Chapter 9
Asymptotic Analysis of Implied Volatility

The implied volatility was first introduced in the paper [LR76] of H.A. Latané and R.J. Rendleman under the name "the implied standard deviation". Latané and Rendleman studied standard deviations of asset returns, which are implied in actual call option prices when investors price options according to the Black–Scholes model. For a general model of call option prices, the implied volatility can be obtained by inverting the Black–Scholes call pricing function with respect to the volatility variable and composing the resulting inverse function with the original call pricing function.

This chapter mainly concerns the asymptotics of the implied volatility at extreme strikes. In Sect. 9.1, we define the implied volatility in general models of call option prices and discuss its elementary properties. Implied volatility models free of static arbitrage are characterized in Sect. 9.2 (see Theorem 9.6). The rest of the chapter is devoted to sharp asymptotic formulas with error estimates for the implied volatility. We discuss asymptotic formulas of various orders, and show how certain symmetries hidden in stochastic asset price models allow to analyze the asymptotic behavior of the implied volatility for small strikes, by using information about its behavior for large strikes. These symmetries become more explicit in the so-called symmetric models, which are also discussed in the present chapter.

9.1 Implied Volatility in General Option Pricing Models

Fix $K > 0$ and $T > 0$. Then the function $\rho(\sigma) = C_{\mathrm{BS}}(T, K, \sigma)$ is increasing on $(0, \infty)$. This follows from the fact that the Greek vega is positive (see Sect. 8.4). If $0 < K < x_0 e^{rT}$, then the range of the function ρ coincides with the interval $(x_0 - Ke^{-rT}, x_0)$, while for $x_0 e^{rT} \geq K$, the range of ρ is the interval $(0, x_0)$.

Definition 9.1 Let C be a call pricing function. For $(T, K) \in (0, \infty)^2$, the implied volatility $I(T, K)$ associated with C is the value of the volatility σ in the Black–Scholes model for which $C(T, K) = C_{\mathrm{BS}}(T, K, \sigma)$. The implied volatility $I(T, K)$ is defined only if such a number σ exists and is unique.

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It follows from the discussion above that if $0 < K < x_0 e^{rT}$, then the condition $x_0 - Ke^{-rT} < C(T, K) < x_0$ is necessary for the existence of the implied volatility $I(T, K)$. Similarly, if $x_0 e^{rT} \le K$, then $I(T, K)$ is defined if and only if $0 < C(T, K) < x_0$. Note that the inequality $C(T, K) < x_0$ holds for all $T \ge 0$ and $K > 0$. Moreover, if $(T, K) \in [0, \infty)^2$, then

$$\left(x_0 - Ke^{-rT}\right)^+ \le C(T, K).$$

In the next definitions, we introduce special classes of call pricing functions.

Definition 9.2 The class PF_∞ consists of all call pricing functions C, for which one of the following equivalent conditions holds:

1. $C(T, K) > 0$ for all $T > 0$ and $K > 0$ with $x_0 e^{rT} \le K$.
2. $P(T, K) > e^{-rT} K - x_0$ for all $T > 0$ and $K > 0$ with $x_0 e^{rT} \le K$.
3. For every $T > 0$ and all $a > 0$ the random variable X_T is such that $\mathbb{P}^*[X_T < a] < 1$.

Definition 9.3 The class PF_0 consists of all call pricing functions C, for which one of the following equivalent conditions holds:

1. $P(T, K) > 0$ for all $T > 0$ and $K > 0$ with $K < x_0 e^{rT}$.
2. $C(T, K) > x_0 - e^{-rT} K$ for all $T > 0$ and $K > 0$ with $K < x_0 e^{rT}$.
3. For every $T > 0$ and all $a > 0$ the random variable X_T is such that $0 < \mathbb{P}^*[X_T < a]$.

Remark 9.4 Suppose the maturity $T > 0$ is fixed, and consider the pricing function C and the implied volatility I as functions of the strike price K. If $C \in PF_\infty$, then the implied volatility $I(K)$ is defined for large values of K. This allows to study the asymptotic behavior of the implied volatility as $K \to \infty$. Similarly, if $C \in PF_0$, then $I(K)$ exists for small values of K. Finally, if $C \in PF_\infty \cap PF_0$, then the implied volatility $I(T, K)$ exists for all $T > 0$ and $K > 0$.

9.2 Implied Volatility Surfaces and Static Arbitrage

Let $I(T, K)$ with $(T, K) \in (0, \infty)^2$ be a positive function of two variables, and suppose we would like to model the implied volatility surface by this function. Then the function $\widetilde{C}$ defined on $[0, \infty)^2$ by

$$\widetilde{C}(T, K) = \begin{cases} C_{\mathrm{BS}}(T, K, I(T, K)), & \text{if } (T, K) \in (0, \infty)^2, \\ (x_0 - K)^+, & \text{if } T = 0,\ K \ge 0, \\ x_0, & \text{if } T \ge 0,\ K = 0, \end{cases} \tag{9.1}$$

where x_0 is the initial price of the asset in the Black–Scholes model, should be a call pricing function. For the sake of simplicity, we will assume that $r = 0$. The next definition concerns the implied volatility in a no-arbitrage environment.

Definition 9.5 It is said that the function I modeling the implied volatility is free of static arbitrage if the model of call prices given by the function $\widetilde{C}$ in (9.1) is free of static arbitrage (see Definition 8.2).

Our next goal is to provide necessary and sufficient conditions for the absence of static arbitrage in a given implied volatility model.

Theorem 9.6 *Suppose the function I models the implied volatility. Suppose also that for every $T > 0$ the function $K \mapsto I(T, K)$ is twice differentiable on $(0, \infty)$. Then I is free of static arbitrage if and only if the following conditions hold*:

1. *For all* $(T, K) \in (0, \infty)^2$,

$$\left(1 - \frac{K}{I}\log\left(\frac{K}{x_0}\right)\frac{\partial I}{\partial K}\right)^2 + TK^2 I\frac{\partial^2 I}{\partial K^2} - \frac{1}{4}T^2K^2I^2\left(\frac{\partial I}{\partial K}\right)^2 + TKI\frac{\partial I}{\partial K} \geq 0. \tag{9.2}$$

2. *For every* $K > 0$ *the function* $T \mapsto \sqrt{T}I(T, K)$ *is increasing on* $(0, \infty)$.
3. *For every* $T > 0$, $\lim_{K\to\infty} d_1(T, K, I(T, K)) = -\infty$.

Proof It suffices to prove that the conditions in Theorem 8.3, formulated for the function $\widetilde{C}$ given by (9.1), are equivalent to conditions 1–3 in Theorem 9.6.

Fix $T > 0$, and differentiate the function $\widetilde{C}$ on $(0, \infty)$ with respect to K. This gives

$$\frac{\partial \widetilde{C}}{\partial K} = \frac{\partial C_{\mathrm{BS}}}{\partial K}\big(T, K, I(T, K)\big) + \frac{\partial C_{\mathrm{BS}}}{\partial \sigma}\big(T, K, I(T, K)\big)\frac{\partial I}{\partial K}.$$

Differentiating again, we obtain

$$\frac{\partial^2 \widetilde{C}}{\partial K^2} = \frac{\partial^2 C_{\mathrm{BS}}}{\partial K^2}\big(T, K, I(T, K)\big) + 2\frac{\partial^2 C_{\mathrm{BS}}}{\partial K \partial \sigma}\big(T, K, I(T, K)\big)\frac{\partial I}{\partial K} + \frac{\partial C_{\mathrm{BS}}}{\partial \sigma^2}\big(T, K, I(T, K)\big)\left(\frac{\partial I}{\partial K}\right)^2 + \frac{\partial C_{\mathrm{BS}}}{\partial \sigma}\big(T, K, I(T, K)\big)\frac{\partial^2 I}{\partial K^2}.$$

Next, taking into account explicit formulas for the Greeks (see Sect. 8.4), we see that for every $T > 0$ the convexity of the function $K \mapsto \widetilde{C}(T, K)$ on $(0, \infty)$ is equivalent to the following inequality:

$$\frac{1}{\sqrt{T}K^2 I} + \frac{2d_1(T, K, I(T, K))}{KI}\frac{\partial I}{\partial K} + \frac{\sqrt{T}d_1(T, K, I(T, K))d_2(T, K, I(T, K))}{I}\left(\frac{\partial I}{\partial K}\right)^2 + \sqrt{T}\frac{\partial^2 I}{\partial K^2} \geq 0, \quad K > 0. \tag{9.3}$$

It is not hard to see, using the definition of d_1 and d_2, that (9.3) is equivalent to (9.2). Hence the convexity of the function $K \mapsto \widetilde{C}(0,K)$ on $(0,\infty)$ is equivalent to the validity of (9.1).

We will next turn our attention to the convexity conditions for the function $K \mapsto \widetilde{C}(T,K)$ on $[0,\infty)$. Let us assume that condition 1 in Theorem 9.6 holds, and put $\varphi(K) = \widetilde{C}(T,K)$. Then the function φ is twice differentiable and convex on $(0,\infty)$. Moreover, the function φ' is increasing on $(0,\infty)$, and it follows from (8.7) that for all $0 < x < y < \infty$,

$$\varphi'(x) \le \frac{\varphi(y) - \varphi(x)}{y - x} \le \varphi'(y). \tag{9.4}$$

Using the definition of the Black–Scholes call pricing function, we see that for all $T > 0$ and $K > 0$,

$$\begin{aligned} \varphi(K) &= \frac{x_0}{\sqrt{2\pi}} \int_{-\infty}^{d_1(T,K,I(T,K))} e^{-\frac{y^2}{2}}\, dy \\ &\quad - \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{d_2(T,K,I(T,K))} e^{-\frac{y^2}{2}}\, dy. \end{aligned} \tag{9.5}$$

It will be shown next that

$$\lim_{K \to 0} d_1\bigl(T, K, I(T,K)\bigr) = \infty. \tag{9.6}$$

Indeed, for small values of K we have

$$d_1(T,K,I) = \frac{\log \frac{x_0}{K} + \frac{1}{2} T I^2}{\sqrt{T} I} \ge \sqrt{2 \log \frac{x_0}{K}}, \tag{9.7}$$

and (9.6) follows. Using (9.5), we obtain the following equality:

$$\lim_{K \to 0} \widetilde{C}(T,K) = x_0. \tag{9.8}$$

Therefore, the function $K \mapsto \widetilde{C}(T,K)$ is continuous on $[0,\infty)$. Our next goal is to prove the differentiability of this function from the right at $K = 0$. It follows from (9.4) that there exists the limit $M = \lim_{K \to 0} \varphi'(K)$. In addition, (9.4) and (9.8) give

$$M \le \frac{\varphi(S) - \varphi(0)}{S} = \frac{\varphi(S) - x_0}{S} \le \varphi'(S)$$

for all $0 < K < S < \infty$. Therefore $M = \varphi'_+(0)$. Moreover, (9.5) and (9.7) imply

$$\begin{aligned} \varphi(K) &= x_0 - K - \frac{x_0}{\sqrt{2\pi}} \int_{d_1(T,K,I(T,K))}^{\infty} e^{-\frac{y^2}{2}}\, dy \\ &\quad + \frac{K}{\sqrt{2\pi}} \int_{d_2(T,K,I(T,K))}^{\infty} e^{-\frac{y^2}{2}}\, dy \end{aligned}$$

$$
\begin{aligned}
&= x_0 - K - \frac{x_0}{\sqrt{2\pi}} \int_{d_1(T,K,I(T,K))}^{\infty} e^{-\frac{y^2}{2}}\, dy + o(K) \\
&= x_0 - K - \frac{x_0}{\sqrt{2\pi}} \int_{\sqrt{2\log \frac{x_0}{K}}}^{\infty} e^{-\frac{y^2}{2}}\, dy + o(K) \\
&= x_0 - K + o(K)
\end{aligned}
$$

as $K \to 0$. Therefore, $M = -1$, and it follows that for every $T > 0$ the function $K \mapsto \widetilde{C}(T, K)$ is convex on $[0, \infty)$ (use affine extrapolation).

The next step in the proof deals with condition 3 in Theorem 9.6. Our goal is to show that

$$
\lim_{K\to\infty} d_1\big(T, K, I(T, K)\big) = -\infty \iff \lim_{K\to\infty} \widetilde{C}(T, K) = 0. \tag{9.9}
$$

We will first prove the following equality:

$$
\lim_{K\to\infty} d_2\big(T, K, I(T, K)\big) = -\infty. \tag{9.10}
$$

Suppose $d_2(T, K, I(T, K))$ does not tend to $-\infty$ as $K \to \infty$. Then there exists a sequence $K_n \uparrow \infty$ such that

$$
\int_{-\infty}^{d_2(T,K_n,I(T,K_n))} e^{-\frac{y^2}{2}}\, dy \geq c > 0
$$

for all $n \geq 1$. It follows from (9.5) that $\widetilde{C}(T, K_n) < 0$ for $n > n_0$, which is impossible. Therefore, (9.10) holds.

It will be shown next that we always have

$$
K \int_{-\infty}^{d_2(T,K,I(T,K))} e^{-\frac{y^2}{2}}\, dy \to 0 \tag{9.11}
$$

as $K \to \infty$. Reasoning as in (9.7), we see that for large values of K,

$$
d_2(T, K, I)^2 \geq 2\log \frac{K}{x_0}. \tag{9.12}
$$

Using formula (8.25), we obtain

$$
F(K) \sim \frac{K}{|d_2(T, K, I(T, K))|} \exp\left\{-\frac{d_2(T, K, I(T, K))^2}{2}\right\},
$$

as $K \to \infty$, where F denotes the function on the left-hand side of (9.11). It follows from (9.10) and (9.12) that (9.11) holds. Now it is clear that (9.5) implies the equivalence in (9.9). Note that the condition on the right-hand side of (9.9) also holds for $T = 0$. This follows from the definition of the function $\widetilde{C}$.

Next, we turn our attention to condition 2 in Theorem 9.6. It is not hard to see that this condition is equivalent to the following:

$$\frac{1}{2\sqrt{T}}+\sqrt{T}\frac{\partial I}{\partial T}\geq 0, \quad T>0. \tag{9.13}$$

On the other hand,

$$\frac{\partial \widetilde{C}}{\partial T}=\frac{\partial C_{\mathrm{BS}}}{\partial T}(T,K,I)+\frac{\partial C_{\mathrm{BS}}}{\partial \sigma}(T,K,I)\frac{\partial I}{\partial T},$$

and using the formulas for the Greeks in Sect. 8.4, we obtain

$$\frac{\partial \widetilde{C}}{\partial T}=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{d_1(T,K,I(T,K))^2}{2}\right\}\left[\frac{1}{2\sqrt{T}}+\sqrt{T}\frac{\partial I}{\partial T}\right].$$

Now (9.13) implies that condition 2 in Theorem 9.6 is equivalent to the following condition. For all $K>0$, the function $T\mapsto \widetilde{C}(T,K)$ is non-decreasing on $(0,\infty)$. For $K=0$, the same conclusion follows from the definition of the function $\widetilde{C}$. In addition, the function $T\mapsto \widetilde{C}(T,K)$ is also non-decreasing on $[0,\infty)$. Indeed, for any volatility parameter σ in the Black–Scholes model, we have $(x_0-K)^+\leq C_{\mathrm{BS}}(T,K,\sigma)$, $T\geq 0$, $K\geq 0$. Therefore, $\widetilde{C}(0,K)\leq \widetilde{C}(T,K)$.

Let us denote by μ_T the second distributional derivative of the function $K\mapsto \widetilde{C}(T,K)$, and suppose that the conditions in the formulation of Theorem 9.6 hold for the function $\widetilde{C}$. Recall that $\varphi_+(0)=x_0$, $\varphi'_+(0)=-1$, and $\lim_{K\to\infty}\varphi(K)=0$. The function φ' is non-decreasing (see (9.4)) and integrable on $[0,\infty)$. Therefore, φ' is non-positive. Our next goal is to prove that

$$\lim_{K\to\infty} K\left|\varphi'(K)\right|=0. \tag{9.14}$$

Using (9.4), we see that $K|\varphi'(2K)|\leq\varphi(K)-\varphi(2K)$, and it is clear that the previous estimate implies (9.14). Next, taking into account (9.14), we obtain

$$\mu_T\big([0,\infty)\big)=\lim_{K\to\infty}\varphi'(K)-\varphi'_+(0)=1.$$

Moreover, the integration by parts formula for Stieltjes integrals implies the following equality:

$$\int_{[0,\infty)} x\,d\mu_T(x)=\varphi_+(0)-\lim_{K\to\infty}\varphi(K)+\lim_{K\to\infty}K\varphi'(K)=x_0.$$

It follows that condition 2 in Theorem 8.3 is valid for the function $\widetilde{C}$, provided that the conditions in the formulation of Theorem 9.6 hold. Finally, it is not hard to see, taking into account what was said above and applying Theorem 8.3, that Theorem 9.6 holds. □

9.3 Asymptotic Behavior of Implied Volatility Near Infinity

In this section, we find sharp asymptotic formulas for the implied volatility $K \mapsto I(K)$ associated with a general call pricing function C. It is assumed that the maturity T is fixed and the implied volatility is considered as a function of the strike price. We also assume that $C \in PF_\infty$. This guarantees the existence of the implied volatility for large values of the strike price.

The next theorem provides an asymptotic formula for the implied volatility associated with a general call pricing function.

Theorem 9.7 *Let $C \in PF_\infty$. Then*

$$I(K) = \frac{1}{\sqrt{T}}\sqrt{2\log K + 2\log\frac{1}{C(K)} - \log\log\frac{1}{C(K)}} - \frac{1}{\sqrt{T}}\sqrt{2\log\frac{1}{C(K)} - \log\log\frac{1}{C(K)}} + O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}}\right) \tag{9.15}$$

as $K \to \infty$.

Theorem 9.7 and the mean value theorem imply the following statement:

Corollary 9.8 *For any call pricing function $C \in PF_\infty$,*

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}}\left[\sqrt{\log K + \log\frac{1}{C(K)}} - \sqrt{\log\frac{1}{C(K)}}\right] + O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}}\log\log\frac{1}{C(K)}\right) \tag{9.16}$$

as $K \to \infty$.

Proof of Theorem 9.7 The next lemma will be needed in the proof of Theorem 9.7.

Lemma 9.9 *Let C be a call pricing function, and fix a positive continuous increasing function ψ, satisfying $\psi(K) \to \infty$ as $K \to \infty$. Suppose ϕ is a positive function such that $\phi(K) \to \infty$ as $K \to \infty$ and*

$$C(K) \approx \frac{\psi(K)}{\phi(K)}\exp\left\{-\frac{\phi(K)^2}{2}\right\}. \tag{9.17}$$

Then the following asymptotic formula holds:

$$I(K)=\frac{1}{\sqrt{T}}\left(\sqrt{2\log\frac{K}{x_0e^{rT}}+\phi(K)^2}-\phi(K)\right)+O\left(\frac{\psi(K)}{\phi(K)}\right) \tag{9.18}$$

as $K\to\infty$.

Remark 9.10 It is easy to see that if (9.17) holds, then $C\in PF_\infty$.

Proof of Lemma 9.9 Let us compare the implied volatility I with a function $\widetilde{I}$ such that

$$0<\widetilde{I}(K)<I(K),\quad K>K_0. \tag{9.19}$$

Our goal is to prove that

$$I(K)=\widetilde{I}(K)+O\left(C(K)\exp\left\{\frac{1}{2}d_1\big(K,\widetilde{I}(K)\big)^2\right\}\right) \tag{9.20}$$

as $K\to\infty$, where $d_1(K,\sigma)$ is defined in (8.19).

It is not hard to see that the function ρ given by

$$\rho(K)=\frac{1}{\sqrt{T}}\sqrt{2\log\frac{K}{x_0e^{rT}}}$$

satisfies the equalities

$$d_1\big(K,\rho(K)\big)=0 \tag{9.21}$$

and

$$d_2\big(K,\rho(K)\big)=\sqrt{T}\rho(K). \tag{9.22}$$

Plugging (9.21) and (9.22) into the Black–Scholes formula (formula (8.22)), we obtain

$$C_{\mathrm{BS}}\big(K,\rho(K)\big)=\frac{x_0}{2}-Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{\sqrt{T}\rho(K)}^{\infty}\exp\left\{-\frac{y^2}{2}\right\}dy\to\frac{x_0}{2} \tag{9.23}$$

as $K\to\infty$. Next, taking into account (9.23) and the fact that

$$C_{\mathrm{BS}}\big(K,I(K)\big)=C(K)\to 0$$

as $K\to\infty$, we see that $C_{\mathrm{BS}}(K,I(K))<C_{\mathrm{BS}}(K,\rho(K))$ for all $K>K_0$. Therefore,

$$I(K)<\rho(K),\quad K>K_0. \tag{9.24}$$

Here we use the fact that for every fixed $K>0$ and $T>0$ the vega is a strictly increasing function of σ.

It is easy to see that for sufficiently large values of K, the function

$$\sigma \mapsto d_1(K,\sigma) \tag{9.25}$$

increases. It follows from (9.21) and (9.24) that

$$d_1\big(K, I(K)\big) < 0, \quad K > K_1. \tag{9.26}$$

Moreover, using the explicit expression for the vega (see Sect. 8.4) and the mean value theorem, we get

$$\begin{aligned} &C_{\mathrm{BS}}\big(K, I(K)\big) - C_{\mathrm{BS}}\big(K, \widetilde{I}(K)\big) \\ &\quad = \frac{x_0\sqrt{T}}{\sqrt{2\pi}}\big(I(K) - \widetilde{I}(K)\big)\exp\left\{-\frac{d_1^2(K,\lambda)}{2}\right\}, \quad K > K_1, \end{aligned} \tag{9.27}$$

where $\widetilde{I}(K) < \lambda < I(K)$. Since the function in (9.25) increases and (9.26) holds,

$$d_1\big(K, \tilde{I}(K)\big) < d_1(K,\lambda) < d_1\big(K, I(K)\big) < 0, \quad K > K_1. \tag{9.28}$$

Now, using (9.27) and (9.28), we establish the validity of formula (9.20).

Let us continue the proof of Lemma 9.9. Suppose $\widetilde{I}$ is a function satisfying the equality

$$d_1\big(K, \tilde{I}(K)\big) = -\phi(K), \quad K > K_0. \tag{9.29}$$

Such a function exists, since for large values of K the function $\sigma \mapsto d_1(K,\sigma)$ increases from $-\infty$ to ∞. It follows from (9.29) and from the definition of d_1 that

$$\widetilde{I}(K) = \frac{1}{\sqrt{T}}\left(\sqrt{2\log\frac{K}{x_0 e^{rT}} + \phi(K)^2} - \phi(K)\right). \tag{9.30}$$

Our next goal is to use formula (9.20) with $\widetilde{I}$ defined in (9.30). However, we have to first prove inequality (9.19). Using (8.22), (8.25), and (9.29), we see that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} &C_{\mathrm{BS}}\big(K, I(K)\big) - C_{\mathrm{BS}}\big(K, \tilde{I}(K)\big) \\ &\quad = C(K) - C_{\mathrm{BS}}\big(K, \tilde{I}(K)\big) \\ &\quad \geq c_1\frac{\psi(K)}{\phi(K)}\exp\left\{-\frac{\phi(K)^2}{2}\right\} - c_2\frac{1}{d_1(K,\widetilde{I}(K))}\exp\left\{-\frac{d_1(K,\widetilde{I}(K))^2}{2}\right\} \\ &\quad = c_1\frac{\psi(K)}{\phi(K)}\exp\left\{-\frac{\phi(K)^2}{2}\right\} - c_2\frac{1}{\phi(K)}\exp\left\{-\frac{\phi(K)^2}{2}\right\}, \quad K > K_2. \end{aligned} \tag{9.31}$$

Since $\psi(K) \to \infty$ as $K \to \infty$ and (9.31) holds, we get

$$C_{\mathrm{BS}}\big(K, I(K)\big) > C_{\mathrm{BS}}\big(K, \tilde{I}(K)\big)$$

for sufficiently large values of K. Using the fact that the vega is an increasing function of σ, we obtain inequality (9.19). Now it is clear that (9.18) follows from (9.17), (9.20), and (9.29).

The proof of Lemma 9.9 is thus completed. □

Let us return to the proof of Theorem 9.7. Let ψ be a positive increasing function such that $\psi(K) \to \infty$ as $K \to \infty$. We also assume that the function $\psi(K)$ tends to infinity slower than the function $K \mapsto \log\log \frac{1}{C(K)}$. Put

$$\phi(K) = \left[2\log \frac{1}{C(K)} - \log\log \frac{1}{C(K)} + 2\log\psi(K)\right]^{\frac{1}{2}}.$$

Then we have

$$\phi(K) \approx \sqrt{2\log \frac{1}{C(K)}}$$

as $K \to \infty$. It follows that

$$\psi(K)\exp\left\{-\frac{\phi(K)^2}{2}\right\}\phi(K)^{-1} \approx C(K)$$

as $K \to \infty$. Using formula (9.18), we obtain

$$\begin{aligned} I(K) &= \frac{1}{\sqrt{T}}\left(\sqrt{2\log \frac{K}{x_0 e^{rT}} + \phi(K)^2} - \phi(K)\right) \\ &\quad + O\left(\left(\log \frac{1}{C(K)}\right)^{-\frac{1}{2}}\psi(K)\right) \end{aligned} \tag{9.32}$$

as $K \to \infty$. Now, it is not hard to see that (9.15) can be derived from (9.32), the mean value theorem, and Lemma 3.1.

This completes the proof of Theorem 9.7. □

9.4 Corollaries

Our objective in this section is to replace the function C in formula (9.15) by another function $\widetilde{C}$.

Corollary 9.11 *Let $C \in PF_\infty$, and suppose $\widetilde{C}$ is a positive function such that $\widetilde{C}(K) \approx C(K)$ as $K \to \infty$. Then*

$$I(K) = \frac{1}{\sqrt{T}}\sqrt{2\log K + 2\log \frac{1}{\widetilde{C}(K)} - \log\log \frac{1}{\widetilde{C}(K)}}$$

$$
\begin{aligned}
&- \frac{1}{\sqrt{T}} \sqrt{2 \log \frac{1}{\widetilde{C}(K)} - \log \log \frac{1}{\widetilde{C}(K)}} \\
&+ O\left(\left(\log \frac{1}{\widetilde{C}(K)} \right)^{-\frac{1}{2}} \right) \qquad (9.33)
\end{aligned}
$$

as $K \to \infty$. Therefore,

$$
\begin{aligned}
I(K) = {} & \frac{\sqrt{2}}{\sqrt{T}} \left[\sqrt{\log K + \log \frac{1}{\widetilde{C}(K)}} - \sqrt{\log \frac{1}{\widetilde{C}(K)}} \right] \\
&+ O\left(\left(\log \frac{1}{\widetilde{C}(K)} \right)^{-\frac{1}{2}} \log \log \frac{1}{\widetilde{C}(K)} \right) \qquad (9.34)
\end{aligned}
$$

as $K \to \infty$.

Formula (9.33) can be established exactly as (9.15). Formula (9.34) follows from (9.33) and the mean value theorem.

We can also replace a call pricing function C in (9.15) by a function $\widetilde{C}$ under more general conditions. However, this may lead to a weaker error estimate. For instance, put

$$
\tau(K) = \left| \log \frac{1}{C(K)} - \log \frac{1}{\widetilde{C}(K)} \right|. \qquad (9.35)
$$

Then the following theorem holds:

Theorem 9.12 *Let $C \in PF_\infty$, and suppose $\widetilde{C}$ is a positive function satisfying the following condition. There exist $K_1 > 0$ and c with $0 < c < 1$ such that*

$$
\tau(K) < c \log \frac{1}{\widetilde{C}(K)} \qquad (9.36)
$$

for all $K > K_1$, where τ is defined by (9.35). Then

$$
\begin{aligned}
I(K) = {} & \frac{1}{\sqrt{T}} \sqrt{2 \log K + 2 \log \frac{1}{\widetilde{C}(K)} - \log \log \frac{1}{\widetilde{C}(K)}} \\
&- \frac{1}{\sqrt{T}} \sqrt{2 \log \frac{1}{\widetilde{C}(K)} - \log \log \frac{1}{\widetilde{C}(K)}} \\
&+ O\left(\left(\log \frac{1}{\widetilde{C}(K)} \right)^{-\frac{1}{2}} \left[1 + \tau(K) \right] \right) \qquad (9.37)
\end{aligned}
$$

as $K \to \infty$.

Proof It is not hard to check that (9.36) implies the formula

$$\log \frac{1}{\widetilde{C}(K)} \approx \log \frac{1}{C(K)}$$

as $K \to \infty$. Now using (9.15), (9.35), and the mean value theorem, we obtain (9.37). □

The next statement follows from Theorem 9.12 and the mean value theorem.

Corollary 9.13 *Let $C \in PF_\infty$, and suppose $\widetilde{C}$ is a positive function satisfying the following condition. There exist $\nu > 0$ and $K_0 > 0$ such that*

$$\left|\log \frac{1}{\widetilde{C}(K)} - \log \frac{1}{C(K)}\right| \le \nu \log\log \frac{1}{\widetilde{C}(K)} \tag{9.38}$$

for all $K > K_0$. Then

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}}\left[\sqrt{\log K + \log \frac{1}{\widetilde{C}(K)}} - \sqrt{\log \frac{1}{\widetilde{C}(K)}}\right] + O\left(\left(\log \frac{1}{\widetilde{C}(K)}\right)^{-\frac{1}{2}} \log\log \frac{1}{\widetilde{C}(K)}\right)$$

as $K \to \infty$.

Remark 9.14 It is not hard to see that if $C(K) \approx \widetilde{C}(K)$ as $K \to \infty$, or if (9.38) holds, then $\log \frac{1}{C(K)} \sim \log \frac{1}{\widetilde{C}(K)}$ as $K \to \infty$.

Corollary 9.15 *Let $C \in PF_\infty$, and suppose $\widetilde{C}$ is a positive function satisfying the condition*

$$\log \frac{1}{C(K)} \sim \log \frac{1}{\widetilde{C}(K)} \tag{9.39}$$

as $K \to \infty$. Then

$$I(K) \sim \frac{\sqrt{2}}{\sqrt{T}}\left[\sqrt{\log K + \log \frac{1}{\widetilde{C}(K)}} - \sqrt{\log \frac{1}{\widetilde{C}(K)}}\right] \tag{9.40}$$

as $K \to \infty$.

Proof It follows from (9.16) that

$$I(K) \sim \frac{\sqrt{2}}{\sqrt{T}}\left[\sqrt{\log K + \log \frac{1}{\widetilde{C}(K)}} - \sqrt{\log \frac{1}{\widetilde{C}(K)}}\right]\Lambda(K) \tag{9.41}$$

where

$$\Lambda(K) = \frac{\sqrt{\log K + \log \frac{1}{\widetilde{C}(K)}} + \sqrt{\log \frac{1}{\widetilde{C}(K)}}}{\sqrt{\log K + \log \frac{1}{C(K)}} + \sqrt{\log \frac{1}{C(K)}}}.$$

We will next prove that $\Lambda(K) \to 1$ as $K \to \infty$. We have

$$\Lambda(K) = \frac{\sqrt{\Lambda_1(K) + \Lambda_2(K)} + \sqrt{\Lambda_2(K)}}{\sqrt{\Lambda_1(K) + 1} + 1}$$

where

$$\Lambda_1(K) = \frac{\log K}{\log \frac{1}{C(K)}} \quad \text{and} \quad \Lambda_2(K) = \frac{\log \frac{1}{\widetilde{C}(K)}}{\log \frac{1}{C(K)}}.$$

It is not hard to show that for all positive numbers a and b,

$$\left|\sqrt{a+b} - \sqrt{a+1}\right| \le \left|\sqrt{b} - 1\right|.$$

Therefore,

$$\begin{aligned}\left|\Lambda(K) - 1\right| &= \frac{|\sqrt{\Lambda_1(K) + \Lambda_2(K)} - \sqrt{\Lambda_1(K) + 1}| + |\sqrt{\Lambda_2(K)} - 1|}{\sqrt{\Lambda_1(K) + 1} + 1} \\ &\le \left|\sqrt{\Lambda_2(K)} - 1\right| \end{aligned} \tag{9.42}$$

for $K > K_0$. It follows from (9.39) and (9.42) that $\Lambda(K) \to 1$ as $K \to \infty$. Next using (9.41) we see that (9.40) holds.

This completes the proof of Corollary 9.15. □

9.5 Extra Terms: First-Order Asymptotic Formulas for Implied Volatility

Formula (9.15) characterizes the asymptotic behavior of the implied volatility in terms of the call pricing function C, while in formula (9.33), the function C is replaced by a function $\widetilde{C}$, equivalent to C in a certain sense. We call these formulas zero-order asymptotic formulas for the implied volatility. In an important recent paper [GL11], K. Gao and R. Lee obtained a hierarchy of higher-order asymptotic formulas generalizing formula (9.15). Note that formula (9.33) cannot be generalized in a similar way.

In the present section we establish a first-order asymptotic formula, which is different from similar first-order formulas obtained in [GL11]. Higher-order asymptotic formulas from [GL11] are discussed in Sect. 9.6. Our proofs of above-mentioned formulas are refinements of the proof of Theorem 9.7 given in Sect. 9.3, and they differ from the proofs given in [GL11].

For the sake of simplicity, we assume $x_0 = 1$ and $r = 0$.

Theorem 9.16 *Let $C \in PF_\infty$, and suppose there exist a number $\lambda > 0$ and a continuous function Λ satisfying the following conditions*:

$$\Lambda(K) = o\left(\log \frac{1}{C(K)}\right)$$

and

$$\log \frac{1}{C(K)} = \lambda \log K + O\big(\Lambda(K)\big) \tag{9.43}$$

as $K \to \infty$. Then

$$\begin{aligned} I(K) = {} & \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log K + \log \frac{1}{C(K)} - \frac{1}{2} \log\log \frac{1}{C(K)} + \log \frac{\sqrt{\lambda+1} - \sqrt{\lambda}}{2\sqrt{\pi}\sqrt{\lambda+1}}} \\ & - \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log \frac{1}{C(K)} - \frac{1}{2} \log\log \frac{1}{C(K)} + \log \frac{\sqrt{\lambda+1} - \sqrt{\lambda}}{2\sqrt{\pi}\sqrt{\lambda+1}}} \\ & + O\left(\Lambda(K)\left(\log \frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) \\ & + O\left(\log\log \frac{1}{C(K)}\left(\log \frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) \end{aligned} \tag{9.44}$$

as $K \to \infty$.

Proof Suppose ψ is a positive slowly increasing function such that $\psi(K) \to \infty$ and

$$\psi(K) \frac{\Lambda(K) + \log\log \frac{1}{C(K)}}{\log \frac{1}{C(K)}} \to 0$$

as $K \to \infty$. Put

$$\begin{aligned} \varphi^2(K) = {} & 2 \log \frac{1}{C(K)} - \log\log \frac{1}{C(K)} + 2 \log A \\ & + 2 \log\left[1 + \psi(K) \frac{\Lambda(K) + \log\log \frac{1}{C(K)}}{\log \frac{1}{C(K)}}\right]. \end{aligned} \tag{9.45}$$

Here $A > 0$ is a constant that will be chosen later. We have

$$\begin{aligned} & \exp\left\{\frac{\varphi^2(K)}{2}\right\} C(K) \\ & = A\left(\log \frac{1}{C(K)}\right)^{-\frac{1}{2}} + A\psi(K)\left[\Lambda(K) + \log\log \frac{1}{C(K)}\right]\left(\log \frac{1}{C(K)}\right)^{-\frac{3}{2}}. \end{aligned} \tag{9.46}$$

Lemma 9.17 *Let $\widetilde{I}$ be the function, for which (9.29) holds with φ given by (9.45). Set*

$$A = \frac{\sqrt{\lambda+1}-\sqrt{\lambda}}{2\sqrt{\pi}\sqrt{\lambda+1}}. \tag{9.47}$$

Then $\widetilde{I}(K) \le I(K)$.

Proof It follows from (8.22) and (8.25) that

$$\exp\left\{\frac{\varphi^2(K)}{2}\right\} C_{\mathrm{BS}}\big(K, \widetilde{I}(K)\big) = \frac{1}{\sqrt{2\pi}}\frac{1}{\varphi(K)} - \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\log K + \varphi^2(K)}} + O\big(\varphi^{-3}(K)\big) \tag{9.48}$$

as $K \to \infty$. Using (9.43), (9.45), and the formula

$$(1+h)^{-\frac{1}{2}} = 1 + O(h), \quad h \to 0,$$

we obtain

$$\frac{1}{\varphi(K)} = \frac{1}{\sqrt{2}}\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}} + O\left(\log\log\frac{1}{C(K)}\left(\log\frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) \tag{9.49}$$

as $K \to \infty$. Moreover, we have

$$\frac{1}{\sqrt{2\log K + \varphi^2(K)}} = \frac{1}{\sqrt{2}}\left(\frac{\lambda}{\lambda+1}\right)^{\frac{1}{2}}\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}} + O\left(\Lambda(K)\left(\log\frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) + O\left(\log\log\frac{1}{C(K)}\left(\log\frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) \tag{9.50}$$

as $K \to \infty$.

Our next goal is to combine formulas (9.46)–(9.50). It is not hard to see that there exists $K_0 > 0$ such that

$$C_{\mathrm{BS}}\big(K, I(K)\big) - C_{\mathrm{BS}}\big(K, \widetilde{I}(K)\big) = C(K) - C_{\mathrm{BS}}\big(K, \widetilde{I}(K)\big) > 0$$

for all $K > K_0$. Now Lemma 9.17 follows from the fact that the vega is an increasing function of σ. □

Let us return to the proof of Theorem 9.16. Since formula (9.27) holds, we have

$$I(K)-\widetilde{I}(K)=O\left(\exp\left\{\frac{\varphi^2(K)}{2}\right\}\left[C(K)-C_{\mathrm{BS}}\left(K,\widetilde{I}(K)\right)\right]\right)$$

as $K\to\infty$. Now using formulas (9.46)–(9.50) again, we obtain

$$\begin{aligned} I(K)&=\widetilde{I}(K)\\ &\quad+O\left(\psi(K)\left[\Lambda(K)+\log\log\frac{1}{C(K)}\right]\left(\log\frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) \end{aligned} \tag{9.51}$$

as $K\to\infty$. It follows from (9.30) and (9.45) that

$$\begin{aligned} \widetilde{I}(K)&=\sqrt{2\log K+2\log\frac{1}{C(K)}-\log\log\frac{1}{C(K)}+2\log A+V(K)}\\ &\quad-\sqrt{2\log\frac{1}{C(K)}-\log\log\frac{1}{C(K)}+2\log A+V(K)}, \end{aligned} \tag{9.52}$$

where

$$\begin{aligned} V(K)&=2\log\left(1+\psi(K)\frac{\Lambda(K)+\log\log\frac{1}{C(K)}}{\log\frac{1}{C(K)}}\right)\\ &=O\left(\psi(K)\frac{\Lambda(K)+\log\log\frac{1}{C(K)}}{\log\frac{1}{C(K)}}\right) \end{aligned} \tag{9.53}$$

as $K\to\infty$. Applying the mean value theorem to (9.52) and taking into account (9.47), (9.51), and (9.53), we obtain (9.44) with an extra factor $\psi(K)$ in the error term. Finally, using Lemma 3.1, we get rid of the extra factor.

This completes the proof of Theorem 9.16. □

Formula (9.44) will be used in Sect. 10.5 to study the asymptotic behavior of the implied volatility in the correlated Heston model.

9.6 Extra Terms: Higher-Order Asymptotic Formulas for Implied Volatility

In this section, we discuss higher-order asymptotic formulas for the implied volatility obtained in [GL11]. We restrict ourselves to second- and third-order formulas, since the higher-order cases can be treated similarly. Note that when the order grows, the formulas become more and more complicated. That is why we decided to use simpler formulas from Sects. 9.3 and 9.4 in the rest of the present book.

Let us begin with a second-order formula (see [GL11], formula (6.2) in Corollary 6.1). Our presentation of this result of Gao and Lee is different from that in [GL11]. The main idea is to replace the constant λ in Theorem 9.16 by the function

$$\lambda(K) = (\log K)^{-1} \log \frac{1}{C(K)}$$

and put $\Lambda(K) = 0$. Then formula (9.47) takes the following form:

$$A(K) = \frac{\sqrt{\log K + \log \frac{1}{C(K)}} - \sqrt{\log \frac{1}{C(K)}}}{2\sqrt{\pi}\sqrt{\log K + \log \frac{1}{C(K)}}}. \tag{9.54}$$

This choice of the function A leads to the cancellation of all the terms in the upper estimate for the function $C(K) - C_{\mathrm{BS}}(K, \widetilde{I}(K))$, except for the higher-order error terms (see the proof of Theorem 9.16). To justify the previous statement, we will need the estimate

$$\begin{aligned} 0 &\le \log \frac{1}{A(K)} \\ &= \log(2\sqrt{\pi}) \\ &\quad + \log \frac{\sqrt{\log K + \log \frac{1}{C(K)}}(\sqrt{\log K + \log \frac{1}{C(K)}} + \sqrt{\log \frac{1}{C(K)}})}{\log K} \\ &= O\left(\log\log \frac{1}{C(K)}\right). \end{aligned} \tag{9.55}$$

Taking into account the previous remarks, we see that the following assertion holds.

Theorem 9.18 *Let $C \in PF_\infty$. Then*

$$\begin{aligned} I(K) &= \frac{\sqrt{2}}{\sqrt{T}}\sqrt{\log K + \log \frac{1}{C(K)} - \frac{1}{2}\log\log \frac{1}{C(K)} + \log A(K)} \\ &\quad - \frac{\sqrt{2}}{\sqrt{T}}\sqrt{\log \frac{1}{C(K)} - \frac{1}{2}\log\log \frac{1}{C(K)} + \log A(K)} \\ &\quad + O\left(\log\log \frac{1}{C(K)}\left(\log \frac{1}{C(K)}\right)^{-\frac{3}{2}}\right) \end{aligned} \tag{9.56}$$

as $K \to \infty$, where the function A is defined by (9.54).

Our next goal is to establish a third-order asymptotic formula for the implied volatility (see formula (9.68) below). The proof of this formula is similar to that of formula (9.56), but is more involved. Put

$$\varphi(K)^2 = 2\log\frac{1}{C(K)} - \log\log\frac{1}{C(K)} + 2\log A(K) + 2\log\left[1 + \frac{B(K)}{\log\frac{1}{C(K)}} + \psi(K)\frac{\log\log^2\frac{1}{C(K)}}{\log^2\frac{1}{C(K)}}\right]. \tag{9.57}$$

In (9.57), ψ is a positive continuous function such that $\psi(K) \to \infty$ and

$$\psi(K)\left(\log\frac{1}{C(K)}\right)^{-1} \to 0 \tag{9.58}$$

as $K \to \infty$. The function B, appearing in (9.57), will be chosen later. This function should satisfy the following condition:

$$B(K) = O\left(\log\log\frac{1}{C(K)}\right) \tag{9.59}$$

as $K \to \infty$. We have

$$\exp\left\{\frac{\varphi(K)^2}{2}\right\}C(K) = \left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}}A(K) \times\left[1 + \frac{B(K)}{\log\frac{1}{C(K)}} + \psi(K)\frac{\log\log^2\frac{1}{C(K)}}{\log^2\frac{1}{C(K)}}\right]. \tag{9.60}$$

On the other hand, using (8.22) and (8.25), we obtain

$$\exp\left\{\frac{\varphi(K)^2}{2}\right\}C_{\mathrm{BS}}\big(K, \widetilde{I}(K)\big) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\varphi(K)} - \frac{1}{(2\log K + \varphi(K)^2)^{\frac{1}{2}}} - \frac{1}{\varphi(K)^3} + \frac{1}{(2\log K + \varphi(K)^2)^{\frac{3}{2}}}\right] + O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{5}{2}}\right) \tag{9.61}$$

as $K \to \infty$. Set

$$h(K) = -\frac{\log\log\frac{1}{C(K)}}{2\log\frac{1}{C(K)}} + \frac{\log A(K)}{\log\frac{1}{C(K)}} + \left(\log\frac{1}{C(K)}\right)^{-1}\log\left[1 + \frac{B(K)}{\log\frac{1}{C(K)}} + \psi(K)\frac{(\log\log\frac{1}{C(K)})^2}{\log^2\frac{1}{C(K)}}\right].$$

Using (9.55), (9.58), and (9.59), we obtain

$$h(K) = O\left(\frac{\log\log\frac{1}{C(K)}}{\log\frac{1}{C(K)}}\right)$$

as $K \to \infty$. Therefore,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}\varphi(K)} &= \frac{1}{2\sqrt{\pi}}\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}}\big(1+h(K)\big)^{-\frac{1}{2}} \\
&= \frac{1}{2\sqrt{\pi}}\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}}\left(1-\frac{1}{2}h(K)+O\big(h(K)^2\big)\right) \\
&= \frac{1}{2\sqrt{\pi}}\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}} + \frac{\log\log\frac{1}{C(K)} - 2\log A(K)}{8\sqrt{\pi}(\log\frac{1}{C(K)})^{\frac{3}{2}}} \\
&\quad + O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{5}{2}}\left(\log\log\frac{1}{C(K)}\right)^2\right)
\end{aligned} \tag{9.62}$$

as $K \to \infty$. Similarly,

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}\sqrt{2\log K + \varphi^2(K)}} \\
&= \frac{1}{2\sqrt{\pi}}\left(\frac{\log\frac{1}{C(K)}}{\log K + \log\frac{1}{C(K)}}\right)^{\frac{1}{2}}\left(\log\frac{1}{C(K)}\right)^{-\frac{1}{2}} + \frac{\log\log\frac{1}{C(K)} - 2\log A(K)}{8\sqrt{\pi}(\log K + \log\frac{1}{C(K)})^{\frac{3}{2}}} \\
&\quad + O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{5}{2}}\left(\log\log\frac{1}{C(K)}\right)^2\right)
\end{aligned} \tag{9.63}$$

as $K \to \infty$. Moreover,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}\varphi(K)^3} &= \frac{1}{4\sqrt{\pi}}\left(\log\frac{1}{C(K)}\right)^{-\frac{3}{2}} \\
&\quad + O\left(\left(\log\frac{1}{C(K)}\right)^{-\frac{5}{2}}\log\log\frac{1}{C(K)}\right)
\end{aligned} \tag{9.64}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}(2\log K + \varphi^2(K))^{\frac{3}{2}}} \\
&= \frac{1}{4\sqrt{\pi}}\left(\frac{\log\frac{1}{C(K)}}{\log K + \log\frac{1}{C(K)}}\right)^{\frac{3}{2}}\left(\log\frac{1}{C(K)}\right)^{-\frac{3}{2}}
\end{aligned}$$

$$+ O\left(\left(\log \frac{1}{C(K)}\right)^{-\frac{5}{2}} \log\log \frac{1}{C(K)}\right) \tag{9.65}$$

as $K \to \infty$.

Our next goal is to combine formulas (9.60)–(9.65). Recalling the cancellation properties of the function A, we see that the correct choice of the function B is as follows:

$$B(K) = \frac{1}{8\sqrt{\pi}} \times \frac{(\log\log \frac{1}{C(K)} - 2\log A(K) - 2)[(\log K + \log \frac{1}{C(K)})^{\frac{3}{2}} - (\log \frac{1}{C(K)})^{\frac{3}{2}}]}{A(K)(\log K + \log \frac{1}{C(K)})^{\frac{3}{2}}}. \tag{9.66}$$

Indeed, it is not hard to see that with this choice of B all the terms in the estimate for the difference $C(K) - C_{\mathrm{BS}}(K, \widetilde{I}(K))$, containing the factor $(\log \frac{1}{C(K)})^{-\frac{3}{2}}$, cancel out. It follows that formula (9.66) can be rewritten in the following form:

$$B(K) = \frac{\log\log \frac{1}{C(K)} - 2\log A(K) - 2}{4(\log K + \log \frac{1}{C(K)})} \times \left(\log K + 2\log \frac{1}{C(K)} + \sqrt{\left(\log K + \log \frac{1}{C(K)}\right) \log \frac{1}{C(K)}}\right). \tag{9.67}$$

Here we take into account (9.54).

It remains to prove that the function B satisfies condition (9.59). It is not hard to see that this condition follows from formulas (9.55) and (9.67). Analyzing the proof sketched above, we see that the following assertion holds.

Theorem 9.19 *Let $C \in PF_\infty$. Then*

$$\begin{aligned} I(K) &= \frac{\sqrt{2}}{\sqrt{T}} \times \sqrt{\log K + \log \frac{1}{C(K)} - \frac{1}{2}\log\log \frac{1}{C(K)} + \log A(K) + \log\left[1 + \frac{B(K)}{\log \frac{1}{C(K)}}\right]} \\ &\quad - \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log \frac{1}{C(K)} - \frac{1}{2}\log\log \frac{1}{C(K)} + \log A(K) + \log\left[1 + \frac{B(K)}{\log \frac{1}{C(K)}}\right]} \\ &\quad + O\left(\left(\log\log \frac{1}{C(K)}\right)^2 \left(\log \frac{1}{C(K)}\right)^{-\frac{5}{2}}\right) \end{aligned} \tag{9.68}$$

as $K \to \infty$, *where the function* A *is defined by* (9.54).

Formula (9.68) is a third-order asymptotic formula for the implied volatility in a general model of call prices.

9.7 Symmetries and Asymptotic Behavior of Implied Volatility Near Zero

In this section, we turn our attention to the asymptotic behavior of the implied volatility as $K \to 0$. It is interesting to mention that one can derive asymptotic formulas for the implied volatility at small strikes from similar results at large strikes, by taking into account certain symmetries existing in the world of stochastic asset price models. We will next describe those symmetries and explain what follows from them.

Let C be a general call pricing function, and let X be the corresponding stock price process. This process is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}^*)$, where $\mathbb{P}^*$ is a risk-neutral probability measure. We assume that the interest rate r and the initial condition x_0 are fixed, and denote by μ_T the distribution of the random variable X_T. Put

$$\eta_T(K) = \left(x_0 e^{rT}\right)^2 K^{-1}.$$

We call η_T a symmetry transformation. It is easy to see that the Black–Scholes pricing function C_{BS} satisfies the following condition:

$$C_{\mathrm{BS}}(T, K, \sigma) = x_0 - K e^{-rT} + \frac{K e^{-rT}}{x_0} C_{\mathrm{BS}}\big(T, \eta_T(K), \sigma\big). \tag{9.69}$$

On the other hand, the put–call parity formula implies that

$$C(T, K) = x_0 - K e^{-rT} + \frac{K e^{-rT}}{x_0} G\big(T, \eta_T(K)\big), \tag{9.70}$$

where G is given by

$$G(T, K) = \frac{K}{x_0 e^{rT}} P\big(T, \eta_T(K)\big). \tag{9.71}$$

It follows from (8.4) and (9.71) that

$$G(T, K) = x_0 \int_0^{\eta_T(K)} d\mu_T(x) - \frac{K}{x_0 e^{2rT}} \int_0^{\eta_T(K)} x \, d\mu_T(x). \tag{9.72}$$

Define a family of Borel measures $\{\tilde{\mu}_T\}_{T \geq 0}$ on $(0, \infty)$ as follows. For every Borel subset A of $(0, \infty)$ put

$$\tilde{\mu}_T(A) = \frac{1}{x_0 e^{rT}} \int_{\eta_T(A)} x \, d\mu_T(x). \tag{9.73}$$

It is not hard to see that $\{\tilde{\mu}_T\}_{T\geq 0}$ is a family of probability measures. Moreover, for all $K > 0$ and $T \geq 0$, we have

$$\int_K^\infty d\tilde{\mu}_T(x) = \frac{1}{x_0 e^{rT}} \int_0^{\eta_T(K)} x\, d\mu_T(x) \tag{9.74}$$

and

$$\int_K^\infty x\, d\tilde{\mu}_T(x) = x_0 e^{rT} \int_0^{\eta_T(K)} d\mu_T(x). \tag{9.75}$$

It follows from (9.72), (9.74), and (9.75) that

$$G(T,K) = e^{-rT} \int_K^\infty x\, d\tilde{\mu}_T(x) - e^{-rT} K \int_K^\infty d\tilde{\mu}_T(x). \tag{9.76}$$

Remark 9.20 Suppose for every $T > 0$ the measure μ_T is absolutely continuous with respect to the Lebesgue measure on $(0,\infty)$. Denote the Radon–Nikodym derivative of μ_T with respect to the Lebesgue measure by D_T. Then, for every $T > 0$ the measure $\tilde{\mu}_T$ admits a density $\widetilde{D}_Y$ given by

$$\widetilde{D}_T(x) = \frac{(x_0 e^{rT})^3}{x^3} D_T\left(\frac{(x_0 e^{rT})^2}{x}\right), \quad x > 0.$$

The next theorem has important consequences. For example, it will allow us to establish a link between the asymptotic behavior of the implied volatility at large and small strikes.

Theorem 9.21 *Let C be a call pricing function and let P be the corresponding put pricing function. Then the function G defined by (9.71) is a call pricing function with the same interest rate r and the initial condition x_0 as the pricing function C. Moreover, if $\widetilde{X}$ is the stock price process associated with G, then for every $T > 0$ the measure $\tilde{\mu}_T$ defined by (9.73) is the distribution of the random variable $\widetilde{X}_T$.*

Proof According to Theorem 8.3, it suffices to prove that conditions 1–5 in the formulation of this theorem are valid for the function G. We have already shown that for every $T \geq 0$, $\tilde{\mu}_T$ is a probability measure. In addition, equality (8.8) holds for $\tilde{\mu}_T$, by (9.75). Put

$$V(T,K) = \int_K^\infty x\, d\tilde{\mu}_T(x) - K \int_K^\infty d\tilde{\mu}_T(x).$$

Then $G(T,K) = e^{-rT} V(T,K)$. Moreover, the function $K \mapsto V(T,K)$ is convex on $[0,\infty)$, since its second distributional derivative coincides with the measure $\tilde{\mu}_T$. This establishes conditions 1 and 2 in Theorem 8.3. The equality $G(0,K) = (x_0 - K)^+$ can be obtained using (9.71). Thus condition 4 holds. Next, we see that (9.76) implies

$$G(T,K) \le e^{-rT} \int_K^\infty x\, d\tilde{\mu}_T(x),$$

and hence $\lim_{K\to\infty} G(T,K) = 0$. This establishes condition 5. In order to prove the validity of condition 3 for G, we notice that (9.70) gives the following:

$$G\big(T, e^{rT}K\big) = \frac{K}{x_0} C\left(T, e^{rT}\frac{x_0^2}{K}\right) + x_0 - K. \tag{9.77}$$

Now it is clear that condition 3 for G follows from the same condition for C. Therefore, G is a call pricing function.

This completes the proof of Theorem 9.21. □

Remark 9.22 It is not hard to see that if the call pricing function C in Theorem 9.21 satisfies $C \in PF_\infty$, then $G \in PF_0$. Similarly, if $C \in PF_0$, then $G \in PF_\infty$.

Let C be a call pricing function such that $C \in PF_\infty \cap PF_0$. Then $G \in PF_\infty \cap PF_0$, and hence the implied volatilities I_C and I_G associated with the pricing functions C and G, respectively, exist for all $T > 0$ and $K > 0$. Replacing σ by $I_C(K)$ in (9.69) and taking into account (9.70) and the equality

$$C_{\mathrm{BS}}\big(T, K, I_C(T,K)\big) = C(T,K),$$

we see that

$$C_{\mathrm{BS}}\big(T, \eta_T(K), I_C(T,K)\big) = G\big(T, \eta_T(K)\big).$$

Therefore, the following lemma holds.

Lemma 9.23 *Let $C \in PF_\infty \cap PF_0$, and let G be defined by* (9.76). *Then*

$$I_C(T,K) = I_G\big(T, \eta_T(K)\big) \tag{9.78}$$

for all $T > 0$ and $K > 0$.

Lemma 9.23 shows that the implied volatility associated with C can be obtained from the implied volatility associated with G by applying the symmetry transformation.

9.8 Symmetric Models

The notion of a symmetric model is based on the symmetry properties of stochastic models discussed in the previous section.

Definition 9.24 A stochastic asset price model is called symmetric if, for every $T > 0$ the distributions μ_T and $\tilde{\mu}_T$ coincide.

Lemma 9.25 *The following statements hold*:

1. *Suppose for every* $T > 0$ *the measure* μ_T *admits a density* D_T. *Then the model is symmetric if and only if for all* $T > 0$,
$$D_T(x) = \left(x_0 e^{rT}\right)^3 x^{-3} D_T\left(\left(x_0 e^{rT}\right)^2 x^{-1}\right) \tag{9.79}$$
almost everywhere with respect to the Lebesgue measure on $(0, \infty)$.
2. *Suppose the asset price process* X *is strictly positive and for every* $T > 0$ *the measure* μ_T *admits a density* D_T. *Define the log-price process by* $X^{\log} = \log X$ *and denote by* $D_T^{\log}$ *the distribution density of* $X_T^{\log}$, $T > 0$. *Then the model is symmetric if and only if*
$$D_T^{\log}(x) = x_0 e^{rT} e^{-x} D_T^{\log}\left(-x + 2\log\left(x_0 e^{rT}\right)\right)$$
almost everywhere with respect to the Lebesgue measure on $\mathbb{R}$.
3. *The model is symmetric if and only if for all* $T > 0$ *and* $K > 0$, $G(T, K) = C(T, K)$.
4. *The model is symmetric if and only if for all* $T > 0$ *and* $K > 0$,
$$C(T, K) = \frac{K}{x_0 e^{rT}} C\left(T, \left(x_0 e^{rT}\right)^2 K^{-1}\right) + x_0 - e^{-rT} K.$$
5. *Let* $C \in PF_\infty \cap PF_0$. *Then the model is symmetric if and only if for all* $T > 0$ *and* $K > 0$,
$$I(T, K) = I\left(T, \left(x_0 e^{rT}\right)^2 K^{-1}\right).$$

Proof Part 3 of Lemma 9.25 follows from (8.3), (9.76), and from the fact that the measures μ_T and $\tilde{\mu}_T$ are the second distributional derivatives of the functions $K \mapsto C(T, K)$ and $K \mapsto G(T, K)$, respectfully. Part 4 can be easily derived from (9.77). As for part 5 of Lemma 9.25, it can be established using part 3 and Lemma 9.23. In addition, part 1 follows from Definition 9.24 and Remark 9.20. Finally, the equivalence 1 $\Leftrightarrow$ 2 follows from the standard equalities $D_T^{\log}(x) = e^x D_T(e^x)$ and $D_T(y) = y^{-1} D_T^{\log}(\log y)$.

This completes the proof of Lemma 9.25. □

Special examples of symmetric models are uncorrelated stochastic volatility models in a risk-neutral setting. Let us consider a stochastic model defined by
$$\begin{cases} dX_t = rX_t\,dt + f(Y_t)X_t\,dW_t, \\ dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dZ_t, \end{cases} \tag{9.80}$$
where W and Z are independent Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}^*)$, and suppose that the measure $\mathbb{P}^*$ is risk-neutral. Suppose also that the solvability conditions

discussed in Sect. 2.1 hold. It is clear that for such a model, formula (9.79) follows from formula (3.6). Therefore, part 1 of Lemma 9.25 shows that the model in (9.80) is symmetric.

Remark 9.26 The symmetry condition for the implied volatility in part 4 of Lemma 9.25 becomes especially simple if the strike K is replaced by the log-moneyness k defined by

$$k = \log \frac{K}{x_0 e^{rT}}, \quad K > 0.$$

In terms of the log-moneyness, the symmetry condition can be rewritten as follows: $I(k) = I(-k)$ for all $-\infty < k < \infty$. For uncorrelated stochastic volatility models, the previous equality was first obtained in [RT96].

In [CL09], P. Carr and R. Lee established that under certain restrictions, stochastic volatility models are symmetric if and only if $\rho = 0$. We will next prove this result of Carr and Lee. We restrict ourselves to models with time-homogeneous volatility equation. However, Theorem 9.27 also holds when volatility equations are inhomogeneous (see [CL09]).

Let us consider the stochastic model given by

$$\begin{cases} dX_t = rX_t\,dt + \sqrt{Y_t}X_t\,dW_t, \\ dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dZ_t. \end{cases} \tag{9.81}$$

It is assumed in (9.81) that $Z = \sqrt{1-\rho^2}\widetilde{Z} + \rho W$, where $\widetilde{Z}$ is a standard Brownian motion independent of W, and the correlation coefficient ρ is such that $-1 \le \rho \le 1$. It is also assumed that the functions b and σ in (9.81) satisfy the linear growth condition and the Lipschitz condition, the function σ is positive, and for every ρ and every positive initial condition y_0 the solution Y to the second equation in (9.81) is a positive process.

Theorem 9.27 *Suppose the model in* (9.81) *satisfies the conditions formulated above. In addition, suppose the discounted price process is a martingale. Then the model is symmetric if and only if* $\rho = 0$.

Remark 9.28 It is worth mentioning that the conditions in Theorem 9.27 are rather restrictive. For example, this theorem is not applicable to the Stein–Stein model, or the Heston model. Indeed, in the Stein–Stein model the volatility process is not positive, while in the Heston model the function σ does not satisfy the Lipschitz condition. On the other hand, Theorem 9.27 can be used to prove that a negatively correlated Hull–White model cannot be symmetric. Indeed, in such a model the volatility process is a geometric Brownian motion, and hence it is a positive process. Moreover, the stock price process is a martingale (use Theorem 2.33). Note also that if a geometric Brownian motion Y is the solution to the equation

$$dY_t = \nu Y_t\,dt + \xi Y_t\,dZ_t$$

with the initial condition $y_0 > 0$, then the process $\widetilde{Y} = \sqrt{Y}$ is also a geometric Brownian motion satisfying the equation

$$d\widetilde{Y}_t = \left(\frac{\nu}{2} - \frac{\xi^2}{8}\right)\widetilde{Y}_t\,dt + \frac{\xi}{2}\widetilde{Y}_t\,dZ_t$$

with the initial condition $\sqrt{y_0}$. Summarizing what was said above, we see that Theorem 9.27 can be applied to the negatively correlated Hull–White model. If the Hull–White model is positively correlated, then Theorem 2.33 implies that the stock price process is not a martingale. Therefore, Theorem 9.27 cannot be applied to such a model. It would be interesting to extend Theorem 9.27 to a larger class of stochastic volatility models.

Proof It has already been established that for $\rho = 0$, the model is symmetric. We will next prove the converse statement. With no loss of generality, we can assume $r = 0$. Fix $\rho > 0$, and suppose the symmetry condition holds for the model given by

$$\begin{cases} dX_t = \sqrt{Y_t}X_t\,dW_t, \\ dY_t = b(Y_t)\,dt + \sqrt{1-\rho^2}\sigma(Y_t)\,d\widetilde{Z}_t + \rho\sigma(Y_t)\,dW_t. \end{cases}$$

Using the Itô formula, we can rewrite the model above in terms of the log-price process defined by $X^{\log} = \log X$ and $X_0^{\log} = \log x_0$. This gives

$$\begin{cases} dX_t^{\log} = -\dfrac{1}{2}Y_t\,dt + \sqrt{Y_t}\,dW_t, \\ dY_t = b(Y_t)\,dt + \sqrt{1-\rho^2}\sigma(Y_t)\,d\widetilde{Z}_t + \rho\sigma(Y_t)\,dW_t. \end{cases} \tag{9.82}$$

Let us fix $T > 0$. Since the process X is a martingale, the measure $\widetilde{\mathbb{P}}$ determined from $d\widetilde{\mathbb{P}} = x_0^{-1}X_T\,d\mathbb{P}$ is a probability measure. Define a new process by

$$\widehat{W}_t = W_t - \int_0^t \sqrt{Y_s}\,ds, \quad 0 \le t \le T.$$

It follows from Girsanov's theorem that the process $(\widehat{W}_t, \widetilde{Z}_t)$, $t \in [0, T]$, is a two-dimensional standard Brownian motion under the measure $\widetilde{\mathbb{P}}$. Therefore, the same is true for the process $(\widetilde{W}_t, \widetilde{Z}_t)$, $0 \le t \le T$, where $\widetilde{W}_t = -\widehat{W}_t$ for all $t \in [0, T]$. It is easy to see that under the measure $\widetilde{\mathbb{P}}$, the system in (9.82) can be rewritten as follows:

$$\begin{cases} d\big(-X_t^{\log}\big) = -\dfrac{1}{2}Y_t\,dt + \sqrt{Y_t}\,d\widetilde{W}_t, \\ dY_t = \Phi(Y_t)\,dt + \sqrt{1-\rho^2}\sigma(Y_t)\,d\widetilde{Z}_t + \rho\sigma(Y_t)\,d\widetilde{W}_t \end{cases} \tag{9.83}$$

where

$$\Phi(u) = b(u) + \rho\sigma(u)\sqrt{u}.$$

Recall that, by our assumption, the model described by (9.82) is symmetric. Using part 2 of Lemma 9.25, we obtain

$$\begin{aligned}\mathbb{E}\big[X_T^{\log}\big] &= \int_{-\infty}^{\infty} x D_T^{\log}(x)\,dx \\ &= -\frac{1}{x_0}\int_{-\infty}^{\infty} ue^u D_T^{\log}(u)\,du + \frac{2\log x_0}{x_0}\int_{-\infty}^{\infty} e^u D_T^{\log}(u)\,du \\ &= -\frac{1}{x_0}\mathbb{E}\big[X_T X_T^{\log}\big] + \frac{2\log x_0}{x_0}\mathbb{E}[X_T].\end{aligned}$$

Next, using the fact that the process X is a martingale, we see that

$$\mathbb{E}\big[X_T^{\log}\big] = -\widetilde{\mathbb{E}}\big[X_T^{\log}\big] + 2\log x_0. \tag{9.84}$$

The next step in the proof is to take the expectation $\mathbb{E}$ in the first stochastic differential equation in (9.82), written in the integral form. This gives

$$\mathbb{E}\big[X_T^{\log}\big] = -\frac{1}{2}\int_0^T \mathbb{E}[Y_t]\,dt + \log x_0. \tag{9.85}$$

Similarly, applying $\widetilde{\mathbb{E}}$ to the first equation in (9.83), we obtain

$$\widetilde{\mathbb{E}}\big[X_T^{\log}\big] = \frac{1}{2}\int_0^T \widetilde{\mathbb{E}}[Y_t]\,dt + \log x_0. \tag{9.86}$$

It follows from (9.84), (9.85), and (9.86) that

$$\int_0^T \mathbb{E}[Y_t]\,dt = \int_0^T \widetilde{\mathbb{E}}[Y_t]\,dt. \tag{9.87}$$

We will next use a coupling argument. Consider the following processes: $X^{\log}$, Y, W, $\widetilde{Z}$ under the measure $\mathbb{P}$ and $-X^{\log}$, Y, $\widetilde{W}$, $\widetilde{Z}$ under the measure $\widetilde{P}$. Applying the lemma formulated on p. 24 of [IW77], we see that there exist a filtered measure space $(\widetilde{\Omega}, \mathcal{F}, \mathcal{F}_t, \widehat{\mathbb{P}})$ and adapted stochastic processes $X^{(1)}$, Y^1, $X^{(2)}$, $Y^{(2)}$, $W^{(1)}$, and $Z^{(1)}$ on $\widetilde{\Omega}$ such that the following conditions hold:

- The processes $(X^{\log}, Y, W, \widetilde{Z})$ and $(X^{(1)}, Y^1, W^{(1)}, Z^{(1)})$ have the same law under the measures $\mathbb{P}$ and $\widehat{\mathbb{P}}$, respectively.
- The processes $(-X^{\log}, Y, \widetilde{W}, \widetilde{Z})$ and $(X^{(2)}, Y^2, W^{(1)}, Z^{(1)})$ have the same law under the measures $\widetilde{\mathbb{P}}$ and $\widehat{\mathbb{P}}$, respectively.
- The process $(W^{(1)}, Z^{(1)})$ is a two-dimensional $\mathcal{F}_t$-Brownian motion under the measure $\widehat{\mathbb{P}}$.

It follows (9.82), (9.83), and the previous statements that under the measure $\widehat{P}$,

$$\begin{cases} dX_t^{(1)} = -\frac{1}{2}Y_t^{(1)}\,dt + \sqrt{Y_t^{(1)}}\,dW_t^{(1)}, \\ dY_t^{(1)} = b\big(Y_t^{(1)}\big)\,dt + \sqrt{1-\rho^2}\sigma\big(Y_t^{(1)}\big)\,dZ_t^{(1)} + \rho\sigma\big(Y_t^{(1)}\big)\,dW_t^{(1)} \end{cases} \tag{9.88}$$

and

$$\begin{cases} dX_t^{(2)} = -\frac{1}{2} Y_t^{(2)}\, dt + \sqrt{Y_t^{(2)}}\, dW_t^{(1)}, \\ dY_t^{(2)} = \Phi\big(Y_t^{(2)}\big)\, dt + \sqrt{1-\rho^2}\sigma\big(Y_t^{(2)}\big)\, dZ_t^{(1)} + \rho\sigma(Y_t)\, dW_t^1. \end{cases} \tag{9.89}$$

Moreover, (9.87) implies that

$$\int_0^T \widehat{\mathbb{E}}\big[Y_t^{(1)}\big]\, dt = \int_0^T \widehat{\mathbb{E}}\big[Y_t^{(2)}\big]\, dt. \tag{9.90}$$

Now we are ready to finish the proof. Applying the strong comparison theorem for stochastic differential equations (Theorem 54 in [Pro04]) to (9.88) and (9.89), we see that

$$Y_t^{(2)} > Y_t^{(1)} \quad \text{for all } 0 < t < T. \tag{9.91}$$

Here we take into account that $b(u) < \Phi(u)$ and the initial condition (x_0, y_0) is the same for the processes $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$. However, (9.91) contradicts (9.90). It follows that if $\rho > 0$, then the model cannot be symmetric. The case where $\rho < 0$ is similar.

This completes the proof of Theorem 9.27. □

9.9 Asymptotic Behavior of Implied Volatility for Small Strikes

Lemma 9.23 and the results obtained in Sect. 9.3 imply sharp asymptotic formulas for the implied volatility as $K \to 0$.

Theorem 9.29 *Let $C \in PF_0$, and let P be the corresponding put pricing function. Suppose*

$$P(K) \approx \widetilde{P}(K) \quad \text{as } K \to 0, \tag{9.92}$$

where $\widetilde{P}$ is a positive function. Then the following asymptotic formula holds:

$$\begin{aligned} I(K) = {} & \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log \frac{1}{\widetilde{P}(K)} - \frac{1}{2} \log\log \frac{K}{\widetilde{P}(K)}} \\ & - \frac{\sqrt{2}}{\sqrt{T}} \sqrt{\log \frac{K}{\widetilde{P}(K)} - \frac{1}{2} \log\log \frac{K}{\widetilde{P}(K)}} \\ & + O\left(\left(\log \frac{K}{\widetilde{P}(K)}\right)^{-\frac{1}{2}}\right) \end{aligned} \tag{9.93}$$

as $K \to 0$.

Corollary 9.30 *The following asymptotic formula holds*:

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}}\left[\sqrt{\log\frac{1}{\widetilde{P}(K)}} - \sqrt{\log\frac{K}{\widetilde{P}(K)}}\right] + O\left(\left(\log\frac{K}{\widetilde{P}(K)}\right)^{-\frac{1}{2}}\log\log\frac{K}{\widetilde{P}(K)}\right)$$

as $K \to 0$.

An important special case of Theorem 9.29 is as follows:

Corollary 9.31 *Let* $C \in PF_0$, *and let* P *be the corresponding put pricing function. Then*

$$I(K) = \frac{\sqrt{2}}{\sqrt{T}}\left[\sqrt{\log\frac{1}{P(K)}} - \sqrt{\log\frac{K}{P(K)}}\right] + O\left(\left(\log\frac{K}{P(K)}\right)^{-\frac{1}{2}}\log\log\frac{K}{P(K)}\right)$$

as $K \to 0$.

Proof of Theorem 9.29 Formulas (9.71) and (9.92) imply that

$$G(K) \approx \widetilde{G}(K) \quad \text{as } K \to \infty$$

where

$$\widetilde{G}(K) = K\widetilde{P}\big(\eta_T(K)\big). \tag{9.94}$$

Next, applying Corollary 9.11 to G and $\widetilde{G}$, we get

$$\frac{\sqrt{T}}{\sqrt{2}}I_G(K) = \sqrt{\log K + \log\frac{1}{\widetilde{G}(K)} - \frac{1}{2}\log\log\frac{1}{\widetilde{G}(K)}} - \sqrt{\log\frac{1}{\widetilde{G}(K)} - \frac{1}{2}\log\log\frac{1}{\widetilde{G}(K)}} + O\left(\left(\log\frac{1}{\widetilde{G}(K)}\right)^{-\frac{1}{2}}\right) \tag{9.95}$$

as $K \to \infty$. It follows from (9.78), (9.94), (9.95), and from the mean value theorem that

$$\frac{\sqrt{T}}{\sqrt{2}}I(K) = \sqrt{\log\frac{(x_0e^{rT})^2}{K} + \log\frac{K}{(x_0e^{rT})^2\widetilde{P}(K)} - \frac{1}{2}\log\log\frac{K}{(x_0e^{rT})^2\widetilde{P}(K)}}$$

$$
\begin{aligned}
&- \sqrt{\log \frac{K}{(x_0 e^{rT})^2 \widetilde{P}(K)} - \frac{1}{2} \log \log \frac{K}{(x_0 e^{rT})^2 \widetilde{P}(K)}} \\
&+ O\left(\left(\log \frac{K}{\widetilde{P}(K)}\right)^{-\frac{1}{2}}\right) \\
&= \sqrt{\log \frac{1}{\widetilde{P}(K)} - \frac{1}{2} \log \log \frac{K}{\widetilde{P}(K)}} - \sqrt{\log \frac{K}{\widetilde{P}(K)} - \frac{1}{2} \log \log \frac{K}{\widetilde{P}(K)}} \\
&+ O\left(\left(\log \frac{K}{\widetilde{P}(K)}\right)^{-\frac{1}{2}}\right)
\end{aligned}
$$

as $K \to 0$.

This completes the proof of Theorem 9.29. □

9.10 Notes and References

- The books [FPS00, Reb04, Haf04, Fen05, Gat06, H-L09], the dissertations [Dur04, Rop09], the surveys [Ski01, CL10], and the papers [SP99, SHK99, Lee01, CdF02, Lee04a, CGLS09, Fri10] are useful sources of information on the implied volatility.
- Section 9.2 is mostly adapted from [Rop10]. However, the conditions in Theorem 9.6 are not exactly the same as in the similar result (Theorem 2.9) in [Rop10]. Moreover, Theorem 9.6 is formulated in terms of the strike price, while the log-moneyness is used in [Rop10].
- The asymptotic formulas for the implied volatility included in Sects. 9.3, 9.4, and 9.9 are taken from [Gul10].
- The material in Sects. 9.7 and 9.8 (symmetries and symmetric models) comes mostly from [Gul10]. We send the interested reader to [CL09, Teh09a, DM10, DMM10] for more information on symmetric models.
- The paper [GL11] of K. Gao and R. Lee is an important recent work on smile asymptotics. In Sects. 9.5 and 9.6 of this chapter, several theorems from [GL11] are presented. These theorems provide higher-order approximations for the implied volatility at extreme strikes. However, we have not touched upon the results in [GL11] characterizing the asymptotic behavior of the implied volatility with respect to the maturity, or in certain combined regimes.

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