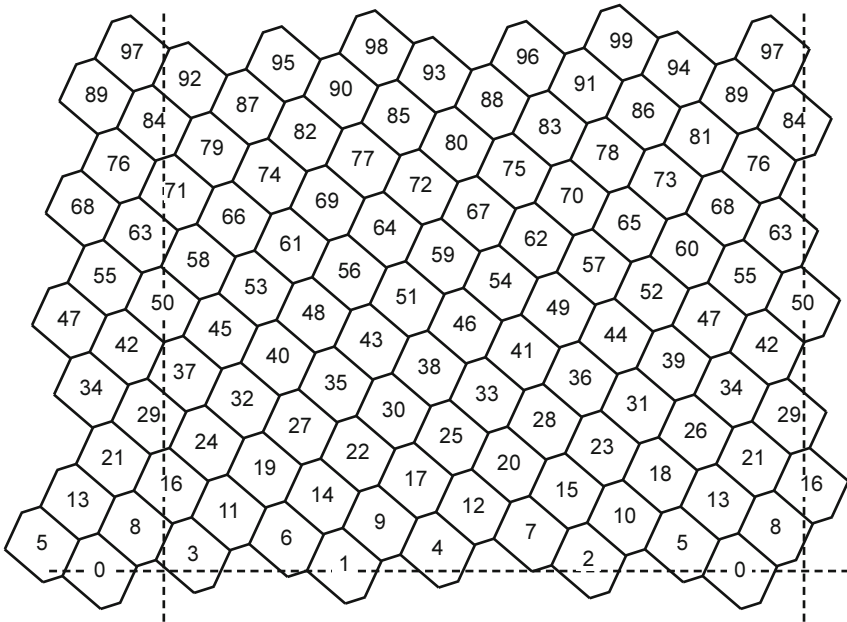


The first thirty points on a cylinder showing spirals of 3, 5 and 8.

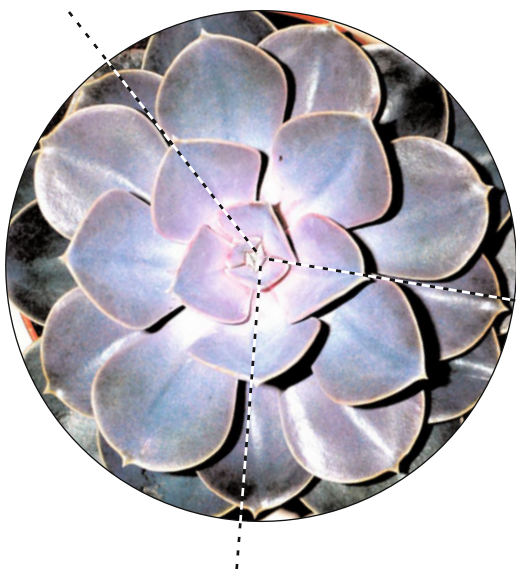
Changing the ratio a bit more produces the next stage as shown above when the spirals are of 3, 5 and 8.

Finally, the pattern below illustrates the case of 5, 8 and 13 spirals which corresponds very closely to the structure of a pineapple – although some pineapples may twist the other way.

We can also change the horizontal and vertical separation the other way which corresponds to very rapid growth and reveals spirals of 1, 2 and 3 and then 1, 1 and 2. Remember that the first two Fibonacci numbers are both 1. Finally, we can obtain spirals of 0, 1 and 1. The spiral of 0 occurs when two regions meet each other in both directions.



The pattern on a pineapple with 5, 8 and 13 spirals.



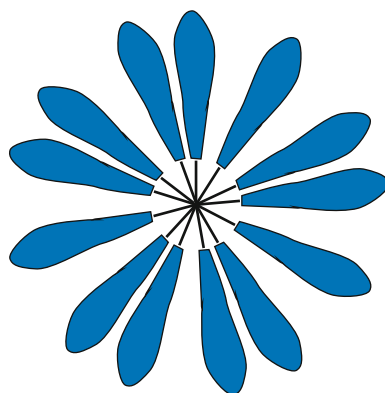
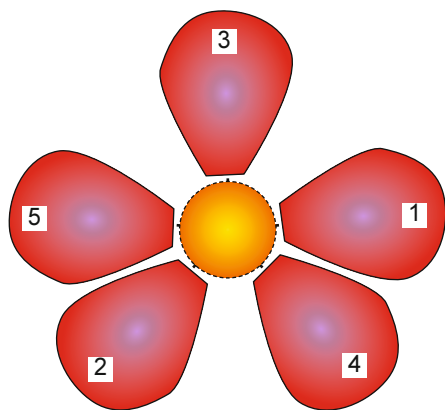
The succulent,  
*Echevaria elegans*.

Three successive  
leaves are marked  
showing the golden  
angles between them.

Another interesting example of the appearance of the golden number is provided by the succulent *Echevaria elegans* shown above. Successive fleshy leaves grow out at golden angles from each other.

We now return to the question of why it is that the number of petals on many flowers are Fibonacci numbers. Again it seems to concern the emission of buds around a central axis with the golden angle of  $137.5^\circ$  between them.

Suppose that buds grow out from the centre but unlike the sunflower head do not move away from the centre but just grow into petals. The picture on the left below shows the first five petals. Note that the first one to grow is in fact that



Flowers with 5 and 13 petals.



A flower of *Echinacea purpurea*.

It has 13 petals which show strong indications of being grouped into pairs and singletons.

on the right and the top petal is the third; we have shown it rotated to reveal the symmetry. The sixth petal would be located between the first and third but clearly there isn't room without overlapping. So maybe that is why many plants stop there.

Of course, if the petals are narrower then more can be placed as in the picture on the right which has 13 petals. Amazingly, if we forbid overlapping it is always the case that the first one that will not fit is that immediately after a Fibonacci number of petals have been placed. We also note that the petals are grouped into pairs and singletons. The picture above shows an excellent specimen of *Echinacea purpurea* which has 13 petals in which some grouping into pairs is evident.

It is fairly easy to show that, as they are added, the nearest petal to the one which is the  $N$ th after a particular Fibonacci number is in fact petal number  $N$ . Thus the nearest petal to number 10 is number 2 (since 10 is 2 more than the



## 24 Gems of Geometry

Fibonacci number 8) and the nearest to 11 is number 3. Moreover, the spacing depends only on the previous Fibonacci number and decreases by a factor of  $\tau$  when the next Fibonacci number is reached. Thus petals 9 to 13 are all about  $20^\circ$  from petals 1 to 5 respectively whereas petals 14 to 21 are all about  $12.4^\circ$  from petals 1 to 8 respectively and so on.

Another intriguing feature is that the petals are indeed grouped into pairs and singletons. If there are exactly  $F_n$  petals where  $F_n$  is the  $n$ th Fibonacci number then there are  $F_{n-2}$  pairs and  $F_{n-3}$  singles. Thus a flower with 13 petals has them grouped into 5 pairs and 3 singletons. Note that this illustrates the fact that for any Fibonacci number

$$F_n = 2F_{n-2} + F_{n-3}$$

For example,  $13 = 2 \times 5 + 3$  and  $55 = 2 \times 21 + 13$ .

In practice of course, nature is not always so precise and especially with the larger numbers of petals some variation is found. Nevertheless, it seems that the golden angle is at the bottom of it all.

The sunflower features in the painting entitled *Virgin of Guadalupe* by Salvador Dali. The head of the Virgin is depicted against a giant sunflower head made of stones with the stones arranged in spirals.

## Further reading

CHAPTER 11 of *Introduction to Geometry* by Coxeter covers the key aspects of the golden number and gives for example the mathematical details of Fibonacci's numbers and the pineapple. Pictures of the divisions of squares into squares will be found in *Tilings and Patterns* by Grünbaum and Shephard and also in *Mathematical Snapshots* by Steinhaus although at the time of writing of the latter the decomposition of the square of side 21 was not known; the relationship with electrical circuits is discussed in detail in *Graph Theory* by Bollabás. There is a good discussion on continued fractions in *The Higher Arithmetic* by H Davenport. *The Book of Numbers* by Conway and Guy covers all sorts of numbers and series and includes material on the Fibonacci numbers and phyllotaxis.

A rather different reference work is *Stamping Through Mathematics* by Robin Wilson. This illustrates almost 400 postage stamps depicting various aspects of mathematics. Those of relevance to this lecture include the golden spiral on an 80 cent Swiss stamp of 1987, the decomposition of a rectangle into eleven squares on a 110 pfennig German stamp of 1998, and a somewhat symbolic pineapple on an Israeli stamp of 1961.

An intriguing reference must surely be *De Divina Proportione* by Paccioli published in 1509 and illustrated with drawings of models by Leonardo da Vinci. Curiously, it is written in old Italian rather than Latin. He describes thirteen



properties of the golden number. The ninth is that the diagonals of a pentagon intersect in the golden ratio. The original text (in a modern font) begins

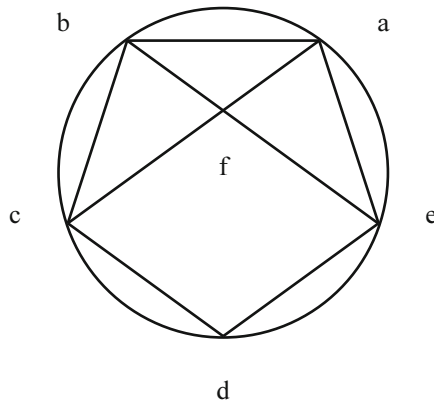
**S**E nel cerchio se formi el pentagono equilatero e ali suoi doi  $\text{ppinqui}$  anguli se subtéda doi linee recte mosse dali termini deli suoi lati de necessita quelle fra loro se divideráno secondo la nostra  $\text{ppportióe}$ . E cadauna dele lor magior parti semp sira el lato del dicto pétagono.

I am grateful to the British Library for permission to quote the above fragment and to Frank Bott of Aberystwyth for providing the following translation.

**I**F an equilateral pentagon be inscribed in the circle and to two of its adjacent vertices two straight lines be subtended to the end of the sides, of necessity they divide each other according to our ratio. And each of their larger parts will always be equal to the side of the said pentagon.

It is unclear why the pentagon needed to be inscribed in a circle but the associated diagram shown below certainly helps with the translation of the rest of it which considers an example where the diagonal of the pentagon is 10 units and then  $CF$  is  $\sqrt{125} - 5$ , and  $AF$  is  $15 - \sqrt{125}$ .

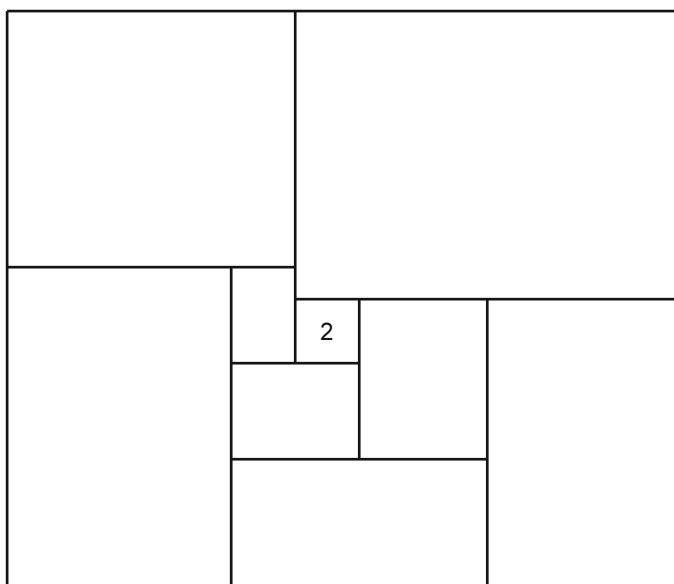
We will encounter other properties of the golden number in later lectures.



Woodcut by Leonardo da Vinci.

## Exercises

- 1 Find the sizes of the squares in the subdivision of a rectangle shown below. They are all different whole numbers and the smallest has side 2. The diagram is distorted so you can't cheat by measuring!
- 2 Draw a flower with eight petals and number them in a similar way to the numbering on the flower with five petals in the notes.
- 3 The continued fraction for  $\pi$  is  $[3; 7, 15, 1, 292, \dots]$ . We saw that the first convergent  $[3; 7]$  is  $22/7$  which is a frequently used approximation. Compute the next two approximations  $[3; 7, 15]$  and  $[3; 7, 15, 1]$ .



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