Chapter 3
Extensions of Valuations to Quantized Algebras

3.1 Extension of Central Valuations

We look at skewfields obtained as total quotient rings of algebras defined by generators and relations. It is of particular interest to consider so-called quantized algebras stemming from noncommutative geometry because we hope to use valuation theory in the construction of a kind of divisor theory in noncommutative geometry.

Consider a field K with valuation ring $O_v \subset K$ having maximal ideal $m_v \subset O_v$ and residue field $k_v = O_v/m_v$. Let A be a connected positively graded K-algebra, $A = K \oplus A_1 \oplus \ldots \oplus A_n \oplus \ldots$, where each A_i is a finite dimensional K-space and $A = K[A_1]$, $A_1 = \oplus_{i=1}^{n} Ka_i$. We view A as an algebra given by generators and relations:

$$0 \to \mathcal{R} \to K < X_1, \ldots, X_n > \xrightarrow{\pi} A \to 0$$

where $K < X_1, \ldots, X_n >$ is the free K-algebra on $\{X_1, \ldots, X_n\}$ and π is given by $\pi(X_i) = a_i, i = 1, \ldots, n$. The ideal of relations $\mathcal{R}$ is homogeneous in the usual gradation of $K < X_1, \ldots, X_n >$. We can also consider the ungraded case where A is a finitely generated K-algebra with generators $a_1, \ldots, a_n$ and π defined as before but then $\mathcal{R}$ is not homogeneous is the usual gradation of $K < X_1, \ldots, X_n >$. Restriction of π to $O_v < \underline{X} >$ defines a graded subring Λ of A with $\Lambda_0 = O_v$

$$0 \to \mathcal{R} \cap O_v < \underline{X} > \to O_v < \underline{X} > \underset{\mathrm{res}\pi}{\longrightarrow} \Lambda \to 0$$

It is clear that π maps $w_v < \underline{X} >$ to $w_v\Lambda$ which is a graded ideal of Λ. We write: $\overline{\Lambda} = \Lambda/w_n\Lambda$ and $\overline{\mathcal{R}} = (\mathcal{R} \cap O_v < \underline{X} >) + w_v < \underline{X} > /w_v < \underline{X} >$, so we arrive at the following commutative diagram with exact rows:

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$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathcal{R}} \arrow[r] & k_r < \underline{X} > \arrow[r, "\overline{\pi}"] & \overline{\Lambda} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{R} \cap O_v < \underline{X} > \arrow[r] \arrow[u] \arrow[d, hook] & O_v < \underline{X} > \arrow[r] \arrow[u] \arrow[d, hook] & \Lambda \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{R} \arrow[r] & K < \underline{X} > \arrow[r, "\pi"'] & A \arrow[r] & 0
\end{tikzcd}$$

When $\mathcal{R}$ is generated by $p_1(\underline{X}), \ldots, p_d(\underline{X})$ as a two-sided ideal, then we may assume $p_i(\underline{X}) \in O_v < \underline{X} >$ up to multiplying by some constant but it does not follow that $\mathcal{R} \cap O_v < \underline{X} >$ is generated as a (two-sided) ideal by $\{p_1(\underline{X}), \ldots, p_d(\underline{X})\}$, nor that $\overline{\mathcal{R}}$ is generated by the reduced expressions $\overline{p}_1(\underline{X}), \ldots, \overline{p}_d(\underline{X})$, obtained by reducing coefficients at m_v.

3.1.1 Definition

We say that $\mathcal{R}$ (or A) **reduces well at** $O_v$or that Λ defines a **good reduction**, if $\overline{\mathcal{R}}$ is generated as an ideal by $\{\overline{p}_1(\underline{X}), \ldots, \overline{p}_d(\underline{X})\}$.

Let us write fK for the Γ-valuation filtration of K associated to v and define a Γ-filtration $fK < \underline{X} >$ by putting: for $\gamma \in \Gamma$, $f_\gamma K < \underline{X} >= (f_\gamma K) < \underline{X} >$. The latter is a strong filtration on $K < \underline{X} >$ with $f_0 K < \underline{X} >$ equal to $O_v < \underline{X} >$. A left ideal J of $O_v < \underline{X} >$ is said to be v**-comaximal** of for all $\gamma \in \Gamma$, $J \cap (f_\gamma K) < \underline{X} >= (f_\gamma K)J$.

3.1.2 Lemma

If the ideal L of $O_v < \underline{X} >$ generated by $p_1(\underline{X})$, $p_d(\underline{X})$ is v-comaximal then $\mathcal{R}$ reduces well at O_v.

Proof. Since $fK < \underline{X} >$ is a strong filtration and for any $r \in f_\gamma K < \underline{X} >$ for some $\gamma \in \Gamma$ yields $f_{\gamma^{-1}} K < \underline{X} > r \in \mathcal{R} \cap f_0 K < \underline{X} >$, we have that $\mathcal{R} = K < \underline{X} > (O_v < \underline{X} > \cap \mathcal{R})$. Let L' be the left ideal in $O_v < \underline{X} >$ generated by $\{p_1(\underline{X}), \ldots, p_d(\underline{X})\}$; then we have $L'K < \underline{X} >= \mathcal{R}$ since $L'K < \underline{X} >$ is the two-sided ideal generated by $\{p_1(\underline{X}), \ldots, p_d(\underline{X})\}$. For $x \in f_0(L'K < \underline{X} >)$ there is a $\gamma \in \Gamma$ such that $xf_{\gamma^{-1}} K < \underline{X} >\subset L$ as well as $xf_{\gamma^{-1}} K < \underline{X} >\subset f_{-\gamma} K < \underline{X} >$ since $x \in f_0 K < \underline{X} >$. Therefore we arrive at $xf_{\gamma^{-1}} K < \underline{X} >\subset L \cap f_{\gamma^{-1}} K < \underline{X} >= (f_{\gamma^{-1}} K)L$ by the v-comaximality of L. From this it follows that$f_\gamma K < \underline{X} > xf_{-\gamma} K < \underline{X} >\subset (f_\gamma K)(f_{\gamma^{-1}} K)L = L$ hence $x \in L$.

Then we obtain:

$$L \subset \mathcal{R} \cap O_v < \underline{X} >= f_0\mathcal{R} = f_0(L'K < \underline{X} >) \subset L$$

arriving at $\mathcal{R} \cap O_v < \underline{X} >= L$ being the two-sided ideal in $O_v < \underline{X} >$ generated by $\{p_1(X), \ldots, p_d(\underline{X})\}$; from this it follows easily that $\overline{\mathcal{R}}$ is the two-sided ideal generated by the reductions $\overline{p}_i(\underline{X})$ of $p_i(\underline{X})$. □

In case the reduced relations $\overline{p}_1(\underline{X}), \ldots, \overline{p}_d(\underline{X})$ determine a simple algebra then the O_v-reduction is necessarily a good reduction, indeed the ideal $(\overline{p}_1(\underline{X}), \ldots, \overline{p}_d(\underline{X}))$ is now maximal in $k_v < \underline{X} >$ hence $\overline{\mathcal{R}} = (\overline{p}_1(\underline{X}), \ldots, \overline{p}_d(\underline{X}))$.

3.1.3 Corollary

If $A = \mathbb{A}_n(K)$ is the n-th Weyl algebra defined as $K < X_i, Y_i, i = 1, \ldots, n > /(Y_iX_i - X_iY_i - 1, X_iX_j - X_jX_i, Y_iY_j - YiY_j)$ then the reduced relations define $\mathbb{A}_n(k_v)$ which is known to be a simple algebra (if $\text{char}(k_v) = 0$) so the reduction at O_v is good if $\text{char}(k_v) = 0$.

As we have already pointed out the results concerning good reduction are valid in the ungraded case, but it is interesting to look at positively filtered algebras since any finitely generated K-algebra inherits a standard filtration via $\pi : K < \underline{X} >\to A, X_i \mapsto a_i$, from the gradation filtration of the free algebra $K < X >$. So, let us assume again that the K-algebra A is given by generators and relation via

$$(*) : 0 \to \mathcal{R} \to K < \underline{X} >\to A \to 0$$

Let FA be the generator filtration of A induced by the gradation filtration of $K < \underline{X} >$ making $\pi : K < \underline{X} >\to A, X_i \mapsto a_i$, into a strict filtered morphism. On $\mathcal{R}$ we may consider the induced filtration $F\mathcal{R} = \mathcal{R} \cap FK < \underline{X} >$. Then (*) is a strict exact sequence that is to say that the image filtration on $\mathcal{R}$ is exactly the filtration induced by $FK < \underline{X} >$ and this yields exactness of $G(*) : 0 \to G(\mathcal{R}) \to G(K < \underline{X} >) \to G_{FA}(A) \to 0$. In fact we have the following:

3.1.4 Lemma

With notation as above, $G(A) = G_{FA}(A)$ is defined by:

$$0 \to \dot{\mathcal{R}} \to K < \underline{X} > \xrightarrow{G(\pi)} G(A) \to 0$$

where $\dot{\mathcal{R}}$ is the left ideal of $K < \underline{X} >$ generated by the $\dot{p}$ for all $p \in \mathcal{R}$ and $\dot{p}$ is the highest degree component of p in the decomposition of p in the gradation of $K < \underline{X} >$. For the Rees ring $\widetilde{A}$ with respect to FA we obtain:

$$0 \to \widetilde{\mathcal{R}} \to K < \underline{X} >^{\sim} \underset{\overline{\pi}}{\longrightarrow} \widetilde{A} \to 0$$

where $\overline{\pi}$ corresponds to $\pi : K < \underline{X} > \to A$ on the Rees object level.

Proof. See e.g. [40, Proposition 1.1.5. p. 10]. □

3.1.5 Theorem

If $G(A)$ reduces well with respect to O_v, say $\dot{\mathcal{R}}$ is generated as a two-sided ideal by $q_1(\underline{X}), \ldots, q_q(\underline{X})$ then there are $p_1(\underline{X}), \ldots, p_d(\underline{X})$ in $K < \underline{X} >$ such that $\mathcal{R} = (p_1(\underline{X}), \ldots p_d(\underline{X}))$ and $\dot{p}_i(\underline{X}) = q_i(X)$ for $i = 1, \ldots, d$, such that $\mathcal{R}$ (i.e. A) reduces well with respect to O_v.

Proof. Choose $p_i'(\underline{X}) \in \mathcal{R}$ such that $q_i(\underline{X})$ is the homogeneous part of highest degree in the decomposition of $p_i'(\underline{X})$, for $i = 1, \ldots, d$. Pick $\mu \in f_{\gamma^{-1}}K$ for $\gamma \in \Gamma_+$ large enough (how large will be clear in the sequel) and replace X_i by $\mu X_i, i = 1, \ldots, d$. Put $\deg q_i(\underline{X}) = m$. Then $\mu^m p_i'(\underline{X}) = q_i(\mu\underline{X}) + \mu\Psi(\mu\underline{X})$ where Ψ has degree lower than m; put this equal to $p_i(\mu\underline{X})$ for $i = 1, \ldots, d$. In the new variables $\mu X_i, i = 1, \ldots, d$, the homogeneous part of highest degree of $p_i(\mu\underline{X})$ is exactly $q_i(\mu\underline{X})$ and $p_i(\mu\underline{X})$ is in $\mathcal{R}$ because $\mu^m p_i'(\underline{X})$ is a relation for A. By choosing γ large enough we may assume that the coefficients appearing in $\mu\Psi(\mu\underline{X})$ are contained in O_v so that $p_i(\mu\underline{X}) \in O_v < \mu\underline{X} >$. Obviously $q_i(\mu\underline{X})$ viewed in $K < \mu\underline{X} >= K < \underline{X} >$ still generates the ideal of relations of $G(A)$. Now consider the two-sided ideal I in $K < \underline{X} >$ generated by $p_i(\underline{X})$, then $I \subset \mathcal{R}$. By construction we have $\dot{I} = \dot{\mathcal{R}}$ so $I \subset \mathcal{R}$ then yields $I = \mathcal{R}$ (for example see [51, 52]). Indeed if $r \in \mathcal{R} - I$ then $\dot{r} = i$ for some $\iota \in I$ hence $r - \iota \in \mathcal{R}$ and in $F_m\mathcal{R}$ with $m < n$ where $r \in F_nR - F_{n-1}R$, thus $(r - \iota)^{\cdot} = i_1$ with $\iota_1 \in F_{m_1}I$ then $r - \iota - \iota_1 \in \mathcal{R}$ and in $F_{m_i}\mathcal{R}$ with $m_1 < m$, and so on, leads to $r - \iota - \iota_1 - \ldots - \iota_t = 0$ since FR is a positive filtration, i.e. $r \in I$ as claimed. The good reduction assumption for $G(A)$ means that $\dot{\mathcal{R}} \cap O_v(\underline{X})$ is generated as a two-sided ideal by $q_1(\underline{X}), \ldots, q_d(\underline{X})$. Taking $(\mathcal{R} \cap O_v < \underline{X} >)^{\cdot}$ in $O_v < \underline{X} >$ we obtain:

$$(\mathcal{R} \cap O_v < \underline{X} >)^{\cdot} \subset \dot{\mathcal{R}} \cap O_v < \underline{X} >$$

Since $q_i(\underline{X})$ is the highest homogeneous part of $p_i(X) \in \mathcal{R} \cap O_v < \underline{X} >$ it is clear that $(\mathcal{R} \cap O_v(\underline{X}))^{\cdot}$ contains $q_i(\underline{X})$ and is an ideal of $O_v(\underline{X})$ (because if $\overline{h}(\underline{X})$ is a homogeneous element of $(\mathcal{R} \cap O_v < \underline{X} >)^{\cdot}$ then it is the leading term of some $h(\underline{X})$ in $R \cap O_v(\underline{X})$ and a nonzero $\overline{h}(\underline{X}).\underline{x}$ for some $\overline{x} \in O_v < \underline{X} >= G(O_v(\underline{X}))$ is the leading term of $h(X)x$ for some x with $\sigma(x) = \overline{x}$ and $h(x)x \in \mathcal{R} \cap O_v < \underline{X} >$). Hence we obtain: $(\mathcal{R} \cap O_v < \underline{X} >)^{\cdot} = \dot{\mathcal{R}} \cap O_v < X >$.

The $O_v < \underline{X} >$-ideal J generated by the $p_i(\underline{X})$ is in $\mathcal{R} \cap O_v < \underline{X} >$ and $q_i(\underline{X}) \in \dot{J} \subset (\mathcal{R} \cap O_v < X >)^{\cdot} = \dot{\mathcal{R}} \cap O_v < \underline{X} >$ yielding: $\dot{J} = (\mathcal{R} \cap O_v < \underline{X} >)^{\cdot}$. As before: $J = \mathcal{R} \cap O_v < \underline{X} >$ follows and this states exactly that $\mathcal{R}$ (hence A) reduces well at O_v.

The filtration fA defined by $f_\gamma A = (F_\gamma K)\Lambda$ will be used for extending the valuation v of K to some quotient ring of Λ.

3.1.6 Lemma

1. Let A be graded and $\Gamma = \mathbb{Z}$ and assume A is gr-simple, then the filtration fA is separated and $G_f(A)$ is strongly graded. If $\overline{\Lambda}$ is a domain then $G_f(A)$ is a domain and Λ is a domain.
2. If A is not graded but simple then the statement of 1 holds too.
3. For a non-discrete Γ assume that A has a PBW-basis $\{a_1, \ldots, a_d\}$ i.e. the $\{a_1, \ldots, a_d\}$ can be ordered such that elements of A have a unique expression as ordered polynomials in the generators $a_1, \ldots, a_d$. Then the statements of 1 are still true.

Proof. 1. Consider $I = \cap\{(f_{\gamma^{-1}}K)\Lambda, \gamma \in \Gamma_+\}$. Clearly $KI \subset I, IK \subset I$ hence $AI \subset I$ and $IA \subset I$ since $K\Lambda = A$. Thus I is a graded ideal of A hence $I = 0$. That $G_f(A)$ is strongly graded follows from fA being a strong filtration. If $\overline{\Lambda}$ is a domain, then from Lemma 1.8.9 it follows that $G_f(A)$ is a domain and then A is domain too.

2. In the ungraded situation but with A simple the statements of 1 follow in an almost identical way.
3. If we can establish that fA is Γ-separated then it is again a strong filtration and the statements in (1) follow in the same way. Suppose fA is not separated, that is there in an $x \in A$ such that for every $\gamma \in \Gamma$ such that $x \in F_\gamma A$ there is a $\delta < \gamma$ in Γ such that $x \in f_\delta A$ too! So for $x \in (f_\gamma K)\Lambda$ this means $x \in (f_\delta K)\Lambda$ for some $\delta < \gamma$. Assume that $\{a_1, \ldots, a_d\}$ is a PBW-basis for A in the ordering given by the indices . Then $x = \sum \xi_{\underline{i}} \underline{a}^{\underline{i}} = \sum \eta_{\underline{i}} \underline{a}^{\underline{i}}$ with $\xi_{\underline{i}} \in f_\gamma K, \eta_{\underline{i}} \in f_\delta K$. Pick $c \in f_{\gamma^{-1}}K$ such that $c\xi_{\underline{i}} \in O_v$ but not all in m_v and $c\eta_{\underline{i}} \in m_v$ (because $\delta < \gamma$). Adapting a common multi-index notation (i.e. inserting some zero-coefficients $\xi_{\underline{i}}$ or $\eta_{\underline{i}}$ when necessary) we obtain $\sum(c\xi_{\underline{i}} - c\eta_{\underline{i}})\underline{a}^{\underline{i}} = 0$. This relation is non-trivial since not all coefficients are in m_v, but that contradicts the PWB-basis property of $\{a_1, \ldots, a_d\}$. Hence such x does not exist so for every $z \in A$ there exists a $\gamma \in \Gamma$ such that $z \in f_\gamma A$ and $z \notin f_\mu A$ with $\mu < \gamma$, or fA is separated. □

Since we consider Γ-valuations on K the Γ-filtration defined on a K-algebra A is not Zariskian i.e. $\widetilde{A}$ need not be Noetherian, so we cannot use results or Zariskian filtration here. We consider a separated Γ-filtration fA on a ring A and S an Ore set of A such that $\sigma(S)$ consists of regular elements of $G_F(A) = G(A)$. We define the localized filtration $FS^{-1}A$ by putting $x \in F_\gamma S^{-1}A$ if there exists an $s \in S$, s of $\deg s = \tau \in \Gamma$, such that $sx \in f_{\tau\gamma}A$.

3.1.7 Proposition

With notation as before, $FS^{-1}A$ is a Γ-filtration of $S^{-1}A$, Γ-separated, inducing fA on A.

Proof. Since $\sigma(S)$ consists of regular elements of $G(A)$ also S consists of regular elements of A. If $x \in F_\gamma S^{-1}A$ then $sx \in f_{\tau\gamma}A$ for some $s \in S$ with $\deg\sigma(s) = \tau$; there is a $\delta \in \Gamma$ such that $sx \in f_\delta A$ but $sx \notin f_{\delta'}A$ for $\delta' < \delta$. Hence $x \in F_{\tau^{-1}\delta}A$ and $x \notin F_{\delta'}A$ with $\delta' < \tau^{-1}\delta$. This follows from the uniqueness of γ, suppose $s_\delta x \in f_{\delta\gamma}A$ and $s_\sigma x \in f_{\sigma\tau}A$ with $\tau \neq \gamma$, say $\tau < \gamma$ in Γ. By the Ore condition there is an s_α such that $s_\alpha s_\delta = a s_\sigma$ with $a \in A$, where the index of the $s's$ refers to the degree of the $\sigma(s)$. Since $\sigma(s_\sigma)$ is regular in $G(A)$ we must have that $\deg\sigma(a) = \alpha\delta\sigma^{-1}$ in Γ. Then $0 \neq s_\alpha s_\delta x = a_{\alpha\delta\sigma^{-1}} s_\sigma x \in a_{\alpha\delta\sigma^{-1}} f_{\sigma\tau}A \subset f_{\alpha\delta\tau}A$; on the other hand we also have that $s_\alpha s_\delta x \in s_\alpha f_{\delta\gamma}A \subset f_{\alpha\delta\gamma}A$, so if we assume $\delta\gamma$ to be the lowest in Γ such that $s_\delta x \in f_{\delta\gamma}A$ then from $\tau < \gamma$ we reach a contradiction because $\alpha\delta\gamma$ is then the lowest containing $s_\alpha s_\delta x$ (as $\sigma(s_\alpha s_\delta x) = \sigma(s_\alpha)\sigma(s_\delta x)$. If $x, y \in F_\gamma S^{-1}A$ then $s_\delta x \in f_{\delta\gamma}A$, $s_\rho y \in f_{\rho\gamma}A$ for some $s_\delta, s_\rho \in S$; then $s_\delta y \in F_{\delta\gamma}S^{-1}A$ for $s_\tau s_\delta = a_{\tau\delta\rho^{-1}} s_\rho$ for some $s_\tau \in S, a_{\tau\delta\rho^{-1}} \in A$ yields $s_\tau s_\delta y = a_{\tau\delta\rho^{-1}} s_\rho y \in f_{\tau\delta\rho^{-1}} A f_{\rho\gamma}A$ hence $s_\tau s_\delta y \in f_{\tau\delta\gamma}A$, consequently: $s_\tau s_\delta(x + y) = s_\tau(s_\delta x) + s_\tau s_\delta y \in f_{\tau\delta\gamma}A$, or $x + y \in F_\gamma S^{-1}A$, proving that the $F_\gamma S^{-1}A$ are additive subgroups. Now for $x \in F_\gamma S^{-1}A, y \in F_\tau S^{-1}A$ we have $s_\alpha x \in f_{\alpha\gamma}A, s_\beta y \in f_{\beta A}$. Write $a_{\alpha\gamma}$ for $s_\alpha x$ and pick $s_\mu \in S$ such that $s_\mu a_{\alpha\gamma} = a' s_\beta$ where $a' \in f_{\mu\alpha\gamma\beta^{-1}}A$ follows from $\sigma(a')\sigma(s_\beta) = \sigma(s_\mu)\sigma(a_{\alpha\gamma})$ and $\deg a_{\alpha\gamma} \leq \alpha\gamma$, hence $\deg\sigma(a') \leq \mu\alpha\gamma\beta^{-1}$. Now $s_\mu s_\alpha xy = s_\mu a_{\alpha\gamma} y = a' s_\beta y$ with $s_\beta y \in f_{\beta\tau}A$ yields $s_\mu s_\alpha xy \in f_{\mu\alpha\gamma\beta^{-1}} A f_{\beta\tau}A \subset f_{\mu\alpha\gamma\tau}A$. Putting $s_\mu s_\alpha = s_{\mu\alpha}$ yields $xy \in F_{\gamma\tau}S^{-1}A$, so $FS^{-1}A$ is a filtration. The filtration is separated because for $x \in S^{-1}A$ there is an $s_\delta \in S$ such that $s_\delta x \in f_{\delta\gamma}A$ and if $\delta\gamma$ is such that $s_\delta x \notin f_{\gamma'}A$ for $\gamma' < \delta\gamma$ then $x \notin F_\tau S^{-1}A$ for $\tau < \gamma$ (observe that $F_\tau S^{-1}A \cap A = f_\tau A$ because for $a \in A \cap F_\tau S^{-1}A$ some $s_\delta a \in f_{\delta\tau}A$ so $\deg\sigma(a) \leq \tau$, hence $FS^{-1}A$ induces fA on A). □

Next we look at the Weyl skewfield $D_1(K)$ and a Γ-valuation O_v in K.

3.1.8 Theorem

Every Γ-valuation O_v of K extends to a noncommutative valuation ring Λ_v of $D_1(K)$.

Proof. In view of Proposition 1.8.10.3 it suffices to construct a separated Γ-filtration on $D_1(K)$ extending the valuation filtration of K such that the associated graded ring is a domain. In fact we only have to construct a Γ-separated filtration on

$\mathbb{A}_1(K)$ extending v on K such that the associated graded ring is a domain because by Proposition 3.1.7 we can extend this to the localized filtration at the Ore set $\mathbb{A}_1(K)^*$ (the Weyl algebra is an Ore domain) provided $\sigma(\mathbb{A}_1(K)^*)$ consists or regular elements. Now $\Lambda = \mathbb{A}_1(O_v)$ defines a good reduction of $\mathbb{A}_1(K)$ at O_v and $\overline{\Lambda} = \mathbb{A}_1(k_v)$ is a Weyl algebra over the residue field k_v, hence a domain. Thus the filtration $f^v\mathbb{A},(K)$ defined by $f^v_\gamma\mathbb{A}_1(K) = (f_\gamma K)\mathbb{A}_1(O_v)$ has the properties mentioned in (3) of Lemma 3.1.6 and the elements of $\sigma(\mathbb{A}_1(K)^*)$ form exactly the set of homogeneous elements of $G_f(\mathbb{A}_1(K)) = \mathbb{A}_1(k_v)\Gamma$ where $G_f(K) = k_v\Gamma$ and these form even an Ore set because $\mathbb{A}_1(k_v)$ is an Ore domain (and $k_v\Gamma$ is central in $G_f\mathbb{A}_1$ and they are certainly regular in $G_f(\mathbb{A}_1(K))$. For the localized filtration $FD_1(K)$ of $f\mathbb{A}_1(K)$ the associated graded $G_F D_1(K)$ is the graded quotient ring of $\mathbb{A}_1(k_v)\Gamma$ which is $\mathbb{D}_1(k_v)\Gamma$ and a domain! □

3.1.9 Observation

In the foregoing Γ is abelian because it comes from O_v on the commutative K. We shall see later that any valuation on $D_n(K)$ is in fact abelian!

We can extend the foregoing theorem to K-algebras with a PBW-basis as follows.

3.1.10 Proposition

Let A be a K-algebra with PBW-basis $\{a_1,\dots,a_d\}$ and $\Lambda = O_v < a_1,\dots,a_d >$. Suppose that A is an Ore domain with skew field of fractions $Q(A)$ and that $\overline{\Lambda} = \Lambda/m_v\Lambda$ is a domain then v extends to a noncommutative valuation of $Q(A)$.

Proof. Define fA by $f_\gamma A = (f_\gamma K)\Lambda$ for every $\gamma \in \Gamma$. Statement (3) from Lemma 3.1.6 yields that fA is a Γ-separated filtration and $G_f(A)$ is a domain. We have that $\sigma(A^*)$ is a graded Ore set of $G_f(A)$ in fact $\sigma(A^*)$ is the set of homogeneous elements (nonzero) of $G_f(A)$; indeed if $\overline{a},\overline{b} \in h(G_f(A))^*$ then there are $a',b' \in A^*$ such that $a'b = b'a$ by the Ore condition for A^* and since $G_f(A)$ is a domain $\sigma(a'b) = \sigma(a')\sigma(b) = \sigma(b')\sigma(a) = \sigma(b'a)$, or $\sigma(a')\overline{b} = \sigma(b')\overline{a}$. For $x \in G_f(A)$ say $x = x_{\gamma_1}+\dots+x_{\gamma_n}$ with $x_{\gamma_i} \in hG_f(A)^*$. There is an $s_1 \in hG_f(A)^*$, $s_1x_{\gamma_1} = y_1\overline{a}$, hence $s_1x = y_1\overline{a} + s_1x_{\gamma_2} + \dots + s_1x_{\gamma_n}$ with $\overline{a} \in hG_f(A)^*$ and $y_1 \in hG_f(A)^*$.

Then take $s_2 \in hG_f(A)^*$ such that $s_2s_1x_{\gamma_2} = y_2\overline{a}$ with $y_2 \in hG_f(A)^*$, then $s_2s_1x = s_2y_1\overline{a} + y_2\overline{a} + s_2s_1x_{\gamma_3} + \dots + s_2s_1x_{\gamma_n}$. Repeating this n times we arrive at $s_1,\dots,s_n \in hG_f(A)^*$ such that $s_n\dots s_1x = y\overline{a}$ with $y \in G_f(A)^*$, so $hG_g(A)^*$ is an Ore set in $G_f(A)$. Thus fA defines $FQ(A)$ by localization and the associated graded ring of $Q(A), G_F(Q)$ is the localization of $G_f(A)$ at $hG_f(A)^*$ which is a domain (and in fact a gr-skewfield $Q_{\text{cl}}(\overline{\Lambda})\Gamma$). Therefore $F_0 Q(A)$ is a Γ-valuation ring extending v on K to $Q(A)$. □

We have a similar result for Dubrovin valuations using now Theorem 1.8.11.

3.1.11 Proposition

Let A be a K-algebra with PBW-basis $\{a_1, \ldots, a_d\}$ and put $\Lambda = O_v\langle a_1, \ldots, a_d\rangle$. If A is a prime Goldie ring such that Λ_{reg} maps to regular elements of $\overline{\Lambda} = \Lambda/m_v$ and $\overline{\Lambda}$ is a prime Goldie ring than v extends to a Dubrovin valuation on the simple Artinian $Q_{\text{cl}}(A)$.

Proof. The filtration fA defined by $f_\gamma A = (f_\gamma K)\Lambda$ is again separated and strong, hence $G_f(A)$ is strongly graded by Γ over $G_f(A)_0 = \overline{\Lambda}$. The homogeneous elements of $G_f(K)$ are central units in $G_f(A)$ and $G_f(A) = G_f(A)_0 G_f(K)$, hence $G_f(A)$ is also a prime Goldie ring. A regular element of A, x say, may be multiplied by a $\lambda \in K$ to a regular element λx of Λ, such that $\lambda x \notin m_v\Lambda$. Hence $\sigma(\lambda x)$ is regular in $\overline{\Lambda}$ hence in $G_f(A)$, since $\sigma(\lambda)$ is regular in $G_f(A), \sigma(\lambda x) = \sigma(\lambda)\sigma(x)$ hence $\sigma(x)$ is regular in $G_f(A)$. Then fA extends to the localized filtration $FS^{-1}A$, where $S = A_{\text{reg}}$ and $S^{-1}A$ is a simple Artinian ring. The associated graded ring of $S^{-1}A$ is $G(S)^{-1}G_f(A)$ which is again a prime Goldie ring as it is an order in the simple Artinian ring $T^{-1}G_f(A)$ where $T = G_f(A)_{\text{reg}}$ ($G_f(A)$ is prime Goldie). In fact $\sigma(S)^{-1}G_f(A) = Q_{\text{cl}}(G_f(A)_0)G_f(K)$ where $Q_{\text{cl}}(G_f(A)_0)$ is simple Artinian. In view of Theorem 1.8.11 we obtain that $F_0 S^{-1}A$ is a Dubrovin valuation ring. □

The extension problem for valuations of K to K-algebra appearing as simple Artinian or skewfield quotient rings of algebras given by generators and relations has now been reduced to finding "good reductions" or more directly to the existence of an O_v-order Λ defining a suitable filtration on A that extends well to a localized filtration of $Q_{\text{cl}}(A)$. This comes down to the verification of domain or prime Goldie properties of the associated graded ring. This method applied to the Weyl field, but also to other interesting examples.

3.1.12 Observation

In all of the following situation the extension result for valuations $O_v \subset K$ to the quotient ring of the K-algebra is valid.

(a) The quantum plane $A = K < X, Y > /(XY - qYX)$ and $O_v \subset K$ such that q is a unit in O_v.
(b) The quantized Weyl algebra $A_1(K, q)$ defined as $K < X, Y > /(XY - qYX - 1)$ and $O_v \subset K$ containing q as a unit.

(c) The enveloping algebra $U(g)$ for a finite dimensional Lie algebra g over K.
(d) Quantum 2×2-matrices defined as $K < a, b, c, d >$ with relations: $ba = q^{-2}ab, ca = q^{-2}ac, bc = cb, db = q^{-2}bd, dc = q^{-2}cd, ad - da = (q^2 - q^{-2})bcx$.
(e) The conformal sl_2-enveloping algebra in the sense of L. Le Bruyn given as $K < X, Y, Z >$ modulo the relations:

$$\begin{cases} XY - aYX = Y, ZX - aXZ = Z \\ YZ - cZY = bX^2 + Y \end{cases}$$

at O_v containing a, b, c as units.
(f) Let A be as in Proposition 3.1.10 but assuming now that $\overline{\Lambda}$ is Auslander regular (cf. [40]) and positively graded over a field of characteristic zero. Then it is known that $\overline{\Lambda}$ is a domain and the extension result follows.

The results in this section open the possibility for developing a valuation and divisor theory on quantized algebras, these are deformations of classical algebras depending on certain parameters (as in Observation 3.1.12 above).

3.2 Discrete Valuations on the Weyl Skewfield

In this section K is a field of characteristic zero and $\mathbb{A}_1(K)$ is the first Weyl algebra, $\mathbb{A}_1(K) = K < x, y >= K < X, Y > /(YX - XY - 1)$. We know that $A_1(K)$ is a simple Noetherian non-Artinian, Ore domain and it has a skewfield of fractions $\mathbb{D}_1(K)$ called the first Weyl field. For $A_n(K) = \mathbb{A}_1(K) \otimes \ldots \otimes \mathbb{A}_1(K)$ we have a skewfield of functions $\mathbb{D}_n(K)$.

The Bernstein filtration of $\mathbb{A}_1(K)$ is defined by putting $\deg x = \deg y = 1$, i.e. $F_0\mathbb{A}_1(K), F_n\mathbb{A}_1(K) = K \oplus Kx \oplus Ky, \ldots, F_n\mathbb{A}_n(K) = (F_1\mathbb{A}_1(K))^n, \ldots$. It is a separated $\mathbb{Z}$-filtration with $G_F\mathbb{A}_1(K) \cong K[X, Y]$, we let σ be the principal symbol map of F. On $\mathbb{D}_1(K)$ we consider the quotient filtration $F\mathbb{D}_1(K)$, then $G_F\mathbb{D}_1(K) = Q^g(K[X, Y])$, the graded quotient field of $K[X, Y]$. Since the latter is a domain we know that $F_0\mathbb{D}_1(K)$ is a valuation ring of $\mathbb{D}_1(K)$ and the valuation filtration of it coincides with $F\mathbb{D}_1(K)$; the corresponding valuation v_B is called the Bernstein valuation ring of $\mathbb{D}_1(K)$. To a discrete valuation v of $\mathbb{D}_1(K)$ there corresponds a noncommutative valuation ring Λ_v and a valuation filtration $f_v\mathbb{D}_1(K)$ with $(f_v\mathbb{D}_1(K))_0 = \Lambda_v$.

3.2.1 Observation

If Λ_v is a valuation ring of a skewfield Δ then the following statements are equivalent for $a, b \in \Delta^*$.

1. $\Lambda_v a \subset \Lambda_v b$.
2. $v(a) \geq v(b)$.
3. $a\Lambda_v \subset b\Lambda_v$.

Proof. (an expansion of Lemma 1.3.2.8).

1. $\Rightarrow$ 2 If $\Lambda_v a \subset \Lambda_v b$ than $a = \lambda b$ for some $\lambda \in \Lambda_v$ then $v(a) = v(\lambda) + v(b)$ yields $v(a) \geq v(b)$.
2. $\Rightarrow$ 3. From $v(a) \geq v(b)$ it follows that $v(b^{-1}a) \geq 0$ or $b^{-1}a \in \Lambda_v$ and $a\Lambda_v \subset b\Lambda_v$ follows.
3. $\Rightarrow$ 2. and 3. $\Rightarrow$ 1. follow by symmetry from the foregoing. □

Recall that two discrete valuations v_1 and v_2 are said to be equivalent if there exist n and m in $\mathbb{Z}$ such that $nv_1(x) = mv_2(x)$ for every $x \in \Delta$.

Let us recall how the valuation function $v : \Delta^* \to \Gamma$ is constructed from a valuation ring Λ of Δ. Put $P \subset \Lambda$ equal to the ideal $P = \{x \in \Lambda, x^{-1} \notin \Lambda\}$. For $\lambda \in \Lambda$ define $(P : \lambda) = \{(a, b) \in \Delta \times \Delta, a\lambda b \in P\}$ and call $\lambda_1 \simeq \lambda_2$ if $(P : \lambda_1) = (P : \lambda_2)$. Let $[P : \lambda]$ denote the class of $(P : \lambda)$ with respect to the foregoing equivalence relation. On the set of equivalence classes Γ introduce the total order induced by the inclusion ordering on the set of $(P : \lambda), \lambda \in \Delta$. The function $v : \Delta^* \to \Gamma, x \mapsto [P : \lambda]$ is well-defined. Multiplication of Δ induces a multiplication in Γ making Γ into a totally ordered group. The valuation ring Λ_v coincides with Λ and $P = w_v$.

3.2.2 Proposition

A valuation v on a skewfield Δ has rank one exactly when Λ_v is maximal as a proper subring of Δ (this extends Proposition 1.2.12 to the noncommutative case).

Proof. If v has rank (1) then Γ is Archimedean (cf. Proposition 1.3.1.4). If Λ_v is not maximal let $\Lambda' \supsetneqq \Lambda_v$ be a proper subring of Δ, suppose $a \in \Lambda' - \Lambda_v$ and consider $b \in \Delta^* - \Lambda'$. Since $v(a), v(b) < 0$, the Archimedean property yields that there is an $n \in \mathbb{N}$ such that $v(a^{-n}) \geq v(b^{-1})$, or $ba^{-n} \in \Lambda_v$ with $a^n \in \Lambda'$ and $b \in \Lambda_v a^n \subset \Lambda'$, contradiction. Conversely, if Λ_v is maximal then $htw_v = 1$. Indeed, any nontrivial prime $P \subsetneqq w_v$ is a completely prime ideal (since left ideals of Λ_v are idreals!). Moreover $S = \Lambda_v - P$ is an Ore set of Λ_v since for given $s \in D, \Lambda \in \lambda_v$ we have $s\lambda \in s\Lambda_v = \Lambda_{v^2}$ or $s\lambda = \lambda' s$ for some $\lambda' \in \Lambda_v$. Now $\Lambda_v \subsetneqq S^{-1}\Lambda_v$. It is clear that $S^{-1}\Lambda_v \neq \Delta$ since $(S^{-1}\Lambda_v)P$ is a proper ideal of $S^{-1}\Lambda_v$. Maximality of Λ_v thus entails $ht(w_v) = 1$. If $rk\Gamma > 1$ then Γ contains a convex subgroup C. Put $P = \{x \in \Lambda, v(a) \in C\}$. It is easily verified that P is a prime ideal of Λ_v and also $P \subsetneqq w_v$ because $C \neq \Gamma^+$, this would contradict $ht(w_v) = 1$. □

A slight extension of the final part of the foregoing proof yields also a proof of the following.

3.2.3 Observation

Nonzero prime ideals P of Λ_v correspond bijectively to nontrivial convex subgroups of Γ. Normal convex subgroups of Γ correspond to prime ideals of Λ_v which are invariant under inner automorphisms of Δ.

When studying valuations on $\mathbb{D}_n(K)$ one may restrict to abelian Γ. It is known that every valuation on a finite dimensional skewfield is abelian but for $\mathbb{D}_n(K)$ this result is somewhat surprising. They are in some sense very noncommutative rings, in fact they even contain free subalgebras of any countable rank! The result is due to J. Shtipel'man but we follow L. Makar-Limanov's proof.

3.2.4 Theorem

Let v be a Γ-valuation on $\mathbb{D}_1(K)$ then Γ is abelian.

Proof. Write $\mathbb{A}_1(K) = K < x, y > \subset \mathbb{D}_1(K) = K(< x, y >)$. Take $r \neq 0$ in $\mathbb{A}_1(K)$ and suppose that $v(xr) \neq v(rx)$, say $v(rx) > v(xr)$ (in the other case the proof is formally similar). Then for $[x, r] = xr - rx$ we have $v([x, r]) = v(xr)$. By an easy induction argument we then obtain: $v([x, -]^n(r)) = v(x^n r)$. For every $r \in \mathbb{A}_1(K)$ there is an $e = e(r)$ such that $[x, -]^e(r) = 0$ (because every $r \in \mathbb{A}_n(K)$ has a unique finite polynomial expression in x and y with powers in x before powers in y and $[x-]$ lowers the y-degree because $xy - yx = -1$). Thus we obtain $v(0) = v([x, -]^e r) = v(x^e r)$ but that is a contradiction since $x^e r \neq 0$. Since Γ is generated as a group by the semigroup $v(\mathbb{A}_1(K))$ it follows from $v(x) + v(r) = v(r) + v(x)$ that $v(x) \in Z(\Gamma)$. In fact the foregoing establishes that $v(f) \in Z(\Gamma)$ for every f such that every $r \in \mathbb{A}_1(K)$ is annihilated by some power of $[f, -]$, in particular this holds for all $f \in K[x]$. Since Γ is totally ordered $\gamma^m\sigma = \sigma\gamma^m$ for some m entails $\gamma\sigma = \sigma\gamma$ hence $Z(\Gamma)$ is root-closed in Γ. Now assume $r \in \mathbb{A}_1(K)$ is such that $v(r) \notin Z(\Gamma)$. Since $GK\dim(\mathbb{A}_1(K)) = 2$ it follows that for any $s \in \mathbb{A}_1(K)$ we have a relation: $\Sigma x_{ij} r^i s^j = 0$ with $x_{ij} \in K[x]$ (the GK dimension bounds the transcendence of the ring, so the r and s cannot be algebraically independent over $K[x]$). At least two monomials in this relation have the same valuation, otherwise $v(\Sigma x_{ij} r^i s^j)$ would necessarily be the valuation of the unique monomial in it having minimal valuation but that could not be equal to $-\infty$. Say $v(x_{i_0 j_0} r^{i_0} s^{j_0}) = v(x_{i_1 j_1} r^{i_s} s^{j_1})$, then either some $v(s^k) \in Z(\Gamma) < v(r) >$ or $v(r^l) \in Z(\Gamma) < v(s) >$ with k, l larger than zero, because $v(x_{ij}) \in Z(\Gamma)$ by foregoing remarks. In either case we obtain that some power of $v(r)$ commutes with some power of $v(s)$. Since Γ is totally ordered ($\gamma^n\sigma = \sigma\gamma^n$ entails $\gamma\sigma = \sigma\gamma$) it then follows that $v(r)$ and $v(s)$ commute. This holds for arbitrary $s \in \mathbb{A}_1(K)$, hence it contradicts $v(r) \notin Z(\Gamma)$. Consequently $Z(\Gamma) = \Gamma$ or Γ is abelian. $\square$

3.2.5 Corollary

Every valuation of $\mathbb{D}_n(K)$ is abelian.

Proof. $\mathbb{D}_n(K)$ is the n-fold tensor product of copies of $\mathbb{D}_1(K)$, its value group is a subgroup of a product of the value groups $v(\mathbb{D}_1(K))$ which is an abelian group. □

3.2.6 Remark and Project

The above proof is elementary except for the key result about the GK-dimension. For the general theory about GKdim we may refer to G. Krause and T. Lenagan, [34] or C. Năstăsescu and F. Van Oystaeyen [53]. It would be an interesting project to relate GKdim and valuation theory further, or perhaps the GKtd (Gelfand–Kirrilov transcendence degree) could be used instead of GKdim. The driving conjecture could be that for a skewfield of GK-dim$\Delta = n$ and a valuation v of Δ of rank m we would have $GK\dim\overline{\Delta} = n-m$, $\overline{\Delta}$, the residue skewfield of v. Also it seems possible to extend the foregoing theorem to skewfields obtained as skewfields of fractions of enveloping algebras of nilpotent Lie algebras.

3.2.7 Lemma

There are no discrete K-valuations of $D_1(K)$ with residue field K.

Proof. Suppose v is a discrete valuation of $\mathbb{D}_1(K)$ with valuation ring Λ_v and $\Lambda_v/w_v = K$. Write $w_v = (\pi)$. If $a, b \in \Lambda_v$ then for each $n \in \mathbb{N}$ there are polynomials $f(\pi)$ and $g(\pi)$ with coefficients in K such that:

$$v(a - f(\pi)) \geq n \text{ and } v(b - g(\pi)) \geq n$$

Since $f(\pi)$ and $g(\pi)$ commute we obtain that $ab - ba$ is in $f^v_{-n}\mathbb{D}_1(K)$ and this holds for all $n \in \mathbb{N}$. Since Λ_v cannot be commutative as it has $\mathbb{D}_1(K)$ for its quotient skewfield $ab \neq ba$ for some $a, b \in \Lambda_v$ and then $ab - ba$ is not in $f^v_{-N}\mathbb{D}_1(K)$ for some $N \in \mathbb{N}$. □

3.2.8 Lemma

For any Γ-valuation v on $\mathbb{D}_1(K)$ we have that $v([x, y]) > v(xy) = v(x) + v(y)$.

Proof. Since Γ is abelian $v(xy) = v(yx)$. Hence for the valuation filtration degree: $\deg\sigma_v(xy - yx) < \deg\sigma_v(xy)$. Therefore $v(xy - yx) > v(xy) = v(x) + v(y)$. □

3.2.9 Corollary

From $xy - yx = -1$ we obtain that $v(x) + v(y) < 0$ (see the foregoing lemma). If $v(x) > v(y)$, then $v(x+y) = v(y)$ and we may generate $\mathbb{A}_1(K)$ as $K < x+y, y >$. In other words we may assume that $\mathbb{A}_1(K)$ is generated by x and y with $[y,x] = 1$ and $v(x) = v(y) < 0$.

The valuation v_B corresponding to the Bernstein filtration factors over the principal symbol map $\sigma_B : \mathbb{D}_1(K) \to Q_g(K[X,Y])$. In fact there is only one such discrete valuation up to equivalence.

3.2.10 Proposition

If $\overline{v}$ is a discrete K-valuation of $K(X,Y)$ such that $\overline{v}\sigma_B$ is a nontrivial discrete K-valuation of $\mathbb{D}_1(K)$ then $\overline{v}\sigma_B$ is equivalent to the valuation v_B induced by the Bernstein filtration.

Proof. Suppose $a,b \in \mathbb{D}_1(K)$ are such that $\deg\sigma_B(b) < \deg\sigma_B(a)$. Then $\overline{v}(\sigma_B(a)) = \overline{v}(\sigma_B(a+b)) = \overline{v}\sigma_B(a+b) \geq \min\{\overline{v}\sigma_B(a), \overline{v}\sigma_B(b)\}$. If $\overline{v}\sigma_B(a) \neq \overline{v}\sigma_B(b)$, then equality holds, thus $\overline{v}\sigma_B(a) < \overline{v}\sigma_B(b)$. In particular when $\sigma_B(b)$ is homogeneous in $K(X,Y)$ of strictly negative degree, then $\overline{v}\sigma_B(b) \geq 0$. For every homogeneous element x of $K(X,Y)$ of degree zero (this is always σ_B of some element of $\mathbb{D}_1(K)$) in $Q^g(K[X,Y])$ we thus have $\overline{v}(x) = 0$. For $a \in \mathbb{D}_1(K)$ we have $\deg\sigma_B(a) = n$, then $\overline{v}\sigma_B(a) = \overline{v}(\sigma_B(ay^{-n})\sigma_B(y^n)) = 0 + n\overline{v}\sigma_B(y)$ as σ_B is multiplicative. Consequently $\overline{v}\sigma_B = (\overline{v}\sigma_B(y)).\deg\sigma_B$ hence $\overline{v}\sigma_B$ is equivalent to the Bernstein valuation defined by $\deg\sigma_B$. □

We write $K(\frac{X}{Y})$ for $G_B(\mathbb{D}_1(K))_0$ and let v be a discrete K-valuation on $\mathbb{D}_1(K)$ with $v(x) = v(y) < 0$. Let Λ_v be the valuation ring of v with maximal ideal w_v and residue skewfield Δ_v. From Lemma 3.2.8 we may derive that Δ_v is commutative; indeed if $a,b \in \Lambda_v - w_v$ then $v([a,b]) > v(a) + v(b) = 0$ or $ab - ba \in w_v$ and thus $\Delta_v = \Lambda_v/w_v$ is commutative. Consequently $G_v(\mathbb{D}_1(K))$ is commutative and of the form $\Delta_v[T,T^{-1}] \cong \Delta_v\mathbb{Z}$.

The valuation filtration $f^v\mathbb{D}_1(K)$ induces a filtration on $G_B(\mathbb{D}_1(K))_0 = K(\frac{X}{Y})$, $f_i^v(K(\frac{X}{Y})) = f_i^v\mathbb{D}_1(K) \cap \Lambda_B/(f_i^v\mathbb{D}_1(K) \cap w_B)$. This is an exhaustive filtration but it need not be separated.

3.2.11 Lemma

With notation as above, $f^vK(\frac{X}{Y})$ is separated if and only if $\sigma_B(z) = 1$ entails $v(z) \leq 0$. In case $f^vK(\frac{X}{Y})$ is not separated then $\cap_i f_i^vK(\frac{X}{Y}) = K(\frac{X}{Y})$.

Proof. Put $I = \cap_i f_i^v K(\frac{X}{Y})$. If $\sigma_B(x) \neq 0$ is in I then there exists a $y \in \mathbb{D}_1(K)$ such that: $\sigma_B(z) = \sigma_B(y)$ and $v(y) > v(z)$ (choose y in $f_i^v \mathbb{D}_1(K)$ for appropriate i.) Then $\sigma_B(yx^{-1}) = 1$ but $v(yz^{-1}) > 0$. Conversely if $z \in \mathbb{D}_1(K)$ is such that $\sigma_B(z) = 1$ and $v(z) > 0$ then for all $n \in \mathbb{N}$ we have: $1 = (1 - z^n) + z^n$. Thus $1 \in f_{-n}^v K(\frac{X}{Y})$, then for all $n \geq 0$ we obtain that $f_{-n}^v K(\frac{X}{Y}) = K(\frac{X}{Y}))$ as desired. □

From here on we assume that K is algebraically closed. We say that v is F^B-**compatible** if $f^v K(\frac{X}{Y})$ is a separated filtration, i.e. $\sigma_B(z) = 1$ yields $v(z) \geq 0$.

3.2.12 Proposition

A discrete K-valuation on $\mathbb{D}_1(K)$ that is F^B-compatible is determined by its restriction to the subfield $K(\frac{X}{Y})$ in $\mathbb{D}_1(K)$.

Proof. A pseudo-homogeneous element of $\mathbb{A}_1(K)$ is one having a homogeneous expressions in x and y (this is not unique since $yx = xy + 1$ but that is harmless here). Consider $f \in \mathbb{A}(K)$ and write $f = f_1 + f_2$ where f_1 is pseudo-homogeneous of degree $\deg\sigma_B(f)$ and $\deg\sigma_B(f_2) < \deg a_{\sigma_B}(f)$. From $\sigma_B(f/f_1) = 1$ it follows that $v(f) \leq v(f_1)$ since v is assumed to be F^B-compatible. Thus, $v(f) \geq \min\{v(f_1), v(f_2)$ with equality whenever $v(f_1) \neq v(f_2)$, yields $v(f) = \min\{v(f_1), v(f_2)\}$.

In case f is pseudo-homogeneous of degree n, say $f = \sum_{i=0}^n a_i x^i y^{n-i}$, then $v(fy^{-n}) = v(g)$ where we put $g = \sum_{i=0}^n a_i (xy^{-1})^i$. If for example $v(fy^{-n}) > v(g)$ then $\sigma_B(fy^{-n}/g) = 1$ and $v(fy^{-n}/g) > 0$ leads to a contradiction. Otherwise look at $\sigma_B(g/fy^{-n}) = 1$. Then $v(f) = v(g) + nv(y)$ and the proof is finished. □

The discrete K-valuations of $K(T)$, $T = \frac{X}{Y}$ are well-known i.e. v is either trivial or v corresponds to an $\alpha \in K$. In the first case v is equivalent to the valuation of the Bernstein filtration, meaning that for all $z \in \mathbb{A}_1(K)$, $v(z) = -\deg\sigma_B(z)v(y)$. By the foregoing proposition the F^B-compatible valuations are determined by three parameters $p = v(x) = v(y) \in \mathbb{Z} - \mathbb{N}$, $q = v(\frac{x}{y} - a) \in \mathbb{N} - \{0\}, a \in K^*$. Given p, q, a then there is at most one discrete K-valuation of $\mathbb{D}_1(K)$ compatible with the Bernstein filtration such that $v(x) = v(y) = p$ and $v(\frac{x}{y} - a) = q$. From foregoing remarks it follows that v, if it exists, may be calculated in the following way. To calculate v of $f \in \mathbb{A}_1(K)$ at first decompose f into pseudo-homogeneous elements, say $f = \sum_{i=0}^n f_i$ and put $v(f) = \min\{v(f_i), i = 0, \ldots, n\}$. To calculate v on a pseudo-homogeneous f_n, of degree n, put $v(f_n) = np + v_a(\sigma_B(f_n))q$ where v_a is the graded K=valuation on $K(\frac{X}{Y})[Y, Y^{-1}]$ such that $v_a(Y) = 0, v_a(\frac{X}{Y} - a) = 1$. It is also possible to define $v(f_n) = v'_a(\sigma_B(f_n))$ where v'_a is he graded valuation of $K(\frac{X}{Y})[Y, Y^{-1}]$ such that $v'_a(\frac{X}{Y} - a) = q, v'_a(Y) = p$. We observe that $K(\frac{X}{Y})[Y, Y^{-1}]$ is a graded field (gr-field) in the sense that every homogeneous element different from 0 is invertible; a gr-valuation is associated to a gr-valuation ring, being a graded subring such that for every homogeneous element of the gr-field, say z, either z

or z^{-1} is in the subring. A gr-valuation of $K(\frac{X}{Y})[Y, Y^{-1}]$ is always induced by a valuation of $K(X, Y)$ which is a graded valuation in the sense that $v(z) \leq 0$ if and only if $v(z_n) \leq 0$ for every homogeneous component n. Graded valuations are not studied in detail in this work, we refer to [38, 66].
We are now ready to prove the existence theorem.

3.2.13 Theorem

Let v be defined by p, q, a as above and assume that $q \leq -p$ then v is a discrete K-valuation of the Weyl field $\mathbb{D}_1, (K)$.

Proof. It suffices to verify the valuation properties on elements of $\mathbb{A}_1(K)$. If f, g are pseudo homogeneous of different degree then: $v(f+g) = \min\{v(f), v(g)\}$ holds by definition; if f, g have the same degree but $f+g \neq 0$, then $v(f+g) = np + v_a(\sigma_B(f+g))q \geq np + \min\{v_a(\sigma_B(f)), v_a(\sigma_B(g))\}q = \min(v(f), v(g)\}$. In the situation that f, g are not pseudo-homogeneous the relation follows by decomposition into pseudo-homogeneous elements and the definition of v.
For pseudo-homogeneous f and g we write:

$$f = \sum_{i=0}^{n} a_i x^i y^{n-i} \text{ and } \sigma_B(f) = \overline{f}$$

$$g = \sum_{i=0}^{m} b_j x^j y^{m-j} \text{ and } \sigma_B(g) = \overline{g}$$

$$fg = \sum_{i=0}^{m+n} \sum_{j=0}^{i} c_{ji} x^j y^{i-j}, h = \sum_{i=0}^{m+n} \sum_{j=0}^{i} c_{ji} X^j Y^{i-j}$$

For the Weyl algebras it is a well-known fact that:

$$h = \sum_{k=0}^{r} (-1)^k \frac{1}{k!} \frac{\partial^k \overline{f} \partial^k \overline{g}}{\partial X^k \partial Y^k}$$

where r is the integral part of $\frac{n+m}{2}$. It is now straightforward to calculate

$$v(fg) = \min\{v \sum_{j=0}^{i} c_{ji} x^j y^{i-j}, i = 0, \ldots, m+n\}$$

$$= \min\{ip + v_a(\sigma_B(\sum_{j=0}^{i} c_{ji} x^j y^{i-j}))q, i = 0, \ldots, m+n\}$$

$$= \min\{(n+m-2k)p + v_a\left(\frac{\partial^k \overline{f}\partial^k \overline{g}}{\partial Y^k \partial X^k}\right)q, k = 0, \ldots, r\}$$

$$= \min\{(n+m+2k)p + (v_a(\overline{f}) - k + v_a(g) - k)q, k = 0, \ldots, r\}$$

$$= (n+m)p + v_a(\overline{f}\overline{g})q + \min\{k(-2p-2q), k = 0, \ldots, r\}$$

$$= v(f) + v(g)$$

The last equality follows from $v_a\left(\frac{\partial^k f \partial^k g}{\partial y^k \partial x^k}\right) \geq v_a(f) - k + v_a(g) - k$. Now more generally, if $f = \sum_{i=0}^{n} f_i, g = \sum_{i=0}^{m} g_i$, then let k and l be maximal such that we have $v(f) = v(f_k), v(f_k), v(g) = r(g_l)$. Then $v(fg) \leq v(\sum_{j=0}^{k+l} f_j g_{k+l-j}) = v(f_k g_l) = v(f_k) + v(g_l) = v(f) + v(g)$.

The other inequality follows from: $v(\sum_{j=0}^{i} f_j g_{i-j}) \geq v(f) + v(g)$ for all $i = 0, \ldots, k+l$. □

It is clear from foregoing proof that v will not be a valuation if $q > -p$, so we suppose hereafter that $q \leq -p$. The valuation filtration f^v defines a commutative associated graded ring and we can calculate this explicitly. For f and g as in the proof we have:

$$fg - gf = \sum_{i,j} d_{ij} x^i y^j$$

Again it follows that:

$$h = \sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k!} \left(\frac{\partial^k \overline{f} \partial^k \overline{g}}{\partial X^k \partial Y^k} - \frac{\partial^k \overline{f} \partial^k \overline{g}}{\partial Y^k \partial X^k} \right)$$

r being the integral part of $\frac{n+m}{2}$. Hence we obtain: $v(fg - gf) = \min\{(n+m-2k)p + v_a(\frac{\partial^k \overline{f}}{\partial X^k}\frac{\partial \overline{g}}{\partial Y^k} - \frac{\partial^k \overline{f}}{\partial Y^k}\frac{\partial^k \overline{g}}{\partial X^k})q, k = 1, \ldots, r\}$.

Now: $v_a\left(\frac{\partial^k \overline{f}}{\partial X^k}\frac{\partial \overline{g}}{\partial Y^k} - \frac{\partial^k \overline{g}}{\partial Y^k}\frac{\partial^k \overline{g}}{\partial X^k}\right) > v_a(\overline{f}) + v_a(\overline{g}) - 2k$, thus $v(fg - gf) > v(f) + v(g)$!

The residual field of an F^B-compatible K-valuation does not only turn out to be commutative, it is actually a purely transcendent extension of degree one of K. From a transcendence (Gelfand–Kirillov) argument it would follow that the transcendent degree is one but for pure transcendence some arithmetical information is needed in the proof.

3.2.14 Theorem

Let $q \leq -p$ and v given by p, q, a. The residue field $\Delta_v \cong K(t)$ with $t = (\frac{x}{y} - a)^l y^k$, where $-kp = lq$ is the least common multiple of $-p$ and q.

Proof. First we observe that v is a $K(t)$ valuation because $v(t-\beta)=0$ for all $\beta \in K$ and K is algebraically closed. Indeed, it is clear that $v(t-\beta) \geq 0$, the leading pseudo homogeneous term of $t-\beta$ is equal to the one of t which is equal to: $\sum_{i=0}^{l}\binom{l}{i}(-a)^{l-i}x^i y^{k-i}$. It follows that at least one of the pseudo-homogeneous terms in the decomposition of $t-\beta$ must have v-value equal to zero, hence $v(t-\beta)=0$. Next we show for $f,g \in \mathbb{A}_1(K)$ such that $v(f)=v(g)$ there is an $h \in K(t)$ such that $v(g^{-1}f-h)>0$. Since $v(f)$ is obtained as $v(f_i)$ for some pseudo homogeneous part f_i of f we may assume that f is pseudo-homogenous. Take F,G in $\mathbb{A}_1(K)$ such that $fG=gF$; there is a pseudo-homogenous $A \in \mathbb{A}_1(K)$ such that $v(A)=v(G)=v(F)$. If there are F_1, G_1 in $K(t)$ such that $v(FA^{-1}-F_1)>0$ and $v(GA^{-1}-G_1)>0$ then $g^{-1}f-F_1G_1^{-1}=g^{-1}(fG_1-gF_1)G_1^{-1}=g^{-1}(g(FA^{-1}-F_1)-f(GA^{-1}-G_1))G_1^{-1}$. Consequently: $v(g^{-1}f-F_1G_1^{-1})>-v(g)+v(g)-v(G_1)=0$. From the foregoing it follows that we may select h such that $v(g^{-1}f-h)>0$ with $v(f)=v(g)$ and both f,g are assumed to be pseudo-homogenous. We arrive at:

$$v(f)=p\deg\sigma_B(f)+qv_a(\sigma_B(f))$$
$$v(g)=p\deg\sigma_B(g)+qv_a(\sigma_B(g))$$

The integer $(\deg\sigma_B(f)-\deg\sigma_B(g))(-p)=v_a(\sigma_B(f(g))q$ is a common multiple of $-p$ and q. Thus we obtained an $n \in \mathbb{Z}$ such that: $nk=\deg\sigma_B(f)-\deg\sigma_B(g), nl=v_a(\sigma_B(f/g))$.

We now calculate $\sigma_B(\frac{f}{g}y^{-nk})=(T-a)^{nl}h(T)$ with $h(T)\in K(T), T=\frac{X}{Y}$, and $\gamma=h(a)\neq 0$.

If $\sigma_B(f/g)\neq\sigma_B(\gamma t^n)$, then $v(\frac{f}{g}-\gamma t^n)=nkp+v_a((h(T)-\gamma)(T-a)^{nl})q$, and the latter is strictly bigger than $nkp+nlq=0$. In case $\sigma_B(f/g)=\sigma_B(\gamma t^n)$ we may assume that $\gamma=1$ and $n\in\mathbb{N}$. Clearly, then $t^n=\sum_{i=0}^{ln}\binom{ln}{i}(-a)^{ln-i}x^i y^{kn-i}$ + terms having a strictly positive value.

Since both f and the term $\sum_{i=0}^{ln}\binom{ln}{i}(-a)^{ln-i}x^i g y^{kn-i}$ are pseudo homogeneous and have the same image under the principal symbol map they must be equal! (in the Weyl algebra every element has a unique pseudo-homogeneous decomposition with powers of x preceding powers of y). Hence, modulo terms with value strictly larger than $v(f)=v(g)$ we obtain:

$$f-gt^n=\sum_{i=0}^{ln}\binom{ln}{i}(-a)^{ln-i}[x^i,g]y^{kn-i}$$

If $g=\sum_{j=0}^{m}a_jx^{m-j}y^j$ then $[x^i,g]$ is equal to

$$\sum_{j=0}^{m}a_j\left(\sum_{k=0}^{\min(i,j)}(-1)^k\frac{i!}{(i-k)!}\binom{j}{k}x^{m-j+i-k}y^{j-k}\right)$$

Therefore: $v(f - gt^n - \sum_{k=1}^{\min(lnm)} R_k > v(f) = v(g)$, where

$$R_k = (-1)^k \sum_{i=k}^{l_n} \binom{ln}{i} (a)^{ln-i} \frac{i!}{(i-k)!} \left(\sum_{j=k}^{m} a_j \binom{j}{k} x^{m-j+i-k} y^{kn-i+j-k} \right)$$

Calculate:

$$v(R_k) = (kn + m - 2k)p + v_a \left(\sum_{i=k}^{ln} \binom{ln}{i} (-a)^{ln-i} \frac{i!}{(i-k)!} T^{i-k} \right) q$$
$$+ v_a \left((-1)^k \sum_{j=k}^{m} a_j \binom{j}{k} T^{m-j} \right) q$$

The first value we need to know is $ln - k$ since we recognize the k^{th}-derivative of $(T-a)^{ln}$. The second value equals:

$$\begin{aligned} & v_a \left(\frac{1}{k!}(-1)^k \sum_{j=k}^{m} a_j \frac{j!}{(j-k)!} T^{-j-1} \right) \\ &= v_a \left(\sum_{j=k}^{m} a_j (k-j-1)\dots(-j+1)(-j) T^{-j-1} \right) \\ &= v_a \left(\frac{a^k}{dT^k} \left(\sum_{i=0}^{m} a_j T^{k-j-1} \right) \right) \\ &\geq v_a \left(\sum_{j=0}^{m} a_j T^{-j} \right) - k = v_a(\sigma_B(g)) - k \end{aligned}$$

So we obtain for all k that: $v(R_k) = v(g) - 2k(p+q)$ which is strictly larger than $v(g)$ if $p + q < 0$. Thus if $p + q < 0$ then $v(\frac{f}{g} - G^n) > 0$.

For the case where $p + q = 0$ and $\sigma_B(f/g) = \sigma_B(t^n)$ we finish the proof by induction on n. Observe that $t = x - ay$ and $k = l - 1$. If $n = 0$ then $\sigma_B(f) = \sigma_B(g)$ and thus $f = g$ since both f and g are pseudo homogeneous. Then suppose $\sigma_B(f/g) = \sigma_B(t^n)$. Previous calculation establishes that: $\frac{f}{g} - t^n = g^{-1} \left(\sum_{k=1}^{\min(n,m)} R_k \right)$ plus terms of strictly positive value.

For each k there is a $\gamma_k \in K(t)$ such that: $v(g^{-1}R_k - \gamma_k) > 0$ either by induction in case $\sigma_B(g^{-1}R_k) = \sigma_B(\beta t^{n-2k})$ for some $\beta \in K^*$ or by the first part of the proof in case $\sigma_B(g^{-1}R_k) \neq \sigma_B(\beta t^{n-2k})$ for any $\beta \in K^*$. Finally we arrive at:

$$\left(t^n + \sum_{k=1}^{\min(n,m)} \gamma_k\right) \in K(t) \text{ and } v(\frac{f}{g} - (t^n + \sum_{k>1}^{\min(n,m)} \gamma_k)) > 0$$

□

3.3 Some Divisor Theory for Weyl Fields Over Function Fields

In this section we let K be an algebraic function field of degree one over an algebraically closed $k \subset K$ of characteristic zero, i.e. K is the function field of a nonsingular projective curve C over k.

Points on the curve C correspond bijectively to the discrete k-valuations of K and each such valuation induces a valuation filtration f^vK on K. This filtration extends to $\mathbb{A}_1(K)$ and to $f^v\mathbb{D}_1(K)$ as observed earlier. The associated graded ring of $f^v\mathbb{D}_1(K)$ is exactly $\mathbb{D}(k)[T, T^{-1}], T$ a central variable. Hence $f^v\mathbb{D}_1(K)$ is a valuation filtration corresponding to a discrete noncommutative valuation ring $f_0^v\mathbb{D}_1(K)$. In a sense the constant field k is now replaced by $\mathbb{D}_1(k)$ but we will point out some essential new features related to this "skewfield of constants".

3.3.1 Proposition

If v is a $\mathbb{D}_1(k)$-valuation of $\mathbb{D}_1(K)$, then:

for $a_{ij} \in K, i = 0\ldots, n, j = 0, \ldots, m$:

$$v\left(\sum_{i,j=0}^{n,m} a_{ij}x^i y^j\right) = \min\{v(a_{ij}); i = 0, \ldots, n, j = 0, \ldots, m\}$$

Proof. Write $\underline{a} = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}x^i y^j$, $p = \min\{v(a_{ij}), i = 0, \ldots, n, j = 0, \ldots, m\}$.

Since $x, y \in \mathbb{D}_1(k)$ the equality $v(\underline{a}) \geq p$ is obvious. Let $\delta(\underline{a}) \in \mathbb{N}$ be the filtration degree of $\underline{a}$ in the Bernstein filtration of $\mathbb{A}_1(K)$. If $\delta(\underline{a}) = 0$ then the claim holds. So we assume that the claim holds for $\underline{b} \in \mathbb{A}_1(K)$ with $\delta(\underline{b}) < d$. We calculate:

$$\begin{aligned} v(x\underline{a} - \underline{a}x) &= v\left(\sum_{i=0}^{n}\sum_{j=0}^{m} ja_{ij}xy^{j-1}\right) \\ &= \min\{v(a_{ij}), j \neq 0\} \end{aligned}$$

$$v(y\underline{a} - \underline{a}y) = v\left(-\sum_{i=0}^{n}\sum_{j=0}^{m} i a_{ij} x^{i-1} y^j\right)$$
$$= \min\{v(a_{ij}, i \neq 0\}$$

Now $v(x\underline{a} - \underline{a}x) \geq v(\underline{a}) \geq p$ and similarly $v(v\underline{a} - \underline{a}y) \geq p$. The foregoing implies that $v(\underline{a})$ is smaller than $\min\{v(y\underline{a} - \underline{a}y), v/(\underline{a} - \underline{a}y)\}$, hence $v(\underline{a}) \leq p$ or $v(\underline{a}) = p$ follows. □

Since we assume that k is algebraically closed in K we have that $k = \cap_v\{O_v, O_v$ a discrete valuation ring of $K\}$. We may look at the ring R obtained as the intersection $\cap_v \Lambda_v$ for all discrete $\mathbb{D}_1(k)$-valuation rings of $\mathbb{D}_1(K)$. For a k-valuation O_v of K we have a nontrivial $\mathbb{D}_1(k)$-valuation Λ_v extending v to $\mathbb{D}_1(K)$ it is given by v on $\mathbb{A}_1(K)$ as in the foregoing proposition.

We have that $\mathbb{D}_1(k) \subset R$ but R is not equal to $\mathbb{D}_1(k)$. Look at $(X + a)^{-1}$ since $v(X+a)$ is at most zero we have that $v(X+a)^{-1}$ is at least zero, hence $(X+a)^{-1} \in R$ for every $a \in K - k$. Clearly $(X + a)^{-1} \in R$ is not invertible in R so R is not a skewfield, yet it has many invertible elements e.g. $(ax + y)(x + ay)^{-1}$. As an intersection of valuation rings R has the property that every one-sided ideal of R is an ideal of R, moreover R is invariant under inner automorphism of $\mathbb{D}_1(K)$. A formal sum $D = \sum'_{v \in C} n_v v$ with $n_v \in \mathbb{Z}$ almost all being zero, is called a **divisor** of C. When we chose Λ_v to represent v of C (C the curve of $K(k)$) we define a **divisor for** $\mathbb{D}_1(K)$ as $\sum'_v n_v v$. To an element $q \in \mathbb{D}_1(K)$ we associate a principal divisor: $\mathrm{div}(q) = \sum_v v(q)v$. Divisors for $\mathbb{D}_1(K)$ are partially ordered by: $\sum'_v n_v v > \sum'_v m_v r$ if and only if $n_v \geq m_v$ for all v. So the ring R may be obtained by putting $R = \{q \in \mathbb{D}_1(K), \mathrm{div}(q) \geq 0\}$. A divisor for $\mathbb{D}_1(K), D$ is said to be **positive** if $D > 0$.

3.3.2 Lemma

A positive divisor is principal, i.e. if $D > 0$ then $D = \mathrm{div}(r)$ for some $r \in R$.

Proof. It will be sufficient to establish that for each $v \in C$ there is an $r \in R$ such that $\mathrm{div}(r) = v$, that is $v(r) = 1$ and $w(r) = 0$ for every $w \neq v$ in C. Take $a \in K$ such that $v(a) = -1$ and write $\mathrm{div}(a) = D_1 - D_2$ with D_1 and D_2 being both positive divisors. The Riemann–Roch theorem on C yields the existence of an integer N such that for any divisor D on C of degree (being the sum of the n_v appearing in the divisor) larger than N we have: $\dim_k L(D) = \deg D + 1 - g$, where $L(D) = \{f \in K, \mathrm{div}(f) + D > 0\}$ and g is the genus of the curve C. Fix a positive divisor D_3 of degree larger than N such that every valuation with nonzero coefficient in D_2 appears with zero coefficient in D_3 (note: v appears in D_2). Then it is easily checked that:

$$\dim_k L(D_3 + v) = 1 + \dim_k L(D_3)$$

So there must be a nonzero b in $L(D_3 + v) - L(D_3)$. Consequently $v(b) = -1 = v(a)$; moreover for all $w \in C - \{v\}$ we have that $\max\{w(a), w(b)\} \geq 0$. Now define: $r = (a^{-1}x + b^{-1})(a^{-1}x + b^{-1} + y)$ in $\mathbb{D}_1(K)$. If w is a k-valuation of K different from v, then:

$$\begin{aligned} w(a^{-1}x + b^{-1}) &= \min\{w(a^{-1}), w(b^{-1})\} \\ &= -\max\{w(a), w(b)\} \\ w(a^{-1}X + b^{-1} + y) &= \min\{w(a^{-1}), w(b^{-1}), 0\} \\ &= -\max\{w(a), w(b)\} \end{aligned}$$

Thus from $w \neq v$ we obtain $w(r) = 0$. On the other hand

$$\begin{aligned} &v(a^{-1}x + b^{-1}) = \min\{1, 1\} = 1 \\ &v(a^{-1}x + b^{-1} + y) = \min\{1, 1, 0\} = 0 \end{aligned}$$

Hence $v(r) = 1$ and $\mathrm{div}(r) = v$ as desired. □

This leads to a rather beautiful structure result on R.

3.3.3 Theorem

The ring R is a principal ideal domain.

Proof. As a first step we establish that the sum of two cyclic ideals is again cyclic. Hence consider Ra and Rb. For all $v \in C$ and $r, r' \in R$ we have: $v(ra + r'b) \geq \min\{v(ra), v(r'b)\} \geq \min\{v(a), v(b)\} = v(c)$ for some $c \in R$. Thus $ra + r'b \in Rc$ or $Ra + Rb \subset Rc$. Write $a = fg^{-1}, b = f'g^{-1}$ with f, f', g in $\mathbb{A}_1(K)$.

Let us fix an integer n larger than the Bernstein filtration degree of f and put: $b' = x^n f' g^{-1}$. Then $Rb = Rb'$ and

$$\begin{aligned} v(a + b') &= v(f + x^n f') - v(g) \\ &= \min\{v(f), v(f')\} - v(g) \\ &= \min\{v(a), v(b)\} = v(c) \end{aligned}$$

Consequently: $Rc \subset R(a + b') \subset Ra + Rb' \subset Ra + Rb$, hence $Ra + Rb = Rc$. From the first step it follows that every finitely generated ideal is cyclic and for every $b \notin Ra$ we have $Ra + Rb = Rc$ with $0 < \mathrm{div}(c) < \mathrm{div}(a)$. Consequently any ascending chain of (left, right) ideals must terminate thus R is Noetherian and then every ideal is finitely generated (on the left) hence principal.

3.3.4 Corollary

The ring R is a Noetherian domain with quotient division ring $\mathbb{D}_1(K)$.

Proof. As observed in the theorem R is a Noetherian domain hence it has a classical ring of fractions which is a skewfield. If $q \in \mathbb{D}_1(K)$, then $\operatorname{div}(q) = D_1 - D_2$ for positive divisors D_1, D_2. From the lemma it follows that there are r_1 and r_2 in R such that $\operatorname{div}(r_1) = D_1, \operatorname{div}(r_2) = D_2$. It follows that $qr_2r_1^{-1} \in R$ is invertible in R as it has the zero divisor for its divisor. Therefore $q \in \mathbb{Q}_{\text{cl}}(R)$ and $\mathbb{D}_1(K) = \mathbb{Q}_{\text{cl}}(R)$ follows. □

To a divisor D on $\mathbb{D}_1(K)$ we associate a space $\mathcal{L}(D) = \{q \in \mathbb{D}_1(K), \operatorname{div}(q) + D > 0\}$. In particular $\mathcal{L}(0) = R$ and each $\mathcal{L}(D)$ is an R-bimodule. The ideals of R are exactly the $\mathcal{L}(D)$ with $D < 0$. The theorem states that any $\mathcal{L}(D) \cong R$ as an R-bimodule. For $D_1 < D_2$ we have $\mathcal{L}(D_1) \subset \mathcal{L}(D_2)$. We let $\Lambda_v = F_0^v\mathbb{D}_1(K)$ be the discrete valuation ring of $\mathbb{D}_1(K)$ corresponding to $F^v\mathbb{D}_1(K)$ and we write $\pi_v \subset \Lambda_v$ for its unique maximal ideal, $m_v = R \cap \pi_v$.

Observe that $m_v = \mathcal{L}(-v)$ is a maximal ideal of R and the correspondence $v \in C \to m_v \subset R$ defines a bijective correspondence between points of C and the set of maximal ideals $\Omega(R)$ of R. The following expands on this relation.

3.3.5 Proposition

With notation as above: $\Lambda_v = R_{m_v}$, the localization of R at the maximal ideal m_v.

Proof. For $q \in \Lambda_v$ write $\operatorname{div}(q) = D_1 - D_2$ where D_1 and D_2 are positive divisors having disjoint supports (no valuation appears with a nonzero coefficient in both D_1 and D_2, this is of course always possible). Again we find r_1 and r_2 in R such that $\operatorname{div}(r_1) = D_1, \operatorname{div}(r_2) = D_2$. Consequently $q = ur_1r_2^{-1}$ for some unit u of R. Since $v(q) \geq 0$ we cannot have $r_2 \in m_v$, hence $q \in R_{m_v}$. On the other hand the $R_{m_v} \subset \Lambda_v$ is obvious so equality follows: □

3.3.6 Theorem

If $D_1 < D_2$ then $\dim_{\mathbb{D},(k)}(\mathcal{L}(D_2)/\mathcal{L}(D_1))$ equals the degree of the divisor $D_2 - D_1$.

Proof. Form the foregoing proposition it follows that π_v is the extension of $m_v \subset R$ to Λ_v and $R/m_v \to \Lambda_v/\pi_v$ is an isomorphism. In particular $R/m_v \cong \mathbb{D}_1(k)$ and $R = \mathbb{D}_1(k) + m_v$. Let us write F^vR for the filtration induced on R by $F^v\mathbb{D}_1(K)$. For the associated graded rings we have: $G_v(\mathbb{D}_1(K)) \cong \mathbb{D}_1(k)[T, T^{-1}]$ and we may restrict this isomorphism to the associated graded ring of F^vR and obtain $G_v(R) \cong D_1(k)[T^{-1}]$. It is straightforward to verify for every $v \in C$ and every divisor D

on $\mathbb{D}_1(K)$ we have: $\mathcal{L}(D) = m_v\mathcal{L}(D+v)$. Then it follows from this that $\mathcal{L}(D+v)/\mathcal{L}(D) \cong R/m_v \otimes_R \mathcal{L}(D+v) \cong R/m_v \cong \mathbb{D}_1(k)$. For $D_1 < D_2$ the left dimension over $\mathbb{D}_1(k)$ of the space $\mathcal{L}(D_2)/\mathcal{L}(D_1)$ may thus be counted as the degree of the divisor $D_2 - D_1$ (note that in a similar way this degree also equals the right $\mathbb{D}_1(k)$-dimension, so that left and right dimension of the bimodule $\mathcal{L}(D_2)/\mathcal{L}(D_1)$ are actually equal in this situation). □

The foregoing Riemann–Roch type theorem is independent of the genus of C, the formula proved actually corresponds to stating that this is a "genus-less" situation, a remark that may be related to the noncommutative geometry of $\mathbb{A}_1(K)$.

If we consider the Bernstein filtration then $\mathbb{D}_1(k)$ is not in $F_0^B R = F_0^B \mathbb{D}_1(K) \cap R$. We may compare the Bernstein filtrations on $\mathbb{D}_1(K)$ and R; in some sense R is a rather big subring of $\mathbb{D}_1(K)$, perhaps unexpected for the intersection of all $\mathbb{D}_1(k)$-valuation (discrete) rings of $\mathbb{D}_1(K)$.

3.3.7 *Proposition*

With respect to the Bernstein filtrations: $G_B(R) = G_B(\mathbb{D}_1(K))$.

Proof. It is known that $G_B(\mathbb{D}_1(K))$ is just the graded quotient field of $G_B(\mathbb{A}_1(K)) = K[X,Y]$. Consider p in $K[X,Y]_m$, $p = \sum_{i=0}^m a_i XY^{m-i}$ and look at $p^{-1} \in G_B(\mathbb{D}_1(K))$. Consider $q = \sum_{i=0}^m a_i xy^{m-i} \in \mathbb{A}_1(K)$ and write $\operatorname{div}(q) = D_1 - D$ with D_1 and D_z positive. Pick $r \in R$ such that $\operatorname{div}(r) = D_2$, say $r = fg^{-1}$ with $f, g \in \mathbb{A}_1(K)$ and let $n \in \mathbb{N}$ be larger than the degree of g in the Bernstein filtration. Now put $a = (rx^n q))(rx^n + 1)^{-1}$. Obviously $\sigma(a) = p$. For all $v \in C$ we obtain: $v(rx^n + 1) = \min\{v(r), 0\}$ because of the choice of n. Then we obtain $\operatorname{div}(a) = D_1$ and thus $a \in R$. Finally, if $m > 0$ then $r' = (q+1)^{-1}$ is in R and we have $\sigma(r') = p^{-1}$. Thus $Q_{\mathrm{Cl}}^g(K[X,Y]) \subset G(R)$ and also $G(\mathbb{D}_1(K)) = Q_{\mathrm{Cl}}^g(K[X,Y])$ entail $G(R) = G(\mathbb{D}_1(K))$. □

The results may be generalized to $\mathbb{D}_n(k)$-valuations of $\mathbb{D}_n(K)$ but we do not go into this here leaving it as an exercise for the zealous reader.

3.4 Hopf Valuation Filtration

The guiding principle in foregoing sections is that an extension of valuation theory to K-algebras can be obtained from a value function on A extending a valuation v of K with corresponding filtrations FA, resp. fK. The ring F_oA, where 0 is the neutral element of the value group Γ, is an order in A over O_v, and it enjoys certain properties like being a separated prime, a Dubrovin valuation, a noncommutative valuation ring, depending on properties of the value function. However we may look at a strong filtration FA and ask other structural properties of A possibly in combination with some properties of the value function, e.g. we may look at Hopf algebras or quantum groups A and ask F_0A to be also a Hopf algebra over O_v. In this way we shall define Hopf valuations and their filtrations.

3.4.1 Definition. Good Γ-Filtrations on K-Vector Spaces

Consider a field K with a separated Γ-filtration fK for some totally ordered group Γ. The category of Γ-filtered vector spaces over K is denoted by K-filt. A Γ-filtration on a K-vector space V, FV is a **good filtration** if there exist sets $\{v_\alpha, \alpha \in \mathcal{A}\}, \{\gamma_\alpha \in \Gamma, \alpha \in \mathcal{A}\}$ such that for $\gamma \in \Gamma$ we have: $F_\gamma V = \sum_{\alpha \in \mathcal{A}} f_{\gamma - \gamma_\alpha} K v_\alpha$. It is clear that $\{v_\alpha, \alpha \in \mathcal{A}\}$ is a set of K-generators for V. In the sequel fK will be a strong filtration, in fact a valuation filtration. Then from $f_{\gamma-\gamma_\alpha} K v_\alpha \in F_\gamma V$ it follows that $v_\alpha \in F_{\gamma_\alpha} V$ for all $\alpha \in \mathcal{A}$; moreover, for every $\gamma \in \Gamma$ we also have that $F_\gamma V = f_\gamma K F_0 V$. Hence, if FV is a good filtration then without loss of generality we may assume that $\{v_\alpha, \alpha \in \mathcal{A}\}$ is taken in $F_0 V$ and for $\gamma \in \Gamma$, $F_\gamma V = \sum_{\alpha \in \mathcal{A}} f_\gamma K v_\alpha$. If $F_0 V$ is free over $F_0 K$ with basis $\{w_i, i \in \mathcal{J}\}$ then FV may be given by $F_\gamma V = \sum_{i \in \mathcal{J}} f_\gamma K w_i$. If fK is a valuation filtration then it is strong and $f_0 K = O_v$ is a valuation ring so torsion free finitely generated O_v-modules will be free.

For detail on Hopf algebras we refer to [20, 31]. We let H be a K-Hopf algebra with counit $\varepsilon : H \to K$, comultiplication $\Delta : H \to H \otimes H$ and antipode $S : H \to H$. A **Γ-filtered Hopf algebra** is a K-Hopf-algebra H with a filtration FH such that ε, Δ, S are filtered morphisms, e.g.

1. $\varepsilon(F_\gamma H) \subset F_\gamma K = f_\gamma K$, for all $\gamma \in \Gamma$.
2. $S(F_\gamma H) \subset F_\gamma H$, for all $\gamma \in \Gamma$.
3. $\Delta(F_\gamma H) \subset \sum_{\sigma+\tau=\gamma} F_\sigma H \otimes F_\tau H$, for all $\gamma \in \Gamma$.

The condition (3) just expresses that Δ is a filtered morphism if $H \otimes H$ is equipped with the tensor filtration defined by putting

$$F_\gamma(H \otimes H) = \sum_{\sigma+\tau=\gamma} F_\sigma H \otimes F_\tau H, \gamma \in \Gamma.$$

We write $F_0^0 H = \sum_{\gamma<0} F_\gamma H$, $F_\tau^0 H = \sum_{\gamma<\tau} F_\gamma H$ for $\tau \in \Gamma$.

3.4.2 Proposition

Let H be a Hopf algebra over K with Hopf filtration FH, then $G(H)$ is a Γ-graded Hopf algebra. If FH extends fK then $G(H) = k\Gamma \otimes_k F_0 H$ with Hopf structure deriving from $F_0 H$ (via $F_0 H / F_0^0 H$) making it into a graded Hopf algebra over the gr-field $k\Gamma$ (where k is the residue field of K).

Proof. Observe that F_0H is a sub-Hopf algebra of H over $f_0K = O_v$. Indeed, $\Delta(F_0H) \subset \sum_{\gamma\in\Gamma} F_\gamma H \otimes F_{-\gamma}H$ but since $F_\gamma H = f_\gamma K \otimes F_0H$ for all $\gamma \in \Gamma$, it follows that $\Delta(F_0H) \subset F_0H \otimes F_0H$; the restriction of ε to F_0H defines the O_v-linear $\varepsilon|F_0H$ and the $k\Gamma$-linear $\overline{\varepsilon}$ on $G(H)$ extending $F_0H/F_0^0H \to K$. Since $S|F_0H$ defines the $k\Gamma$-linear $\overline{S} : G(H) \to G(H)$, all claims in the proposition follows easily. □

Note that for a Hopf filtration FH the inclusion $K \hookrightarrow H$ is a filtered morphism; for a strong Hopf filtration FH the condition of extending fK is equivalent to $F_0H \cap K = f_0K$.

Now we consider a Hopf algebra H over K with a Γ-valuation ring $O_v = D$ in K; the valuation of O_v is $v : K^* \to \Gamma$ and we also write $v : K \to \Gamma \cup \{\infty\}$ by putting $v(0) = \infty$. The residue field of v will be denoted by k.

A **Hopf valuation function** extending v is a function $-\xi : H \twoheadrightarrow \Gamma \cup \{\infty\}$, usually viewed as a **Hopf valuation filtration function** $\xi : H \twoheadrightarrow \Gamma \cup \{-\infty\}$, satisfying:

HV.1 We have $\xi(h) = -\infty$ if and only if $h = 0$.
HV.2 We have $\xi(1) = 0$.
HV.3 For $h \in H, \lambda \in K, \xi(\lambda h) = \xi(\lambda h) = \xi(h) - v(\lambda)$.
HV.4 For $g, h \in H, \xi(gh) \le \xi(g) + \xi(h)$.
HV.5 For $g, h \in H, \xi(g + h) \le \max\{\xi(g), \xi(h)\}$.
HV.6 For $h \in H, \xi(S(h)) \le \xi(h), \xi(\varepsilon(h)) \le \xi(h)$.
HV.7 For $h \in H$, $\Delta(h) = \Sigma h_1 \otimes h_2$ (Sweedler notation) $\xi(h) \ge \inf\{\max_\Sigma\{\xi(h_1) + \xi(h_2)\}\}$, where $\max_\Sigma$ is taken over the terms in a fixed expression of $\Delta(h)$ while inf is over all possible decompositions of $\Delta(h)$. By $\xi(h) \ge \inf\{\gamma, \gamma \in \mathcal{A} \subset \Gamma\}$ we just mean that if $\sigma \le \gamma$ for $\gamma \in \mathcal{A}$ then $\xi(h) \ge \sigma$.

3.4.3 Proposition

If ξ is a Hopf valuation function then we have equality in HV.7, in fact:

$$\begin{aligned}\xi(h) &= \inf\{\max_\Sigma\{\xi(h_1) + \xi(h_2)\}\} = \inf\{\max_\Sigma\{\xi(\varepsilon(h_1)) + \xi(h_2)\}\} \\ &= \inf\{\max_\Sigma|\xi(h_1) + \xi(\varepsilon(h_2))\}\end{aligned}$$

Proof. Indeed, from $\Delta(h) = \Sigma h_1 \otimes h_2$ we may derive: $h = \Sigma\varepsilon(h_1)h_2 = \Sigma h_1\varepsilon(h_2)$. Applying HV.3 leads to:

$$\xi(h) \le \max_\Sigma\{\xi(\varepsilon(h_1)h_2)\} = \max_\Sigma\{\xi(\varepsilon(h_1)) + \xi(h_2)\} \qquad (*)$$

Since (*) holds for any decomposition of $\Delta(h)$ we obtain:

$$\xi(h) \le \inf\{\max_\Sigma\{\xi(\varepsilon(h_1)) + \xi(h_2)\}\}$$

and by this we just mean that $\xi(h) \leq \max_\Sigma(\ldots)$ for all possible decompositions of $\Delta(h)$. Now using HV.6 this leads to: $\xi(\varepsilon(h_1)) + \xi(h_2) \leq \xi(h_1) + \xi(h_2)$, or $\xi(h) \leq \inf\{\max_\Sigma(\xi(\varepsilon(h_1)) + \xi(h_2)\}$. This proves:

$$\begin{aligned}\xi(h) &= \inf\{\max_\Sigma\{\xi(h_1) + \xi(h_2)\}\} \\ &= \inf\{\max_\Sigma\{\xi(\varepsilon(h_1)) + \xi(h_2)\}\} \\ &= \inf\{\max_\Sigma\{\xi(h_1) + \xi(\varepsilon(h_2))\}\}\end{aligned}$$

where now we may interpret inf in the classical way, because $\xi(h)$ is smaller than all $\max\{\ldots\}$ and bigger than all $\sigma \in \Gamma$ smaller than all $\max_\Sigma\{\ldots\}$. Hence HV.7 entails the existence of the inf as defined. The last equality following by using $h = \Sigma h_1 \varepsilon(h_2)$. □

3.4.4 Theorem

Hopf filtration functions $\xi : H \to \Gamma \cup \{-\infty\}$, satisfying HV.1,…,HV.7, correspond bijectively to the separated Hopf filtrations FH extending the valuation filtration fK of the valuation v.

Proof. Start from a Hopf valuation filtration function $\xi : H \twoheadrightarrow \Gamma \cup \{-\infty\}$ satisfying HV.1,…,HV.7. For $\gamma \in \Gamma$ put $F_\gamma H = \{h \in H, \xi(h) \leq \gamma\}$ Properties HV.5 and HV.3 entail that $F_\gamma H$ is an additive subgroup of H, containing 0 because of HV.1. From HV.2, HV.4 and HV.5 it follows that FH is a filtration of the ring H. Putting $h = 1$ in HV.3 entails $\xi(\lambda) = -v(\lambda)$ for $\lambda \in K$, hence FH extends the valuation filtration fK corresponding to v. For $h \in H$ we have $h \in F_{\xi(h)}H$, hence FH defines an exhaustive filtration of H. From HV.6 we obtain that S and ε are filtered maps of degree zero with respect to FH. In case FH would not be separated, then there is a nonzero $z \in H$ such that for every $\gamma \in \Gamma$ such that $z \in F_\gamma H$ we have $z \in F_\gamma^0 H$. In any case $z \in F_{\xi(z)}H$ but if $z \in F_\gamma H$ with $\gamma < \xi(z)$, then by definition of $F_\gamma H$ it means that $\xi(z) \leq \gamma$, contradiction. Thus FH is separated. Now consider $h \in H, \Delta(h) = \Sigma h_1 \otimes h_2$, then Proposition 3.4.3 entails: $\xi(h) = \inf\{\max_\Sigma\{\xi(h_1) + \xi(h_2)\}\}$. For $\Delta(F_\gamma H) \subset \sum_{\tau\in\Gamma} F_\tau H \otimes F_{\gamma-\tau} H$ it suffices to establish this for $\gamma = \xi(h)$, any $h \in H$ (indeed ξ like v is assumed to be surjective). If $\delta > \xi(h) = \gamma$, then for some decomposition $\Delta(h) = \sum h_1 \otimes h_2$ we have $\delta \geq \max_\Sigma\{\xi(h_1) + \xi(h_2)\}$, and this comes down to:

$$\Delta(h) \in \sum_{\tau\in\Gamma} F_\tau H \otimes F_{\delta-\tau} H \tag{$*$}$$

Recall that FH is a strong filtration hence for every $\gamma \in \Gamma$, $F_\gamma H = F_\gamma K F_0 H = f_\gamma K F_0 H$. For the tensor filtration on $H \otimes H$ defined by FH we have:

$$F_\gamma(H \otimes H) = \sum_{\sigma\in\Gamma} F_{\gamma-\sigma} H \otimes F_\sigma H = \sum_{\sigma\in\Gamma} F_{\gamma-\sigma} K F_\sigma K F_0 H \otimes F_0 H = F_\gamma H \otimes F_0 H$$

From $(*)$ we obtain $\Delta(h) \in F_\delta H \otimes F_0 H$ for every $\delta > \xi(h)$. As a D-module $F_0 H$ is flat, indeed over a valuation domain every finitely generated torsion free modules is projective and every projective is free. Thus $F_0 H$ is the direct limit of free modules of finite rank, thus flat. Consequently: $\cap_\delta (F_\delta H \otimes_D F_0 H) = (\cap_\delta F_\delta) \otimes_D F_0 H$. Either $\xi(h)$ is equal to some $\max_\Sigma \{\xi(h_1) + \xi(h_2)\}$ for a certain decomposition $\Delta(h) = \Sigma h_1 \otimes H_2$ in which case $(*)$ applies with $\delta = \xi(h)$ and there is nothing left to prove, or else $\xi(h)$ appears as the inf of elements $\max_\Sigma \{\xi(h_1) + \xi(h_2)\} \in \Gamma$ (observe that $\max_\Sigma$ is over a finite set). In view of the remark after this proof, separatedness of FH yields $F_{\xi(h)} = \cap_{\delta > \xi(h)} F_\delta H$. Therefore we obtain $\Delta(h) \in F_{\xi(h)} H \otimes F_0 H$ and it follows that FH is a Hopf filtration. Conversely if FH is a separated Hopf filtration extending fK_0, then for $x \neq 0$ in H there is a unique $\gamma \in \Gamma$ such that $x \in F_\gamma H - F_\gamma^0 H$. The function $\xi : H \to \Gamma \cup \{-\infty\}$ defined by $\xi(0) = -\infty$ and for $x \neq 0$, $\xi(x) = \inf\{\tau \in \Gamma, \inf_\tau H\}$ is well-defined and surjective since FH extends fH and v is surjective. We may view $\xi(x)$ as the "filtration degree" with respect to FH. Verifying the properties HV.1,...,HV.7 is easy enough. □

3.4.5 Remark

If FR is a separated filtration on a ring R and $\mathcal{A}$ is a subset of Γ such that $\gamma = \inf\{\alpha, \alpha \in \mathcal{A}\} \in \Gamma \cup \{-\infty\}$, then $F_\gamma R = \cap_{\alpha \in \mathcal{A}} F_\alpha R$. Indeed if $x \notin F_\gamma R$ there is a unique $\delta \in \Gamma$ such that $x \in F_\delta R$ but $x \notin F_{\delta'} R$ for any $\delta' < \delta$ in view of the separatedness. Then $\gamma < \delta$ because $\delta \leq \gamma$ and $x \in F_\delta R$ leads to $x \in F_\gamma R$ which is excluded. Hence $\alpha_0 < \delta$ for sone $\alpha_0 \in \mathcal{A}$ and thus $x \notin F_{\alpha_0} R$ yields $x \notin \cap_{\alpha \in \mathcal{A}} F_\alpha R$.

From now on we consider a separated Hopf filtration FH extending fK associated to a valuation v of R with associated Hopf valuation function ξ. We have seen that $F_0 H$ is a Hopf algebra over $D = f_0 K$ and it is an order of H in the sense that $KF_0 H = K \otimes_D F_0 H = H$. For $h \in H$ we define $I_h \subset K$ by putting $I_h = \{\lambda \in K, \lambda h \in F_0 H\}$. The next proposition establishes that ξ may be calculated from data in K.

3.4.6 Proposition

For $h \in H$, $\xi(h) = v(I_h)$, in particular for $h = \lambda \in K, \xi(\lambda) = -v(\lambda)$. If ξ_1, ξ_2 correspond to Hopf filtrations $F^1 H$ resp. $F^2 H$, then $F_0^1 H \subset F_0^2 H$ is equivalent to $\xi_2 \leq \xi_1$.

Proof. Let FH be the Hopf filtration corresponding to ξ; note that surjectivity of ξ implies $F_\gamma H \neq F_\tau H$ for $\gamma \neq \tau$ in Γ. Indeed if $\sigma \in \Gamma$ hen $\sigma = \xi(z)$ for some $z \in H$ and $z \in F_{\xi(z)} H$ but $z \notin F_{\xi(h)} H$, i.e. $F_\sigma H \neq F_\tau H$ for all $\tau < \sigma$. Now take $h \in H$ then $h \in F_\gamma H - F_\gamma^0 H$ for some $\gamma \in \Gamma$, in fact $\gamma = \delta(h)$. If $\lambda \in K$ is such that $\lambda h \in F_0 H$ then $\xi(\lambda h) = \xi(h) - v(\lambda) \leq 0$. From $\gamma = \xi(h) \leq v(\lambda)$ it follows that $\xi(\lambda) \leq -\gamma$ or $I_h \subset f_{-\gamma}K$. Since $f_{-\gamma}K$ is obviously in $I_h = f_{-\gamma}K$, so $v(I_h) = \gamma = \delta(h)$ follows (in general we may define $v(L)$ for an O_v-submodule L of K as $-\max\{\sigma_v(\lambda), \lambda \in L\}$ if the maximum exists in Γ where σ_v is the principal symbol map for fK). For the second statement observe that F^1H and F^2H are strong filtrations hance $F_0^1H \subset F_0^2H$ entails $F_\gamma^1H \subset F_\gamma^2H$ for all $\gamma \in \Gamma$ and $\xi_2 \leq \xi_1$ follows. Conversely from $\xi_2 \leq \xi_1$ it follows that $F_0^1H = \{h \in H, \xi_1(h) \leq 0\} \subset F_0^2H = \{h \in H = \{h \in H, \xi_2(h) \leq 0\}$. □

For $h \in F_0H$ we have $\varepsilon(h) \in D$ and if $v(\varepsilon(h)) = \gamma$ then we may divide h by $\lambda \in K$ with $v(\lambda) = \gamma$ and we still have that $\varepsilon(\lambda^{-1}) \in D$ but we do not know whether $\lambda^{-1}h \in F_0H$. If a suitable set of K-generators for H, B say, can be selected such that the D-module generated by $\{\lambda_i^{-1}h_i, h_i \in B\}$ is a D-ring then we may obtain a method to construct D-orders in H. Elements of H which are candidates for the ones with best divisibility properties are those $h \in H$ such that $\varepsilon(h) = 0$, i.e. the elements of the augmentation ideal. The advantage of Theorem 3.4.4 is that ξ is known if we know the Hopf order F_0H and conversely. In case $H = KG$ for a finite group G the knowledge of a Zassenhaus valuation on G does not determine unambiguously a Hopf valuation on KG nor a Hopf order of KG. Indeed the construction of Larson orders different from DG in KG, cf.[35], exactly shows that different orders may be constructed from the same Zassenhaus valuation of G. Theorem 3.4.4 applied to $H = KG$ entails that DG and a nontrivial Larson order $\mathcal{L}$, $DG \subsetneqq \mathcal{L} \subsetneqq KG$, correspond to different Hopf valuation functions on KG but these take the same values on some K-basis of the Hopf algebra, e.g. G in KG. It turns out that some basis is better than another! In the sequel we obtain a construction method for (maximal) orders of Larson-type in any finite dimensional (semisimple) Hopf algebra, and a description in terms of some suitably selected basis; we include some examples with number theoretical flavour.

Consider the K-space H/K and define $d_\xi : H/K \to \Gamma \cup \{-\infty\}$ by putting $d_\xi(\overline{h}) = \xi(h - \varepsilon(h))$, where $\overline{h}$ is the class of h in H/K. We may define d_ξ on H by putting $d_\xi(h) = \xi(h - \varepsilon(h))$. Taking into account that $d_\xi(\lambda) = -\infty$ for every $\lambda \in K$. We call d_ξ the **derived valuation function** of ξ.

3.4.7 Lemma

With notation as above, either $\xi(h) = \xi(\varepsilon(h))$ or $\xi(h) = d_\xi(h)$, in other words $d_\xi(h) < \xi(h)$ only if $\xi(h) = \xi(\varepsilon(h))$ and $\varepsilon(h) \neq 0$.

Proof. From $h = (h - \varepsilon(h)) + \varepsilon(h)$ if $\varepsilon(h) \neq 0$, we obtain: $\xi(h) \leq \max\{d_\xi(h), \xi(\varepsilon(h))\}$. By definition of d_ξ we also have: $d_\xi(h) \leq \max\{\xi(h), \xi(\varepsilon(h))\}$. Combination of these inequalities yields either $\xi(h) = \xi(\varepsilon(h))$ or else $\xi(h) = d_\xi(h)$. The second statement in the lemma is now clear. Recall that ε is a filtered morphism, hence $\xi(\varepsilon(h)) \leq \xi(h)$. □

The properties of d_ξ are modifications of those of ξ.

3.4.8 Proposition

With notation and conventions as before the function d_ξ satisfies the following properties.

DV.1 For $h \in H$, $d_\xi(h) = -\infty$ if and only if $h \in K$.
DV.2 For $\lambda \in K, h \in H$, $d_\xi(\lambda h) = d_\xi(h) - v(\lambda)$.
DV.3 For $g, h \in H, d_\xi(gh) \leq \max\{d_\xi(g) + \xi(h), \xi(g) + d_\xi(h)\}$. In case $\varepsilon(g) = \varepsilon(h) = 0$, hen $d_\xi(gh) \leq d_\xi(g) + d_\xi(h)$.
DV.4 For $g, h \in H$, $d_\xi(g + h) \leq \max\{d_\xi(g), d_\xi(h)\}$.
DV.5 For $h \in H$, $d_\xi(Sh) \leq d_\xi(h)$.
DV.6 For $h \in H - K$ with $\varepsilon(h) = 0$ and $\Delta(h) = \Sigma h_1 \otimes h_h$, $d_\xi(h) \geq \inf\{\max_\Sigma\{d_\xi(h_1) + d_\xi(h_2)\}$.

Proof. The proof of DV.1, DV.2, DV.4 is straightforward.

DV.3 The first statement follows from:

$$gh - \varepsilon(gh) = (g - \varepsilon(g))h + \varepsilon(g)(h - \varepsilon(h))$$

Hence: $\xi(gh) \leq \max\{\xi(g-\varepsilon(g)h), \xi(\varepsilon(g))(h-\varepsilon(h))\}$. Now HV.4 and $\xi(\varepsilon(g)) \leq \xi(g)$ yields the statement. In case $\varepsilon(g) = \varepsilon(h) = 0$ we may assume $g \notin K$ and we have $d_\xi(g) = \xi(g)$, $d_\xi(h) = \xi(h)$ and HV.4 applies.
DV.5 Follows from $d_\xi(Sh) = \xi(Sh - \varepsilon(Sh)) \leq \xi(h - \varepsilon(h)) = d_\xi(h)$.
DV.6 Since $\varepsilon(h) = 0$, $d_\xi(h) = \xi(h)$ and the claim follows from [HV.7]. □

To a Hopf valuation filtration function δ there corresponds a Hopf order $H(\xi) = \{h \in H, \xi(h) \leq 0\} = F_0 H$, to d_ξ we may associate $H(d_\xi) = D \oplus \{h \in \varepsilon^{-1}(D) : -\infty < d_\xi(h) \leq 0\} \cup \{0\}$.

3.4.9 Observation

With notation as above: $H(d_\xi) = H(\xi)$. Indeed it is clear if $h \in H(d_\xi)$ with $h \in \mathrm{Ker}\varepsilon$ then $\xi(h) \leq 0$, hence $h \in H(\xi)$; if $h \notin \ker\varepsilon$ then $\xi(h) = \xi(\varepsilon(h))$ entails $\xi(h) \leq 0$ because $\varepsilon(h) \in D$. hence $H(d_\xi) \subset H(\xi)$. Conversely if $\xi(h) \leq 0$ then $\xi(h - \varepsilon(h)) \leq \max\{\xi(h), \xi(\varepsilon(h)\} = \xi(h) \leq 0$, thus $H(\xi) \subset H(d_\xi)$ follows, hence $H(\xi) = H(d_\xi)$.

3.4.10 Definition

A D-order in a finite dimensional K-algebra A, Λ say, is called a moderate order if it is integral over D and the prime radical $\mathrm{rad}(\Lambda)$ is a finite D-module.

Note

In the definition of moderate order given in [2] it is forgotten to remind that orders are assumed to be integral! The term "Hopf order" used before only refers to the property $KH(\xi) = H$ but from now on we shall restrict attention to ξ **corresponding to orders $H(\xi)$ which are integral over D.**

So we look at Hopf valuation functions $\xi \leq \xi_0$ with associated $H(\xi_0) \subset H(\xi)$ and assuming $H(\xi)$ is a moderated order. Let us write $m = m_v$ for the maximal ideal of $D = O_v$. Pick a K-basis $B = \{b_1 = 1, b_2, \ldots, b_n\}$ in $H(\xi_0)$ and we may assume without loss of generality that $\varepsilon(b_2) = \ldots = \varepsilon(b_n) = 0$.

Define $H_B(\xi)$ to be the D-algebra generated by $1, m^{d_\xi(b_i)}b_i$, $i = 1, \ldots, n$. Since $\xi(m^{d_\xi(b_i)}b_i) = 0$, it follows that $H_B(\xi) \subset H(\xi)$ hence $H_B(\xi)$ is integral over D. Since 1 and the $m^{d_\xi(b_i)}b_i$ for $i = 2, \ldots, n$ are again a K-basis for H it follows that $KH_B(\xi) = H$, hence $H_B(\xi)$ is a D-order of H.

We are now interested in the discrete case, so we suppose D is a discrete valuation ring of K from hereon, we may also assume $ch(K) = 0$ but that is not really necessary.

3.4.11 Proposition

With notation as before, if D is a discrete valuation ring of K then $H(\xi_0)$, $H_B(\xi)$, $H(\xi)$ are finite D-modules.

Proof. Since $K\mathrm{rad}H(\xi)$ is a nil ideal of H it is in $\mathrm{rad}H$ and therefore nilpotent hence $\mathrm{rad}H(\xi) = H(\xi) \cap \mathrm{rad}H$ and similar for $\mathrm{rad}H(\xi_0)$, $\mathrm{rad}H_B(\xi)$. Thus $\mathrm{rad}(\xi_0) = H(\xi_0) \cap \mathrm{rad}H$, $\mathrm{rad}H_B(\xi) = H_B(\xi) \cap \mathrm{rad}H(\xi)$ and we obtain $\overline{H(\xi_0)} \subset \overline{H(\xi)}$ and $\overline{H_B(\xi)} \subset \overline{H(\xi)} \subset \overline{H}$, where we denoted $\overline{R}$ for $R/\mathrm{rad}R$. Now $\overline{H}$ is semisimple Artinian with center $L_1 \oplus \ldots \oplus L_d$ say, each L_i the center of a simple component $\overline{S}_i$ of $\overline{H}$. The center of $\overline{H(\xi)}$, also for $\overline{H(\xi_0)}$ and $\overline{H_B(\xi)}$ is integral over D hence in the integral closure of D in $L_1 \oplus \ldots L_d$ and therefore it is a finitely generated D-module, thus also Noetherian. In any case for all three D-orders it follows by a result of G. Cauchon that they are finitely generated D-modules because they are finite over their center since it is Noetherian (P.I. rings with Noetherian center) and the center is a finite D-module as observed above. Since $\mathrm{rad}(\xi)$ is a finite D-module, so are $\mathrm{rad}H(\xi_0)$ and $\mathrm{rad}H_B(\xi)$, consequently the statement of the proposition follows. $\square$

3.4.12 Corollary

There is a K-basis B in $H(\xi_0)$ such that $H_B(\xi) = H(\xi)$.

Proof. Since $H(\xi)$ is finitely generated as a D-module it is free of finite rank over D and it has a basis, B say. Without loss of generality we may assume $B = \{b_0 = 1, b_2, \ldots, b_n\}$ with $\varepsilon(b_2) = \varepsilon(b_3) = \ldots = \varepsilon(b_n) = 0$. Now (writing $m = (\pi) \subset D$) look at $\pi^{\beta_i} b_i$ where $\beta_i = \xi_0(b_i)$. This yields a K-basis in $H(\xi_0)$, say B_0, such that $H_{B_0}(\xi)$ contains B hence $D[B]$, consequently $H_{B_0}(\xi) = H(\xi)$. □

From the foregoing it follows that if there is a (finitely generated) Hopf order over D, say H', containing $H(\xi_0)$ then we may construct it from some K-basis contained in $H(\xi_0)$. Now let us forget $H(\xi)$, but start from a D-basis B of $H(\xi_0)$, this is also a K-basis for H, and make the D-order $H_B(\xi_0)$ and then try to check we obtain a finitely generated D-module which is a Hopf-order. In case $H(\xi_0)$ is not a maximal Hopf order, the foregoing corollary suggested we can find a maximal one by the H_B-construction but for some K-basis B in $H(\xi_0)$, not necessarily for a D-basis of $H(\xi_0)$! Nevertheless we shall show in a series of examples that this method leads to new Hopf orders in many cases.

3.4.13 Example. The Sweedler Hopf Algebra

Consider the Sweedler Hopf algebra over the rational fields Q, say $H = Q[x, y]$ with relations $x^2 = 1$, $y^2 = 0$ and $xy + yx = 0$, and put:

$$\varepsilon(x) = 1, S(x) = x, \Delta(x) = x \otimes x,$$
$$\varepsilon(y) = 0, S(y) = xy, \Delta(y) = 1 \otimes y + y \otimes x$$

Let $D = \mathbb{Z}_p$ the localization of $\mathbb{Z}$ at the prime ideal (p) and $m = (p)$. Define $H(m) = \mathbb{Z}_p + \mathbb{Z}_p(x-1) + n^{-n}y + m^{-n}xy$, for every $n \geq 0$, $H(0) = H_{\mathbb{Z}_p}$. It is easily verified that $H(n)$ is a Hopf order in H. If $H_{\mathbb{Z}_p} = \mathbb{Z}_p[x, y]$ is in some Hopf order $H(\xi)$ then $\xi(x-1) \leq 0$ and from $\xi(S(h)) \leq \xi(h)$ it follows that $\xi(xy) \leq \xi(y)$ and $\xi(y) \leq \xi(xy)$ by taking $h = y$ resp. $h = xy$. Thus $\xi(xy) = \xi(y)$. So we put $\xi(y) = \xi(xy) = -n$ with $n \geq 0$ and $\xi(x-1) = 0$ is then the $\mathbb{Z}_p$-subalgebra of H generated by $b^{\xi(b)}(b - \varepsilon(b))$ for b in the chosen basis for $H_{\mathbb{Z}_p}$, has $\mathbb{Z}_p$-basis $a, x-1, \pi^{-n}y, \pi^{-n}xy$ and it is exactly the $H(n)$ we defined.

3.4.14 Example. The Taft Algebras Over $\mathbb{Q}$

Let $H_T(n)$ be the Taft algebra over $K = \mathbb{Q}$, i.e. $H_T(n) = \mathbb{Q}[x, y]$ with $x^n = 1$, $y^n = 0$ and $xy + yx = 0$, where we put:

$$\varepsilon(x) = 1, S(x) = x^{n-1}, \Delta(x) = x \otimes x,$$
$$\varepsilon(y) = 0, S(y) = -x^{n-1}, \Delta(y) = 1 \otimes y + y \otimes x$$

Then $H_T(n)(-1) = \mathbb{Z}_p + \sum_{i=1}^{n-1} \mathbb{Z}_p(x-1) + \sum_{i=0,j=1}^{n-1} \mathbb{Z}_p \pi^{-j} xy^j$ is a Hopf order on $H_T(n)$ with $\mathbb{Z}_p$-basis $\{x(\pi^{-1}y)^j, i, j = 0, \ldots, n-1\}$. We can construct

$H_T(n)(-1)$ from $H_T(n)$ by constructing ξ by putting $\xi(x^{n-1}-1)=0$, $i=1,\ldots,n-1$, $\xi(y^j)=1$, $j=1,\ldots,n-1$. In checking that $H_T(n)(-1)$ is an Hopf order only $\Delta(\pi^{-j}x^jy^j)\in H_T(n)(-1)\otimes H_T(n)(-1)$ needs some work; this follows from the following lemma.

3.4.15 Lemma

In $H_T(n)$, from all $0\le i,j<n$ we have: $\Delta(x^i,y^i)=x^iy^i\otimes(x^i-1)+x^iy^i\otimes 1+\sum_{\tau=1}^{j}\alpha_\pi\binom{j}{\pi}x^{i+r+j-v}\otimes x^iy^r+1\otimes x^iy^j+(x^{i+j}-1)\otimes x^iy^j$, where: for j even, $\alpha_r=0$ when r is odd and $\alpha_\tau=\frac{(r-1)(r-3)\ldots}{(j-1)(j-3)(j-r+1)}$ when r is even, for j odd, $\alpha_r=\frac{r(r-2)\ldots}{j(j-2)\ldots(j-r+1)}$ if r is odd and $\alpha_r=\frac{(r-1)(r-3)\ldots}{j(j-2)\ldots(j-1+2)}$ if r is even.

Proof. Since Δ is an algebra morphism $\Delta(x^iy^j)=\Delta(x)^i\Delta(y)^j=(x^i\otimes x^i)(y\otimes 1-x\otimes y)^j$. Applying the binomial formula and taking into account that $yx=-xy$, yields the result in a straightforward way. □

The group-like elements we have to consider are just the $g-1$ ($\varepsilon(g)=1$). Therefore the following numerical lemma will be useful.

3.4.16 Lemma

Let g be any element in a Q-algebra A, then for any natural number n we have the equality:

$$\begin{aligned}(g-1)^n=g^n&+\alpha_{n-1}(-1)(g-1)^{n-1}+\alpha_{n-2}(-1)^2(g-1)^{n-2}\\&+\ldots+\alpha_{n-i}(-1)^i(g-1)^{n-i}+\ldots\\&+\alpha_2(-1)^{n-2}(g-1)^2+\alpha_1(-1)^{n-1}(g-1)+c_n(-1)^n\end{aligned}$$

where

$$\begin{aligned}\alpha_{n-1}&=\binom{n}{1},\alpha_{n-2}=\binom{n}{2}-\alpha^{n-1}\binom{n}{1},\ldots\\ \alpha_{n-i}&=\binom{n}{i}-\alpha_{n-1}\binom{n-1}{i-1}-\alpha_{n-2}\binom{n-2}{i-2}-\ldots\\&\qquad-\alpha_{n-(i-1)}\binom{n-(i-1)}{1}\text{ and }c_n=1-\alpha_{n-1}-\alpha_{n-2}-\alpha_{n-3}-\ldots-\alpha_1\end{aligned}$$

Proof. The proof is by induction on n. If $n=2$, then

$$(g-1)^2=g^2+\alpha_1(-1)(g-1)+c_2(-1)^2$$

$\alpha_1=\binom{2}{1}$ and $c_2=1-\alpha_1=1-2=-1$.

Suppose now that it is true for all $n \leq k$, it is true for $n = k+1$. Since it is true for $n = k$, then we can write:

$$\begin{aligned}(g-1)^{k+1} = {} & g^{k+1} + (\alpha'_{k-1}+1)(-1)(g-1)^k + \\ & (\alpha'_{k-2} - \alpha'_{k-1})(-1)^2(g-1)^{k-1} + \ldots + \\ & (\alpha'_{k-(i+1)} - \alpha'_{k-1})(-1^{i+1})(g-1)^{k-i} + \ldots + \\ & \alpha_2(-1)^{n-2}(g-1)^2 + (\alpha'_1 - \alpha'_2)(-1)^{k-i}(g-1)^2 + \\ & (c_k - \alpha'_1)(-1)^k(g-1) - c_k(-1)^{k+1}\end{aligned}$$

where

$$\alpha'_{k-1} = \binom{k}{1}, \alpha'_{k-2} = \binom{k}{2} - \alpha'_{k-1}\binom{k}{1}, \ldots,$$
$$\alpha'_{k-i} = \binom{k}{i} - \alpha'_{k-1}\binom{k-1}{i-1} - \alpha'_{k-2}\binom{k-2}{i-2} - \ldots - \alpha'_{k-(i-1)}\binom{k-(i-1)}{1}$$

and $c_k = 1 - \alpha'_{k-1} - \alpha'_{k-1} - \alpha'_{k-2} - \alpha'_{k-3} - \ldots - \alpha'_1$. To complete the proof we prove $\alpha_{k-i} - \alpha'_{k-(i+1)} - \alpha'_{k-1}, \alpha_k = \alpha'_{k-1} + 1$ and $\alpha_1 = c_k - \alpha'_1$. First we prove by induction that $\alpha_{k-i} = \alpha'_{k-(i+1)} - \alpha'_{k-i}$. Since $\alpha_k = \binom{k+1}{1}, \alpha'_{k-1} = \binom{k}{1}$, then $\alpha_k = \alpha'_{k-1} + 1$. Since

$$\alpha_{k-1} = \binom{k+1}{2} - \alpha_k\binom{k}{1} = \binom{k+1}{2} - \alpha'_{k-1}\binom{k}{1} - \binom{k}{1} = \binom{k}{2} - \alpha'_{k-1}\binom{k}{1}$$

and

$$\alpha'_{k-2} - \alpha'_{k-1} = \binom{k}{2} - \alpha'_{k-1}\binom{k}{1}$$

then $\alpha_{k-2} - \alpha'_{k-1}$. Suppose now this is true for all $j < i$, then

$$\begin{aligned}\alpha_{k-1} = {} & \binom{k+1}{i+1} - \alpha_k\binom{k}{i} - \alpha_{k-1}\binom{k-1}{i-1} - \ldots - \alpha_{k-(i-1)}\binom{k-(i-1)}{1} \\ = {} & \binom{k+1}{i+1} - \binom{k}{i} - \alpha'_{k-1}\left[\binom{k}{i} - \binom{k-1}{i-1}\right] \\ & - \alpha'_{k-2}\left[\binom{k-1}{i-1} - \binom{k-2}{i-2}\right] - \ldots \\ & - \alpha'_{k-(i-1)}\left[\binom{k-i}{2} - \binom{k-(i-1)}{1}\right] - \alpha'_{k-i}\binom{k-(i-1)}{1} \\ \alpha_{k-i} = {} & \binom{1}{i+1} - \alpha'_{k-1}\binom{k-1}{i} - \alpha'_{k-2}\binom{k-2}{i-1} - \ldots\end{aligned}$$

$$-\alpha'_{k-(i-1)}\binom{k-(i-1)}{2}-\alpha'_{k-i}\binom{k-(i-1)}{1}$$
$$=\alpha'_{k-(i-1)}-\alpha'_{k-i}$$

Using $\alpha_{k-i}=\alpha'_{k-(i+1)}-\alpha'_{k-i}$ we can prove that $\alpha_1=c_k-\alpha'$, then

$$\begin{aligned}\alpha &= \binom{k+1}{k}-\alpha_k\binom{k}{k-1}-\alpha_{k-1}\binom{k-1}{k-2}-\dots\\ &\quad -\alpha_{k-1}\binom{k-i}{k-(i-1)}-\dots-\alpha_2\binom{2}{1}\\ &= 1-\alpha'_{k-1}-\alpha'_{k-2}-\alpha'_{k-3}-\dots-\alpha'_{k-i}-\dots-2\alpha'_1\\ &= c_k-\alpha'_1\end{aligned}$$

Using $\alpha_{k-1}=\alpha'_{k-(i+1)}-\alpha'_{k-i}$ we find that $\alpha_1=c_k-\alpha'_1$, then $c_{k+1}=1-\alpha_k-\alpha_{k-1}-\alpha_{k-2}-\dots-\alpha_1=c_k$.

3.4.17 Remark

1. $\alpha_{n-i}=(-1)^{i+1}\binom{n}{i}$.
2. $\alpha_{n-i}=(-1)^{i+1}\alpha_i$.
3. If $n=p_1^{i_1}p_2^{i_2}\dots p_s^{i_s}$ where $p_1,\dots,p_s$ are nonequal prime numbers, then $p_j^{i_j}|\alpha_{p_l^{i_l}}, 0\le j,l\le s$ and $j\ne l$ and $p_j^{i_j}\not|\alpha_{p_j^{i_j}}$.
4. If $n=p^s, s\ge 1$ then $p^{s-i}|\alpha_{p^i},\alpha_{p^i(p^{s-i}-1)}$ and $p^{s-i+1}\not|\alpha_{p^i},\alpha_{p^i(p^{s-i}-1)}$.

Proof. 1. We prove it by induction.

If $i=1$, then $\alpha_{n-1}=\binom{n}{1}=(-1)^2\binom{n}{1}$. Suppose it is true for all $i\le k-1$, then

$$\begin{aligned}\alpha_{n-k} &= \binom{n}{k}-\alpha_{n-1}\binom{n-1}{k-1}-\alpha_{n-2}\binom{n-2}{k-2}-\dots\\ &\quad -\alpha_{n-(k-1)}\binom{n-(k-1)}{1}\\ \alpha_{n-k} &= \binom{n}{k}-k\binom{n}{k}+k(k-1)/2\binom{n}{k}-\dots\\ &\quad \dots-(-1)^k k\binom{n}{k}\end{aligned}$$

If k is odd, then: $\alpha_{n-k}=(-1)^{k+1}\binom{n}{k}$. If $k=2s$ is even, then:

$$\begin{aligned}\alpha_{n-k} &= \binom{n}{k}-2k\binom{n}{k}+2k(k-1)/2\binom{n}{k}-\dots\\ &\quad +2k(k-1)(k-2)\dots(k/2)/((k/2)-1)!\binom{n}{k}\\ &\quad - \quad 2k(k-1)(k-2)\dots((k/2)+1)/(k/2)!\binom{n}{k}\\ &= \binom{n}{k}\left(1-2\binom{k}{1}+2\binom{k}{2}-\dots+2\binom{k}{s-1}-\binom{k}{s}\right)\end{aligned}$$

2. If $n = pq$ then $\alpha_p = (-1)^{n-p+1} pq(pq-1)\dots(pq-p+1)/p! = qz$. In fact, no factor $pq - i, 1 \leq i \leq p-1$ can be divided by p, otherwise $i = sp$, a contradiction.
3. If $n = p^s$ then $\alpha_{p^i} = (-1)^{n-p^i+1} p^s(p^s-1)\dots(p^s-p^i+1)/p^i! = p^{s-i}z$. In fact, no factor $p^i - t$, $1 \leq t \leq p^i - 1$ can be divided by p^l with $l \leq i$ without reducing p^l by a factor of $(p^i-1)!$, otherwise $i = kp^i$, a contradiction.
4. Similar to (3). □

3.4.18 *Proposition*

Consider a numberfield $K/\mathbb{Q}$ and let D be a discrete valuation ring of K extending $\mathbb{Z}_p \subset Q$ (here $\mathbb{Z}_p$ is the location of $\mathbb{Z}$ with respect to the prime p). Let $e = v(p)$ be the absolute ramification index of K. Consider a finite dimensional Hopf algebra H over K and let $G = G(H)$ be its finite group of group-like elements. If ξ is any Hopf valuation filtration function on H extending v then we have:

1. $\xi(g) = 0$ for $g \in G$ such that the order $0(g)$ of g in G is not a power of p.
2. $\xi(g) \leq e(p^s - p^{s-1})^{-1}$ if the order of g is p^s.

Proof. This follows from (3) and (4) in Remark 3.4.17. Indeed if $0(g) \neq p^s$ then it is a multiple of at least two different primes and none of these can divide all α's in the formula given in Lemma 3.4.16, hence in that case $\xi(g) = 0$. On the other hand if $n = p^s$ then p will be a divisor of all α's in the formula in Lemma 3.4.16 but p^2 will not divide $\alpha_{p^{s-1}}$. From Lemma 3.4.16 we may obtain an expression for $(\pi^{\xi(g)}(g-1))^n$, i.e.: with $n = p^s$:

$$(\pi^{\xi(g)})^n = \alpha_{n-1}(-1)\pi^{\xi(g)}(\pi^{\xi(g)}(g-1))^{n-1} + \dots$$
$$\dots + \alpha_{p^{s-i}}\pi^{(n-p^{s-i})\xi(g)}(-1)^{p^{s-i}}(\pi^{\xi(g)}(g-1))^{p^{s-i}} + \dots$$

It follows from this that $p = d\pi^{(n-p^{s-1})\xi(g)}$ for some $d \in D$, therefore $e \geq (p^s - p^{s-1})\xi(g)$. □

3.4.19 *Remark*

1. For $H = KG$, G a finite group, and ξ corresponding to a Larson order (i.e. a Hopf order $H_B(\xi)$ corresponding to the basis $\{1, 1-g, g \in G\}$), then the conditions in Proposition 3.4.17 do reduce to the conditions also found by Larson in [35]. Note that in [35] the author proves that the constructed orders (of Larson-type) in RG are in fact finitely generated D-modules without the assumption that

they are constructed in an integral D-order. The proof is more combinatorial in nature.
2. The conditions (2) in Proposition 3.4.18 make it clear that the realization of a certain ξ forces rather demanding ramification properties of v, e.g. for $\xi(g) = 1$ one needs $e \geq p^s - p^{s-1}$, $p^s = 0(g)$.

   Recall the definition of the generalized Taft algebra with respect to a root of unity ρ, say $\rho^n = 1$. Put $H_T(n)$ equal to the K-algebra generated by x and y satisfying $x^n = 1$, $y^n = 0$ and $xy = \rho yx$, with Hopf algebra structure given by:

$$\varepsilon(x) = 1, S(x) = x^{n-1}, \Delta(x) = x \otimes x$$

$$\varepsilon(y) = 1, S(y) = -\rho^{-1}x^{n-1}y, \Delta(y) = 1 \otimes y \otimes y \otimes x$$

3.4.20 Example

Let D be a discrete valuation ring of $K, \rho \in D$.

In case $n \neq p^s$ for $x \geq 1$, then

$$H_T(-n) = D + \sum_{i=1}^{n-1} D(x-1)^i + \sum_{i=0, j=1}^{n-1} D\pi^{-jn}(x-1)^i y^j$$

is a Hopf order.

In case $n = p^s$, $\pi^{m(p^s - p^{s-1})} | p$ and $\pi^m | (\rho - 1)$ in D, then

$$H_T(-p^s) = D + \sum_{i=1}^{n-1} D\pi^{-im}(x-1)^i + \sum_{i=0, j=1}^{n-1} D\pi^{-im-jn}(x-1)^i y^j$$

is a Hopf order.

Observe that $(\rho - 1)^{p^s} = p \sum_{i=1}^{p^s - 1} (\alpha_i / p)(\rho - 1)^i$ (Lemma 3.4.16) and then $(\rho - 1)^{p^s} \in (p) \subset \pi^{(p^s - p^{s-1})m}$, where $p^s m < e + p^{s-1}m, e = v(p)$.

A full proof of the claims can be obtained via the quantum binomial formula (see [31]) applied to $f = \pi^{-m}(g-1)$, and via a careful coefficient calculation in an expression for $X^t f^s$, $t, s \in \mathbb{N}$. We omit these technical details here. Let us just provide a concrete case where all of the above phenomena are clear.

3.4.21 Example

Take $p = 2$ and look at the localization of $\mathbb{Z}_2[\rho]$ at $\rho - 1$ where $\rho = \sqrt[4]{-1} = \sqrt{i}$. For D we take $\mathbb{Z}[(\rho - 1)^{1/5}]_{((\rho-1)^{1/5})}$. In this case $\pi = (\rho - 1)^{1/5}$, $(2) = (\pi^5)$,

$v(\pi) = 1$, $e = v(2) = 5$, $v(\rho - 1) = 5$, $\rho^8 = 1$. So we have $(\rho - 1)^8 \in (2)$ and $(2) \subset (\rho - 1) \subset (\pi) \subset D$. The constructions in Example 3.4.20 apply in this case.

Let us conclude with an example showing the effect of base change.

3.4.22 Example

We start with the situation of Example 3.4.13 but with K a number field such that $\pi^m | 2$ in D. Put $H(m, n) = D[f, \chi]$, $f = \pi^{-m}(g - 1)$, $\chi = \pi^{-n}h$. Then $H(m, n)$ is a Hopf algebra of rank 4 over D with:

$$\begin{aligned} \Delta(f) &= f \otimes g + 1 \otimes f \\ f\chi + \chi f &= \nu\chi, \text{ where } 2 = \nu\chi^m \text{ with } \nu \in D \\ \Delta(f\chi) &= f\chi \otimes 1 + 1 \otimes f\chi + f \otimes g\chi + \chi \otimes fg \\ \Delta(\chi) &= 1 \otimes \chi + \chi \otimes g \\ f\chi - \chi &= \nu g\chi \end{aligned}$$

We may also define $H(n)$ as in Example 3.4.13, it is of rank 4 over D with basis $\{1, g-1, \pi^{-n}h, \pi^{-n}gh\}$. Both $H(m, n)$, $H(n)$ contain the Hopf order $D[g, h]$ (viewed as $H(\xi_0)$ in Proof of 3.4.12).

Now $H(n)$ is of the form $H_B|\xi|$ with respect to $B = \{1, g - 1, h, gh\}$ and $H(m, n)$ with respect to $B' = \{1, g-1, h, (g-1)h\}$, and these orders are obviously different.

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